risks

MDPI

Article

Risk-Adjusted Kelly Investing Under Non-Homogeneous Reward and Risk

Sagara Dewasurendra [1,*], Pedro Júdice [2] and Qiji Jim Zhu [3]

[1] School of Natural Science, Indiana University Southeast, New Albany, IN 47150, USA
[2] ISCTE Business Research Unit, Avenida das Forças Armadas, 1649-026 Lisbon, Portugal; pedro.judice@iscte-iul.pt
[3] Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA; qiji.zhu@wmich.edu
* Correspondence: sadewasu@iu.edu

Abstract

In financial applications, the Kelly criterion is a well-known strategy for maximizing long-term growth, but is often criticized for its high-risk approach. To mitigate this effect, previous research proposed two risk-adjusted Kelly criteria in a finite investment horizon: the inflection point, which identifies optimal marginal return, and the optimal reward-to-risk ratio. These approaches maintain the core benefits of the Kelly criterion while reducing risk. As the investment size is used as a proxy for risk, the theory is developed under a linear and smooth setting. However, in practical cases such as transaction costs, reward and risk become non-homogeneous and non-smooth, thus requiring the development of a novel non-smooth analysis framework for risk-adjusted Kelly strategies, which we conduct in this paper. After developing the theory, we apply it to several practical settings, and extend the framework to multiple assets. The research concludes with an empirical analysis of the risk and reward of the risk-adjusted Kelly strategies applied to transaction costs, confirming the superiority of the proposed strategies to the classical Kelly criterion.

Keywords: Kelly criterion; non-homogeneous reward and risk; transaction costs

1. Introduction

1.1. Overview of the Problem

Rooted in the works (Cover 1991; Kassouf and Thorp 1967; Kelly 1956; Lintner 1965; Shannon 1948; Thorp 1971), Growth Optimal Portfolio Theory (GOP) gained prominence through its applications in various financial domains, including blackjack money management in (Thorp 1962), stock trading (Thorp 2018), and warrant trading in (Kassouf and Thorp 1967). The theory revolves around maximizing the asymptotic growth function to determine an optimal portfolio that exhibits the fastest growth in terms of expected compounded returns compared to other portfolios. The approach is particularly suitable for long-term investors who seek to maximize long-term wealth gains.

However, the inherent attractiveness of this unique, fastest-growth optimal portfolio, is counterbalanced by its recognized high level of risk in practical applications, particularly considering that estimating the probabilities of wins and losses is less stable in finance than other settings such as blackjack. To address this issue, Vince and Zhu (2015) have developed two alternative strategies based on the Kelly criterion, under a finite investment horizon, and taking risk into account: the inflection point, which determines the size of the portfolio that reaches the optimal marginal return per unit of risk, and the optimal reward-to-risk

Check for updates

Academic Editor: Alejandro Balbás

Received: 1 July 2026
Revised: 4 September 2026
Accepted: 9 September 2026
Published: 15 September 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

size. These alternative risk-adjusted strategies make a better balance between reward and risk. In their framework, the level or risk is assessed via the investment size. The framework was subsequently extended to portfolio analysis involving multiple assets in (Prado et al. 2019) and is applied to provide a realistic estimate of the appropriate level of leverage for the banking sector in a study conducted by (Dewasurendra et al. 2019).

The main goal of this paper is to extend the framework developed in (Prado et al. 2019; Vince and Zhu 2015) to portfolio analysis in the presence of rebalancing costs and other practical settings. It is worth emphasizing that the framework in (Prado et al. 2019; Vince and Zhu 2015) requires not only first-order but also second-order derivative information of the reward–risk function. In the classical GOP setting, this does not pose significant technical difficulties, since the reward–risk function is smooth. However, once rebalancing costs are introduced, the reward–risk function generally loses smoothness, leading to additional analytical challenges. These difficulties go beyond those encountered in related studies such as (Feng et al. 2011) and necessitate the use of tools from nonsmooth analysis (see, e.g., Borwein and Zhu 2005). A more detailed discussion of the historical background and motivation is provided in Section 2.

1.2. Contributions and Organization of the Paper

The contributions of the paper are as follows. First, we develop the theory of risk-adjusted Kelly strategies (the optimal reward-to-risk and the inflection points) under the non-smooth and nonhomogeneous setting, and analyze the properties and the existence of these strategies. Second, we use the extended framework in a number of applications in addition to transaction costs, which we explore in Section 5, such as asymmetric costs as in the case of banks, and decreasing returns to scale. The approach is subsequently extended to multiple assets. Finally, we conduct an empirical analysis of the risk-adjusted Kelly strategies with particular cases that are susceptible to high transaction costs such as physical gold and real estate, with an out-of-sample test that shows that the risk-adjusted Kelly strategies significantly outperform the classical Kelly criterion.

In the subsequent sections of this research paper, in Section 2 we delve into the foundational concepts of growth portfolio optimization and the intricacies of modeling the investment, with a particular focus on addressing single-asset problems. In Sections 3 and 4, we not only expound upon the theoretical framework but also provide a detailed examination of the calculations for three pivotal points: the inflection point, the reward-risk ratio maximizing point, and Kelly's optimal point. Section 5 highlights the applications of the framework to several practical cases. Moving forward to the Section 5,we extend our analysis to encompass the realm of multiple asset problems. In Section 7, and final section, we demonstrate the practical applications of our research by exploring optimal investment sizes in assets with particularly high transaction costs, such as physical gold and real estate, covering both single and multiple-asset cases. Furthermore, the same section offers comprehensive insights into the computational results for these cases, thus presenting a holistic view of our findings.

Sections 1 and 2 introduce and motivate the problem in detail; Sections 3–7 consist of new results that extend the theory developed by Vince and Zhu (2015) to the non-smooth and non-homogeneous setting, allowing the application to several real-world scenarios such as transaction costs and decreasing returns to scale.

2. History and Motivation

2.1. History

The growth optimal portfolio (GOP) is inspired by Kelly's pioneering work (Kelly 1956), which was originally introduced to explain Shannon's information rate in information

theory (Shannon 1948). In the context of investment, Kelly's idea is most clearly illustrated by the following example.

Example 1. *Suppose an investor repeatedly plays a game in which a fixed fraction s of total capital is wagered on the flip of a biased coin. The probability of obtaining a head is p, in which case the investor wins the bet, while with probability $q = 1 - p$ the investor loses the bet. Such an investment is meaningful only when $p > 1/2$. Kelly asserted that the optimal strategy is to choose the betting fraction s that maximizes the log-return function*

$$l(s) = p \ln(1 + s) + q \ln(1 - s),$$

which yields the optimal betting fraction $\kappa := p - q = 2p - 1$.

To interpret this result financially, suppose the investor plays the game n consecutive times, obtaining p_n heads and $q_n = n - p_n$ tails. The resulting total capital is

$$(1 + s)^{p_n}(1 - s)^{q_n}.$$

The average growth factor per play is therefore

$$(1 + s)^{p_n/n}(1 - s)^{q_n/n}.$$

Since $p_n/n \to p$ and $q_n/n \to q$ almost surely as $n \to \infty$, Kelly argued that the optimal betting fraction should maximize $(1 + s)^p(1 - s)^q$, or equivalently the log-return function $l(s)$.

More generally, consider investing in a single risky asset with one-period return represented by a random variable X. In the long run, such an investment is worthwhile if and only if $E[X] > 0$. In this case, the optimal fraction s to invest in the risky asset is obtained by maximizing the expected log return

$$l(s) := E[\ln(1 + sX)].$$

As noted in the introduction, this allocation guarantees the fastest asymptotic growth rate of capital among all admissible strategies, which motivates the name growth optimal portfolio.

Despite its strong theoretical appeal, the GOP is widely regarded as excessively risky in practice. To illustrate this, consider Example 1 with $p = 0.9$. The optimal betting fraction is then $\kappa = 0.8$. However, encountering two consecutive losses—which occurs with probability 1%—results in a loss of approximately 96% of total capital. Such drawdowns are clearly unacceptable for most investors.

What causes this theoretically attractive strategy to be impractical? Vince and Zhu (2015) point out that the classical GOP is optimal only in the limit of an infinite investment horizon. In practice, investors operate over finite horizons. In this setting, the appropriate objective becomes the total return over a finite horizon of Q investment periods, which can be approximated by

$$\mu_Q(s) = \exp\big(Ql(s)\big) - 1. \tag{1}$$

Figure 1 illustrates a typical shape of the function $\mu_Q(s)$.

The peak of the curve corresponds to the GOP or the Kelly portfolio. However, the S-shaped region to the left of this peak suggests two alternative strategies. The first is the point ξ_Q, which maximizes the ratio $\mu_Q(s)/s$, a proxy for the reward–risk trade-off. This point is characterized by the first-order condition

$$\mu_Q'(s)s - \mu_Q(s) = 0. \tag{2}$$

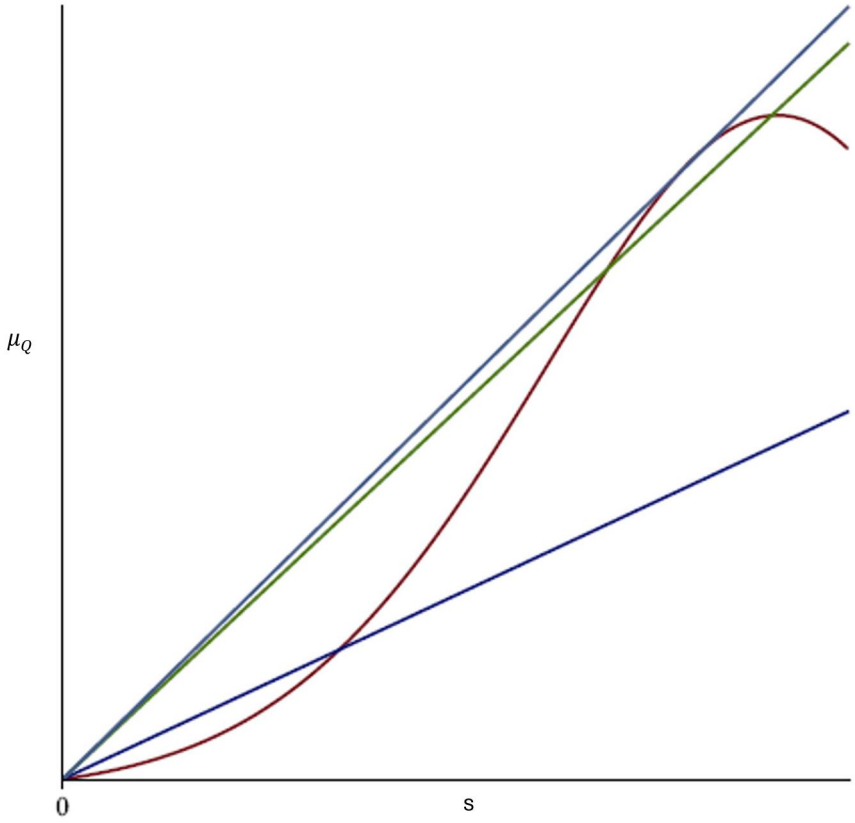


Figure 1. Reward function $\mu_Q(s)$, shows in dark red, for a finite investment horizon Q. The Kelly investment size is attained where $\mu_Q(s)$ reaches a maximum (intersection of the green straight line and $\mu_Q(s)$). The optimal reward-to-risk ratio (intersection of the light blue straight line and $\mu_Q(s)$) corresponds to the crossing of the straight lines shown with the function $\mu_Q(s)$ for which the slope is the highest. The inflection point of $\mu_Q(s)$ (intersection of the dark blue straight line and $\mu_Q(s)$) corresponds to the size s for which the function $\mu_Q(s)$ switches from convex to concave.

In Figure 1, the optimal reward-to-risk ratio corresponds to the crossing of the straight lines shown with the function $\mu_Q(s)$ for which the slope is the highest.

