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Robust strategy for a member of defined contribution pension plan when asset prices can jump

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ABSTRACT

This paper investigates the robust optimal investment for an ambiguity averse member of a defined contribution (DC) pension plan in a fully-fledged, time consistent mean-variance modeling framework. In particular, the paper extends the literature on defined contribution pension plans in three directions: (1) We relax its assumption of purely continuous stock and/or contribution processes, which allows to introduce the effects of news, job loss, macroeconomic conditions, etc., into the model; (2) Unlike most studies in DC pension plans, we allow for ambiguity about both the mean arrival rate and jump size distribution of the stock returns and contribution rate processes of the member; (3) Ambiguity in our setting is time-varying. The model thus features stochastic stock volatility, stochastic interest rate, stochastic contribution rate, jumps in both stock and contribution rate processes, and time-varying ambiguity about diffusion parameters. Welfare analysis indicates that ignoring ambiguity can be very costly to the member. The framework proposed in this paper is general and adds significant realism to existing models in the literature.

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1. Introduction

Defined contribution (DC) pension plans play a pivotal role in retirement planning as they offer flexibility in wealth allocation and empower individuals to take control of their financial futures. These plans allow participants to make contributions from their salaries, which are then invested in a range of assets such as stocks, bonds, and mutual funds. This investment flexibility enables individuals to build a diversified portfolio tailored to their risk tolerance, investment goals, and time horizon. A closely related term is the defined benefit (DB) pension plan. The main difference between the two is that in the DB plan, the benefits are fixed by the sponsor from the outset while the contribution is adjusted to maintain the fund balance. Consequently, the investment risk and the member's longevity risk are undertaken by the sponsor in the DB pension plan, while transferred to the member in the DC pension plan.

In many countries, the aging population and the longevity risk pose a threat to DB pension systems. Thereby, there has been a burgeoning interest on DC pension plans over their DB counterparts as a mechanism to mitigate this risk to the sponsor. Given the investment risk underlying DC plans, a realistic and predictive model capable of accommodating key price features becomes essential for informed and efficient decision-making.

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This paper investigates the optimal investment problem for a member of DC pension plan that is not fully confident about her model estimates and, thereby, also considers alternative models that are robust to misspecification.[1] The paper sets itself apart from the rest of the literature in three directions. First, in addition to stochastic volatility, interest rate and contribution rate processes, we allow for jumps with random sizes in both stock price and contribution rate of the ambiguity averse member (AAM) of DC pension plan. Although the introduction of jumps in stock returns complicates the analysis significantly, discrete shocks in prices are ubiquitous in finance and widely accepted as an important driving factor of asset returns – see, for instance, Eraker et al. (2003), Eraker (2004), Broadie et al. (2007), and Shaliastovich (2015). Hence, our framework introduces an additional layer of realism to existing models by extending the literature to include both stochastic opportunity set and jumps.

For the sake of tractability, most studies – see, for instance, Cairns et al. (2006), Han and Hung (2012), Guan and Liang (2014), Sun et al. (2017), Wang and Li (2018), Zhang et al. (2020), and Wang et al. (2021) – abstract away from jump risks in the stock process. Although this assumption facilitates the calculations significantly, it renders the model less effective in capturing periods characterized by significant prices drops such as the Great Recession of 2007. Other studies do include jumps in their analyses (e.g. Sun et al., 2016), but assume either deterministic interest rates or constant volatility (or both).

Second, the AAM is ambiguous not only about the diffusive components of stock volatility, and bond and stock returns, but also about the jump size distribution and jump intensity parameters of the stock process. Due to analytical tractability, most studies on DC pension plans focus purely on either diffusive ambiguity or jump intensity ambiguity, or both, with jump size ambiguity being mostly overlooked. But more often than not, AAM and investors in general are not only uncertain about jump intensity, but also about the distribution of jump size. Consequently, a model able to take this into account can give important insights on the role of ambiguity on the AAM's decision making problems. Ambiguity in our framework is based on robust control theory, pioneered by Hansen and Sargent, with preferences being in the class of recursive multiple priors of Epstein and Schneider (2003) – see, for instance, Hansen and Sargent (2001), Anderson et al. (2003), Drechsler (2013), Aït-Sahalia and Matthys (2019), and Glória et al. (2024).

Third, ambiguity in our framework is time-varying, consistent with empirical studies (e.g. Drechsler, 2013; Shi, 2019). In particular, we allow ambiguity about bond returns to increase with interest rates, ambiguity about stock returns to increase with stock volatility and ambiguity about stock volatility to increase with stock volatility itself. These relations are well understood in the literature and, thus, add more realism to our model. The relationship between volatility and ambiguity captures the idea that the future growth of the economy is especially misty during recessions and extremely volatile periods (Sbuelz & Trojani, 2008). Similarly, in addition to causing the bond price to go down and increasing its volatility, high interest rates may signal concerns about inflationary pressures, tightening monetary policy or financial instability and, thus, increase the AAM ambiguity about her model.

Our paper also contributes more broadly to the literature on portfolio allocation by studying robust decision problems in a setting with stochastic stock returns, stochastic interest rates, stochastic volatility and jumps in stock returns. To the best of our knowledge, no previous work has undertaken a similar analysis. Branger and Larsen (2013) and Aït-Sahalia and Matthys (2019) analyze the optimal investment problem of a robust investor in a setting with jumps and constant opportunity set. Flor and Larsen (2014) consider the case of a robust investor that has access to bonds, stock and cash, where the interest rate follows the Vasicek (1977) model. Guan et al. (2022) introduce model uncertainty into the strategic interaction between firms and unions in overfunded DB pension plans. Zhang et al. (2025) introduce model uncertainty into a competitive DC pension setting,

[1] In this paper, we use the terms ambiguity and model uncertainty interchangeably.

modeling two ambiguity-averse fund managers who seek robust strategies under worst-case scenarios. They demonstrate that ambiguity significantly alters optimal portfolio decisions compared to traditional, non-robust models.

Escobar et al. (2015) and Wei et al. (2023) extend Branger and Larsen (2013) to include stochastic volatility and deterministic jump size. Escobar et al. (2018) study the optimal portfolio of a robust investor in a setting with stochastic interest rates, stock returns and volatility. Our paper is closely related to Escobar et al. (2018), but has important differences. First, ambiguity in our setting is time-varying. Second, stock returns in our framework are subject to jumps with random sizes while in theirs are purely diffusive. Third, in our model, the robust investor is not only uncertain about diffusive parameters of the relevant processes dynamics, but also about jump size and jump intensity parameters.

The framework in this paper is developed in a continuous-time version of Markowitz (1952) mean-variance framework. More specifically, the member aims to find an equilibrium strategy that maximizes the expected value of terminal surplus and, at the same time, minimize its variance, two conflicting objectives given the well-known mean-variance tradeoff. In order to be in line with what is observed in practice, the strategy must be time consistent and robust to model misspecification. However, the variance term in the standard (Markowitz, 1952) mean-variance problem implies that the investment problems are time inconsistent in the sense that Bellman optimality principle does not hold.

The literature deals with this problem in two different ways. One strand of the literature employs the so-called precommitment strategy, wherein it is assumed that the AAM pre-commits to a strategy from the beginning and does not change her decisions as time passes. However, given the long investment horizon of DC pension plans, precommitment strategy is a strong assumption. The other strand considers investment problems within a game theoretic framework of Björk and Murgoci (2010) and seeks an equilibrium strategy by solving the extended Hamilton-Jacobi-Bellman (HJB) system of equations. In the latter, the investment strategy is time consistent. Our paper falls within the second group inasmuch as the AAM wants decisions rules that are optimal not only at a single point in time, but also during the full accumulation phase. By applying the game theoretic framework of Björk and Murgoci (2010) together with our specification of rectangular set of priors, we obtain strategies that are time consistent and robust to model misspecification. For more on the mean-variance framework see, for instance, Bian et al. (2018), Sun et al. (2016), Wang et al. (2021) and references therein.

The rest of the paper is organized as follows. Section 2 lays out the model primitives and formulates the problem. Section 3 presents the solutions to the auxiliary and the original problems. Section 4 presents some comparative statics results and Section 5 concludes.

2. Model primitives

We assume a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, \mathbb{P})$, satisfying the usual conditions, where $0 < T < \infty$ is the investment horizon (i.e. the period $[0, T]$ represents the accumulation phase of the member). All stochastic processes below are defined on this probability space. $\mathbb{P}$ denotes the member's reference probability measure which, given the historical information available to him, represents his best estimates of the data generating process. In addition, $\mathbb{Q}$ denotes the, not necessarily unique, equivalent martingale measure that has the money market account as numeraire.

2.1. Reference model

The investment opportunity set available to the member at time $t(\geq 0)$ consists of three assets: a risk-free asset with price $S_0(t)$, a stock with price $S_1(t)$, and a rolling bond with price $B_K(t)$, where $K \geq t$ denotes the bond maturity.

The instantaneous interest rate $r(t)$ and stock volatility $V(t)$ evolve according to[2]

$$\mathrm{d}r(t) = (a - br(t))\,\mathrm{d}t - \sqrt{k_1 r(t) + k_2}\,\mathrm{d}W_r(t), \quad r(0) = r_0, \tag{1}$$

$$\mathrm{d}V(t) = \alpha(\delta - V(t))\,\mathrm{d}t + \sigma_V\sqrt{V(t)}\left(\rho\,\mathrm{d}W_S(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_V(t)\right), \quad V(0) = V_0, \tag{2}$$

where $\{W_r(t) : t \geq 0\}$, $\{W_V(t) : t \geq 0\}$ and $\{W_S(t) : t \geq 0\}$ are three mutually independent Brownian motions, a, b, k_1 and k_2 are non-negative constants, and α, σ_V and δ are positive constants, and $\rho \in [0, 1]$. The affine process of the interest rate includes the Vasicek (1977) ($k_1 = 0$) and Cox and Ingersoll (1985b) (CIR) ($k_2 = 0$) as special cases.

As pointed out in Cox and Ingersoll (1985b), Equation (1) (when $k_2 = 0$) and (2) have a stationary Gamma distribution, provided that $b > 0$ and $a/b > 0$, and $\alpha > 0$ and $\delta > 0$, respectively. In the case of $V(t)$, the stationary Gamma has shape parameter $2\alpha\delta/V(t)$ and scale parameter $\sigma_V^2/(2\alpha)$. If $2\alpha\delta > \sigma_V^2$, the Feller (1951) condition is then satisfied and the distribution is finite at zero. Since it is not realistic to assume the existence of zero-coupon bonds with any maturity $t > 0$, we follow Boulier et al. (2001) and assume a constant maturity K for the rolling bond.

