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SEMIGROUP REAL CHARACTERS GENERATED BY QUASICHARACTERS

GASTÃO BETTENCOURT AND SÉRGIO MENDES

Abstract. Let S be an infinite, finitely generated semigroup, endowed with a probability measure. Based on the work of Erschler and Karlsson [3], we construct a real character of S using a random walk approach. In our construction, the word length used in Erschler and Karlsson's work is replaced by a quasicharacter.

1. Introduction

Let S be a semigroup. A real character of S is a semigroup homomorphism $S \to \mathbb{R}$ to the additive semigroup of real numbers $(\mathbb{R}, +)$. In [4, p.126] Grigorchuk defines a quasicharacter of S to be a function $f : S \to \mathbb{R}$ whose deviation from being a homomorphism is bounded, that is

$$|f(ab) - f(a) - f(b)| \leqslant D_f, \ \forall a, b \in S$$

for some constant $D_f \geqslant 0$. Quasicharacters are also known by the name of quasimorphisms. Following [4], we adopt throughout the text the terminology quasicharacter. A quasicharacter $f : S \to \mathbb{R}$ is called homogeneous if $f(g^n) = nf(g)$, for all $g \in S$ and for all $n \in \mathbb{N}$ ($n \in \mathbb{Z}$ if S is a group). Bounded cohomology of discrete groups is part of the theory of cohomology of topological groups and Banach algebras. It turns out that quasicharacters are useful to compute bounded cohomology in degree 2. Working in a rather general setting where S can be either a group or a semigroup, Grigorchuk proved that $H^2_b(S, \mathbb{R})$ contains a certain subspace $H^2_{b,2}(S, \mathbb{R})$. Moreover, [4, Theorem 3.5] shows that a homogeneous quasicharacter that is not a homomorphism determines a nonzero class in $H^2_{b,2}(S, \mathbb{R})$.

The use of random walks to construct homomorphisms has been considered by several authors. Karlsson and Ledrappier in [5] proved that if a finite first moment random walk has zero entropy and positive drift, then there exists a non-trivial homomorphism to $\mathbb{R}$. Erschler and Karlsson [3] used a random walk approach to establish the existence of such homomorphism in the case where the drift can also be zero. For a brief discussion of random walks see the Introduction in [1] and the bibliography therein. In this note, we show that Erschler and Karlsson's construction still works

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when we replace a finitely generated group by a finitely generated semigroup and the length function by a quasicharacter. We now give a brief description of the paper. Section 2 reviews some concepts on semigroups and probability measures on finitely generated semigroups. The class of measure with finite first moment is defined (see Definition 2.1). In section 3, a certain map, which we call the kernel (see (3.1)), is introduced and its basic properties are proved. In section 4 we construct out of a quasicharacter ϕ of the semigroup S, a sequence $(\chi_n)_n$ using the kernel of the previous section. In general, χ_n is not a quasicharacter. However, in the event that ϕ is a genuine character of S then χ_n coincide with ϕ, for every n. An important inequality (Proposition 4.5) is proved, which will be crucial in next section. Our main result is contained in section 5:

Theorem 5.1. Suppose that μ is non-degenerate, has finite second moment and for some sequence n_k it holds that $\Delta_{n_k}\Phi > 0$ and

$$\lim_{k\to+\infty} \frac{\beta(n_k)}{\Delta_{n_k}\Phi} = 0.$$

Then S admits a non-trivial real character.

The paper ends with a brief discussion about our construction in some classes of semigroups.

2. Preliminaries

To make the paper reasonably self-contained we start with some basic definitions about semigroups and probability measures. For an account on algebraic semigroups see [2].

A semigroup is a nonempty set S with associative multiplication. If X is a nonempty subset of S we denote by $\langle X\rangle$ the subsemigroup generated by X. We say that S is finitely generated if there is a nonempty finite set $X \subset S$ such that $S = \langle X\rangle$.

An element $e \in S$ is called a left (right) identity element if, for every $s \in S$, $es = s$ ($se = s$). A two-sided identity element of S, or simply identity of S, is an element that is both a right and a left identity. Although a semigroup can have several left (right) identities, the identity is unique if it exists. A semigroup with an identity element is called a monoid. Given a semigroup S without identity we may form a monoid by adjoining an identity element: $S^1 = S \cup \{e\}$.