The second is the inflection point ν_Q, defined by the second-order condition

$$\mu''_Q(s) = 0, \tag{3}$$

or equivalently, in terms of the log-return function,

$$Q\big(l'(s)\big)^2 + l''(s) = 0. \tag{4}$$

The inflection point represents the level of investment at which the marginal reward with respect to risk (proxied by the investment fraction) changes from increasing to decreasing. In Figure 1, it corresponds to the size s for which the function $\mu_Q(s)$ switches from convex to concave.

Both ξ_Q and ν_Q are more conservative than the GOP and provide a better balance between reward and risk. These strategies have been shown to be more practical alternatives to the GOP in (Prado et al. 2019; Vince and Zhu 2015).

2.2. Motivation

We now consider the effect of rebalancing costs. Suppose a portfolio consists of cash and a single risky asset, and trading the risky asset incurs a transaction cost of λ per share.

Assume that at the beginning of an investment period, the investor has one unit of capital, with fraction s invested in the risky asset and $1 - s$ held in cash. At the end of the period, the portfolio value becomes

$$(1 - s) + s(1 + X) = 1 + sX,$$

where X denotes the excess return of the risky asset relative to cash.

To rebalance the portfolio and restore the investment fraction s, the investor must hold $s(1 + sX)$ in the risky asset. Since the current position in the risky asset is $s(1 + X)$, the investor must buy or sell $s|1 - s||X|$ shares. The resulting rebalancing cost is therefore $\lambda s|1 - s||X|$. Consequently, the log-return function becomes

$$l(s) := E\big[\ln\big(1 + sX - \lambda s|1 - s||X|\big)\big]. \tag{5}$$

We can see that the presence of transaction costs introduces nonsmoothness into the log-return function, requiring analytical tools capable of handling nonsmooth optimization. Moreover, in the preceding analysis, the investment fraction was used as a proxy for risk. In practice, however, investors are often concerned with risk measures such as value at risk (VaR) and conditional value at risk (CVaR), which are typically nonlinear functions of the investment size or, in multi-asset settings, of the portfolio weights. These considerations motivate a more general framework for analyzing possibly nonsmooth reward–risk manifolds in reward–risk space, which we develop in the next section. As we will see below, the general framework also can be used for other practical applications which we explore later in the paper.

3. Risk-Adjusted Kelly Investment Size with Nonhomogenous Return and Risk

Motivated by the previous section, we lay out the foundations for the general theory for risk-adjusted Kelly criteria (the inflection point and the optimal reward-to-risk ratio) when the return function is a general function of the investment size s given by $g(s, X)$, instead of sX as in the classical case, and when risk is also given by a function $\gamma(s)$ which may be nonhomogeneous as well. In Equation (5), we have the return given by $g(s, X) = sX - \lambda s|1 - s||X|$. As we will show in the application section, the general framework encompasses also other practical cases.

We now define the space of the returns for the financial asset available to the investor and the probabilities for the returns on the asset. For this matter, we consider the space of random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a measurable space, $\mathbb{P}$ is its associated probability measure and $\mathcal{F}$ is a σ-field. We denote by $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the space of L^∞ random variables in that space. The returns on the asset are given by a random variable $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the investor is presented with the choice of the investment percentage s, which is nonnegative and can be greater than 1 (in which case the investment is leveraged). The return for the investor, as we will define below, will be a function of the return X on the asset, the allocation level s, and other effects which will be characterized below.

For the returns to make sense and avoid arbitrage, we assume that $\mathbb{P}(X < 0) > 0$ and $\mathbb{P}(X > 0) > 0$. The assumption of bounded random variables results from the fact that in practice returns are bounded. In particular, in the case of securities investments, an investor will never lose more than the amount he invested. This assumption will also ensure that the log return will be well defined within its domain and that we can use the dominated convergence theorem in the proofs below. We also assume that $\mathbb{P}(X \geq -1) = 1$, i.e., the loss on the assets cannot be more than the amount invested.

We now define the expected log returns, a generalization of the log-returns in the case of the Kelly criterion, but now considering a general return function $g(s, X)$ instead of sX as in the Kelly criterion. This formulation will be fundamental for defining the risk-adjusted Kelly criteria, namely the optimal reward-to-risk ratio and the inflection point.

Definition 1 (Concave log net average return)**.** *Let s be the percentage investment proportion on an investment opportunity. Assume a return function is $g(s, X)$ which is continuous in s and X, piece-wise C^2 in $s \geq 0$, and $g(0, X) = 0$ for X almost everywhere. The log net average returns are given by*

$$l(s) = E[\ln(1 + g(s, X))], \tag{6}$$

where we assume that $l(s)$ is strictly concave and has a maximum at $\kappa > 0$.

We also need to define a risk measure in one dimension, but as we highlighted before, we drop the homogeneity property. We also drop convexity as a property, since the inclusion of market frictions such as transaction costs yield non-convex functions, as we elaborate in Section 5.

Definition 2 (Nonhomogeneous Risk Measure)**.** *A nonhomogeneous risk measure is a function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which quantifies the risk associated with an investment size s, such that $\gamma(0) = 0$, is strictly increasing.*

In order to generalize the results of the inflection point and reward-to-risk point to nonhomogenous settings, we need to define the reward function and the risk-reward curve. The reward function and the risk-reward curve for the Q investment horizon can be defined below.

Definition 3 (Reward Function for a Finite Investment Horizon)**.** *Let Q be the investment horizon. Assume that the investment size is s on an asset with a return random variable X as defined before. Then we define the reward function for a horizon Q as*

$$\mu_Q(s) = e^{Ql(s)} - 1, \tag{7}$$

where $l(s) = E[\ln(1 + g(s, X))]$ as per Definition 1.

We proceed to define the risk-reward curve, which generalizes the risk-reward curve that is shown in Figure 2, and may be nonsmooth.

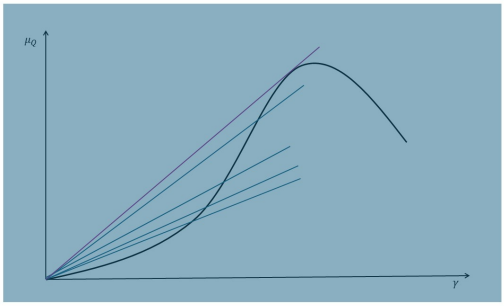


Figure 2. Illustration of the optimal reward to risk point: it corresponds to the point in the risk-reward curve where the line $\mu_Q = k\gamma$, passing at zero, is tangent to the curve (in purple; other colors are reward-to-risk lines but are not optimal).

Definition 4 (Risk-Reward Curve)**.** *Let $\mu_Q(s)$ be the reward function, and $\gamma(s)$ be the risk measure as defined before. Then, the risk-reward curve is defined as*

$$C := \{(\mu_Q(s), \gamma(s)) | s \in [0, +\infty)\}. \tag{8}$$

It is straightforward to show that maximizing $\mu_Q(s)$ in Definition 1 is the same as maximizing $l(s)$. Since the maximizer of $l(s)$ when $g(s, X) = sX$ is the Kelly point, we generalize the definition of the Kelly point to the non-homogeneous setting as follows.

Definition 5 (Kelly point)**.** *The Kelly point κ is defined to be the unique investment size s where curve C attains a maximum on the (μ, γ) plane.*

The uniqueness of the Kelly point follows from the strict concavity of $l(s)$ in Definition 1 and the strictly increasing property of $\gamma(s)$ in Definition 2.

Motivated by Section 2.1, we proceed to define the inflection point in our general setting. The inflection point corresponds to the investment size after which the marginal return per unit of risk starts decreasing, so that it is more rational to pursue other investment opportunities.

Definition 6 (Inflection point)**.** *If it exists, the inflection point ν is the first point where the risk-reward curve (Definition 4) changes from convex to concave, when going from 0 to the supremum of the domain of $l(s)$. This definition only applies to risk-reward curves that are convex at zero. If the risk-reward curve is concave at zero, by definition we state that there is no inflection point.*

Note that in our setting, there are situations in which the concavity or convexity of $\mu_Q(s)$ may change more than once, as we illustrate below in the concrete situation of Example 2, thus the need to specify that the inflection point is the first of such points where there is change from convexity to concavity.

Next, we will define the point or investment size ξ that corresponds to the optimal reward-risk ratio.

Definition 7 (Optimal reward-risk ratio point)**.** *The optimal reward to risk ratio point is the investment size ξ that maximizes the ratio $\mu_Q(s)/\gamma(s)$.*

Finally, we point out that both the inflection point and the optimal reward-to-risk point depend both on $\mu_Q(s)$ and thus on the investment horizon.

4. Existence and Properties of the Optimal Points

We now analyze the properties and the existence of the inflection point and the optimal reward-to-risk ratio point in the general setting.

4.1. Inflection Point

The inflection point may not always exist as illustrate in a particular example later in the paper (Example 3), but the following proposition guarantees its existence, provided we have twice differentiability at 0 and κ, and Q is large enough. Notice that smoothness in the whole curve is not required, as per Definition 1. The proposition is proved in the Appendix A.

Proposition 1. *Let us assume that $l(s)$ and $\gamma(s)$ are twice differentiable at 0 and κ. The inflection point exists if the investment horizon, Q, is sufficiently large.*

We note that if the inflection point occurs where $l(s)$ and $\gamma(s)$ are twice differentiable, then it is given by the point where the second derivative of μ_Q with respect to γ is zero:

$$\begin{aligned}\frac{d^2\mu_Q}{d\gamma^2}(\nu) &= \frac{d}{d\gamma}\frac{\mu_Q'(\nu)}{\gamma'(\nu)} = \frac{d}{ds}\frac{\mu_Q'(\nu)}{\gamma'(\nu)}/\gamma'(\nu) \\ &= \frac{\mu_Q''(\nu)\gamma'(\nu) - \mu_Q'(\nu)\gamma''(\nu)}{(\gamma'(\nu))^3} = 0.\end{aligned} \tag{9}$$

Historical investment data show variations in the returns, influencing the return-risk curve, which is also dependent upon the investment horizon Q. As we have described before, the inflection point is not always guaranteed. In the following theorem, we establish the minimum horizon Q that guarantees the existence of the inflection point, under certain weak conditions. We emphasize that we only assume smoothness conditions at 0 and the Kelly Point κ, allowing nonsmoothness to occur between these two points. Notice that this assumption fits most of the application examples discussed below. The proof of the theorem is in the Appendix A.

Theorem 1. *Let $\mu_Q(s)$ be the reward function as in Definition 3, where $l(s) = E[\ln(1+g(s,X))]$. Suppose that $l'(0) > 0$, and that $l(s)$ and $\gamma(s)$ are twice differentiable at 0 and the Kelly point κ. The inflection point exists if and only if the investment horizon satisfies the condition $Q > Q_{min}$, where*

$$Q_{min} = \frac{l'(0)\gamma''(0) - l''(0)\gamma'(0)}{(l'(0))^2\gamma'(0)}. \tag{10}$$

4.2. Optimal Reward-Risk Ratio

In this section, we prove the existence of the optimal reward to risk ratio, under mild conditions. Let us first look at the graphical illustration in Figure 2, the risk-reward curve, whose shape depends the investment horizon Q. The figure shows the lines, $\mu_Q = k\gamma$, passing through the origin, where k is a constant. The optimal risk-reward ratio point is given where the straight line is tangent to the curve μ_Q in the space (μ_Q, γ). We now show that under reasonable, yet weak assumptions, an optimal reward-to-risk ratio point exists. The smoothness of the curve μ_Q is not required. The proof is in the Appendix A.