The stock price is assumed to follow a jump-diffusion model in order to incorporate discrete shocks that are frequently observed in prices (see, for instance, Broadie et al., 2007; Eraker, 2004). Under the reference belief $\mathbb{P}$, the three assets have the following dynamics

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\,\mathrm{d}t, \quad S_0(0) > 0, \tag{3}$$

$$\begin{aligned}\frac{\mathrm{d}S_1(t)}{S_1(t^-)} &= \left[r(t) + \sigma_S\lambda_r(k_1 r(t) + k_2) + \nu V(t) + (k_S l - k_S^{\mathbb{Q}} l^{\mathbb{Q}})V(t)\right]\mathrm{d}t \\ &\quad + \sigma_S\sqrt{k_1 r(t) + k_2}\,\mathrm{d}W_r(t) + \sqrt{V(t)}\,\mathrm{d}W_S(t) \\ &\quad + (e^{Z(t)} - 1)\,\mathrm{d}N(t) - lV(t)k_S\,\mathrm{d}t, \quad S_1(0) > 0,\end{aligned} \tag{4}$$

$$\frac{\mathrm{d}B_K(t)}{B_K(t)} = r(t)\,\mathrm{d}t + f(K)\lambda_r(k_1 r(t) + k_2)\,\mathrm{d}t + f(K)\sqrt{k_1 r(t) + k_2}\,\mathrm{d}W_r(t), \quad B_K(K) = 1, \tag{5}$$

where σ_S captures the sensitivity of stock volatility to interest rate risk, $\lambda_r\sqrt{k_1 r(t) + k_2}$ is the market price of Brownian motion risk W_r, $\nu\sqrt{V(t)} > 0$ represents the market price of risk $W_S(t)$, $\{N(t) : t \geq 0\}$ is a Poisson process with time-varying intensity $\{lV(t) : t \geq 0\}$ for $l \geq 0$ and independent of all Brownian shocks in this paper, $\{Z(t) : t \geq 0\}$ is a set of normal random variables all with mean μ_S and variance Σ^2 under $\mathbb{P}$ for all $t \geq 0$, and $\mu_S^{\mathbb{Q}}$ and Σ^2 under $\mathbb{Q}$, $k_S = E[e^{Z(t)} - 1] = e^{\mu_S + \frac{1}{2}\Sigma^2} - 1$, $(k_S l - k_S^{\mathbb{Q}} l^{\mathbb{Q}})V(t)$ is the premium associated with jump risk, $f(t) = \frac{2(e^{mt}-1)}{m-(b-k_1\lambda_r)+e^{mt}(m+b-k_1\lambda_r)}$, $m = \sqrt{(b - k_1\lambda_r)^2 + 2k_1}$. For convenience, we assume that $Z(t)$ has time-invariant distribution under $\mathbb{P}$ and $\mathbb{Q}$ and that its outcomes are independent of the other random variables.

Finally, we consider a DC pension plan member that is entitled to a labor income (i.e. salary) and that deposits a stochastic fraction thereof $\{C(t) : t \in [0, T]\}$ into her pension account before

[2] With a slight abuse of terminology, we use the term stock volatility to denote the diffusion volatility associated with the stock idiosyncratic risk W_S, not the total stock volatility. In addition, while we assume independence between $r(t)$ and $V(t)$, it is important to note that correlation between $r(t)$ and the total stock volatility is already guaranteed given that total instantaneous return variance is given by: $\sigma_S^2(k_1 r(t) + k_2) + V(t) +$ jumpsvariance (see Equation (4)) and, hence, our model is able to capture the correlation between stock total volatility and interest rates, as documented in Bian et al. (2025).

retirement time T. $C(t)$ is called the contribution rate and is assumed to evolve according to

$$\begin{aligned}\frac{\mathrm{d}C(t)}{C(t^-)} &= \Big[\vartheta + \sigma_{C_1}\lambda_r(k_1 r(t)+k_2) + \sigma_{C_2}\nu V(t) + \sigma_{C_2}V(t)(k_S l - k_S^{\mathbb{Q}} l^{\mathbb{Q}})\Big]\,\mathrm{d}t \\ &\quad + \sigma_{C_1}\sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r(t) + \sigma_{C_2}\sqrt{V(t)}\,\mathrm{d}W_S(t) + \sigma_{C_2}(e^{Z(t)}-1)\,\mathrm{d}N(t) \\ &\quad - \sigma_{C_2} l k_S V(t)\,\mathrm{d}t, \quad C(0) = C_0, \end{aligned} \tag{6}$$

where ϑ is a constant, σ_{C_1} and σ_{C_2} are scale parameters measuring how different diffusive and jump risks stemming from the interest rate and stock affect salary.

It is easy to see that $C(t)$ is perfectly correlated with stock, which implies that it can be spanned by the financial market. The specification is motivated by tractability. Introducing non-hedgeable shocks would significantly complicate the analysis and is thus left for future research.

2.2. Perturbed models and maximization problem

This paper departs from traditional frameworks of the DC pension plan, where the member is assumed to be uncertainty neutral. In that case, she has full confidence in the model. In many occasions, however, the member does not know or has just a hunch regarding the true data generating process underlying her investment problems. As a result, she might want decision rules that work relatively well not only when her reference belief $\mathbb{P}$ turns out to be true, but also in cases that it happens to be misspecified.

She thus considers a new probability measure $\mathbb{P}^\eta$ that is equivalent to $\mathbb{P}$, where $\{\eta(t) : t \geq 0\}$ denotes the likelihood ratio process that allows measure change from $\mathbb{P}$ to $\mathbb{P}^\eta$. Fear of model misspecification induces the AAM to consider alternative models that can insure her against the unpleasant outcomes of choosing the wrong model, which take the form

$$\mathrm{d}r(t) = (a - br(t) - \sqrt{k_1 r(t)+k_2}\,h_1(t))\,\mathrm{d}t - \sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r^\eta(t), \tag{7}$$

$$\begin{aligned}\frac{\mathrm{d}B_K(t)}{B_K(t)} &= \Big[r(t) + f(K)\lambda_r(k_1 r(t)+k_2) + f(K)\sqrt{k_1 r(t)+k_2}\,h_1(t)\Big]\,\mathrm{d}t \\ &\quad + f(K)\sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r^\eta(t), \end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\mathrm{d}S_1(t)}{S_1(t^-)} &= \Big[r(t) + \sigma_S\lambda_r(k_1 r(t)+k_2) + \sigma_S\sqrt{k_1 r(t)+k_2}\,h_1(t) + (l^\eta(t)k_S^\eta(t) - l^{\mathbb{Q}}k_S^{\mathbb{Q}})V(t) \\ &\quad + \nu V(t) + \sqrt{V(t)}h_2(t)\Big]\,\mathrm{d}t + \sigma_S\sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r^\eta(t) + \sqrt{V(t)}\,\mathrm{d}W_S^\eta(t) \\ &\quad + (e^{Z^\eta(t)}-1)\,\mathrm{d}N^\eta(t) - l^\eta(t)V(t)k_S^\eta(t)\,\mathrm{d}t, \end{aligned} \tag{9}$$

$$\begin{aligned}dV(t) &= \Big[\alpha(\delta - V(t)) + \sigma_V\sqrt{V(t)}\rho h_2(t) + \sqrt{1-\rho^2}\sigma_V\sqrt{V(t)}h_3(t)\Big]\,\mathrm{d}t \\ &\quad + \sigma_V\sqrt{V(t)}\left(\rho\,\mathrm{d}W_S^\eta(t) + \sqrt{1-\rho^2}\,\mathrm{d}W_V^\eta(t)\right), \end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\mathrm{d}C(t)}{C(t^-)} &= \Big[\vartheta + \sigma_{C_1}\lambda_r(k_1 r(t)+k_2) + \sigma_{C_2}\nu V(t) + \sigma_{C_1}\sqrt{k_1 r(t)+k_2}\,h_1(t) \\ &\quad + \sigma_{C_2}(l^\eta(t)k_S^\eta(t) - l^{\mathbb{Q}}k_S^{\mathbb{Q}})V(t) + \sigma_{C_2}\sqrt{V(t)}h_2(t)\Big]\,\mathrm{d}t + \sigma_{C_1}\sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r^\eta(t) \\ &\quad + \sigma_{C_2}\sqrt{V(t)}\,\mathrm{d}W_S^\eta(t) + \sigma_{C_2}(e^{Z^\eta(t)}-1)\,\mathrm{d}N^\eta(t) - \sigma_{C_2}l^\eta(t)k_S^\eta(t)V(t)\,\mathrm{d}t. \end{aligned} \tag{11}$$

The set of alternative models differs from the reference model described in Section 2.1 in some important ways. First, the drift now includes additional terms that are linear functions of $\mathcal{F}(t)$-measurable

perturbation parameters $h(t) = [h_1(t), h_2(t), h_3(t)]$. The new drifts are results of the application of Girsanov's theorem. Thus, $W^\eta = [W_r^\eta, W_S^\eta, W_V^\eta]$ is a Brownian motion under $\mathbb{P}^\eta$. Second, N^η is a Poisson process with mean arrival rate $l^\eta(t)V(t)$ under $\mathbb{P}^\eta$. Third, the random variable $Z^\eta(t)$ also follows a disturbed normal distribution. Note that $k_S^\eta(t) = E^\eta[e^{Z(t)} - 1]$ is now a function of time, which highlights the non-stationarity of Z^η.

Following, for instance, Drechsler (2013) and Glória et al. (2024), we assume that the jump intensity under $\mathbb{P}^\eta$ is given by

$$l^\eta(t)V(t) = e^{a_l(t)}lV(t), \tag{12}$$

where $\{a_l(t) \in \mathbb{R}^+ : t \in [0, T]\}$ is a process that increases or decreases jump intensity at each time t. If $a_l(t) > 0$ for all $t \in [0, T]\mathbb{P}$-a.s., then $l^\eta(t) > l$ and jumps under $\mathbb{P}^\eta$ occur at higher frequency than under the reference belief. When $a_l(t) = 0$ for all $t \in [0, T]$, we go back to l. $Z^\eta(t)$ is assumed to have a distorted density function $\zeta^\eta(Z(t))$ under $\mathbb{P}^\eta$

$$Z^\eta(t) \sim \mathcal{N}(\mu_S + \Delta\mu_S(t), s_\sigma(t)\Sigma^2), \tag{13}$$

where $\mathcal{N}(\mu, \sigma^2)$ is a normal law with mean μ and variance σ^2, and $\{\Delta\mu_S(t) \in \mathbb{R} : t \in [0, T]\}$ and $\{s_\sigma(t) \in \mathbb{R}^{++} : t \in [0, T]\}$ are mean and variance perturbations. When $\Delta\mu_S(t) > 0$ and $s_\sigma(t) > 1$ for all $t \in [0, T]$, jumps (in absolute value) under $\mathbb{P}^\eta$ are larger on average than under $\mathbb{P}$ and extreme values are more common. When $\Delta\mu_S(t) = 0$ and $s_\sigma(t) = 1$ for all $t \in [0, T]$, Z^η distribution collapses to the member's distribution under the reference belief. Given that $W(t)$ and $N(t)$ are independent, we have

$$\frac{\mathrm{d}\mathbb{P}^\eta}{\mathrm{d}\mathbb{P}}\bigg| \mathcal{F}(t) = \eta(t) = \eta^D(t)\eta^J(t),$$

where $\eta^D(t)$ is a $(\mathcal{F}(t), \mathbb{P})$-martingale that allows the change of measure of the diffusive components, and is given by

$$\begin{aligned}\eta^D(t) = \exp\Bigg\{ &\int_0^t h_1(s)\,\mathrm{d}W_r(s) - \frac{1}{2}\int_0^t (h_1(s))^2\,\mathrm{d}s + \int_0^t h_2(s)\,\mathrm{d}W_S(s) \\ &-\frac{1}{2}\int_0^t (h_2(s))^2\,\mathrm{d}s + \int_0^t h_3(s)\,\mathrm{d}W_V(s) - \frac{1}{2}\int_0^t (h_3(s))^2\,\mathrm{d}s\Bigg\},\end{aligned} \tag{14}$$

and $\eta^J(t)$ is a $(\mathcal{F}(t), \mathbb{P})$-martingale that allows the change of measure of the jump part. The appropriate process for η^J takes the form

$$\eta^J(t) = e^{\int_0^t V(s)[l - l^\eta(s)]\,\mathrm{d}s} \prod_{n=1}^{N(t)} \frac{l^\eta(t_n)}{l} \frac{\zeta^\eta(Z(n))}{\zeta(Z(n))}, \quad E[\eta^J(t)] = 1, \quad \mathbb{P}-\text{a.s.}. \tag{15}$$

It immediately follows that

$$\begin{aligned}\eta^J(t) = \exp\Bigg\{ &\int_0^t \left(a_l(s) + b_2(s)Z(s)^2 + b_1(s)Z(s) - \frac{1}{2}\left[\frac{(\mu_S + \Delta\mu_S(s))^2}{s_\sigma(s)\Sigma^2} - \frac{\mu_S}{\Sigma^2} + \ln s_\sigma(s) \right] \right) \\ &\times \mathrm{d}N(s) - \int_0^t lV(s)(\exp(a_l(s)) - 1)\,\mathrm{d}s \Bigg\}, \quad E[\eta^J(t)] = 1,\ \mathbb{P}-\text{a.s.},\end{aligned} \tag{16}$$

where $b_1(t) = \frac{\mu_S(1 - s_\sigma(t)) + \Delta\mu_S(t)}{s_\sigma(t)\Sigma^2}$ and $b_2 = \frac{1}{2\Sigma^2}(1 - \frac{1}{s_\sigma(t)})$. In order to control the extent to which the alternative models differ from the reference one, we use relative entropy. The growth in entropy of $\mathbb{P}^\eta$ to $\mathbb{P}$ between t and $t + \Delta t$ is defined as

$$G(t, t + \Delta t) \equiv E_t^\eta \left[\log \frac{\eta(t + \Delta t)}{\eta(t)} \right], \tag{17}$$

where $E_t^\eta[\cdot]$ denotes the $\mathcal{F}(t)$-conditional expectation under the probability measure $\mathbb{P}^\eta$.