A left (right) zero element of S is an element $z \in S$ such that, for every $s \in S$, $zs = z$ ($sz = z$). If z is both a left zero and a right zero of S, we simply call it a zero of S. If it exists, the zero is unique.

An element $a \in S$ is idempotent if $a^2 = a$. Zeros and identities are idempotent. When the semigroup S is commutative, the set $E(S)$ of idempotent elements of S form a subsemigroup, but noncommutative semigroups can have idempotent sets which are not subsemigroups.

A nonempty subset A of S is called a left (right) ideal if $SA \subseteq A$ ($AS \subseteq A$). A nonempty subset I is a two-sided ideal (or just ideal) of S if it is both a right and a left ideal, that is, $SI \subseteq I$, $IS \subseteq I$. It follows immediately that every left, right or two-sided ideal of S is a subsemigroup of S. The semigroup S is said to be left (right) simple if it contains no proper left (right) ideal. Similarly, S is called simple if the only ideal of S is S itself. A left (right) ideal is a principal left (right) ideal if it is of the form $\{a\} \cup Sa$ ($\{a\} \cup aS$) for some $a \in S$.

The semigroup S is said be left (right) cancellative if for any $a \in S$, the equation $ax = ay$ ($xa = ya$) in S implies $x = y$.

If S is a semigroup and $aba = a$, $bab = b$ for $a, b \in S$ then b is called an inverse for a. A semigroup is regular if each of its elements has an inverse. An inverse semigroup is a semigroup in which each of its elements $x \in S$ has a uniquely determined inverse (denoted by x^{-1}).

From now on we restrict to infinite, finitely generated semigroups S, endowed with a probability measure μ. Let $X = \{x_1, \cdots, x_p\} \subset S$ be a finite set of generators. For each generator $x \in X$ we assign a measure $\mu(x) > 0$. Note that, since $X \subset \operatorname{supp} \mu$, it follows that the support of μ generates the semigroup S.

The length of an element a of S is defined to be

$$l(a) = \min\{n \in \mathbb{N} : a = x_1 \cdots x_n,\, x_i \in X\}.$$

The length $l : S \to \mathbb{R}$ is a subadditive function

$$l(ab) \leqslant l(a) + l(b). \tag{2.1}$$

Let a, b be elements of S. We define the measure $a\mu$ by

$$a\mu(b) = \mu(a^{-1}b) = \mu(\{t \in S : b = at\})\,, \tag{2.2}$$

where $a^{-1}b = \{t \in S : b = at\}$. Since μ is a probability measure on S and $a^{-1}b \subseteq S$, then

$$0 \leq a\mu(b) = \mu(a^{-1}b) \leq \mu(S) = 1.$$

Similarly, given $a, b, c \in S$,

$$ab\mu(c) = \mu(b^{-1}a^{-1}c) = \mu(\{t \in S : c = abt\}).$$

Since μ is a probability measure, we have

$$\sum_{b \in S} \mu(b) = \sum_{u \in aS} \mu(a^{-1}u) = \sum_{v \in Sa} \mu(va^{-1}) = 1\,, \tag{2.3}$$

where $a^{-1}u = \{b \in S : u = ab\}$ and $va^{-1} = \{b \in S : v = ba\}$.

Given probability measures μ and ν on S define the convolution measure as

$$\begin{aligned} \mu * \nu(b) &= \sum_{a \in S} \mu(a) a\nu(b) \\ &= \sum_{u \in aS} \mu(a)\nu(a^{-1}u) \end{aligned}$$

Denote by μ^n the n-fold convolution.

We need to impose a compatibility condition on the measure μ.

Definition 2.1. We say that μ has finite first moment if

$$\sum_{g\in G} l(g)\mu(g) < +\infty$$

Assuming that μ has finite first moment, define a sequence L_n using the length l as

$$L_n = \sum_{b\in S} l(b)\mu^n(b). \tag{2.4}$$

The sequence L_n is subadditive: $L_{n+m} \leqslant L_n + L_m$. Hence, the classical Fekete lemma ensures that the limit

$$\ell_\mu = \lim_{n\to+\infty} \frac{L_n}{n}$$

exists in $\mathbb{R}\bigcup\{-\infty, +\infty\}$. Since $l \geqslant 0$ and L_n is subadditive we have

$$L_n \leqslant nL_1,$$

and since μ has finite first moment it follows that $\ell_\mu < +\infty$.