Proposition 2 (Existence of maximum of μ_Q/γ)**.** *Let us assume that $l(s)$ and $\gamma(s)$ are differentiable at zero. Then there exists an optimal reward-to-risk ratio point. In other words, the maximum of the ratio μ_Q/γ_S is attained within the interval $[0, \kappa]$.*

The computation of the optimal reward to risk ratio becomes easier in case the risk-reward curve is smooth. In that case, the first derivative of this ratio with respect to γ is given by

$$\frac{d}{d\gamma}\left(\frac{\mu_Q}{\gamma}\right)(s) = \frac{\mu_Q'(s)\gamma(s) - \mu_Q(s)\gamma'(s)}{\gamma^3(s)}.$$

The maximizer of this ratio, ξ, can be derived by solving the equation

$$\mu_Q'(s)\gamma(s) - \mu_Q(s)\gamma'(s) = 0.$$

4.3. Order of the Three Points

If the three optimal points (the inflection point, the maximum of the return-risk ratio, and Kelly point) exist, the following proposition establishes the order of these points. The investment size corresponding to the inflection point is lower than the investment size associated with the optimal reward to risk ratio, which in turn is less than the Kelly point. As a

consequence, in the general setting, the risk-adjusted Kelly investment size decrease the risk compared to the Kelly investment size. Again, the proof can be read in the Appendix A.

Proposition 3. *Let us assume that a maximum reward-to-risk point exists. Denote $\xi = \xi_Q$ the maximum point for μ_Q/γ. If an inflection point ν exists, then $\nu \leq \xi < \kappa$.*

5. Applications

In this section, we show particular applications of the theory to several different cases. The no-transaction-cost case is a natural baseline. We then explore the case involving transaction costs, both symmetric and asymmetric. Finally, we analyze the case of decreasing returns to scale, which often occur in private equity.

5.1. No Transaction Costs

In the no-transaction cost case, we recover the classical Kelly criterion, the inflection and optimal risk-reward point as highlighted in Section 2.1.

Let us examine the risk-reward curve in this case. The reward is given by $\mu_Q(s) = exp(Ql(s)) - 1$ where

$$l(s) = E(\ln(1 + sX)), \tag{11}$$

i.e., we have $g(s, X) = sX$.

We now proceed to compute $\gamma(s)$. Let us compute it for two cases: the standard deviation and conditional value at risk. Assume that the standard deviation of the random variable X is σ_X. If the investor invests a proportion of s, the standard deviation of the returns will be will be $\sigma_X s$. Therefore, the associated risk measure is $\gamma(s) = \sigma_X s$. The standard deviation of the investment size is thus linear in s, A similar reasoning applies to $CVaR$ of the investment size. If $CVaR_X$ is the conditional value at risk of X, which is a constant given X, then the conditional value at risk of sX is given by $s\,CVaR_X$. In this case, the risk measure is given by $\gamma(s) = CVaR_X s$. Both the standard deviation and conditional value at risk are thus linear in s.

As a consequence, the risk reward curve is thus given by the points $(\mu_Q(s), s)$, for $s \geq 0$. Since maximizing $\mu_Q(s)$ is the same as maximizing $l(s)$, the maximum point in the risk reward curve is given the Kelly criterion. Thus, using Definition 5, we recover the classical Kelly criterion. We also recover the optimal reward-to-risk ratio and inflection point as given in Equations (2) and (3).

5.2. Transaction Costs

In Section 2.2 we showed an example of $g(s, X)$ when one considers transaction costs. In that case,

$$g(s, X) = sX - \lambda s|1 - s||X|. \tag{12}$$

We can compute $l(s)$ and $\mu_Q(s)$ by numerical integration. If X is a discrete distribution, these are straightforward to evaluate.

Now let us compute the associated risk measure $\gamma(s)$, considering the cases of value at risk or conditional value-at risk. Let VaR_X and $CVaR_X$ be the value at risk and conditional value at risk for X. We show in the Appendix A that the Value at Risk for the return function $g(s, X)$, as a function of s, is given by

$$\gamma(s) = (s + \lambda s|1 - s|)\ VaR_X.$$

One can also derive the Conditional Value at Risk for Y as a function of s:

$$\gamma(s) = (s + \lambda s|1 - s|)\ CVaR_X. \tag{13}$$

The step by step derivations are performed in the Appendix A. We can thus determine the risk-reward curve and calculate the Kelly point κ (Definition 5), the optimal reward-to-risk point ξ (Definition 7) and the inflection point ν (Definition 6) using straightforward numerical routines.

We now show two examples that we referred to prior to introducing the inflection point (Definition 6) and proving sufficient conditions for its existence (Proposition 1).

In the first example, we show that there may be more than one point where the risk-reward curvature changes, thus the need to specify the inflection point as the first point where this change happens, as in Definition 6.

Example 2. *In the setting defined in this section, namely by Equations (12) and (13), consider a random sample of return from a normal distribution with a mean of 0.02 and standard deviation of 0.15 with* $Q = 500$ *and* $\lambda = 0.001$. *Figure 3 shows the second derivative of the risk-reward curve, two points where the concavity/convexity changes, and a discontinuity at* $s = 1$.

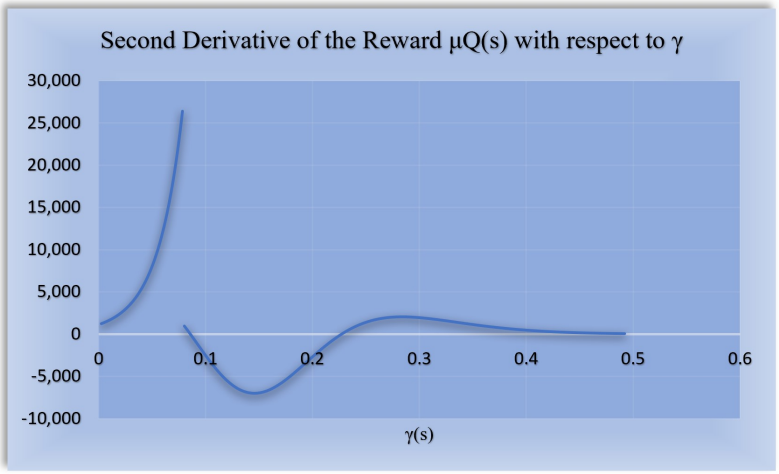


Figure 3. Second derivative of the risk−reward curve $(\gamma(s), \mu_Q(s))$.

In the second example, we show a counterexample for the existence of an inflection point, thus the need to specify the conditions in Proposition 1, namely the existence of a derivative at the Kelly point κ.

Example 3. *We now show an example of a case where there exists no inflection point. Assume again the setting defined by this section, as in Equations (12) and (13). Assume also that the random variable X is drawn from a normal distribution with mean* 0.02 *and standard deviation* 0.15, *with* $\lambda = 0.05$. *Then,* $l(s) = E[\ln(1 + sX - \lambda s|1 - s||X|)]$. *In this case,* $l(s)$ *achieves its maximum at* $s = 1$, *as does* $\mu_Q(s)$, *even though it is not smooth at* $s = 1$. *In the Appendix A, we show that for* Q *large enough there is no inflection point for* $\mu_Q(s) = e^{Ql(s)} - 1$.

5.3. Asymmetric Transaction Costs

In the case of banks, when asset prices fall, the cost of deleveraging due to falling asset prices is much higher than the cost of gaining leverage, as deleveraging usually entails closing business lines, liquidating assets at fire-sale prices, and paying severance packages.

Let us explain this effect in greater detail. As banks have leverage levels which are higher than one, if asset prices fall, then the bank is forced to sell assets. For example, for a bank which is leveraged ten times, if its assets present a negative return of 1%, equity will be depleted by 10%. To maintain the target leverage level of 10, the bank is forced to sell 10% of its assets at fire-sale prices with significant discounts, which are further amplified by systemic shocks and spillover effects (Greenwood et al. 2015). Thus, falling asset prices typically magnify

losses with fire-sale discounts. In contrast, gaining leverage in the short term is not as costly, although the added risk of leveraging may increase the risk and thus the potential losses.

Assume that $\lambda_1 > 0$ is the cost for engaging in new loans, which is lower than the cost of deleveraging $\lambda_2 > 0$ (due to effects such as fire-sales or paying severance packages). In this case, the function $g(s, X)$ will be given by

$$g(s, X) = sX - \lambda_1(s(s-1)X)^+ - \lambda_2(s(1-s)X)^+.$$

The derivation of this formula is in the Appendix A.

From this specification, one can derive the risk-reward curve, similarly to the previous section, and compute the inflection point ν, the optimal reward-to-risk ratio point ξ, and the Kelly Point κ.

5.4. Decreasing Return to Scale

Let us consider the case that marginal returns are decreasing as a function of the investment size. The negative scale effect is present in some contexts, such as private equity, as documented by (Braun et al. 2017). Specifically, let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be C^2, such that the derivative $G'(s)$ is positive and decreasing in s. Suppose also that $E(X) > 0$. We define

$$g(s, X) = G(s)X.$$

One example of such functions is $G(s) = s^\alpha$, where $\alpha \in [0, 1]$. In the Appendix A, we show that this specification of $g(s, X)$ fits Definition 1. We can obtain $l(s)$ and $\mu_Q(s)$ by numerical integration.

For the risk measure $\gamma(s)$, we can consider again conditional value at risk. The computation is similar to the previous sections. If we again consider $CVaR_X$ to be the conditional value at risk of X, the conditional value at risk associated with the return function $g(s, X)$ is equal to

$$G(s)CVaR_X =: \gamma(s). \tag{14}$$

Again, this example γ fits Definition 2.

From these specifications of $\mu_Q(s)$ and $\gamma(s)$, one can determine the inflection point ν, the optimal reward-to-risk ratio point ξ and the Kelly point κ that we defined before.

6. Multiple Assets

We now extend our framework to multiple risky assets. We will detail the theory and also make concrete applications as we go along. As we have highlighted in the previous sections, the framework has other applications in addition to transaction costs.

Consider n risky assets whose return relative to the riskless bond is represented by the random vector $X = (X_1, X_2, ..., X_n)$. Let the portfolio weights on these risky assets be denoted by the vector $\mathbf{s} = (s_1, s_2, ..., s_n)$. In the absence of rebalancing costs, the portfolio return is given by $\mathbf{s} \cdot X$. However, when rebalancing costs or other return structures are taken into account, the net return becomes a nonlinear and often nonsmooth function $g(\mathbf{s}, X)$, as previously discussed. As in the case with one asset, we start with the following assumption.

Assumption 1 (Concave log net average returns)**.** *Assume that $g(\mathbf{s}, X)$ is continuous in $(\mathbf{s}, X)$, piece-wise C^2 in $\mathbf{s}$, $g(\mathbf{0}, X) = 0$ for X almost everywhere, and $\min_X g(\mathbf{s}, X) \to -\infty$ as $|\mathbf{s}| \to \infty$. We assume that the expected log return function*

$$l(\mathbf{s}) := E[\ln(1 + g(\mathbf{s}, X))] \tag{15}$$

is strictly concave in $\mathbf{s}$.

Remark 1. *Property* $\min g(\mathbf{s}, X) \to -\infty$ *implies that* $l(\mathbf{s})$ *attains a unique maximum which we denote* $\boldsymbol{\kappa}$.

We show below that the proportional transaction costs case is a particular case of Assumption 1.

Example 4. *Consider a portfolio that contains n risky assets whose returns relative to the riskless bond are represented by the random vector X. Assume that the portfolio maintains a constant weight* $\mathbf{s}$ *with rebalancing costs proportional to the total shares traded, excluding the portion allocated to the riskless bond, with a proportional constant* $\lambda \geq 0$ *(*$\lambda = 0$ *corresponds to the case when rebalancing cost is ignored). Start with* 1 *unit of investment capital. In the end of the investment period the portfolio value is*

$$1 + \sum_{i=1}^{n} s_i X_i.$$

The amount the gain (or loss) $\sum_{i=1}^{n} s_i X_i$ *needs to be reallocated to the risky assets. The amount of asset i we need to buy or sell is the difference between the amount of asset we have and the amount we need to invest out of the total assets we have, i.e.,* $\left| s_i(1 + X_i) - s_i(1 + \sum_{j=1}^{n} s_j X_j) \right| = s_i \left| X_i - \sum_{j=1}^{n} s_j X_j \right|$. *We can see that the rebalancing cost will be* $\lambda \tilde{g}(s) = \lambda \sum_{i=1}^{n} s_i |X_i - \sum_{j=1}^{n} s_j X_j|$ *and the net return function is*

$$g(\mathbf{s}, X) = \sum_{i=1}^{n} s_i X_i - \lambda \tilde{g}(s).$$

In this case g is piecewise C^2 *but nonhomogeneous and nonsmooth. With* $l(s) = E[1 + g(s, X)]$, *we can calculate*

$$l_{s_j}(s) = E\left[\frac{X_j - \lambda \tilde{g}_{s_j}}{1 + \sum_{i=1}^{n} s_i X_i - \lambda \tilde{g}(s)} \right],$$

and

$$l_{s_j s_k}(s) = E\left[\frac{-X_j X_k - \lambda O(s, X)}{(1 + \sum_{i=1}^{n} s_i X_i - \lambda \tilde{g}(s))^2} \right],$$

where O is a function of s and X involving the derivatives of $\tilde{g}$ *and is bounded. Thus, when* λ *is small enough, the Hessian of l is a negative definite matrix, i.e., l is concave.*

Given an investment horizon Q, we extend the definition of the reward over the horizon (Definition 3) to the case of multiple assets.