Taking the limit as $\Delta t \to 0$, we obtain the instantaneous growth rate of relative entropy at time t

$$\mathcal{R}(\eta(t)) \equiv \lim_{\Delta t \to 0} \frac{G(t, t+\Delta t)}{\Delta t}. \tag{18}$$

We restrict the AAM's set of alternative models by imposing an upper bound on $\mathcal{R}$. The set of admissible alternative models is then

$$\{\eta(t) : \mathcal{R}(\eta(t)) \leq \varphi(t),\ \forall\, t \geq 0, \varphi(t) \geq 0\}, \tag{19}$$

where $\varphi(t)$ will be defined shortly. Full confidence in the model (i.e. no ambiguity) is characterized by $\varphi(t) = 0$ for all $t \in [0, T]$, whereby $\mathcal{R}(\eta(t)) = 0$ and, hence, $\mathbb{P} = \mathbb{P}^{\eta}$. As $\varphi(t)$ increases, so does the set of alternative models. This characterizes a situation in which the AAM has increasing ambiguity about the model. Due to the independence of N and W, relative entropy can be written in the following way: $\mathcal{R}(\eta(t)) = \mathcal{R}(\eta^D(t)) + \mathcal{R}(\eta^J(t))$. The decomposition of relative entropy into diffusive and jump parts implies that the entropy growth bound can also be divided into two parts: $\varphi(t) = \varphi^D(t) + \varphi^J(t)$. The independence of the Brownian motions further implies that we can write $\varphi^D(t) = \varphi_1^D(t) + \varphi_2^D(t) + \varphi_3^D(t)$.

Assumption 2.1: The AAM has time-varying ambiguity about the diffusive components of the interest rate, stock and stock volatility processes, characterized by

$$\varphi_1^D(t) = \omega_1 \frac{k_1 r(t) + k_2}{k_1 a/b + k_2}, \quad \varphi_2^D(t) = \omega_2 \frac{V(t)}{\delta}, \quad \varphi_3^D(t) = \omega_3 \frac{V(t)}{\delta},$$

where $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^+$. Moreover, the bound on jump entropy growth is constant, that is $\varphi^J(t) = \varphi^J \in \mathbb{R}^+$.

Assumption 2.1 implies that ambiguity about stock, interest rates and volatility is time-varying with mean $\omega_i, i \in \{1, 2, 3\}$. In particular, uncertainty about interest rates (resp., stock returns and volatility) increases in periods of high interest rates (resp., volatility). The relationship between volatility and ambiguity captures the idea that the future growth of the economy is especially misty during recessions and extremely volatile periods (Sbuelz & Trojani, 2008). Similarly, in addition to causing the bond price to go down and increasing its volatility, high interest rates may signal concerns about inflationary pressures, tightening monetary policy or financial instability and, thus, increase the AAM ambiguity about her model.

Remark 2.2: Although the bound on jump entropy growth is constant, our framework still implies time-varying jump perturbation parameters. This follows because the opportunity set is stochastic. In particular, time variation in the expected return, volatility and jump intensity cause the jump perturbation parameters to change through time.

The relative entropy growth in this framework can be obtained in closed form.

Lemma 2.3: *The relative entropy growth is given by*

$$\begin{aligned}\mathcal{R}(\eta(t)) = \mathcal{R}(\eta^D(t)) + \mathcal{R}(\eta^J(t)) &= \frac{1}{2}h_1(t)^2 + \frac{1}{2}h_2(t)^2 + \frac{1}{2}h_3(t)^2 \\ &+ lV(t)\left(e^{a_l(t)}\left(a_l(t) + \frac{\Delta\mu_S(t)^2}{2\Sigma^2} + \frac{s_\sigma(t)}{2} - \frac{1}{2}\ln s_\sigma(t) - \frac{3}{2}\right) + 1\right).\end{aligned} \tag{20}$$

Proof: Apply Itô's lemma to Equation (16), and once again to $\log \eta(t)$, integrate and take expectation under $\mathbb{P}^{\eta}$. The result follows by dividing by Δt and taking limit as $\Delta t \to 0$. ■

Assume an initial wealth of x_0 in the member's pension account. Furthermore, let $\pi_B(t)$, $\pi_S(t)$ and $1 - \pi_B(t) - \pi_S(t)$ be the fraction of wealth invested in the rolling bond, the stock and the risk-free asset, respectively, at time t. Let $\pi \equiv \{(\pi_S(t), \pi_B(t)) : t \in [0, T]\}$. Finally, let $\{X^{\pi}(t) : t \in [0, T]\}$ be the wealth process associated with π. Under the perturbed measure $\mathbb{P}^{\eta}$, $X^{\pi}(t)$ evolves according to

$$
\begin{aligned}
\mathrm{d}X^{\pi}(t) &= (1 - \pi_B(t) - \pi_S(t))X^{\pi}(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_B(t)X^{\pi}(t)\frac{\mathrm{d}B_K(t)}{B_K(t)} \\
&\quad + \pi_S(t^-)X^{\pi}(t^-)\frac{\mathrm{d}S_1(t)}{S_1(t^-)} + C(t)\,\mathrm{d}t \\
&= X^{\pi}(t)\Bigg[r(t) + \theta_r(t)\lambda_r(k_1 r(t) + k_2) + \theta_r(t)\sqrt{k_1 r(t) + k_2}h_1(t) + \pi_S(t)\nu V(t) \\
&\quad + \pi_S(t)(l^{\eta}(t)k_S^{\eta}(t) - l^{\mathbb{Q}}k_S^{\mathbb{Q}})V(t) + \pi_S(t)\sqrt{V(t)}h_2(t) + \frac{C(t)}{X^{\pi}(t)}\Bigg]\mathrm{d}t \\
&\quad + X^{\pi}(t)\theta_r(t)\sqrt{k_1 r(t) + k_2}\,\mathrm{d}W_r^{\eta}(t) + X^{\pi}(t)\pi_S(t)\sqrt{V(t)}\,\mathrm{d}W_S^{\eta}(t) \\
&\quad + X^{\pi}(t^-)\pi_S(t^-)(e^{Z^{\eta}(t)} - 1)\,\mathrm{d}N^{\eta}(t) - X^{\pi}(t)\pi_S(t)l^{\eta}(t)k_S^{\eta}(t)V(t)\,\mathrm{d}t, \quad X^{\pi}(0) = x_0, \quad (21)
\end{aligned}
$$

where $\theta_r(t) = \pi_B(t)f(K) + \pi_S(t)\sigma_S$.

Remark 2.4: It is readily seen that θ_r and π_S capture the AAM's wealth exposure to Brownian motion risks $W_r^{\eta}(t)$ and $W_S^{\eta}(t)$, respectively. Our contribution is that π_S also represents the AAM's wealth exposure to jump risk $N(t)$ from stock.

Definition 2.5: Let Π be the set of all admissible strategies $\pi \equiv \{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$. A strategy $(\pi_B(t), \pi_S(t))$ is admissible if

(1) it is progressively measurable with respect to the filtration $\mathcal{F}(t)$ and càdlàg (i.e. right continuous with left limit);
(2) X^{π} is integrable, i.e. $E_t^{\eta^*}\int_0^T [X^{\pi}(t)^2\theta_r(t)^2(k_1 r(t) + k_2) + \pi_S(t)^2 X^{\pi}(t)^2 V(t)]\,\mathrm{d}t < \infty$, where η^* denotes the worst-case probability measure;
(3) Equation (21) has a unique strong solution $\{X^{\pi}(t) : t \in [0, T]\}$ given a set of initial conditions $(t, r(t), V(t), X^{\pi}(t)) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$.

Following, for instance, Guan and Liang (2014), we assume that the terminal benefits are converted into a life annuity. In order to maintain the member's standard of living after retirement, a target level is defined by the pension fund. Thus, the AAM strives to accumulate a terminal wealth above the target retirement benefit. After retirement, the target annuity stream for the member at time $t \in [T, T']$ is $q(t)$, where T' is the random time of death. Therefore, the actuarial present value of the target at time T is given by

$$
\mathcal{Q}(T) = \int_T^w q(s)B(T, s)_{s-T}p_T\,\mathrm{d}s, \quad (22)
$$

where w is the maximum survival age and $q(s) = q(T)e^{I(s-T)}$ and I is the inflation rate. ${}_{s-T}p_T$ is the probability that the member will survive to s conditional on being alive at T, and is given by ${}_{s-T}p_T = e^{-\int_T^s \lambda(u)\,\mathrm{d}u}$, where λ is a deterministic mortality force function. For simplicity, we follow

Wang et al. (2021) and assume that the mortality force is described by the Gompertz mortality function $\lambda(t) = w_1 e^{w_2 t}$, where $w_1 > 0$ and $w_2 > 0$ are the Gompertz mortality shape and scale parameters, respectively. This implies that

$$ {}_{s-T}p_T = \exp\left(-\int_T^s \lambda(u)\,\mathrm{d}u\right) = \exp\left[\frac{w_1}{w_2}(e^{w_2 T} - e^{w_2 s})\right]. $$

$B(t,s)$ is the time-t price of a zero coupon bond with maturity s and face value of \$1, given by

$$ B(t,s) = \exp\left(-f(s-t)r(t) + f_1(s-t)\right), $$

where

$$ f_1(t) = \frac{k_2}{k_1}t - \frac{k_2}{k_1}f(t) + \left(a + \frac{k_2}{k_1}b\right)\frac{2}{k_1}\log\left(\frac{2me^{t(m+b-k_1\lambda_r)/2}}{m-(b-k_1\lambda_r)+e^{mt}(m+b-k_1\lambda_r)}\right). $$

The present value of the target retirement benefit $\mathcal{Q}(T)$ at time t is given by

$$ \mathcal{Q}(t) = B(t,T)\mathcal{Q}(T) = \int_T^w q(s)B(t,s)\,{}_{s-T}p_T\,\mathrm{d}s, \quad t \le T, \tag{23} $$

or, in differential form,

$$ \frac{\mathrm{d}\mathcal{Q}(t)}{\mathcal{Q}(t)} = r(t)\,\mathrm{d}t + \frac{\sqrt{k_1 r(t)+k_2}}{\mathcal{Q}(t)}\left[\mathrm{d}W_r(t) + \lambda_r\sqrt{k_1 r(t)+k_2}\right]\int_T^w {}_{s-T}p_T q(s)B(t,s)f(s-t)\,\mathrm{d}s. \tag{24} $$

The pension plan member aims to maximize the expected terminal wealth over the time-T present value of target annuity payment and, at the same time, minimize its variance, two conflicting objectives given the well-known risk-return tradeoff in portfolio theory. Assuming ambiguity neutrality, the problem can be formulated in a mean-variance framework:

$$ \max_{\pi\in\Pi}\left\{E[X^\pi(T) - \mathcal{Q}(T)] - \frac{\gamma}{2}\mathrm{Var}[X^\pi(T) - \mathcal{Q}(T)]\right\}, \tag{25} $$

subject to (1), (2), (6), and

$$ \begin{aligned} \frac{\mathrm{d}X^\pi(t)}{X^\pi(t^-)} &= \left[r(t) + \theta_r(t)\lambda_r(k_1 r(t)+k_2) + \pi_S(t)\nu V(t) + \pi_S(t)(lk_S - l^{\mathbb{Q}}k_S^{\mathbb{Q}})V(t) + \frac{C(t)}{X^\pi(t)}\right]\mathrm{d}t \\ &\quad + \theta_r(t)\sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r(t) + \pi_S(t)\sqrt{V(t)}\,\mathrm{d}W_S(t) \\ &\quad + \pi_S(t^-)(e^{Z(t)}-1)\,\mathrm{d}N(t) - \pi_S(t)lk_S V(t)\,\mathrm{d}t, \end{aligned} \tag{26} $$

and where $\gamma > 0$ is the risk aversion coefficient.