A quasicharacter of a semigroup S is a function $\phi : S \to \mathbb{R}$ verifying

$$|\phi(ab) - \phi(a) - \phi(b)| \leqslant D_\phi$$

for some real constant $D_\phi \geq 0$. We note that using the triangular inequality we have

$$\big|\,|\phi(ab) - \phi(b)| - |\phi(a)|\,\big| \leqslant |\phi(ab) - \phi(a) - \phi(b)| \leqslant D_\phi.$$

Hence,

$$|\phi(ab) - \phi(b)| \leqslant D_\phi + |\phi(a)|. \tag{2.5}$$

Similarly, we have

$$\big|\,|\phi(ab)| - |\phi(b) + \phi(a)|\,\big| \leqslant |\phi(ab) - \phi(a) - \phi(b)| \leqslant D_\phi,$$

which implies that

$$|\phi(ab)| \leqslant D_\phi + |\phi(a)| + |\phi(b)|.$$

More generally, this last inequality can be written as

$$|\phi(a_1a_2\cdots a_m)| \leqslant (m-1)D_\phi + \sum_{i=1}^{m} |\phi(a_i)|. \tag{2.6}$$

Given a quasicharacter ϕ, we define in analogy with (2.4),

$$\Phi_n = \sum_{a\in S} \phi(a)\mu^n(a) \tag{2.7}$$

From (2.3), we conclude that

$$\sum_{b\in S}\phi(ab)\mu^n(b)=\sum_{b\in S}\phi(b)a\mu^n(b)=\sum_{u\in aS}\phi(u)\mu^n(a^{-1}u)\,, \tag{2.8}$$

where $a^{-1}u=\{b\in S : u=ab\}$.

The next result shows that Φ_n verifies a generalized subadditivity property.

Proposition 2.2. *Let Φ_n be defined as above. Then $\Phi_{m+n}\leq\Phi_m+\Phi_n+D_\phi$.*

Proof. We first observe that

$$\Phi_{m+n}=\sum_{b\in S}\sum_{a\in S}\phi(ab)\mu^n(a)\mu^m(b).$$

In fact, we have

$$\begin{aligned}\sum_{a\in S}\sum_{b\in S}\phi(ab)\mu^n(a)\mu^m(b)&=\sum_{a\in S}\left(\sum_{b\in S}\phi(ab)\mu^m(b)\right)\mu^n(a)\\&=\sum_{a\in S}\left(\sum_{b\in S}\phi(b)a\mu^m(b)\right)\mu^n(a)\\&=\sum_{b\in S}\phi(b)\sum_{a\in S}\mu^n(a)a\mu^m(b)\\&=\sum_{b\in S}\phi(b)\mu^{m+n}(b)\\&=\Phi_{m+n}.\end{aligned}$$

Now, since ϕ verifies (2.5),

$$\begin{aligned}\Phi_{m+n}&\leq\sum_{b\in S}\sum_{a\in S}(\phi(a)+\phi(b)+D_\phi)\mu^n(a)\mu^m(b)\\&=\sum_{a,b\in S}\phi(a)\mu^n(a)\mu^m(b)+\sum_{a,b\in S}\phi(b)\mu^n(a)\mu^m(b)+D_\phi\sum_{a,b\in S}\mu^n(a)\mu^m(b)\\&=\sum_{a\in S}\phi(a)\mu^n(a)+\sum_{b\in S}\phi(b)\mu^m(b)+D_\phi\\&=\Phi_m+\Phi_n+D_\phi.\end{aligned}$$

□

Remark 2.3. From the above we conclude that $\Phi_n\leqslant n\Phi_1+(n-1)D_\phi$. In view of this we will consider quasicharacters ϕ for which $\Phi_1<+\infty$, that is, quasicharacters lying in the space $\ell^1(S,\mu)$.

3. The kernel

Given $a\in S$, define for each $n\in\mathbb{N}$ a map $K_a^n : S\to\mathbb{R}$ by

$$K_a^n(b)=a\mu^n(b)-\mu^n(b) \tag{3.1}$$

The map K_a^n plays the role of a kernel and verifies the following properties.