Definition 8 (Reward Function on a Finite Investment Horizon)**.** *Let Q be the investment horizon, investing* $\mathbf{s}$ *on n assets with a random return vector* $X = (X_1, X_2, ..., X_n)$, *where* X_i *corresponds to asset i. Then the reward function on a Q period is*

$$\mu_Q(\mathbf{s}) = e^{Ql(\mathbf{s})} - 1, \tag{16}$$

where $l(\mathbf{s})$ *is the log return function defined in (15).*

Remark 2. *Since the exponential function is strictly monotonically increasing, we can see that* $\mu_Q(\mathbf{s})$ *also attains a unique maximum at* $\boldsymbol{\kappa}$, *which in this case will have dimension* $\boldsymbol{n}$. *We call* $\boldsymbol{\kappa}$ *the Kelly point as in Definition 5.*

We now examine the trade-off between reward and risk. For a given set of portfolio weights $\mathbf{s}$, let the return of the portfolio be $g(\mathbf{s}, X)$. We denote the aggregate risk associated

with this return as $m(g(\mathbf{s}, X))$. Since the set of risky assets is assumed to be fixed, the risk is treated as a function of $\mathbf{s}$, denoted by $\gamma(\mathbf{s}) := m(g(\mathbf{s}, X))$. This approach focuses on the trade-off between the reward function $\mu_Q(\mathbf{s})$ and a single aggregated risk $\gamma(\mathbf{s})$, analyzed in the two-dimensional reward-risk space. Specifically, the focus is on the set $\mathcal{A} := \{(\gamma(\mathbf{s}), \mu_Q(\mathbf{s})) : \mathbf{s} \in R^n\}$. A convenient way to study this trade-off is by leveraging the results from Section 2. To facilitate this, we introduce the concept of the reward/risk path below.

Definition 9 (Reward/risk path)**.** *Consider the model defined in the beginning of this section. Let X be the vector of random variables representing the return of a group of risky assets relative to a riskless bond and let $\mathbf{s} \in R^n$ be the vector of portfolio weights. Suppose that Assumption 1 is satisfied. A mapping $\mathbf{s} : [a, b] \to R^n$ is called reward/risk path with respect to X, if it has the following properties:*

1. *$\mathbf{s}(\cdot)$ is continuous and piecewise C^2.*
2. *$\mathbf{s}(a) = 0$ and $\mathbf{s}(b) = \kappa$.*
3. *$t \mapsto l(\mathbf{s}(t))$ is an increasing function on $[a, b]$.*
4. *$t \mapsto \gamma(\mathbf{s}(t))$ is an increasing function on $[a, b]$.*

The idea is that at $\mathbf{s} = \kappa$, we achieve the maximum possible reward, while at $\mathbf{s} = \mathbf{0}$, we have the minimum risk. The reward/risk path is a parameterized trajectory that strikes a balance between these two extremes. Ideally, this path should lie on the Pareto-efficient frontier of the set $\mathcal{A}$, i.e., $\mathbf{s} \in \mathcal{A}$ such that no $\mathbf{s}' \in \mathcal{A}$ satisfies $\mu_Q(\mathbf{s}') \geq \mu_Q(\mathbf{s})$, $\gamma(\mathbf{s}') \leq \gamma(\mathbf{s})$, with at least one inequality being strict. However, in practice, it often makes sense to use the idea of Fractional Kelly, as discussed in (MacLean et al. 1992, 2011a, 2011b), and choose $\mathbf{s}(t) = t\kappa$.

When a reward/risk path is defined, the problem of finding the GOP or its alternative for portfolios with multiple risky assets reduces to a problem discussed in Section 2. If the chosen reward/risk path is $\mathbf{s}(t) = t\kappa$, the problem becomes equivalent to the single risky asset case, with the return of the risky asset characterized by $\kappa \cdot X$. As a result, we can define the inflection point and the optimal reward-to-risk point. Also, we can generalize the properties derived in Section 2 to the multi-asset case.

We now illustrate the reward/risk path in the following simple example, considering the case of no transaction costs.

Example 5. *Now we focus on a concrete scenario of Example 4 with $n = 2$, and X_1, X_2 are specified as in Table 1. We consider a simplified version with $\lambda = 0$ for ease of exposition; this will be the no-transaction-costs case.*

Table 1. Specification of two risky assets. Each asset has two outcomes, with different probabilities.

Risky Assets	Prob.	Return	Prob.	Return
X_1	70%	1	30%	-1
X_2	50%	1	50%	-0.5

Then

$$\begin{aligned} l(s_1, s_2) &= 0.35\ln(1 + s_1 + s_2) + 0.35\ln(1 + s_1 - 0.5s_2) \\ &\quad +0.15\ln(1 - s_1 + s_2) + 0.15\ln(1 - s_1 - 0.5s_2). \end{aligned} \tag{17}$$

Solving $\nabla l = 0$ *we derive* $\kappa = (0.36, 0.44)$. *Consider an investment of* $Q = 50$ *cycles and use* $\gamma(\mathbf{s}) = s_1 + s_2$ *as a risk measure. The reward/ risk path on the efficient frontier can be determined from* $\max\{\mu_Q \mathbf{s}) : \gamma(\mathbf{s}) \leq t\}$. *Since exponential function is strictly increasing, this is equivalent to*

$$\max\{l(\mathbf{s}) : \gamma(\mathbf{s}) \leq r\}. \tag{18}$$

Using a Lagrange multiplier α *we have*

$$0 = \nabla l(\mathbf{s}) - \alpha(1, 1),$$

or

$$\frac{\partial l(\mathbf{s})}{\partial s_1} = \frac{\partial l(\mathbf{s})}{\partial s_2}.$$

Setting $s_2 = t \in [0, 0.44]$, *we obtain*

$$s_1 = 0.208 * t + 0.583 - 0.042\sqrt{217t^2 - 196t + 100}.$$

We thus get the reward/risk path determined by the Pareto front.

On the other hand, the reward/ risk path that follows the idea of Fractional Kelly approach, using the same parameters, is given by $\mathbf{s}(t) = (\frac{0.36}{0.44}t, t)$. *The reward-risk curves corresponding to these two reward/ risk paths are shown together in Figure 4. As observed, following the reward/ risk path along the efficient frontier results in a slight improvement in rewards at the alternative optimal investment portfolios* ξ_Q *and* ν_Q, *as expected. However, this improvement comes at the cost of considerable additional effort. Consequently, in practical problems, the Fractional Kelly reward/risk path is often a more pragmatic choice.*

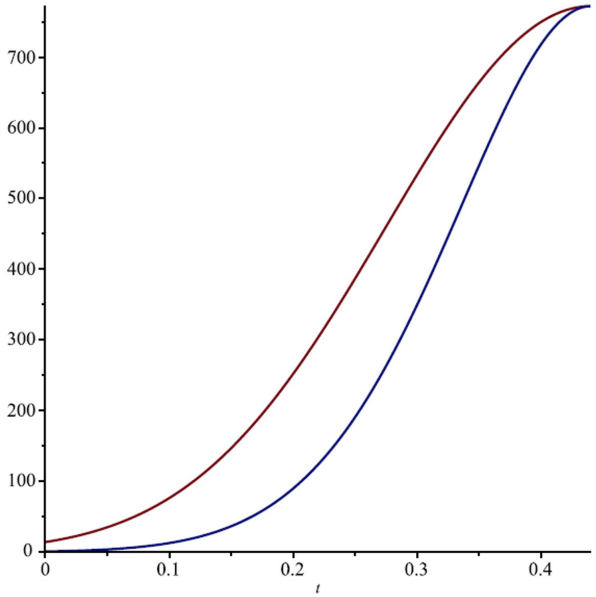


Figure 4. Two reward/risk paths, using the path along the efficient frontier (in red) or a path using a Fractional Kelly approach (in blue).

7. Empirical Analysis

In this Section we conduct several computational tests, in order to assess the results and the robustness of our framework. As the inspiration for this framework was the inclusion of transaction costs in risk-adjusted Kelly strategies, we conduct the tests on the transaction costs case that was discussed in Sections 5.2 and 6. Real estate and physical gold are two instances where the transaction costs can become quite high, and so are ideal assets to test the robustness of the strategies. We compare the three

strategies (inflection point, optimal reward to risk, and Kelly point), and analyze the impact of transaction costs and the investment horizon.

We address the multi-asset case, where we use the methodology devised in Section 6 to a two-asset case of real estate and gold. Finally, we conduct out-of-sample tests on a portfolio comprised of real estate and physical gold, and compare the results with those produced with the Kelly strategies that do not account for transaction costs, showing the significant outperformance of our methodology against classical Kelly.

7.1. Data Sources and Return Calculations

In the following two subsections, we use publicly available data for the U.S. real estate index and gold prices, using monthly data. The real estate data are obtained from the Federal Reserve Economic Data (FRED) database using the S&P CoreLogic Case-Shiller U.S. National Home Price Index (https://fred.stlouisfed.org/series/CSUSHPINSA, 27 February 2026) while historical gold price data are obtained from Yahoo Finance (https://finance.yahoo.com/quote/GC%3DF/history/, 2 February 2023 and updated on 9 February 2026).

For each asset, we calculate the monthly return series from the corresponding price or index level. Let P_t denote the price or index level at time t. The return R_t for period t is defined as

$$R_t = \frac{P_{t-1}}{P_t} - 1.$$

To account for the prevailing risk-free interest rate, we construct the excess return X_t used in our analysis as

$$X_t = R_t - \frac{T_t}{12},$$

where T_t denotes the yield on the 1-month U.S. Treasury Bill at Constant Maturity for period t, obtained from FRED (https://fred.stlouisfed.org/series/DGS1MO, 28 January 2026). Thus, X_t represents the return on the respective asset above the short-term U.S. Treasury rate.

Finally, at the end of Section 7.2, we compare the results for real estate and gold with those of a traditional equity asset class. For this purpose, we use the iShares Core S&P Small-Cap ETF (IJR) as a proxy for U.S. small-cap equities, with historical price data obtained from Yahoo Finance (https://finance.yahoo.com/quote/IJR/, 9 February 2023 and updated on 1 February 2026). This comparison is conducted under the assumption of very low, effectively negligible, transaction costs for the real estate index and gold series considered in the analysis.

7.2. Single Asset Problem

In this case, we consider the case of an investor that is trading a single asset class, which we consider to be either real estate or physical gold. The investor wants to determine the optimal investment size, but needs to take into account the impact of transaction costs. In the spirit of the Kelly criterion, he wants to keep his size constant throughout the investment, but as we have seen, that entails impacts from rebalancing the portfolio.

Our baseline case is described in Table 2. In our baseline scenario, we assume an investment horizon of 30 years and severe transaction costs of 5% on the assets, which are typical in real estate and physical gold.

Monthly gold prices from September 2011 through November 2025 were obtained and transformed into monthly returns. For the real estate asset, the monthly S&P Case-Shiller U.S. National Home Price Index was used and similarly converted into

monthly returns over the same period. Due to the presence of autocorrelation in real estate returns, we unsmoothed the returns according to the procedure of (Geltner 1991; Giliberto 1990). We considered a rental yield of 5% in line with current estimates to account for income in real estate, since the Case-Shiller index is a price-only index. The returns on physical gold and real estate were computed against the 1-month Treasury bill return, i.e., we used excess returns against the risk-free asset as in the standard procedure.