The variance term in the maximization problem above implies that the outlined problem is not dynamically consistent (also known as time consistent). Simply put, this means that a strategy that is optimal at the initial time may no longer be optimal at a later time. This complicates the analysis because the Bellman optimality principle does not hold and it becomes infeasible to use the dynamic programing approach. The literature usually employs one of the two workarounds. In some cases, it seeks the precommitment strategy in which a strategy, deem optimal at the initial time, is assumed to be followed by the member no matter what happens in the future until maturity. Underlying this assumption is the fact that the member does not update her beliefs and tastes as time passes. The control problem in this case is of the form

$$ \sup_{\pi\in\Pi}\left\{E_0[X^\pi(T) - \mathcal{Q}(T)] - \frac{\gamma}{2}\mathrm{Var}_0[X^\pi(T) - \mathcal{Q}(T)]\right\}, \tag{27} $$

where $E_0[\cdot] = E[\cdot|x_0, V_0, r_0, C_0]$ and $\mathrm{Var}_0[\cdot] = \mathrm{Var}[\cdot|x_0, V_0, r_0, C_0]$. Given the long maturity of DC pension contracts, spanning several decades, precommitment strategy is a strong assumption.

The problem can also be handled by seeking an equilibrium strategy without precommitment in which the member updates her target at each time t with objective function

$$\sup_{\pi \in \Pi} \left\{ E_t[X^\pi(T) - \mathcal{Q}(T)] - \frac{\gamma}{2} \mathrm{Var}_t[X^\pi(T) - \mathcal{Q}(T)] \right\}, \tag{28}$$

subject to Equations (1), (2), (6), and (26).

In this paper, we adopt the equilibrium approach which is dynamically consistent and, thereby, allows us to employ the recursive multiple priors framework of Epstein and Schneider (2003).

The problem formulated in (28) describes a situation wherein the member is ambiguity neutral. However, the robust member is not fully confident in her model. As a result, she takes investment decisions that work well not only when the reference model is true, but also in cases that it happens to be the wrong model. Her optimization problem has the following program. She first chooses the worst-case perturbation parameters $h, a_l, \Delta\mu_S$ and s_σ. Next, given the worst-case parameters $h^*, a_l^*, \Delta\mu_S^*$, and s_σ^*, she selects π that maximizes the mean-variance payoff. She has the following equilibrium optimization problem

$$\sup_{\pi \in \Pi} \inf_{\{h, a_l, \Delta\mu_S, s_\sigma\}} \left\{ E_t^\eta \left[X^\pi(T) - \mathcal{Q}(T) \right] - \frac{\gamma}{2} \mathrm{Var}_t^\eta \left[X^\pi(T) - \mathcal{Q}(T) \right] \right\}, \tag{29}$$

subject to (7), (10), (11), (21), and

$$\mathcal{R}(\eta(t)) \leq \varphi(t).$$

3. Equilibrium investment strategy

As mentioned before, the precommitment strategy is time inconsistent and, as a result, it is not appealing from a realistic point of view. Moreover, recursive multiple priors framework requires dynamic consistency of preferences, which further increases the motivation to consider strategies that are time consistent. The problem can be solved within the game theoretic framework of Björk and Murgoci (2010). More precisely, the problem can be viewed as a non-cooperative game where we have one player at each point in time.

We can interpret this player, named player t, as a future incarnation of the AAM. Hence, the objective of the AAM updates as state $(t, X^\pi(t), V(t), r(t), C(t))$ changes, and the AAM's decision process looks like a game process between an infinite number of different players. Before looking for the subgame perfect Nash equilibrium strategy, we first solve an auxiliary problem that transforms the original problem into a self financing one.

3.1. The auxiliary problem

The problem (29) is not a self financing one because there is a continuous injection of contribution C from the member. This increases the dimensionality of the problem and it becomes difficult to fully separate the variables in the HJB equations. Therefore, we follow previous literature and transform the original problem into a self financing one.

Following, for instance, Han and Hung (2012) and Guan and Liang (2014), we simplify the optimization problem by constructing an auxiliary problem based on derivatives pricing theory. We aim to replicate the continuous contribution rate $C(t)$. To this end, let $D(t, s)$ denote the time-t price of a

s-claim paying $C(s)$ at maturity s with $t \le s$. Then

$$D(t,s) = E_t^{\mathbb{Q}}[e^{-\int_t^s r(u)\,\mathrm{d}u} C(s)] \equiv D(t,s,r(t),C(t)). \tag{30}$$

The likelihood ratio process that allows measure change from $\mathbb{P}$ to $\mathbb{Q}$ takes the form

$$\begin{aligned}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}(t)} = \exp\Bigg\{ &-\frac{1}{2}\int_0^t [\lambda_r^2(k_1 r(s)+k_2) + \nu^2 V(s)]\,\mathrm{d}s - \int_0^t \lambda_r\sqrt{k_1 r(u)+k_2}\,\mathrm{d}W_r(u) \\ &- \int_0^t \nu\sqrt{V(u)}\,\mathrm{d}W_S(u)\Bigg\} e^{\int_0^t V(u)(l-l^{\mathbb{Q}})\,\mathrm{d}u} \prod_{n=1}^{N(t)} \frac{l^{\mathbb{Q}}}{l}\frac{\zeta^{\mathbb{Q}}(Z(n))}{\zeta(Z(n))}.\end{aligned} \tag{31}$$

We recall that $\lambda_r\sqrt{k_1 r(t)+k_2}$ is the market price of risk $W_r(t)$, and $\nu\sqrt{V(t)}$ is the market price of risk $W_S(t)$. Thus, under $\mathbb{Q}$, we have

$$\mathrm{d}r(t) = [a - br(t) + (k_1 r(t)+k_2)\lambda_r]\,\mathrm{d}t - \sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r^{\mathbb{Q}}(t),$$

and

$$\begin{aligned}\frac{\mathrm{d}C(t)}{C(t^-)} = {} & \vartheta\,\mathrm{d}t + \sigma_{C_1}\sqrt{k_1 r(t)+k_2}\,\mathrm{d}W_r^{\mathbb{Q}}(t) + \sigma_{C_2}\sqrt{V(t)}\,\mathrm{d}W_S^{\mathbb{Q}}(t) + \sigma_{C_2}(e^{Z^{\mathbb{Q}}(t)}-1)\,\mathrm{d}N^{\mathbb{Q}}(t) \\ & - \sigma_{C_2} V(t) l^{\mathbb{Q}} k_S^{\mathbb{Q}}\,\mathrm{d}t,\end{aligned}$$

where $W_r^{\mathbb{Q}}(t) = W_r(t) + \int_0^t \sqrt{k_1 r(u)+k_2}\lambda_r\,\mathrm{d}u$ and $W_S^{\mathbb{Q}}(t) = W_S(t) + \int_0^t \sqrt{V(\tau)}\nu\,\mathrm{d}\tau$ are two independent Brownian motions under $\mathbb{Q}$, and $N^{\mathbb{Q}}(t)$ is a Poisson process with mean arrival rate $l^{\mathbb{Q}}V(t)$. $Z^{\mathbb{Q}}(t)$ is a normal random variable with mean $\mu_S^{\mathbb{Q}}$ and variance Σ^2 under $\mathbb{Q}$. Given $r(t) = r$ and $C(t) = C$, $D(t,s,r,C)$ satisfies the following partial-integro differential equation (PIDE):

$$\begin{cases} D_t + D_C C\vartheta + D_r(a-br) + \frac{1}{2}D_{rr}(k_1 r + k_2) - D_{Cr}C\sigma_{C_1}(k_1 r + k_2) - D_C C k_S^{\mathbb{Q}} l^{\mathbb{Q}} V\sigma_{C_2} \\ \quad + l^{\mathbb{Q}} V\sigma_{C_2} E^{\mathbb{Q}}[D(t,s,r,C + C\sigma_{C_2}(e^Z - 1)) - D] = -\frac{1}{2}D_{CC}C^2[\sigma_{C_1}^2(k_1 r(t) + k_2 + \sigma_{C_2}^2 V)] \\ \quad + rD - \lambda_r D_r(k_1 r(t) + k_2) \\ D(s,s,r,C) = C \end{cases}, \tag{32}$$

where subscripts denote partial derivatives. The solution to (32) is given by

$$D(t,s) = C(t)e^{\beta_1(s-t)+\beta_2(s-t)r(t)}, \tag{33}$$

where

$$\begin{aligned}\beta_1(t) &= \int_0^t [a\beta_2(s) + \frac{1}{2}k_2\beta_2(s)^2 + (\lambda_r - \sigma_{C_1})k_2\beta_2(s) + \vartheta]\,\mathrm{d}s, \\ \beta_2(t) &= \frac{-2e^{\sqrt{\mathcal{C}}t} + 2}{(b - \lambda_r k_1 + \sigma_{C_1}k_1 + \sqrt{\mathcal{C}})e^{\sqrt{\mathcal{C}}t} + \sqrt{\mathcal{C}} - (b - \lambda_r k_1 + \sigma_{C_1}k_1)}, \\ \mathcal{C} &= (b - \lambda_r k_1 + \sigma_{C_1}k_1)^2 + 2k_1.\end{aligned}$$

Define $F(t,T) = \int_t^T D(t,s)\,\mathrm{d}s$. Hence, we can interpret $D(t,s)$ as the price today of a claim to C paid $s-t$ years in the future and $F(t,T)$ as the price today of a claim paying C continuously from t to T.

Then

$$\begin{aligned} \mathrm{d}F(t,T) = &-C(t)\,\mathrm{d}t + r(t)F(t,T)\,\mathrm{d}t + \int_t^T D(t,s)(\sigma_{C_1} - \beta_2(s-t))\,\mathrm{d}s\sqrt{k_1 r(t) + k_2}\,\mathrm{d}W_r^{\mathbb{Q}}(t) \\ &+ \sigma_{C_2}F(t,T)\sqrt{V(t)}\,\mathrm{d}W_S^{\mathbb{Q}}(t) + F(t^-,T)\sigma_{C_2}(e^{Z^{\mathbb{Q}}(t)} - 1)\,\mathrm{d}N^{\mathbb{Q}}(t) \\ &- \sigma_{C_2}F(t,T)V(t)l^{\mathbb{Q}}k_S^{\mathbb{Q}}\,\mathrm{d}t. \end{aligned} \tag{34}$$

Despite market incompleteness and the fact that $F(t,T)$ is not directly traded in the financial market, Proposition 3.1 shows that it can be replicated via a combination of traded assets in the market.