Proposition 3.1. *Let (S, μ) be a finitely generated semigroup with a probability measure. We have:*

(*i*) *If S has a left identity e then $K_e^n(a) = 0$, $\forall a \in S$;*
(*ii*) *If S is an inverse semigroup then $K^n_{aa^{-1}}(a) = K^n_{a^{-1}a}(a^{-1}) = 0$, $\forall a \in S$;*
(*iii*) *$K_e^n(b) = 0$, $\forall b \in eS$ and $\forall e \in E(S)$;*
(*iv*) *$\sum_{b\in S} K_a^n(b) = 0$;*
(*v*) *$\sum_{b\in S} |K_a^n(b)| \leq 2$;*
(*vi*) *$\sum_{a\in S} K_a^n(b)\mu(a) = \mu^{n+1}(b) - \mu^n(b)$.*

Proof. (i), (ii) and (iii) are simple computations.

We have:

$$\begin{aligned}\sum_{b\in S} K_a^n(b) &= \sum_{b\in S}(a\mu^n(b) - \mu^n(b)) = \sum_{b\in S} a\mu^n(b) - \sum_{b\in S}\mu^n(b)\\ &= \sum_{u\in aS}\mu^n(a^{-1}u) - \sum_{b\in S}\mu^n(b) = 1 - 1 = 0\,,\end{aligned}$$

which proves (iv). To establish (v) notice that

$$\begin{aligned}\sum_{b\in S}|K_a^n(b)| &= \sum_{b\in S}|a\mu^n(b) - \mu^n(b)|\\ &\leq \sum_{b\in S}(a\mu^n(b) + \mu^n(b))\\ &\leq 1 + 1 = 2.\end{aligned}$$

Let us prove (vi). We have,

$$\begin{aligned}\sum_{a\in S} K_a^n(b)\mu(a) &= \sum_{a\in S}(a\mu^n(b) - \mu^n(b))\mu(a)\\ &= \sum_{a\in S} a\mu^n(b)\mu(a) - \sum_{a\in S}\mu^n(b)\mu(a)\\ &= \sum_{a\in S} a\mu^n(b)\mu(a) - \mu^n(b)\sum_{a\in S}\mu(a)\\ &= \mu^{n+1}(b) - \mu^n(b).\end{aligned}$$

□

4. The sequence χ_n

From now on we fix a semigroup quasicharacter ϕ. Define a sequence of maps $\chi_n : S \to \mathbb{R}$ as follows

$$\chi_n(a) = \sum_{b\in S}(\phi(ab) - \phi(b))\mu^n(b). \tag{4.1}$$

The sequence χ_n may be rewritten using the kernel. In fact, by (2.8), we have

$$\begin{aligned}\chi_n(a) &= \sum_{b\in S}(\phi(ab)-\phi(b))\mu^n(b)\\ &= \sum_{b\in S}\phi(ab)\mu^n(b) - \sum_{b\in S}\phi(b)\mu^n(b)\\ &= \sum_{b\in S}\phi(b)(a\mu^n(b)-\phi(b)\mu^n(b)).\end{aligned}$$

Hence,

$$\chi_n(a) = \sum_{b\in S}\phi(b)K_a^n(b). \tag{4.2}$$

The following result collects some properties of the sequence χ_n.

Proposition 4.1. *Let χ_n be defined as above. We have*

(*i*) *If S has a left identity e then $\chi_n(e)=0$;*
(*ii*) *If $E(S)\neq\emptyset$ and S is left simple then $\chi_n(e)=0$, for every $e\in E(S)$;*
(*iii*) $|\chi_n(a)|\leqslant D_\phi + |\phi(a)|$
(*iv*) $\sum_{a\in S}\chi_n(a)\mu(a) = \Phi_{n+1}-\Phi_n$.

Proof. The first and second assertions follow from (*i*) and (*iii*) of proposition 3.1 respectively. To prove property (*iii*) just use the bound in (2.5). Finally, property (*iv*) follows from (*vi*) of proposition 3.1. □

In general, the sequence χ_n is not a semigroup homomorphism. The next result measures how far is χ_n of being a homomorphism. For the sake of readability we will use the following notation

$$\mathrm{Def}_{\chi_n}(a_1,a_2) = \chi_n(a_1a_2)-\chi_n(a_1)-\chi_n(a_2). \tag{4.3}$$

Lemma 4.2. *We have*

$$|\mathrm{Def}_{\chi_n}(a_1,a_2)| \leqslant (D_\phi + |\phi(a_1)|)\sum_{b\in S}|K_{a_2}^n(b)|. \tag{4.4}$$