Table 2. Baseline parameters for the investment in the two asset classes.

	Transaction Costs	**Investment Horizon Q (Years)**
Physical Gold	5%	30
Real Estate	5%	30

We start by computing the optimal investment strategies (inflection point, optimal reward-to-risk and Kelly point), by calibrating the model in the transaction cost framework described in Section 5.2. In Figure 5, we illustrate the risk-reward curve and show the three different strategies. We can immediately see that Proposition 3 is verified: the inflection point is lower than the optimal reward-to-risk size, which is lower than the Kelly investment size. Figure 6 shows the risk reward curve for real estate, where we verify again the order of the three points. In both figures, we observe that the risk-reward curve has a cusp and thus is not smooth.

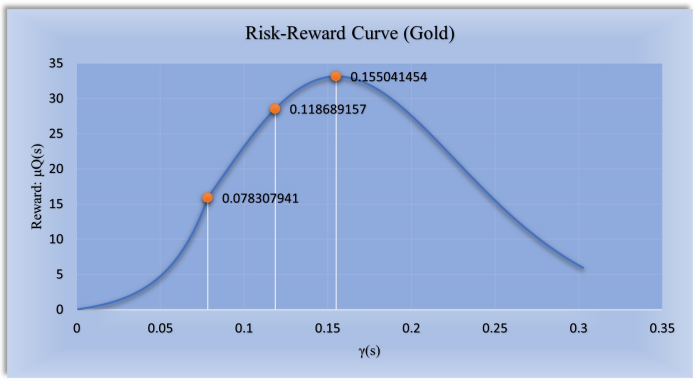


Figure 5. Risk-reward curve $\mu_Q(s)$ vs. the risk $\gamma(s)$, for single asset allocation. The first point is the inflection point ($\gamma(s) = 0.078\%$, $s = 100\%$), the second is the point the maximizes the optimal reward-to-risk ratio ($\gamma(s) = 0.119\%$, $s = 144\%$).

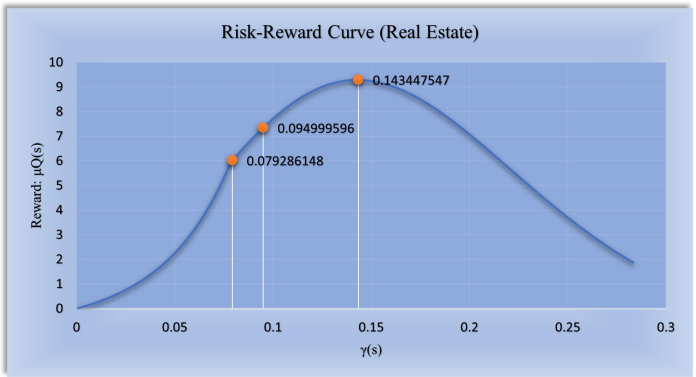


Figure 6. Cumulative return curve $\mu_Q(s)$ vs. the risk $\gamma(s)$, for single asset allocation. The first point is the inflection point ($\gamma(s) = 0.0792\%$, $s = 100\%$), the second is the point the maximizes the optimal reward-to-risk ratio ($\gamma(s) = 0.095\%$, $s = 118\%$).

Tables 3 and 4 show the sensitivity of the investment points to the investment horizon Q, for the case of real estate and physical gold. The Kelly investment size stays always constant, since it is independent of the investment horizon. The optimal reward-to-risk size gets higher with higher investment horizons. The inflection point stays at 100%, due to the discontinuity in the smoothness for $s = 1$ in the return function $g(s, X)$ that we described in Section 5.2. In financial terms, the marginal return increases before $s = 1$ but decreases immediately after $s = 1$, as higher investment sizes entail greater rebalancing, and the transaction costs effect starts hampering marginal returns.

Table 3. Investment sizes for the three criteria, for different horizons, in the base case for physical gold.

Q (Years)	Kelly	RR Max.	Inflection
30	181%	144%	100%
25	181%	137%	100%
20	181%	125%	100%
15	181%	105%	100%
10	181%	100%	100%
5	181%	100%	100%

Table 4. Investment sizes determined by the Kelly, optimal reward-to-risk, and inflection points, for different investment horizons Q, assuming the baseline case for real estate.

Q (Years)	Kelly	RR Max.	Inflection
30	172%	118%	100%
25	172%	107%	100%
20	172%	100%	100%
15	172%	100%	100%
10	172%	100%	100%
5	172%	100%	100%

In Table 5 we document the investment sizes for a much more liquid asset class than real estate or physical gold, the S&P small-cap index, assuming negligible transaction costs. In comparison to physical gold and real estate, we can readily see that the investment size is significantly higher, due to the presence of the above-mentioned frictions on physical gold and real estate transactions.

Table 5. Investment sizes for the three criteria, for different horizons, in the base case for S&P Small-Cap ETF (IJR) where transaction cost is approximately zero. DNE: Does Not Exist.

Q (Years)	Kelly	RR Max.	Inflection
30	240%	198%	150%
25	240%	189%	141%
20	240%	175%	128%
15	240%	152%	109%
10	240%	105%	76%
5	240%	DNE	DNE

Tables 6 and 7 show the sensitivity of the investment sizes to transaction costs. As the transaction costs become lower, the investment sizes for gold and real estate become higher. Lower rebalancing costs yield more leveraged strategies.

Table 6. Investment sizes for the three criteria (Kelly, optimal reward-to-risk, and inflection point), for different transaction costs levels λ, assuming the baseline case for physical gold.

λ	Kelly	RR Max.	Inflection
0.05	181%	144%	100%
0.04	200%	162%	100%
0.03	223%	185%	100%
0.01	302%	263%	204%
0.001	361%	323%	257%
0.0001	368%	331%	263%
0.00001	368%	331%	264%

Table 7. Investment sizes for the three criteria (Kelly, optimal reward-to-risk, and inflection point), for different transaction costs levels λ, assuming the baseline case for real estate.

λ	Kelly	RR Max.	Inflection
0.05	172%	119%	100%
0.04	190%	135%	100%
0.03	214%	157%	100%
0.01	302%	241%	179%
0.001	382%	320%	242%
0.0001	393%	331%	251%

7.3. Empirical Analysis with Multiple Assets

In this subsection we look at the case of medium to long term investor which allocates capital to real estate and physical gold, with transaction costs, in line with what we described in Section 6.

We start by determining the Kelly portfolio $\kappa = (\kappa_1, \kappa_2)$ for this setting. In Figure 7, there is a clear peak at Kelly investment size at $\kappa = (\kappa_1, \kappa_2) = (91\%, 157\%)$, which is represented by a bell-shaped 3D curve.

Next, we proceed to calculate the inflection point portfolio and the optimal reward-to-risk portfolio, using the straight line that connects $(0,0)$ to the Kelly portfolio, as discussed in Section 6. Suppose $b = \kappa_2/\kappa_1$. Then, we can parametrize the curve $l(s)$ along the line passing through $(0,0)$ and κ using $s_1 = t$ and $s_2 = bt$ to determine the inflection and the optimal reward-to-risk point. As in Section 6, we denote $l(t) = l(s(t)) = l(t, bt)$. We thus obtain

$$l(t) = E[\ln(1 + tX_1 + btX_2 - \lambda(t|X_1 - tX_1 - btX_2| + bt|X_2 - tX_1 - btX_2|))].$$

As before, $\mu_Q(s) = exp(Ql(s)) - 1$.

The risk function $\gamma(s)$ is computed in the same fashion as in Equation (13). The reward-risk curve is thus determined and we can compute the inflection point and optimal reward-to-risk point. Again, the inflection point can be found by solving the equation

$$\frac{d^2\mu_Q(t)}{d\gamma(t)^2} = 0. \tag{19}$$

Figure 8 shows the reward-risk curve $(\gamma(s), \mu_Q(s))$, using the straight line passing through $(0,0)$ and the Kelly point $\kappa = (\kappa_1, \kappa_2)$. The inflection point is at $\gamma(t) = 0.0936$, which translates into a portfolio $(s_1, s_2) = (61\%, 105.24\%)$. We also plot the optimal reward-to-risk point, attained at $\gamma(t) = 0.1184$, which translates into a portfolio $(s_1, s_2) = (76\%, 131.12\%)$.

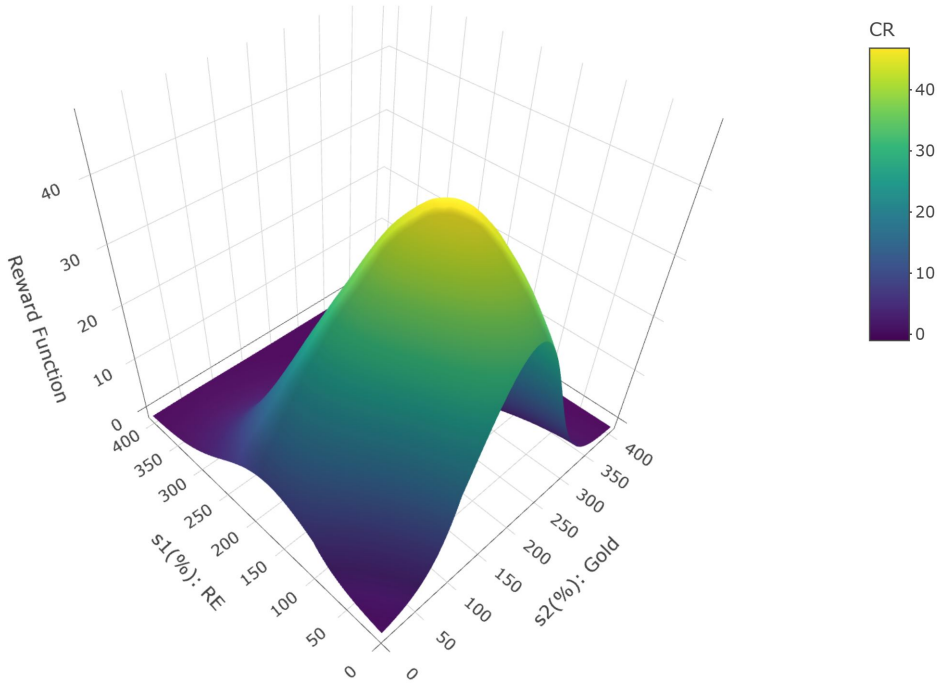


Figure 7. Reward function $\mu_Q(s_1, s_2)$, for two-asset allocation.

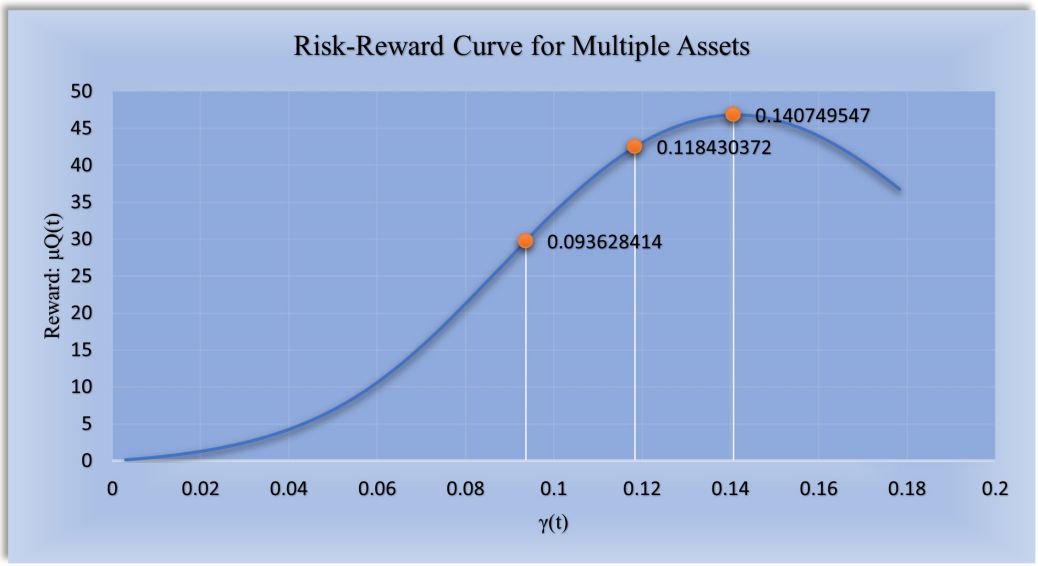


Figure 8. Risk-reward curve along the straight line from $(0, 0)$ to the Kelly portfolio $\kappa = (\kappa_1, \kappa_2)$.