Proposition 3.1: *The process $F(t,T)$ can be replicated by the cash, rolling bond and stock in the market:*

$$\frac{\mathrm{d}F(t,T) + C(t)\,\mathrm{d}t}{F(t,T)} = \pi_0^F(t)\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_B^F\frac{\mathrm{d}B_K(t)}{B_K(t)} + \pi_S^F(t)\frac{\mathrm{d}S_1(t)}{S_1(t^-)}, \tag{35}$$

where

$$\begin{aligned} \pi_0^F(t) &= 1 - \pi_B^F(t) - \pi_S^F(t), \\ \pi_B^F(t) &= \frac{\int_t^T D(t,s)(\sigma_{C_1} - \beta_2(s-t))\,\mathrm{d}s - \sigma_S\sigma_{C_2}F(t^-,T)}{f(K)F(t^-,T)}, \\ \pi_S^F(t) &= \sigma_{C_2}. \end{aligned} \tag{36}$$

These results hold for any measure $* \in \{\mathbb{P}, \mathbb{P}^\eta, \mathbb{Q}\}$.

Proof: The result follows by comparing the coefficients of Equation (35) with those of Equations (4) and (5). The results under the perturbed and reference measures are similar and can be obtained by comparing the coefficients of $F(t,T)$, $B_K(t)$ and $S_1(t)$ under $\mathbb{P}^\eta$ and $\mathbb{P}$, respectively, instead of $\mathbb{Q}$. ■

Given that the minimum guarantee $\mathcal{Q}$ is only correlated with the interest rate risk, it follows that it too can be replicated by cash and bonds.

Proposition 3.2: *Given that the $\mathcal{Q}$ is only correlated with the interest rate risk, it can be replicated by holding a proportion $\pi_B^{\mathcal{Q}}$ of the rolling bond and a proportion $1 - \pi_B^{\mathcal{Q}}$ of cash, i.e.*

$$\frac{\mathrm{d}\mathcal{Q}(t)}{\mathcal{Q}(t)} = \pi_0^{\mathcal{Q}}\frac{\mathrm{d}S_0(t)}{S_0(t)} + \pi_B^{\mathcal{Q}}\frac{\mathrm{d}B_K(t)}{B_K(t)}, \tag{37}$$

where

$$\begin{aligned} \pi_0^{\mathcal{Q}}(t) &= 1 - \pi_B^{\mathcal{Q}}(t), \\ \pi_B^{\mathcal{Q}}(t) &= \frac{\int_T^w {}_{s-T}p_T q(s)B(t,s)f(s-t)\,\mathrm{d}s}{f(K)\mathcal{Q}(t)}. \end{aligned}$$

These results hold for any measure $* \in \{\mathbb{P}, \mathbb{P}^\eta, \mathbb{Q}\}$.

Proof: The result follows by comparing the coefficients of Equation (24) with those of Equation (5). Results under $\mathbb{P}^\eta$ or $\mathbb{Q}$ are similar and can be obtained using the $\mathbb{P}^\eta$- or $\mathbb{Q}$-dynamics of $B_K(t)$ instead of $\mathbb{P}$.

■

We are now ready to define the surplus process which reduces the number of state variables to consider in the optimization problem. Define the surplus process $Y^{\mathbf{u}}(t) = X^{\pi}(t) + F(t,T) - \mathcal{Q}(t)$. Then, it is easy to see that

$$\begin{aligned}\frac{\mathrm{d}Y^{\mathbf{u}}(t)}{Y^{\mathbf{u}}(t^-)} &= r(t)\,\mathrm{d}t + [u_B(t)f(K) + u_S(t)\sigma_S]\sqrt{k_1 r(t) + k_2}\,\mathrm{d}W_r^{\mathbb{Q}}(t) + u_S(t)\sqrt{V(t)}\,\mathrm{d}W_S^{\mathbb{Q}}(t)\\ &\quad + u_S(t^-)(e^{Z^{\mathbb{Q}}} - 1)\,\mathrm{d}N^{\mathbb{Q}}(t) - u_S(t^-)V(t)l^{\mathbb{Q}}k_S^{\mathbb{Q}}\,\mathrm{d}t, \end{aligned} \tag{38}$$

$$\begin{aligned} u_0(t) &= 1 - u_B(t) - u_S(t),\\ u_B(t) &= \frac{\pi_B(t)X^{\pi}(t) + \pi_B^F(t)F(t,T) - \pi^{\mathcal{Q}}(t)\mathcal{Q}(t)}{Y^{\mathbf{u}}(t)},\\ u_S(t) &= \frac{\pi_S(t)X^{\pi}(t) + \pi_S^F(t)F(t,T)}{Y^{\mathbf{u}}(t)}, \end{aligned} \tag{39}$$

and $\mathbf{u} = \{(u_B(t), u_S(t)) : t \in [0,T]\}$. Note that $\mathbf{u}$ is admissible if π associated with $\mathbf{u}$ is admissible. With some abuse of notation, we say that $\mathbf{u} \in \Pi$ if it is admissible. By construction, $Y^{\mathbf{u}}(T) = X^{\pi}(T) + F(T,T) - \mathcal{Q}(T) = X^{\pi}(T) - \mathcal{Q}(T)$. We can thus transform the condition $X^{\pi}(T) \geq \mathcal{Q}(T)$ into $Y(T) \geq 0$ since $Y^{\mathbf{u}}(T) = X^{\pi}(T) - \mathcal{Q}(T)$ at T. Given that $Y^{\mathbf{u}}(t)$ is self-financing, we require that $Y^{\mathbf{u}}(0) = X^{\pi}(0) - F(0,T) - \mathcal{Q}(0) \geq 0$. This condition is necessary and sufficient to ensure that there exists an admissible investment strategy $\mathbf{u}$ such that $Y^{\mathbf{u}}(T) \geq 0$. Let $Y^{\mathbf{u}}(t) = y$ and $J(t,y,V,r,\mathbf{u})$ denote the member's value function. Thus, the optimization problem (29) becomes:

$$J(t,y,V,r,\mathbf{u}^*) = \sup_{\mathbf{u}\in\Pi}\inf_{\{h,a_l,\Delta\mu_S,s_\sigma\}}\left\{E_t^{\eta}[Y^{\mathbf{u}}(T)] - \frac{\gamma}{2}\mathrm{Var}_t^{\eta}[Y^{\mathbf{u}}(T)]\right\}, \tag{40}$$

subject to Equations (7), (10), (38) (but now under $\mathbb{P}^{\eta}$) and $\mathcal{R}(\eta(t)) \leq \varphi(t)$.

An admissible strategy $\mathbf{u}^*$ is called an efficient strategy if there exists no other admissible strategy $\mathbf{u}$ such that $E^{\eta}[Y^{\mathbf{u}}(T)] \geq E^{\eta}[Y^{\mathbf{u}^*}]$ and $\mathrm{Var}^{\eta}[Y^{\mathbf{u}^*}(T)] \geq \mathrm{Var}^{\eta}[Y^{\mathbf{u}}(T)]$, with at least one inequality.

Assumption 3.3: $J(t,y,V,r) \in C^{1,2,2,2}([0,t] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$, that is, it is once continuously differentiable on $[0,T]$, and twice continuously differentiable for y on $\mathbb{R}$, and V and r on $\mathbb{R}^+$, respectively.

Assumption 3.3 allows us to employ the HJB equation for the member's problem. To this end, define the Dynkin operator $\mathcal{A}^{\mathbf{u}}$ for any function $\psi(t,y,V,r)$ with $\psi \in C^{1,2,2,2}([0,T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$, by

$$\begin{aligned}\mathcal{A}^{\mathbf{u}}\psi(t,y,V,r) &= \frac{\partial\psi}{\partial t} + \Big[r + (u_B f(K) + u_S\sigma_S)(k_1 r + k_2)\lambda_r + u_S\nu V\\ &\quad + (u_B f(K) + u_S\sigma_S)\sqrt{k_1 r + k_2}h_1 + u_S\sqrt{V}h_2 - u_S k_S^{\mathbb{Q}} l^{\mathbb{Q}}\Big] y\frac{\partial\psi}{\partial y}\\ &\quad + \left[a - br - \sqrt{k_1 r + k_2}h_1\right]\frac{\partial\psi}{\partial r} + [\alpha(\delta - V) + \sigma_V\sqrt{V}\rho h_2\\ &\quad + \sqrt{(1-\rho^2)V}\sigma_V h_3]\frac{\partial\psi}{\partial V} + \frac{1}{2}y^2\left[(u_B f(K) + u_S\sigma_S)^2(k_1 r + k_2) + u_S^2 V\right]\\ &\quad \times \frac{\partial^2\psi}{\partial y^2} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2\psi}{\partial V^2} + \frac{1}{2}(k_1 r + k_2)\frac{\partial^2\psi}{\partial r^2} - y[u_B f(K) + u_S\sigma_S](k_1 r + k_2)\frac{\partial^2\psi}{\partial y\partial r}\\ &\quad + yu_S\sigma_V V\rho\frac{\partial^2\psi}{\partial y\partial V} + l^{\eta}VE_t^{\eta}\Big[\psi(t, y + yu_S(e^Z - 1), V, r) - \psi\Big]. \end{aligned}$$

3.2. Problem formulation and verification theorem

This subsection is mainly concerned with the extended HJB system of equations for the optimization problem (40) of the AAM in the subgame perfect Nash equilibrium (see Björk & Murgoci, 2010). Before introducing the extended HJB system of equations, we present the definition of the equilibrium strategy for the problem.

Definition 3.4 (Equilibrium Strategy): For an admissible strategy $\mathbf{u}^* = \{(u_B^*(t), u_S^*(t)), t \in [0, T]\}$, define the following strategy for any given $\bar{u}_B \in \mathbb{R}$ and $\bar{u}_S \in \mathbb{R}$

$$\mathbf{u}_\Delta(\epsilon) = \begin{cases} (\bar{u}_B, \bar{u}_S), & t \le \epsilon < t + \Delta t \\ \mathbf{u}^*(\epsilon), & t + \Delta t \le \epsilon \le T \end{cases}, \tag{41}$$

where $\Delta \in \mathbb{R}^+$. If

$$\liminf_{\Delta t \to 0} \frac{J(t, y, V, r, \mathbf{u}^*) - J(t, y, V, r, \mathbf{u}_\Delta)}{\Delta t} \ge 0,$$

then $\mathbf{u}^*$ is called an equilibrium strategy with corresponding equilibrium value function given by

$$J(t, y, V, r) = J(t, y, V, r, \mathbf{u}^*). \tag{42}$$

The equilibrium strategy from the above definition is time consistent. Assuming rectangularity of beliefs – see, for instance, Epstein and Schneider (2003) and Chen and Epstein (2002) – the extended HJB system of equations for the AAM can now be obtained. The member aims to find an equilibrium strategy. Let

$$\begin{aligned} J(t, y, V, r) &= E_t^\eta[Y^{\mathbf{u}^*}(T)] - \frac{\gamma}{2}\mathrm{Var}_t^\eta[Y^{\mathbf{u}^*}(T)] \\ &= E_t^\eta\left[Y^{\mathbf{u}^*}(T) - \frac{\gamma}{2}(Y^{\mathbf{u}^*}(T))^2\right] + \frac{\gamma}{2}\left[E_t^\eta[Y^{\mathbf{u}^*}(T)]\right]^2 \qquad (43) \\ &= E_t^\eta[F(Y^{\mathbf{u}^*}(T))] + G(E_t^\eta[Y^{\mathbf{u}^*}(T)]), \qquad (44) \end{aligned}$$

with

$$F(x) = x - \frac{\gamma}{2}x^2,$$
$$G(x) = \frac{\gamma}{2}x^2.$$

Theorem 3.5 (Verification Theorem): *Assume rectangularity of beliefs. Furthermore, suppose that there exist two real-valued functions* $J(t, y, r, V), g(t, y, r, V) \in C^{1,2,2,2}([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$ *solving the extended HJB system of equations defined as follows for* $t \in [0, T]$:

$$\sup_{\boldsymbol{u} \in \Pi} \inf_{\{h, a_l, \Delta\mu_S, s_\sigma\}} \left\{\mathcal{A}^{\boldsymbol{u}} J(t, y, V, r) - \mathcal{A}^{\boldsymbol{u}}(G \diamond g)(t, y, V, r) + \mathcal{H}^{\boldsymbol{u}} g(t, y, V, r)\right\} = 0, \tag{45}$$

$$\mathcal{A}^{\boldsymbol{u}^*} g(t, y, V, r) = 0, \quad 0 \le t \le T, \tag{46}$$

$$J(T, y, V, r) = y, \tag{47}$$

$$g(T, y, V, r) = y, \tag{48}$$

subject to

$$\mathcal{R}(\eta(t)) \le \varphi(t),$$

where the infimization part is realized at $h^*(t) = [h_1^*(t), h_2^*(t), h_3^*(t)]$, a_l^*, $\Delta\mu_S^*$, *and* s_σ^*, *and where the supremum in* (45) *is obtained with the strategy* $\boldsymbol{u}^*(t, y, V, r, h^*, a_l^*, \Delta\mu_S^*, s_\sigma^*)$ *for all* $(t, y, V, r) \in [0, T] \times$