Proof. We begin by observing that

$$\begin{aligned}\sum_{b\in S}\phi(a_1a_2b)\mu^n(b) &= \sum_{u\in a_1a_2S}\phi(u)\mu^n(a_2^{-1}a_1^{-1}u)\\ &= \sum_{v\in a_2S}\phi(a_1v)\mu^n(a_2^{-1}v)\\ &= \sum_{b\in S}\phi(a_1b)a_2\mu^n(b).\end{aligned}$$

We may now prove our claim.

$$|\mathrm{Def}_{\chi_n}(a_1,a_2)| = \left|\sum_{b\in S}(\phi(a_1a_2b)-\phi(a_1b)-\phi(a_2b)+\phi(b))\mu^n(b)\right|$$

$$
\begin{aligned}
&= \left| \sum_{b\in S} ((\phi(a_1a_2b) - \phi(a_2b))\mu^n(b) - \sum_{b\in S} (\phi(a_1b) - \phi(b))\mu^n(b) \right| \\
&= \left| \sum_{b\in S} (\phi(a_1b) - \phi(b))a_2\mu^n(b) - \sum_{b\in S} (\phi(a_1b) - \phi(b))\mu^n(b) \right| \\
&= \left| \sum_{b\in S} (\phi(a_1b) - \phi(b))K^n_{a_2}(b) \right| \\
&\leqslant \sum_{b\in S} |\phi(a_1b) - \phi(b)|\, |K^n_{a_2}(b)| \\
&\leqslant (D_\phi + |\phi(a_1)|) \sum_{b\in S} |K^n_{a_2}(b)|. \qquad \square
\end{aligned}
$$

Remark 4.3. As mentioned before, in general χ_n is not a homomorphism. Lemma 4.2 shows that, in general, χ_n is not even a quasicharacter.

Now, since

$$\sum_{b\in S} |K^n_a(b)| \leqslant 2,$$

we can consider a sequence $\beta(n)$ such that for every $a \in S$ there exists $C(a)$ with

$$\sum_{b\in S} |K^n_a(b)| \leqslant C(a)\beta(n). \tag{4.5}$$

The measure μ is adapted, i.e. the support generates the semigroup S. Therefore, we may take

$$\beta(n) = \max_{x_i\in X} \sum_{b\in S} |K^n_{x_i}(b)|.$$

The next result shows that the constant $C(a)$ in (4.5) can be made more explicit.

Lemma 4.4. *There exists a constant C_o such that*

$$\sum_{b\in S} |K^n_a(b)| \leqslant l(a)C_o\beta(n).$$

Proof. First note that for any pair $a_1, a_2 \in S$ we have

$$
\begin{aligned}
\sum_{b\in S} (a_1a_2\mu^n(b) - a_1\mu^n(b)) &= \sum_{u\in a_1a_2S} \mu^n(a_2^{-1}a_1^{-1}u) - \sum_{b\in S} a_1\mu^n(b) \\
&= \sum_{u\in a_1S} a_2\mu^n(a_1^{-1}u) - \sum_{u\in a_1S} \mu^n(a_1^{-1}u) \\
&= \sum_{u\in a_1S} (a_2\mu^n(a_1^{-1}u) - \mu^n(a_1^{-1}u)) \\
&= \sum_{b\in S} (a_2\mu^n(b) - \mu^n(b)).
\end{aligned}
$$

Suppose now that a is such that $l(a) = m$. We have $a = x_1 \cdots x_m$, for some $x_i \in X$, and

$$\begin{aligned}
\sum_{b\in S} |K_a^n(b)| &= \sum_{b\in S} |K_{x_1\cdots x_m}^n(b)| \\
&= \sum_{b\in S} |x_1\cdots x_m\mu^n(b) - \mu^n(b)| \\
&= \sum_{b\in S} |x_1\cdots x_m\mu^n(b) - x_1\cdots x_{m-1}\mu^n(b) + x_1\cdots x_{m-1}\mu^n(b) + \cdots \\
&\quad \cdots + x_1\mu^n(b) - \mu^n(b)| \\
&\leqslant \sum_{b\in S} |x_1\cdots x_m\mu^n(b) - x_1\cdots x_{m-1}\mu^n(b)| + \\
&\quad + \sum_{b\in S} |x_1\cdots x_{m-1}\mu^n(b) - x_1\cdots x_{m-2}\mu^n(b)| + \cdots \\
&\quad \cdots + \sum_{b\in S} |x_1\mu^n(b) - \mu^n(b)| \\
&\leqslant \sum_{b\in S} |x_m\mu^n(b) - \mu^n(b)| + \sum_{b\in S} |x_{m-1}\mu^n(b) - \mu^n(b)| + \cdots \\
&\quad \cdots + \sum_{b\in S} |x_1\mu^n(b) - \mu^n(b)| \\
&= \sum_{i=1}^m \sum_{b\in S} |K_{x_i}^n(b)|.
\end{aligned}$$