In Table 8 we show how, in the transaction cost setting, the portfolios determined by the Kelly criterion, the optimal reward-to-risk, and the inflection point, vary as a function of the investment horizon. Again, the Kelly portfolio does not depend on the investment horizon. As the investment horizon Q gets larger, the allocations determined by the inflection point and reward-to-risk portfolios become larger. We also observe that the investment sizes determined by the inflection point and optimal reward-to-risk are significantly smaller than those given by the Kelly criterion.

Table 8. Portfolio allocations for gold and real estate across horizons Q in the base case scenario for two assets, gold and real estate.

Q	Kelly (G, RE)	RR Max (G, RE)	Inflec (G, RE)
30	(157%, 91%)	(131.12%, 76%)	(105.24%, 61%)
20	(157%, 91%)	(120.76%, 70%)	(89.71%, 52%)
15	(157%, 91%)	(112.14%, 65%)	(84.53%, 49%)
10	(157%, 91%)	(98.34%, 57%)	(77.63%, 45%)
5	(157%, 91%)	(82.81%, 48%)	(62.11%, 36%)

Table 9 shows the response of risk-adjusted Kelly strategies to the level of transaction costs. As transaction costs grow, rebalancing costs reduce investment size. Again, we observe that optimal reward-to-risk and inflection portfolios yield more conservative allocations than the Kelly criterion for the transaction costs setting.

Table 9. Portfolio sizes for gold and real estate across transaction cost levels λ.

λ	Kelly (G, RE)	RR Max (G, RE)	Inflection (G, RE)
0.05	(157%, 91%)	(82.81%, 48%)	(62.11%, 36%)
0.04	(169%, 113%)	(145.07%, 97%)	(110.67%, 74%)
0.03	(186%, 145%)	(161.63%, 126%)	(126.99%, 99%)
0.01	(259%, 244%)	(238.83%, 225%)	(194.25%, 183%)
0.001	(319%, 336%)	(303.81%, 320%)	(255.39%, 269%)
0.0001	(327%, 348%)	(311.96%, 332%)	(263.10%, 280%)
0.00001	(327%, 349%)	(312.95%, 334%)	(264.22%, 282%)

7.4. Out-of-Sample Tests

In this section, we conduct an out-of sample analysis for physical gold to compare the performance of our our methodology. We compare the following strategies: the Kelly criterion assuming transaction costs in-sample, the inflection point and the optimal reward-to-risk assuming in-sample transaction costs, as developed in this paper. The strategies are benchmarked against the classical Kelly criterion, the inflection point, and the optimal reward-to-risk as developed by Vince and Zhu (2015), which assume no transaction costs when computing the investment size. When applicable, the strategies are calibrated to an investment horizon Q of 30 years and transaction costs of 5% as in the baseline case in the previous subsections.

We follow a 10-year rolling window testing methodology. For each month, the investment size for physical gold is calibrated to the previous 10 years of data. The out-of-sample return for the next month is calculated taking into account the actual return on physical gold and the rebalancing impact, assuming a level of transaction costs of 5% for all the strategies. The cumulative wealth of the investor is obtained by compounding the monthly out-of-sample returns.

The results are shown in Figure 9 and Table 10. The classical Kelly criterion (assuming no transaction costs in the calibration) has very poor results, due to high leverage and the effect of transaction costs out-of-sample, with a drawdown of 94% and a negative annualized return of -5.17%, i.e., the investor loses that percentage per year. The overly aggressive nature of the classical Kelly criterion entails severe rebalancing costs out-of-sample, which translate into a pronounced loss of wealth. The inflection point and optimal reward-to-risk as developed by Vince and Zhu (2015) (which we call no-transaction-cost inflection point and no-transaction-cost optimal reward-for-risk) also don't provide adequate returns in this setting of high transaction costs.

In sharp constrast, the strategies we propose (denominated transaction-cost optimal reward-to-risk and transaction-cost inflection point) yield significant excessive returns on an annual basis, with drawdowns that are much lower than the classical Kelly criterion. The results confirm the suitability of our methodology for a long-term investor who still wants to control risks.

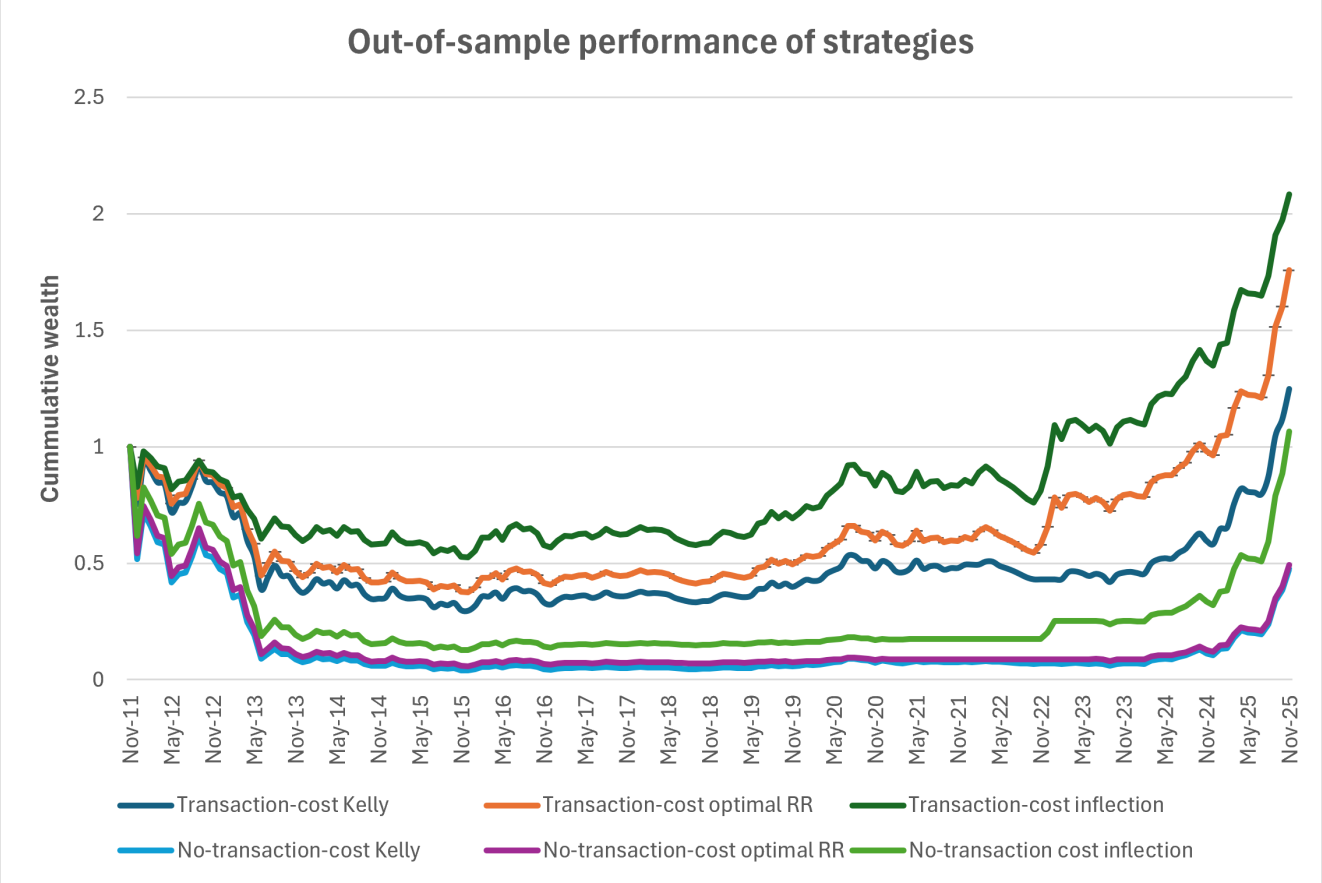


Figure 9. Out-of-sample performance of six strategies: transaction-cost Kelly: Kelly criterion assuming transaction costs in-sample; transaction-cost optimal RR: optimal reward-to-risk ratio, using the new methodology developed in this paper, with a calibration assuming transaction costs; transaction-cost inflection: inflection point, calibrated assuming transaction costs in-sample, as in the new methodology developed in this paper; no-transaction Kelly: Kelly criterion calibrated in-sample assuming no transaction costs (classical Kelly criterion); no transaction-cost optimal RR: optimal reward-to-risk ratio, calibrated without assuming transaction costs, as in Vince and Zhu (2015); no transaction-cost inflection: inflection point, calibrated without assuming transaction costs, as in Vince and Zhu (2015). For each month, the optimal points are calibrated to the data over the previous ten years. For all strategies, we assume transaction costs of 5% out-of-sample.

Table 10. Out-of-sample performance of six strategies, using several metrics: the compounded annual growth rate (CAGR), maximum drawdown, mean monthly return, the standard deviation, the Sharpe Ratio, the skewness and the kurtosis. The tested strategies are: transaction-cost Kelly: Kelly criterion assuming transaction costs in-sample; transaction-cost optimal RR: optimal reward-to-risk ratio, using the new methodology developed in this paper, with a calibration assuming transaction costs; transaction-cost inflection: inflection point, calibrated assuming transaction costs in-sample, as in the new methodology developed in this paper; no-transaction Kelly: Kelly criterion calibrated in-sample assuming no transaction costs (classical Kelly criterion); no transaction-cost optimal RR: optimal reward-to-risk ratio, calibrated without assuming transaction costs, as in Vince and Zhu (2015); no-transaction-cost inflection: inflection point, calibrated without assuming transaction costs, as in Vince and Zhu (2015). For each month, the optimal points are calibrated to the data over the previous ten years. For all strategies, we assume transaction costs of 5% out-of-sample.

	Transaction Cost Kelly	Transaction Cost Optimal RR	Transaction Cost Inflection Point	No Transaction Cost Kelly	No Transaction Cost Optimal RR	No Transaction Cost Inflection Point
CAGR	1.59%	4.11%	5.38%	−5.17%	−4.92%	0.45%
Maximum drawdown	70%	62%	47%	96%	94%	87%
Mean monthly return	0.35%	0.52%	0.56%	0.38%	0.48%	0.48%
Monthly standard deviation	6.57%	6.00%	4.96%	12.32%	9.15%	9.15%
Sharpe ratio (annualized)	0.19	0.30	0.39	0.11	0.18	0.18
Monthly skewness	−0.3	−0.1	0.4	−0.4	−0.3	−0.3
Monthly kurtosis	3.2	3.0	2.1	3.9	5.7	5.7

8. Conclusions

As we have discussed, Kelly criterion maximizes long-term reward, but suffers from excessive risk, and do not take into account transaction costs, or other settings such as decreasing returns to risk which are present in the case of private equity. These motivations led us to pursue a general theory for risk-adjusted Kelly strategies for non-smooth and non-homogeneous reward and risk, and the development of the alternative criteria, the inflection point, and the optimal reward-to-risk in this setting.

We characterized the existence of the investment sizes that satisfy these risk-adjusted Kelly criteria in the general setting, and documented their properties. We subsequently applied the theory to several practical cases, namely the transaction cost setting and diminishing returns to scale. We also developed the theory for the multiple-asset case.