$\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$(G \diamond g)(t, y, V, r) = G(g(t, y, V, r)), \tag{49}$$

$$\mathcal{H}^{\boldsymbol{u}} g = G_x(g(t, y, V, r))\mathcal{A}^{\boldsymbol{u}} g(t, y, V, r), \tag{50}$$

$$G_x(x) = \frac{\mathrm{d}G}{\mathrm{d}x}(x) = \gamma\, x. \tag{51}$$

Then $g(t, y, V, r) = E_t^{\eta}[Y^{\boldsymbol{u}^*}(T)]$, $\boldsymbol{u}^*$ *is the equilibrium strategy, and* $J(t, y, V, r)$ *is the corresponding value function.*

Proof: The infimization part follows from the rectangular set of priors captured by our specification of the set of alternative models (see Chen & Epstein, 2002; Epstein & Schneider, 2003 for details). Given the worst-case perturbation parameters h^*, a_l^*, $\Delta\mu_S^*$, and s_σ^*, the proof regarding the supremum part is similar to that of Björk and Murgoci (2010) and is therefore omitted. ■

Equation (45) can be written as a Lagrangian with the associated Lagrange-multiplier $\Lambda(t)$. Given that relative entropy can be decomposed into diffusive and jump parts, it immediately follows that we can also split the Lagrange multiplier into a diffusive, $\Lambda^D(t)$, and jump, $\Lambda^J(t)$, component. Armed with these results, we have all the machinery required to solve the AAM problem.

3.3. Solution to the auxiliary problem

Before solving the extended HJB system of equations, we first simplify the expression in Theorem 3.5.

Proposition 3.6: *The extended HJB system of equations in Theorem* 3.5 *can be simplified as*

$$\begin{aligned}
&\sup_{\boldsymbol{u}\in\Pi}\inf_{\{h,a_l,\Delta\mu_S,s_\sigma\}}\Big\{J_t + [r + (u_B f(K) + u_S\sigma_S)(k_1 r + k_2)\lambda_r + u_S\nu V + (u_B f(K) + u_S\sigma_S)\\
&\quad\times\sqrt{k_1 r + k_2}h_1 + u_S\sqrt{V}h_2 - u_S k_S^{\mathbb{Q}} l^{\mathbb{Q}} V]yJ_y + \Big[a - br - \sqrt{k_1 r + k_2}h_1\Big]J_r\\
&\quad+\Big[\alpha(\delta - V) + \sigma_V\sqrt{V}\rho h_2 + \sigma_V\sqrt{V(1-\rho^2)}h_3\Big]J_V + \frac{1}{2}y^2[(u_B f(K) + u_S\sigma_S)^2(k_1 r + k_2)\\
&\quad+ u_S^2 V](J_{yy} - \gamma g_y^2) + \frac{1}{2}\sigma_V^2 V[J_{VV} - \gamma g_V^2] + \frac{1}{2}(k_1 r + k_2)[J_{rr} - \gamma g_r^2] - y(u_B f(K) + u_S\sigma_S)\\
&\quad\times(k_1 r + k_2)[J_{yr} - \gamma g_y g_r] + yu_S\sigma_V V\rho[J_{yV} - \gamma g_y g_V] + Vl^{\eta}E_t^{\eta}[J(t, y + yu_S(e^Z - 1), V, r)\\
&\quad- J] - \frac{\gamma}{2}l^{\eta}VE_t^{\eta}\Big[g(t, y + yu_S(e^Z - 1), V, r)^2 - g^2\Big] + \gamma gVl^{\eta}E_t^{\eta}[g(t, y11 + yu_S(e^Z - 1), V, r)\\
&\quad- g]\Big\} = 0,
\end{aligned} \tag{52}$$

$$\begin{aligned}
&g_t + \Big[r + (u_B^* f(K) + u_S^*\sigma_S)(k_1 r + k_2)\lambda_r + u_S^*\nu V + (u_B^* f(K) + u_S^*\sigma_S)\sqrt{k_1 r + k_2}h_1^*\\
&\quad+ u_S^*\sqrt{V}h_2^* - u_S^* k_S^{\mathbb{Q}} l^{\mathbb{Q}} V\Big]yg_y + \Big[a - br - \sqrt{k_1 r + k_2}h_1^*\Big]g_r + [\alpha(\delta - V) + \sigma_V\sqrt{V}\rho h_2^*\\
&\quad+\sqrt{(1-\rho^2)V}\sigma_V h_3^*]g_V + \frac{1}{2}y^2\left[(u_B^* f(K) + u_S^*\sigma_S)^2(k_1 r + k_2) + (u_S^*)^2 V\right]g_{yy} + \frac{1}{2}\sigma_V^2 V g_{VV}
\end{aligned}$$

$$+\frac{1}{2}(k_1 r + k_2) g_{rr} - y[u_B^* f(K) + u_S^* \sigma_S](k_1 r + k_2) g_{yr} + y u_S^* \sigma_V V \rho g_{yV}$$
$$+ l e^{a_l^*} V E_t^{\eta^*}\left[g(t, y + y u_S^*(e^Z - 1), V, r) - g\right] = 0.$$
$$J(T, y, c, V, r) = y,$$
$$g(T, y, c, V, r) = y. \tag{53}$$

Proof: Recalling that $G(x) = \frac{\gamma}{2}x^2$, $G_x(x) = \gamma x$ and $G_{xx}(x) = \gamma$ and combining with $\mathcal{A}^{\mathbf{u}} J(t, y, V, r)$, $\mathcal{H}^{\mathbf{u}} g(t, y, V, r)$ and $\mathcal{A}^{\mathbf{u}^*} g(t, y, c, V, r)$, Proposition 3.6 then follows. ■

By solving the extended HJB system simplified in Proposition 3.6, we obtain the optimal portfolio weights for the auxiliary problem and the worst-case perturbation parameters. Theorem 3.7 summarizes these results.

Theorem 3.7: *The robust equilibrium strategy* $\boldsymbol{u}^*$ *for the problem* (40) *is given by*

$$u_B^*(t) = \frac{\lambda_r - \sqrt{2\omega_1 b/(k_1 a + k_2 b)} - \mathcal{B}(t) + B_4(t)}{y\gamma f(K) B_1(t) e^{\mathcal{B}(t) r}} + \frac{\mathcal{B}(t)}{f(K)}$$
$$- \frac{\sigma_S(e^{a_l^*(t)} l k_S^{\eta^*}(t) - k_S^{\mathbb{Q}} l^{\mathbb{Q}} + \nu - \sqrt{2\omega_2/\delta} - \sigma_V \rho B_3(t))}{y\gamma f(K) B_1(t) e^{\mathcal{B}(t) r} L(t)}, \tag{54}$$

$$u_S^*(t) = \frac{e^{a_l^*(t)} l k_S^{\eta^*}(t) - k_S^{\mathbb{Q}} l^{\mathbb{Q}} + \nu - \sqrt{2\omega_2/\delta} - \sigma_V \rho B_3(t)}{y\gamma\, B_1(t) e^{\mathcal{B}(t) r} L(t)}, \tag{55}$$

where

$$L(t) \equiv L(t; a_l^*, \Delta\mu_S^*, s_\sigma^*)$$
$$= 1 + l e^{a_l^*(t)} + l e^{2(\mu_S + \Delta\mu_S^*(t)) + a_l^*(t) + 2\Sigma^2 s_\sigma^*(t)} - 2 l e^{\mu_S + \Delta\mu_S^*(t) + \frac{1}{2}\Sigma^2 s_\sigma^*(t) + a_l^*(t)}. \tag{56}$$

The corresponding equilibrium value function and expected terminal surplus are given by, respectively,

$$J(t, y, V, r) = A_1(t) e^{\mathcal{B}(t) r} y + \frac{A_3(t)}{\gamma} V + \frac{A_4(t)}{\gamma} r + \frac{A_5(t)}{\gamma}, \tag{57}$$

$$g(t, y, V, r) = E_t^{\eta^*}[Y^{\boldsymbol{u}^*}(T)] = B_1(t) e^{\mathcal{B}(t) r} y + \frac{B_3(t)}{\gamma} V + \frac{B_4(t)}{\gamma} r + \frac{B_5(t)}{\gamma}, \tag{58}$$

where

$$A_1(t) = B_1(t) = e^{\int_t^T [(\lambda_r k_2 + a)\mathcal{B}(u) - \frac{k_2}{2}\mathcal{B}(u)^2]\, \mathrm{d}u}, \tag{59}$$

$$\mathcal{B}(t) = \begin{cases} \dfrac{\chi_1 \chi_2 - \chi_1 \chi_2 e^{-\frac{k_1(\chi_1 - \chi_2)}{2}(T-t)}}{\chi_2 - \chi_1 e^{-\frac{k_1(\chi_1 - \chi_2)}{2}(T-t)}} & k_1 \neq 0 \\ \dfrac{1}{b}\left[1 - e^{-b(T-t)}\right] & k_1 = 0 \end{cases}, \tag{60}$$

$$A_3(t) = e^{\xi_1(T-t)} \int_t^T e^{-\xi_1(T-s)} \xi_2(s)\,\mathrm{d}s, \tag{61}$$

$$A_4(t) = e^{-(b-k_1\sqrt{2\omega_1 b/(k_1a+k_2)})(T-t)} \int_t^T e^{(b-k_1\sqrt{2\omega_1 b/(k_1a+k_2b)})(T-u)} \times \left(\frac{k_1}{2}\left[\lambda_r - \sqrt{2\omega_1 b/(k_1a+k_2b)} - \mathcal{B}(u)\right]^2 + k_1 B_4(u)\left[\lambda_r - \sqrt{2\omega_1 b/(k_1a+k_2b)} - \mathcal{B}(u)\right]\right)\mathrm{d}u, \tag{62}$$

$$A_5(t) = \int_t^T \left(k_2\left[\lambda_r - \sqrt{2\omega_1 b/(k_1a+k_2b)} - \mathcal{B}(u)\right]B_4(u) + aA_4(u) + \alpha\delta A_3(u) + k_2\sqrt{2\omega_1 b/(k_1a+k_2b)}A_4(u) + \frac{1}{2}k_2\left[\lambda_r - \sqrt{2\omega_1 b/(k_1a+k_2b)} - \mathcal{B}(u)\right]^2\right)\mathrm{d}u, \tag{63}$$

$$B_3(t) = e^{\int_t^T \xi_3(u)\,\mathrm{d}u} \int_t^T e^{-\int_u^T \xi_3(s)\,\mathrm{d}s} \xi_4(s)\,\mathrm{d}s,$$

$$B_4(t) = e^{\int_t^T [-b+k_1(\lambda_r - \mathcal{B}(u))]\,\mathrm{d}u} \int_t^T k_1\left[\lambda_r - \sqrt{2\omega_1 b/(k_1a+k_2b)} - \mathcal{B}(u)\right]^2 \times e^{-\int_u^T [-b+k_1\mathcal{B}(s)+\lambda_r k_1]\,\mathrm{d}s}\,\mathrm{d}u, \tag{64}$$

$$B_5(t) = \int_t^T \left[B_4(u)(\lambda_r k_2 - \mathcal{B}(u)k_2 + a) + \alpha\delta B_3(u) + k_2\left(\lambda_r - \sqrt{2\omega_1/(k_1a+k_2b)} - \mathcal{B}(u)\right)^2\right]\mathrm{d}u, \tag{65}$$

with

$$\chi_{1,2} := \frac{k_1\lambda_r - b \pm \sqrt{(k_1\lambda_r - b)^2 + 2k_1}}{k_1}, \tag{66}$$

$$\xi_1 = -\alpha - \sigma_V\rho\sqrt{2\omega_2/\delta} + \sigma_V\sqrt{(1-\rho^2)2\omega_3/\delta}, \tag{67}$$

$$\xi_2(t) = -\frac{1}{2}\sigma_V^2 B_3(t)^2 + \frac{\left[le^{a_l^*(t)}k_S^{\eta^*} - k_S^{\mathbb{Q}}l^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu - \sigma_V\rho B_3(t)\right]^2}{2L(t)}, \tag{68}$$

$$\xi_3(t) = -\alpha - \sigma_V\rho\sqrt{2\omega_2/\delta} + \sigma_V\sqrt{(1-\rho^2)2\omega_3/\delta} - \frac{\sigma_V\rho}{L(t)}(le^{a_l^*(t)}k_S^{\eta^*}(t) - k_S^{\mathbb{Q}}l^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu) \tag{69}$$

$$\xi_4(t) = \frac{1}{L(t)}(le^{a_l^*(t)}k_S^{\eta^*}(t) - k_S^{\mathbb{Q}}l^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu)^2. \tag{70}$$