By (4.5), $\sum_{b\in S} |K_a^n(b)| \leqslant C(a)\beta(n)$, so

$$\begin{aligned}
\sum_{i=1}^m \sum_{b\in S} |K_{x_i}^n(b)| &\leqslant \sum_{i=1}^m C(x_i)\beta(n) \\
&\leqslant \max_{x_i\in X} C(x_i) \sum_{i=1}^m \beta(n) \\
&\leqslant \max_{x_i\in X} C(x_i)\beta(n)m.
\end{aligned}$$

Write $C_o = \max_{x_i\in X} C(x_i)$. Since $m = l(g)$ the result follows. □

Let us define the sequence $\gamma_X(n) := \gamma(n)$, which depends on the choice of a generating set X of the semigroup S, by

$$\gamma(n) = \max_{x\in X} |\chi_n(x)|. \tag{4.6}$$

We have the following.

Proposition 4.5. *Let $\chi_n(a)$ be defined as above, μ a probability measure on S and $a \in \langle supp\,\mu\rangle$. We have*

$$|\chi_n(a)| \leqslant C_2\beta(n)l^2(a) + l(a)\gamma(n), \tag{4.7}$$

with $C_2 = C_2(\phi, a)$.

Proof. Let $a \in \langle \operatorname{supp} \mu \rangle$. Since S is generated by the finite set $X = \{x_1, \cdots, x_p\}$ we can write $a = x_1 \cdots x_m$ where $m = l(a)$. By the previous Lemmas, and using (2.6) we have

$$\begin{aligned} |\mathrm{Def}_{\chi_n}(x_1 \cdots x_{m-1}, x_m)| &\leqslant (D_\phi + |\phi(s_1 \cdots s_{m-1})|) \sum_{b \in S} |K^n_{x_m}(b)| \\ &\leqslant ((m-1)D_\phi + \sum_{i=1}^{m-1} |\phi(x_i)|) C_o \beta(n). \end{aligned}$$

Write $\phi_o = \max_{x \in X} |\phi(x)|$. We get

$$|\mathrm{Def}_{\chi_n}(x_1 \cdots x_{m-1}, x_m)| \leqslant ((m-1)D_\phi + (m-1)\phi_o)\, C_o \beta(n)$$

and writing C_1 as $C_1 = \max\{D_\phi, \phi_o\}$ we get

$$\begin{aligned} |\mathrm{Def}_{\chi_n}(x_1 \cdots x_{m-1}, x_m)| &\leqslant 2C_1(m-1)C_o\beta(n) \\ &= C_2(m-1)\beta(n). \end{aligned}$$

Therefore,

$$\begin{aligned} |\chi_n(a)| &= |\chi_n(x_1 \cdots x_m)| \\ &= |\chi_n(x_1 \cdots x_m) - \chi_n(x_1 \cdots x_{m-1}) - \chi_n(x_m) + \\ &+ \chi_n(x_1 \cdots x_{m-1}) - \chi_n(x_1 \cdots x_{m-2}) - \chi_n(x_{m-1}) + \cdots \\ &\cdots + \chi_n(x_1 x_2) - \chi_n(x_1) - \chi_n(x_2) + \sum_{i=1}^{m} \chi_n(x_i)| \\ &\leqslant C_2\beta(n)((m-1) + (m-2) + \cdots + 1) + \sum_{i=1}^{m} |\chi_n(x_i)| \\ &\leqslant C_2\beta(n)m^2 + m\gamma(n). \end{aligned}$$

□

As a special case, suppose the quasicharacter $\phi : S \to \mathbb{R}$ is a genuine homomorphism into the additive semigroup of the real numbers. This is, of course, a particular case of a quasicharacter where $D_\phi = 0$. We have

$$\begin{aligned} \chi_n(a) &= \sum_{b \in S} (\phi(ab) - \phi(b)) \mu^n(b) \\ &= \sum_{b \in S} (\phi(a) + \phi(b) - \phi(b)) \mu^n(b) \\ &= \phi(a) \sum_{b \in S} \mu^n(b) \\ &= \phi(a). \end{aligned}$$

So, in this case, $\chi_n \equiv \phi$ is a semigroup homomorphism.