Our approach was subsequently tested in the transaction costs case, documenting a reduction in investment size compared to the Kelly criterion, combining two factors: the incorporation of transaction costs in the return function $g(s, X)$, and the risk-adjusted criteria (inflection point and optimal reward-to-risk), which lead to a further reduction in the size. The inflection point and the optimal reward-to-risk sizes perform significantly better out-of-sample than the classical Kelly criterion (not incorporating transaction costs), and the Kelly criterion incorporating transaction costs.

In terms of policy implications, our approach supports a very significant size reduction compared to the pure Kelly criterion due to the incorporation of transaction costs and the risk-adjusted Kelly criteria. Adding transaction costs makes the investment sizes more

conservative, avoiding overleverage in the financial system. We hope this research fosters the practical application of Kelly-type strategies to realistic settings, which allow for the effective implementation of this kind of strategies, which are suitable for institutional investors in illiquid asset classes that face high transaction costs and decreasing returns to scale.

Author Contributions: All authors contributed equally to Conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing–original draft preparation, writing—review and editing, visualization, supervision, and project administration. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this study are available in provided URLs in the Section 7.1.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Proof of Proposition 1.

Proof. Since $\mu_Q'(\kappa) = 0$ and $\mu_Q''(\kappa) < 0$, we have

$$\frac{d^2\mu_Q}{d\gamma^2}(\kappa) = \frac{\mu_Q''(\kappa)}{(\gamma'(\kappa))^2} < 0.$$

On the other hand, since both μ_Q and γ are C^2 at 0, we can compute that

$$\frac{d^2\mu_Q}{d\gamma^2}(0) = Q\frac{Q[l'(0)]^2\gamma'(0) + l''(0)\gamma'(0) - l'(0)\gamma''(0)}{(\gamma(0))^3}.$$

Since $[l'(0)]^2\gamma'(0) > 0$, when Q is sufficiently large

$$\frac{d^2\mu_Q}{d\gamma^2}(0) > 0.$$

This means that the continuous parametric curve (γ, μ_Q) is convex at 0 and concave at κ. Thus, there is an inflection point $\nu \in (0, \kappa)$. □

Proof of Theorem 1.

Proof. To prove the "if" part, we will show that if the condition (10) holds, the risk-reward curve changes convex to concave between 0 and κ.

Since we assume the twice differentiability of γ and μ_Q in 0 and κ, we have $\frac{d^2\mu_Q}{d\gamma^2}(s) = \frac{\gamma'(s)\mu_Q''(s) - \mu_Q'(s)\gamma''(s)}{[\gamma'(s)]^3}$, at $s = 0, \kappa$.

The proof of the existence of the inflection point relies on showing that the derivative above is positive at 0 and negative at κ, thus ensuring that there will be a point where the function will switch from convex to concave and thus guaranteeing the existence of the inflection point.

First, observe that, by the strictly increasing property of $\gamma(s)$, we have $\gamma'(s) > 0$ for $s = 0, \kappa$. Let $h(s) = \gamma'(s)\mu_Q''(s) - \mu_Q'(s)\gamma''(s)$.

By Definition 2.5 at Kelly's optimal point $s = \kappa$, $\mu''_Q(\kappa) < 0$. Then, $h(\kappa) = \gamma'(\kappa)\mu''_Q(\kappa) < 0$. Since γ is an increasing function, $\gamma'(\kappa) > 0$. Therefore we have $h(\kappa) < 0$. i.e., $\frac{d^2\mu_Q}{d\gamma^2}(\kappa) < 0$. We thus show that μ_Q is concave at κ.

Now we want show that μ_Q is convex at zero. We notice that the condition $h(0) > 0$ is equivalent to $Qe^{Ql(s)}\left(l''(0) + Q(l'(0))^2\right)\gamma'(0) - Ql'(0)e^{Ql(s)}\gamma''(0) > 0$. This condition is equivalent to $Q > Q_{min}$, where Q_{min} is defined by Equation (10). Therefore, if $Q > Q_{min}$, we guarantee that μ_Q is convex at zero. As a result, the risk-reward curve changes from convex to concave between zero and κ and we guarantee the existence of the inflection point. Thus, we prove that that $Q > Q_{min}$ is a sufficient condition for the existence of the inflection point.

To prove that $Q > Q_{min}$ is a necessary condition for the existence of the inflection point, by Definition 6 the risk-reward curve is convex at zero. As a result, $\frac{d^2\mu_Q}{d\gamma^2}(0) = \frac{\gamma'(0)\mu''_Q(0) - \mu'_Q(0)\gamma''(0)}{[\gamma'(0)]^3} > 0$, which implies

$$Qe^{Ql(s)}\left(l''(0) + Q(l'(0))^2\right)\gamma'(0) - Ql'(0)e^{Ql(s)}\gamma''(0) > 0, \tag{A1}$$

or $Q > Q_{min}$. Thus we show that $Q > Q_{min}$ is a necessary condition for the existence of the inflection point. □

Proof of Proposition 2.

Proof. First, we will show that we only need to inspect for the maximum of $\mu_Q(s)/\gamma(s)$ within $[0, \kappa]$.

Second, we will show that $\mu_Q(s)/\gamma(s)$ is continuous in $[0, \kappa]$. The existence of the maximum will follow from a result from real analysis that states that every continuous function over a compact interval attains a maximum within that compact set.

To prove the first assertion, we first observe that $\mu_Q(s)$ obtains its maximum at κ, by Definition 1 and the fact that $\mu_Q(s) = exp(Q\,l(s)) - 1$. Therefore, for $s \geq 0$, we have

$$\mu_Q(s) \leq \mu_Q(\kappa). \tag{A2}$$

Also, since $\gamma(s)$ is non-decreasing, for $s \geq \kappa$ we have

$$\gamma(s) \geq \gamma(\kappa). \tag{A3}$$

Equations (A2) and (A3) imply that for $s \geq \kappa$, we have

$$\frac{\mu_Q(s)}{\gamma(s)} \leq \frac{\mu_Q(\kappa)}{\gamma(\kappa)}. \tag{A4}$$

As a consequence, we only need to look for the maximum within the interval $[0, \kappa]$.

To prove the second part, we need to show the continuity of $\mu_Q(s)/\gamma(s)$ within $[0, \kappa]$. The continuity of μ_Q/γ in $[0, \kappa]$ follows from the fact that $\mu_Q(s)$ is continuous (by applying the dominated convergence theorem, which holds due to the boundedness of X), the continuity of $\gamma(s)$, and the fact that $\gamma(s) > 0$ for $s > 0$, since $\gamma(0) > 0$ and the non-decreasing assumption on γ.

We need to show that $\mu_Q(s)/\gamma(s)$ is also continuous at zero. Since both the numerator and denominator are continuous, the only issue to be proved is that the ratio does not explode at $s = 0$. If $\gamma(0) > 0$, the ratio will not explode. Therefore, we only need to analyze the case that $\gamma(0) = 0$. Since $l(s)$ is differentiable at zero, and $\mu_Q(s)$ is also differentiable

at zero. The assumption that $\mu_Q(0) = 0$ (since we are assuming that $l(0) = 0$), and the differentiability of μ_Q and γ at zero imply that there exist δ and ϵ, such that for $s \in [0, \delta]$ we have

$$\frac{\mu_Q(s)}{s} - \mu_Q'(0) \leq \epsilon \tag{A5}$$

$$\frac{\gamma(s)}{s} - \gamma'(0) \geq -\epsilon \tag{A6}$$

We can rearrange the previous equations to get

$$\frac{\mu_Q(s)}{s} \leq \mu_Q'(0) + \epsilon =: K_1 \tag{A7}$$

$$\frac{\gamma(s)}{s} \geq \gamma'(0) - \epsilon =: K_2 \tag{A8}$$

Since $\gamma'(0)$, we can choose ϵ arbitrarily small to make $K_2 > 0$.

This implies that for $s \in [0, \delta]$ we have that

$$\frac{\mu_Q(s)}{\gamma(s)} = \frac{\mu_Q(s)}{s}\frac{s}{\gamma(s)} \leq \frac{K_1}{K_2}. \tag{A9}$$

As a consequence, $\mu_Q(s)/\gamma(s)$ will not explode at $s = 0$. Therefore, the function $\mu_Q(s)/\gamma(s)$ is continuous at zero. Since it is continuous in $[0, \kappa]$, the compactness of the interval ensures the existence of the maximum within $[0, \kappa]$. □

Proof of Proposition 3.

Proof. For any $s < \nu$, letting $\lambda = \gamma(s)/\gamma(\nu)$, we have

$$\gamma(s) = \lambda\gamma(\nu) + (1 - \lambda)\gamma(0), \text{ here } \gamma(0) = 0.$$

Since the curve $(\gamma(s), \mu_Q(s))$ is strictly convex in the interval $[0, \nu]$ and $\mu_Q(0) = 0$, we have

$$\mu_Q(s) < \lambda\mu_Q(\nu) + (1 - \lambda)\mu_Q(0) = \lambda\mu_Q(\nu).$$

It follows that

$$\mu_Q(s)/\gamma(s) < \mu_Q(\nu)/\gamma(\nu).$$

Thus, $\xi \geq \nu$. We already know that $\xi < \kappa$. The proof is completed. □

Derivation of the risk function $\gamma(s)$ in Section 5.2.

Proof. For ease of explanation, let us suppose that X is a random variable with a continuous and strictly increasing cumulative distribution function, which holds in many instances, and simplifies our exposition. In this example, we compute the $CVaR$ of Y, which will yield a function $\gamma(s)$ as a function of leverage.

We first start by showing that $g(s, X) \geq 0$, for $X \geq 0$ for reasonable values of λ, which will enable us to inspect the region $X \leq 0$ for the computation of $\gamma(s)$. Subsequently, we will compute the VaR and the $CVaR$ of Y to get $\gamma(s)$.

Assume again that VaR_X and $CVaR_X$ are the Value at Risk and the Conditional value at risk of X, i.e., VaR_X is obtained by solving

$$\mathbb{P}(X < -VaR_X) = 1 - \alpha, \tag{A10}$$

for $\alpha \in (0,1)$ is the confidence level, and

$$CVaR = \mathbb{E}(-X|X < -VaR_X). \tag{A11}$$

For ease of exposure, we assume that $VaR_X > 0$, which holds for reasonable values of α, given the assumption $\mathbb{P}(X < 0) > 0$.

First, we show that under reasonable assumptions we have $sX - \lambda s|1-s||X| \geq 0$ for $X \geq 0$. This step will allow us to compute the VaR and $CVaR$ of Y in the negative region determined by $X < 0$. Let $\bar{s}$ denote the supremum of the leverage allowed so that the expected log function (6) is well defined, i.e.,

$$\bar{s} = \sup_{s \geq 0}\{s : s\underline{X} - \lambda s|1-s||\underline{X}| \geq -1\}. \tag{A12}$$

where $\underline{X} = \inf X$ is the (essential) infimum value attained by the random variable X. We notice that $\bar{s}$, the maximum leverage allowed, is lower or equal than $1/(-\underline{X})$, which is follows from the definition above.

Suppose that $\lambda < 1$, which always holds in practice. We notice that if

$$(1 - \lambda(\bar{s} - 1)) \geq 0, \tag{A13}$$

then

$$sX - \lambda s|1-s||X| \geq 0 \tag{A14}$$

for $X \geq 0$. Equation (A13) is equivalent to

$$\lambda \leq \frac{1}{\bar{s} - 1}. \tag{A15}$$

Under this assumption, we guarantee that for non-negative X, the value of the adjusted returns is always non-negative, so the VaR and the $CVaR$ of the adjusted returns (12) do not lie in this region. The assumption holds in practice: for example, if the minimum return on X is equal to -10%, the maximum leverage allowed is less than $1/10\% = 10$, and Equation (A15) holds if $\lambda \leq 1/9$. This transaction cost value holds very often in practice, even for assets with high transaction costs such as physical gold that we use in the computational tests in Section 7.

Having shown that $Y = g(s, X) \geq 0$ for $X \geq 0$ in this example, the computation of Var and $CVaR$ of Y is greatly simplified, as we only need to inspect the values for $X \leq 0$. We will start by computing the Value at Risk of Y and then proceed to calculate the conditional value at risk.