The first order conditions for drift perturbations can be solved explicitly and are given by

$$h_1^*(t) = -\sqrt{\frac{2\omega_1(k_1r(t)+k_2)b}{k_1a+k_2b}}, \quad h_2^*(t) = -\sqrt{\frac{2\omega_2 V(t)}{\delta}}, \quad h_3^*(t) = \sqrt{\frac{2\omega_3 V(t)}{\delta}}. \tag{71}$$

The optimal jump size and jump intensity perturbations are the solutions to

$$a_l^*(t) = -\frac{k_S^{\eta^*}(t)}{\Lambda^J(t)\gamma L(t)}\left(le^{a_l^*(t)}k_S^{\eta^*}(t) - l^{\mathbb{Q}}k_S^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu - \sigma_V\rho B_3(t)\right)$$
$$+ \frac{e^{2(\mu_S+\Delta\mu_S^*(t))+2\Sigma^2 s_\sigma^*(t)} - 2e^{\mu_S+\Delta\mu_S^*+\frac{1}{2}\Sigma^2 s_\sigma^*(t)} + 1}{2\Lambda^J(t)\gamma L(t)^2}\left(le^{a_l^*(t)}k_S^{\eta^*}(t) - l^{\mathbb{Q}}k_S^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu\right.$$
$$\left. - \sigma_V\rho B_3(t)\right)^2 - \frac{\Delta\mu_S^*(t)^2}{2\Sigma^2} - \frac{s_\sigma^*(t)}{2} + \frac{1}{2}\ln s_\sigma^*(t) + \frac{1}{2}, \tag{72}$$

$$\frac{e^{\mu_S+\Delta\mu_S^*(t)+\frac{1}{2}\Sigma^2 s_\sigma^*(t)}}{\gamma L(t)}\left(le^{a_l^*(t)}k_S^{\eta^*} - l^{\mathbb{Q}}k_S^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu - \sigma_V\rho B_3(t)\right) \tag{73}$$
$$- \frac{e^{2(\mu_S+\Delta\mu_S^*(t))+2\Sigma^2 s_\sigma^*(t)} - e^{\mu_S+\Delta\mu_S^*(t)+\frac{1}{2}\Sigma^2 s_\sigma^*(t)}}{\gamma L(t)^2}\left(le^{a_l^*(t)}k_S^{\eta^*} - l^{\mathbb{Q}}k_S^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu\right.$$
$$\left. - \sigma_V\rho B_3(t)\right)^2 + \Lambda^J(t)\frac{\Delta\mu_S^*(t)}{\Sigma^2} = 0, \tag{74}$$

$$\frac{\Sigma^2 e^{\mu_S+\Delta\mu_S^*(t)+\frac{1}{2}\Sigma^2 s_\sigma^*(t)}}{2\gamma L(t)}\left(le^{a_l^*(t)}k_S^{\eta^*} - l^{\mathbb{Q}}k_S^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu - \sigma_V\rho B_3(t)\right) \tag{75}$$
$$- \frac{2\Sigma^2 e^{2(\mu_S+\Delta\mu_S^*(t))+2\Sigma^2 s_\sigma^*(t)} - \Sigma^2 e^{\mu_S+\Delta\mu_S^*(t)+\frac{1}{2}\Sigma^2 s_\sigma^*(t)}}{2\gamma L(t)^2}\left(le^{a_l^*(t)}k_S^{\eta^*} - l^{\mathbb{Q}}k_S^{\mathbb{Q}} - \sqrt{2\omega_2/\delta} + \nu\right.$$
$$\left. - \sigma_V\rho B_3(t)\right)^2 + \frac{\Lambda^J(t)}{2}\left(1 - \frac{1}{s_\sigma^*(t)}\right) = 0, \tag{76}$$

where the Lagrange multiplier $\Lambda^J(t)$ is the solution to

$$lV(t)e^{a_l^*(t)}\left(a_l^*(t) + \frac{\Delta\mu_S^*(t)^2}{2\Sigma^2} + \frac{s_\sigma^*(t)}{2} - \frac{1}{2}\ln s_\sigma^*(t) - \frac{3}{2}\right) + lV(t) - \varphi^J = 0. \tag{77}$$

***Proof*:** The proof is similar to that of Drechsler (2013) and Wang et al. (2021) and is, therefore, relegated to the Internet Appendix. The proof regarding the solvability and uniqueness of solutions to Equations (72)–(77) is presented in Appendix A of the paper. ■

Remark 3.8: Examination of optimal policies in Equations (54) and (55) uncovers the effects of jump risk and ambiguity on the AAM investment decisions. The first two terms in Equation (54) capture the demand for the interest rate risk. Since the interest rate process is purely diffusive in our framework, this term is not affected by jumps. Nonetheless, the effect of ambiguity is clear. In particular, ambiguity about interest rates induce the AAM to reduce her proportion of wealth surplus in the rolling bond. Intuitively, ambiguity makes risk premia that is accessible through the bond less certain. The AAM fears this uncertainty and takes less aggressive position in the bond. The third term in (54) stems from the correlation between the interest rate risk and the stock price risk. This term is affected by both diffusive and jump ambiguity.

In turn, the optimal proportion of surplus in stock is affected by both diffusive ambiguity and jump ambiguity. Specifically, diffusive ambiguity acts by reducing $u_S^*(t)$. The effect of jump ambiguity is not clear since it hinges on specific values of a_l^*, $\Delta\mu_S^*$ and s_σ^*. If, for example, $a_l^* > 0$ and $\Delta\mu_S^* < 0$, jump ambiguity may induce the AAM to take more conservative investment decisions by reducing her exposure to stock. The term $-\sigma_V\rho B_3(t)$ captures the hedge demand for stock due to the presence of stochastic volatility and it vanishes if volatility is deterministic ($\sigma_V = 0$).

Finally, in line with Wang et al. (2021), interest rate affects both u_B^* and u_S^* highlighting, therefore, the difference between the pre-commitment strategy (adopted by Guan & Liang, 2015) and the time consistent one.

Remark 3.9: Note that we need the perturbation parameters from time t onward to solve $B_3(t)$ but, at the same time, need $B_3(t)$ to obtain the values of the perturbation parameters at t. This can be solved by starting with the terminal condition $B_3(T) = 0$ and solving the system of nonlinear equations for the perturbation parameters and B_3 recursively back to time t.

3.4. Solution to the original optimization problem

The auxiliary optimization problem (40) is equivalent to the original problem and can thus be solved accordingly. In particular, the equilibrium value function is equivalent under both problems and the equilibrium strategies are linked via Equation (39). The next corollary summarizes the results.

Corollary 3.10: *The equilibrium strategies for problem* (29) *are given by*

$$\begin{aligned}
\pi_0^*(t) &= 1 - \pi_B^*(t) - \pi_S^*(t),\\
\pi_B^*(t) &= \frac{Y^{\boldsymbol{u}^*}(t)}{X^{\pi^*}(t)} u_B^*(t) - \frac{F(t,T)}{X^{\pi^*}(t)} \pi_B^F(t) + \frac{\mathcal{Q}(t)}{X^{\pi^*}(t)} \pi_B^{\mathcal{Q}}(t),\\
\pi_S^*(t) &= \frac{Y^{\boldsymbol{u}^*}(t)}{X^{\pi^*}(t)} u_S^*(t) - \frac{F(t,T)}{X^{\pi^*}(t)} \pi_S^F(t).
\end{aligned} \tag{78}$$

Assume that $J_1(t,x,V,r)$ *is the equilibrium value function for the problem* (29). *Then, equivalence between problems* (29) *and* (40) *implies that*

$$J_1(t,x,V,r) = J(t,y,V,r).$$

Finally, we have that

$$\mathrm{Var}_t^{\eta}[Y^{\boldsymbol{u}^*}(T)] = \frac{M(t)}{[B_3(t)V + B_4(t)r + B_5(t)]^2} \left(E_t^{\eta}[Y^{\boldsymbol{u}^*}(T)] - B_1(t)e^{\mathcal{B}(t)r}y\right)^2, \tag{79}$$

where $M(t) = 2[(B_3(t) - A_3(t))V + (B_4(t) - A_4(t))r + (B_5(t) - A_5(t))]$.

Proof: The proof is similar to that of Wang et al. (2021) and is, therefore, relegated to the Internet Appendix. ■

When $t = 0$, Equation (79) is known as the robust equilibrium efficient frontier for the AAM's portfolio problem. The equilibrium efficient frontier captures the relationship between the expected value and the variance of terminal surplus under the time consistent strategy π^*. A quick look at Equation (79) reveals that the efficient frontier is a straight line in the mean-standard deviation frontier plane emanating from the point $(B_1(0)e^{\mathcal{B}(0)r_0}, 0)$. Ambiguity affects the efficient frontier through its effects on A_i and B_i, $i \in \{3,4,5\}$.

4. Sensitivity analysis

This section is concerned with some numerical examples designed to illustrate the effects of ambiguity and model parameters on the equilibrium strategy.[3] In addition, we also study the effect of robustness parameters on the utility loss from ignoring model uncertainty. The baseline parameters are presented in Table 1.

Table 1. Baseline parameters.

a	b	r_0	k_1	k_2	λ_r	K	σ_S	v	V_0	α	δ	σ_V
0.02	0.2	0.05	0.007	0	1	10	0.2	3	0.15^2	5	0.13^2	0.25
l	$l^{\mathbb{Q}}$	μ_S	$\mu_S^{\mathbb{Q}}$	Σ	σ_{C_1}	σ_{C_2}	γ	ρ	T	t	x_0	C_0
5.92	11.84	−0.1062	−0.12	0.04	0	0	0.5	−0.4	40	0	1	0.15
ϑ	I	w_1	w_2	$q(T)$	w	ω_1		ω_2		ω_3		φ^J
0.02	0	6.08×10^{-5}	0.08	4	120	0.0024		0.0036		0.0012		0.0048

The parameters from the table are in line with those usually employed in the literature – see, for instance, Liu and Pan (2003), Guan and Liang (2014, 2015), Escobar et al. (2015), Wang and Li (2018), and Wei et al. (2023). Our baseline parameters imply that the stock has about a −10% (i.e. $k_S \approx -10\%$) jump with an average frequency of 10 years. Similar to Liu and Pan (2003), we assume that $l^{\mathbb{Q}} = 2l$. We assume a total average ambiguity, $\omega_1 + \omega_2 + \omega_3 + \varphi^J$, of 0.012, which is close to the calibrated value of 0.0136 of Aït-Sahalia and Matthys (2019). Recall that total ambiguity must be small so that they are statistically difficult to distinguish from the reference model (see Anderson et al., 2003). Moreover, we follow Guan and Liang (2014) and use the CIR model to describe the evolution of the interest rate (i.e. $k_2 = 0$).