More general, let $f : S \to \mathbb{R}$ be a semigroup homomorphism to $\mathbb{R}$ and fix a pair of real numbers $\alpha, \beta \in \mathbb{R}$. The map $\phi(a) = \alpha f(a) + \beta$ is a quasicharacter, for

$$|\phi(ab) - \phi(a) - \phi(b)| = |-\beta| = |\beta| = D_\phi.$$

Again, we have

$$\begin{aligned}\chi_n(a) &= \sum_{b\in S}(\phi(ab) - \phi(b))\mu^n(b)\\ &= \alpha f(a)\sum_{b\in S}\mu^n(b)\\ &= \alpha f(a).\end{aligned}$$

We conclude the following

Proposition 4.6. *Let $\phi(a) = \alpha f(a) + \beta$, where $f : S \to \mathbb{R}$ is a semigroup homomorphism and α, β is a pair of real numbers. Then, the associated sequence $\chi_n = \alpha f$ is a semigroup homomorphism.*

5. Real characters generated by quasicharacters

We begin this section by explaining why the homomorphism to $\mathbb{R}$ we are about to obtain in the construction below in nontrivial. Suppose the generating set X is chosen so that the sequence (4.6) verifies $\gamma(n) \neq 0$. We may then put $\alpha(n) = 1/\gamma(n)$. For $a \in S$ with $a = x_1x_2 \dots x_m$, where $x_i \in X$ and $l(a) = m$, Proposition 4.5 allows one to conclude that

$$\alpha(n)|\chi_n(a)| \leqslant C_2\alpha(n)\beta(n)m^2 + m.$$

If the product $\alpha(n)\beta(n)$ remains bounded along some infinite subsequence, take a subsequence $n_k \to +\infty$ of our subsequence such that the limit

$$\chi = \lim_{k\to+\infty} \alpha(n_k)\chi_{n_k}(a)$$

exists, for every $a \in S$. Then, for every n there exists $x \in X$ such that $\alpha(n)\chi_n(x) = 1$. By construction, the limit χ is nontrivial.

Now, upon some conditions on the measure μ, if $\alpha(n)\beta(n) \to 0$, Lemmas (4.2) and (4.4) imply that χ is a homomorphism to $\mathbb{R}$. Summing up, to ensure χ is a nontrivial semigroup homomorphisms to $\mathbb{R}$, it is sufficient to know that there exists a generating set X and a subsequence $(n_k)_k$ such that

(i) $\gamma(n_k) \neq 0$ and (ii) $\dfrac{\beta(n_k)}{\gamma(n_k)} \to 0$ as $k \to +\infty$.

Define $\Delta_n\Phi = \Phi_{n+1} - \Phi_n$, with Φ_n as in (2.7). We may now state and prove our main result.

Theorem 5.1. *Suppose that μ is non-degenerate, has finite second moment and for some sequence n_k it holds that $\Delta_{n_k}\Phi > 0$ and*

$$\lim_{k\to+\infty} \frac{\beta(n_k)}{\Delta_{n_k}\Phi} = 0.$$

Then S admits a non-trivial real character.

Proof. By Proposition (4.5) we have

$$|\chi_n(a)| \leqslant C_2\beta(n)l^2(a) + l(a)\gamma(n).$$

So

$$\begin{aligned}\sum_{a\in S}|\chi_n(a)|\mu(a) &\leqslant C_2\beta(n)\sum_{a\in S}l^2(a)\mu(a) + \gamma(n)\sum_{a\in S}l(a)\mu(a)\\ &= C_3\beta(n) + \gamma(n)C_4.\end{aligned}$$

Using the hypotheses of this proposition and lemma (4.1) we have

$$\begin{aligned}0 &\leqslant \Delta_{n_k}\Phi\\ &= \sum_{a\in S}\chi_{n_k}(a)\mu(a)\\ &= |\sum_{a\in S}\chi_{n_k}(a)\mu(a)|\\ &\leqslant \sum_{a\in S}|\chi_{n_k}(a)|\mu(a)\\ &\leqslant C_3\beta(n_k) + \gamma(n_k)C_4.\end{aligned}$$