For $X \leq 0$, we have that $Y = X(s + \lambda s|1-s|)$. Since we assume continuity of the distribution of X in this example, the Value at Risk VaR_Y of Y is obtained by solving the equation

$$\mathbb{P}(X(s + \lambda s|1-s|) \leq -VaR_Y) = 1 - \alpha, \tag{A16}$$

or

$$\mathbb{P}\left(X \leq \frac{-VaR_Y}{s + \lambda s|1-s|}\right) = 1 - \alpha. \tag{A17}$$

Since the cumulative distribution function of X is strictly increasing and continuous in this example, from Equation (A10) we get

$$VaR_X = \frac{-VaR_Y}{s + \lambda s|1-s|}, \tag{A18}$$

or

$$VaR_Y = (s + \lambda s|1 - s|)VaR_X. \tag{A19}$$

We now proceed to calculate the $CVaR$ of Y, which will yield $\gamma(s)$ in this example. This can be computed as

$$\gamma(s) := -\mathbb{E}(Y|Y \leq -VaR_Y), \tag{A20}$$

having the subsequent computations

$$\gamma(s) = \tag{A21}$$
$$= -\mathbb{E}(X(s + \lambda s|1 - s|) \mid X(s + \lambda s|1 - s|) \leq -VaR_X(1 + \lambda s|1 - s|)) \tag{A22}$$
$$= -(s + \lambda s|1 - s|)\,\mathbb{E}(X \mid X \leq -VaR_X) \tag{A23}$$
$$= (s + \lambda s|1 - s|)\,CVaR_X. \tag{A24}$$

As we are assuming that $\lambda < 1$, this function is strictly increasing. We thus obtain a risk measure that is compliant with Definition 2. □

Proof of no inflection point in Example 3.

Proof. Consider that the random variable X is drawn from a normal distribution with mean 0.02 and standard deviation 0.15, with $\lambda = 0.05$. Let $l(s) = E[\ln(1 + sX - \lambda s|1 - s||X|)]$. In this case, $l(s)$ achieves its maximum at $s = 1$, as does $\mu_Q(s)$, even though it is not smooth at $s = 1$.

In this example, $l'(s) > 0$ for $s \in [0, 1)$. Figure A1 shows that the first derivative of $l(s)$ does not exist at $s = 1$, where μ_Q attains its maximum.

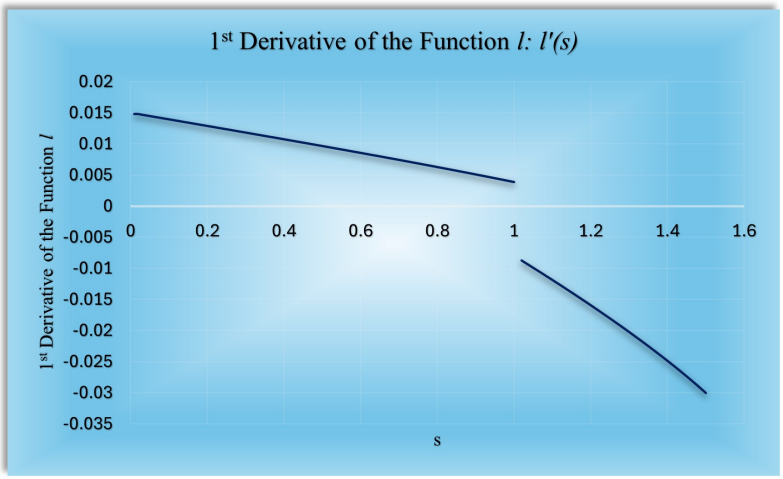


Figure A1. First derivative of $l(s) = E[\ln(1 + sX - \lambda s|1 - s||X|)]$, where X is normally distributed with mean 0.02 and standard deviation 0.15, with $\lambda = 0.05$.

Next, we examine $\mu''_Q(s) = Q[Q(l'(s))^2 + l''(s)]$. Define $K_1 = inf_{s\in[0,1)}|l'(s)|$. Numerical inspection reveals that this value is 0.005, strictly greater than zero. Define $K_2 = inf_{s\in[0,1)}l''(s)$. Numerical inspection shows that this value is -0.013.

Recall that by Equation (9), for $s \in (0, 1)$, we have that

$$\frac{d^2\mu_Q}{d\gamma^2}(s) = \frac{\mu''_Q(s)\gamma'(s) - \mu'_Q(s)\gamma''(s)}{(\gamma'(s))^3}. \tag{A25}$$

We now show that for Q large enough, this expression is bounded below by a positive number for $s \in (0, 1)$, resulting in the absence of an inflection point. Note that for $s \in (0, 1)$, $\gamma(s) = (s + \lambda s(1 - s))CVaR_X$, by Equation (A24), which yields $(1 - \lambda)CVaR_X \leq \gamma'(s) \leq$

$(1+\lambda)CVaR_X$. As a consequence, the denominator of expression (A25) is bounded above and positive, and we only need to show that the numerator is bounded below by a positive number for large enough Q.

Now, the numerator is given by

$$Q\,e^{Ql}\,[Q(l')^2 + l'']\,\gamma' - Q\,e^{Ql}\,l'\,\gamma'', \tag{A26}$$

where we have suppressed the dependency on s for ease of notation. For Q large and $s \in (0,1)$, the expression is bounded below by

$$e^{QK_1}\Big(Q(QK_1^2 + K_2)(1-\lambda) + 2\lambda Q K_1\Big).$$

Taking Q large enough, this expression is positive, guaranteeing that $\frac{d^2\mu_Q}{d\gamma^2}(s)$ is bounded below by a positive number, showing the absence of an inflection point. □

Derivation of $g(s,X)$ for Section 5.3.

Proof. Suppose that s is the amount of leverage allocated to the risky asset, which in this case we assume to be a general loan. The return on the loan is equal to X, and includes interest earned net of credit losses. Suppose the bank has a leverage of s times on loans, meaning that it allocates s times its capital to loans and $1-s$ to the riskless asset. As often happens, s can be higher than 1, in which case this represents short-term funding from the bank.

After the realization of X, the bank has $1-s$ in the riskless asset and $s(1+X)$ in loans. Since its new capital base is now $1+sX$, to maintain the leverage, the bank needs to change its amount in loans from $s(1+X)$ to $s(1+sX)$.

If $s > 1$, the bank will need to engage on new loans at the amount of

$$s(1+sX) - s(1+X) = s(s-1)X. \tag{A27}$$

If the cost per loan of engaging on new loans is λ_1, the bank will have an additional cost of $\lambda_1 s(s-1)X$.

Likewise, if $s < 1$, the bank will need to liquidate loans at the amount of

$$s(1+X) - s(1+sX) = s(1-s)X. \tag{A28}$$

If the cost of liquidating loans is equal to λ_2, the bank incurs in an additional cost of $\lambda_1 s(s-1)X$.

This rationale leads to the formulation of the $g(s,X)$ function as follows

$$g(s,X) = sX - \lambda_1(s(s-1)X)^+ - \lambda_2(s(1-s)X)^+.$$

□

Technical proof for Section 5.4.

Proof. Using the specification of $g(s,X) = G(s)X$ as in Section 5.4, under the assumptions referred in the Example, we show that this function matches Definition 1 and thus is a particular case of our framework.

We now show that

$$l(s) = E[\ln(1+g(s,X))] \tag{A29}$$

is concave. If we take the second derivative of $l(s)$, we get

$$l'' = E\left(\frac{G''X(1+GX) - (G'X)^2}{(1+GX)^2}\right) \tag{A30}$$

$$= G''E\left(\frac{X}{(1+GX)}\right) - (G')^2 E\left(\frac{(X)^2}{(1+GX)^2}\right), \tag{A31}$$

where for ease of notation we have suppressed the input s from G and l. The second term of the expression (A31) is non-positive, since $(G')^2$ is nonnegative, and

$$E\left(\frac{(X)^2}{(1+GX)^2}\right)$$

is also non-negative. To prove that l'' is nonpositive and show concavity, since G'' is also non-positive, it suffices to show that

$$E\left(\frac{X}{(1+GX)}\right)$$

is non-negative. Since the function

$$\phi(x) = \frac{x}{1+G(s)x}$$

is convex, Jensen's inequality yields

$$E\left(\frac{X}{1+G(s)X}\right) \geq \frac{E(X)}{1+G(s)E(X)}.$$

As we are assuming that $E(X) > 0$ and $G(s) \geq 0$, we have that

$$E\left(\frac{X}{1+G(s)X}\right) \geq 0,$$

thus proving the concavity of $l(s)$. □

References

Borwein, Jonathan M., and Qiji Jim Zhu. 2005. *Techniques of Variational Analysis*. New York: Springer.

Braun, Reiner, Tim Jenkinson, and Christoph Schemmerl. 2017. Giants at the Gate: Investment Returns and Diseconomies of Scale in Private Equity. *Journal of Financial Economics* 122: 585–604.

Cover, Thomas M. 1991. Universal Portfolios. *Mathematical Finance* 1: 1–29. [CrossRef]

Dewasurendra, Sagara, Pedro Júdice, and Qiji Jim Zhu. 2019. The Optimum Leverage Level of the Banking Sector. *Risks* 7: 51. [CrossRef]

Feng, Yu, Matúš Medo, Liang Zhang, and Yi-Cheng Zhang. 2011. Transaction Fees and Optimal Rebalancing in the Growth-Optimal Portfolio. *Physica A: Statistical Mechanics and Its Applications* 390: 1635–45. [CrossRef]

Geltner, David Michael. 1991. Smoothing in Appraisal-Based Returns. *Journal of Real Estate Finance and Economics* 4: 327–45. [CrossRef]

Giliberto, S. Michael. 1990. Equity Real Estate Investment Trusts and Real Estate Returns. *Journal of Real Estate Research* 5: 259–63. [CrossRef]

Greenwood, Robin, Augustin Landier, and David Thesmar. 2015. Vulnerable Banks. *Journal of Financial Economics* 115: 471–85. [CrossRef]

Kassouf, Sheen T., and Edward O. Thorp. 1967. *Beat the Market*. New York: Random House.

Kelly, John L., Jr. 1956. A New Interpretation of Information Rate. *Bell System Technical Journal* 35: 917–26. [CrossRef]

Lintner, John. 1965. The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets. *The Review of Economics and Statistics* 47: 13–37. [CrossRef]

MacLean, Leonard C., Edward O. Thorp, and William T. Ziemba, eds. 2011a. Introduction to the Good and Bad Properties of Kelly. In *The Kelly Capital Investment Criterion: Theory and Practice*. Singapore: World Scientific Publishing, pp. 459–64.

MacLean, Leonard C., Edward O. Thorp, and William T. Ziemba, eds. 2011b. *The Kelly Capital Growth Investment Criterion*. Singapore: World Scientific, vol. 3.

MacLean, Leonard C., William T. Ziemba, and George Blazenko. 1992. Growth versus Security in Dynamic Investment Analysis. *Management Science* 38: 1562–85. [CrossRef]

Prado, Marcos L. D., Ralph Vince, and Qiji Jim Zhu. 2019. Optimal Risk Budgeting under a Finite Investment Horizon. *Risks* 7: 86. [CrossRef]

Shannon, Claude E. 1948. A Mathematical Theory of Communication. *Bell System Technical Journal* 27: 379–423. [CrossRef]

Thorp, Edward O. 1962. *Beat the Dealer: A Winning Strategy for the Game of Twenty-One*. New York: Vintage Books.

Thorp, Edward O. 1971. Portfolio Choice and Kelly Criterion. In *Proceedings of the Business and Economics Section of the American Statistical Association*. Washington, DC: American Statistical Association, pp. 215–24.

Thorp, Edward O. 2018. Words From the Wise: An AQR Interview with Ed Thorp. Interviewed by Aaron Brown, Antti Ilmanen, and Rodney N. Sullivan. London: AQR Capital Management.

Vince, Ralph, and Qiji Jim Zhu. 2015. Optimal Betting Sizes for the Game of Blackjack. *Journal of Investment Strategies* 4: 53–75. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.