Again using the hypotheses we have that for any $\varepsilon > 0$ there exists n_k such that $\beta(n_k) \leqslant \varepsilon\,\Delta_{n_k}\Phi$ and we have

$$\begin{aligned}0 < \Delta_{n_k}\Phi &\leqslant C_3\varepsilon\,\Delta_{n_k}\Phi + \gamma(n_k)C_4.\\ (1 - C_3\varepsilon)\Delta_{n_k}L &\leqslant \gamma(n_k)C_4.\end{aligned}$$

Choosing ε such that $C_3\varepsilon < 1/2$ we have, for n_k big enough

$$0 < (1/2)\Delta_{n_k}\Phi \leqslant \gamma(n_k)C_4. \tag{5.1}$$

Now, define the coefficient $\alpha(n_k) = \dfrac{1}{\gamma(n_k)}$. We have

$$0 < \alpha(n_k)\,\Delta_{n_k}\Phi \leqslant 2C_4.$$

Therefore,

$$0 \leqslant \alpha(n_k)\beta(n_k) = \alpha(n_k)\,\Delta_{n_k}\Phi\,\frac{\beta(n_k)}{\Delta_{n_k}\Phi} \leqslant 2C_4\frac{\beta(n_k)}{\Delta_{n_k}\Phi}$$

and using our convergence hypothesis we conclude that

$$\alpha(n_k)\beta(n_k) \to 0.$$

Using lemma (4.2) we can say that

$$\alpha(n_k)|\mathrm{Def}_{\chi_{n_k}}(a_1, a_2)| \leqslant (D_\phi + |\phi(a_1)|)C(a_2)\beta(n_k)\alpha(n_k).$$

Hence,

$$\lim_{k\to+\infty}\alpha(n_k)\mathrm{Def}_{\chi_{n_k}}(a_1, a_2) = 0$$

which amounts to say that

$$\chi(a_1a_2) = \chi(a_1) + \chi(a_2),$$

with $\chi(a) = \lim_{k\to+\infty} \alpha(n_k)\chi_{n_k}(a)$. □

We end with a brief discussion of Theorem 5.1 for certain classes of semigroups.

First of all, the construction in this note is completely general for finitely generated semigroups with a probability measure. In particular, it includes the case of finitely generated groups, which was treated in the first part of [1].

Suppose S is a semigroup such that every element is idempotent. Then, a real character $\chi : S \to \mathbb{R}$ is necessarily trivial, for

$$\chi(a) = \chi(a^2) = \chi(a) + \chi(a)$$

and so $\chi(a) = 0$, for all $a \in S$. An example of such a semigroup is a right (left) zero semigroup, that is, a semigroup satisfying

$$ab = b \ \ (ab = a) \ \forall\, a, b \in S.$$

Now, suppose S is a right group. Recall that a semigroup is called a right group if it is right simple and left cancellative. A left group is defined dually. According to [2, Theorem 1.27], S is a right group if and only if S is a direct product $G \times E$ of a group G and a right zero semigroup E. Let $\chi : G\times E \to \mathbb{R}$ be a real character of a right group. Let e denote the identity of G. We have:

$$\chi((g, a)(e, b)) = \chi(g, a) + \chi(e, b) = \chi(g, b). \tag{5.2}$$

On the other hand,

$$\chi((e, b)(g, a)) = \chi(e, b) + \chi(g, a) = \chi(g, a). \tag{5.3}$$

Hence,

$$\chi(g, a) = \chi(g, b) \ , \ \forall a, b \in E \, , \forall g \in G. \tag{5.4}$$

We conclude that every real character $\chi : G \times E \to \mathbb{R}$ is determined by the group G. Thus, the results of [1] apply to the class of right groups $G \times E$ where G is a finitely generated group and E is a right zero semigroup, not necessarily finitely generated. In fact, in view of (5.4), it is enough to define the sequence χ_n to be constant on E

$$\chi(g, a) = \sum_{h\in G} (\phi(gh) - \phi(h)\mu^n(h), \tag{5.5}$$

where $\phi : G \to \mathbb{R}$ is a quasimorphism of groups and μ is a finitely generated measure on G. Thus, we are actually working in the setting of finitely generated groups with a probability measure as in [1].

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