4.1. Effects of model parameters on the robust equilibrium strategy

Figure 1 depicts the effect of the robustness parameter ω_1 and the risk aversion coefficient γ on the robust equilibrium strategy for bonds π_B^*. Since ω_1 does not have any impact on u_S^*, there is no reason whatsoever to expect it to affect π_S^*. This is a direct consequence of the fact that only the bond is used to hedge the interest rate risk of the portfolio.

From Figure 1, we find that π_B^* decreases with ambiguity and increases with risk aversion. This happens because high risk aversion increases the AAM willingness to reduce her portfolio risk and therefore acts by inducing more conservative investment decisions. Since interest rate risk can be hedged by bonds, she will therefore increase her investment proportion on these assets. On the other hand, a rise in ω_1 indicates higher level of ambiguity aversion about bond dynamics. As a consequence, the AAM reduces her investment proportion on bonds so that her wealth is less sensitive to the unpleasant consequences of potential model misspecification. The effects seem to be more pronounced when γ is low, as the AAM with low ambiguity and low risk aversion tends to hold a large proportion of bonds in her portfolio.

Figure 2 shows the effects of ω_2 and γ on the robust equilibrium strategy (π_B^*, π_S^*).

From Figure 2, π_S^* decreases with γ and with ω_2, while π_B^* slightly increases with the same parameters. This follows because risk and ambiguity aversion increase the AAM willingness to reduce her portfolio risk and to guard herself against some worst-case scenario, respectively, and, as a result, she shies away from stock (which risk cannot be hedged) and increases her wealth proportion on bonds, which can hedge the interest rate risk. Moreover, due to the small correlation of stock with interest rates, the effect on bonds proportion is very small. These results are similar to those of Flor and Larsen (2014) and Escobar et al. (2015).

Figure 3 illustrates the effects of ω_3 and φ^J on the robust equilibrium strategy.

As expected, π_S^* decreases with φ^J and increases with ω_3. Similar to Escobar et al. (2018), the effect of ω_3 does not seem to be very pronounced on π_S^*. The effect of ω_3 on π_B^* is even more insignificant, and hardly noticeable, whilst the effect of φ^J is clear. Thus, the results suggest that as the ambiguity about jump parameters rises, the AAM reduces the proportion of her wealth on stock and increases her investment on the rolling bond. Note that ω_3 affects π_S^* through its effect on the hedge demand for stock (i.e. the term $-\sigma_V \rho B_3(t)$). The results imply that as ω_3 increases, the hedge demand for stock also rises. To put it another way, as the volatility dynamics becomes more ambiguous, the

[3] In the Internet Appendix, we also present the effects of ambiguity and model parameters on the efficient frontier.

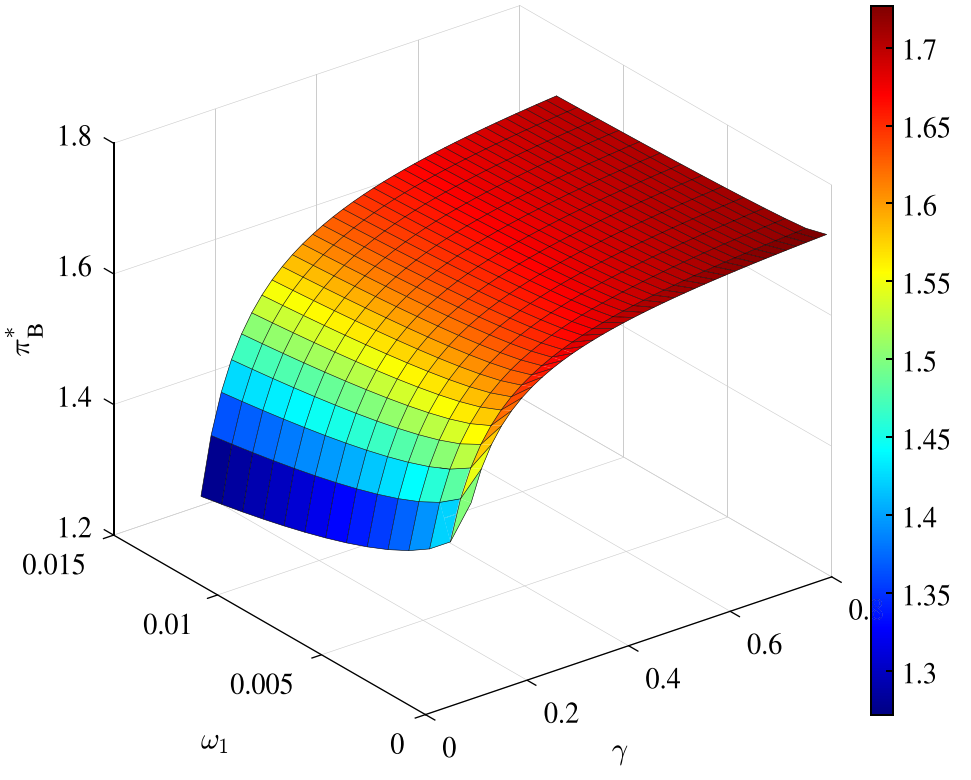


Figure 1. Effect of ω_1 and γ on π_B^*.

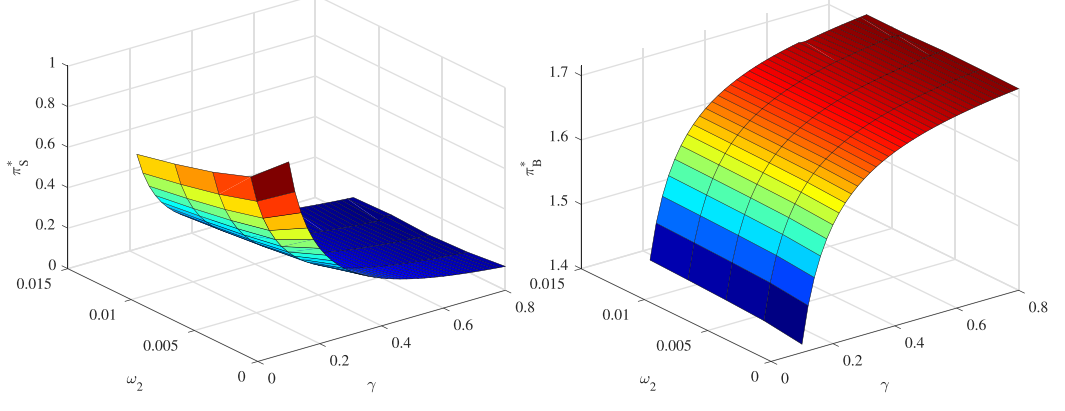


Figure 2. Effects of ω_2 and γ on π_S^* and π_B^*.

AAM increases the proportion of her wealth on stocks in order to curb losses arising from potential misspecification of stock volatility dynamics.

We next analyze the effects of jump parameters μ_S and l on the robust equilibrium strategy. Figure 4 displays the results.

The optimal proportion on stock π_S^* decreases with jump intensity parameter l and increases with μ_S. As l increases, the negative jumps become more likely to occur and, thereby, the risk of the stock rises. In order to reduce the overall risk of her portfolio, the AAM thus decreases the proportion of her wealth on stock. Moreover, lower μ_S implies that jumps (when they happen) destroy a larger

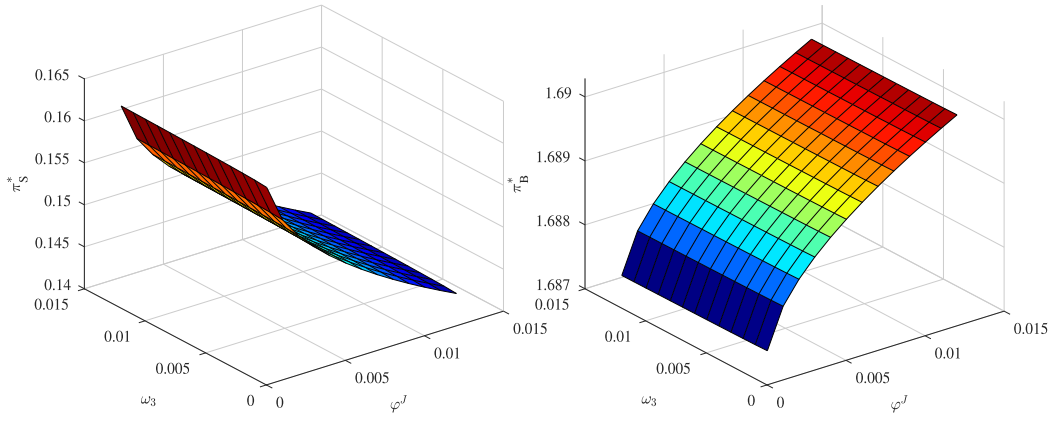


Figure 3. Effects of ω_3 and φ^J on π_S^* and π_B^*.

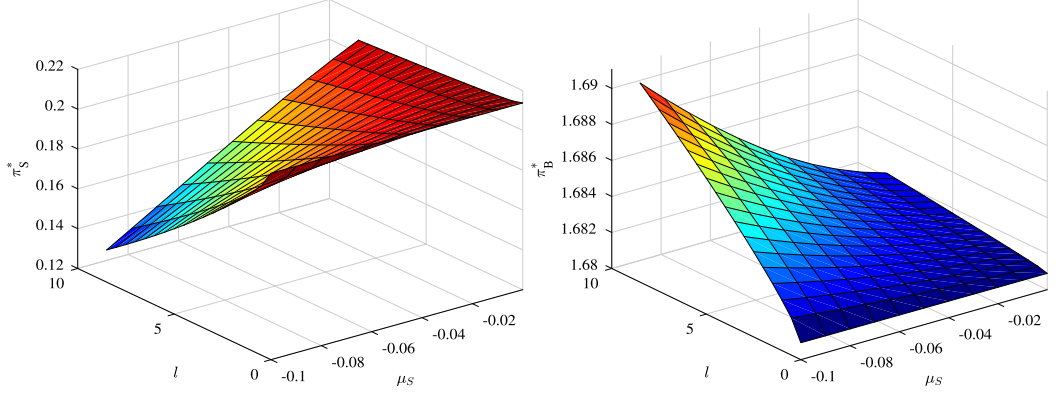


Figure 4. Effects of l and μ_S on π_S^* and π_B^*.

proportion of AAM investment. The member thus acts by choosing a lower proportion of her wealth on stock so that this risk is mitigated. Due to the negative correlation between stock and bond markets, she takes the opposite position on bonds. The figure indicates that jumps can have a very significant effect on the equilibrium strategy.

4.2. Effect of ambiguity parameters on utility loss

The literature on DC pension plans reveals that the AAM incurs a significant utility loss when ignoring model uncertainty. In this section, we present the effects of the ambiguity parameters on the member's utility loss.[4] The utility loss from ignoring model uncertainty can be small at the beginning when the investment horizon is large (greater than, say, 10 years) as the AAM might still have the opportunity to correct some of her investment mistakes (including model misspecification) in the future. Hence, in this subsection, we set $t = 10$ and study the effect of robustness parameters on the utility loss. Figure 5 displays the results.

Figure 5(a) shows the sensitivity of utility loss to t and ω_2. We set $Y^{\mathbf{u}^*}(t) = Y^{\mathbf{u}^*}(10)$ for all $t \in [10, T]$. The figure indicates that the utility loss from ignoring ambiguity increases with t until year 30 approximately. After that, it starts to decrease as we approach the retirement time T. Figure 5 also

[4] Details about the computation of the utility loss can be found in the Internet Appendix.

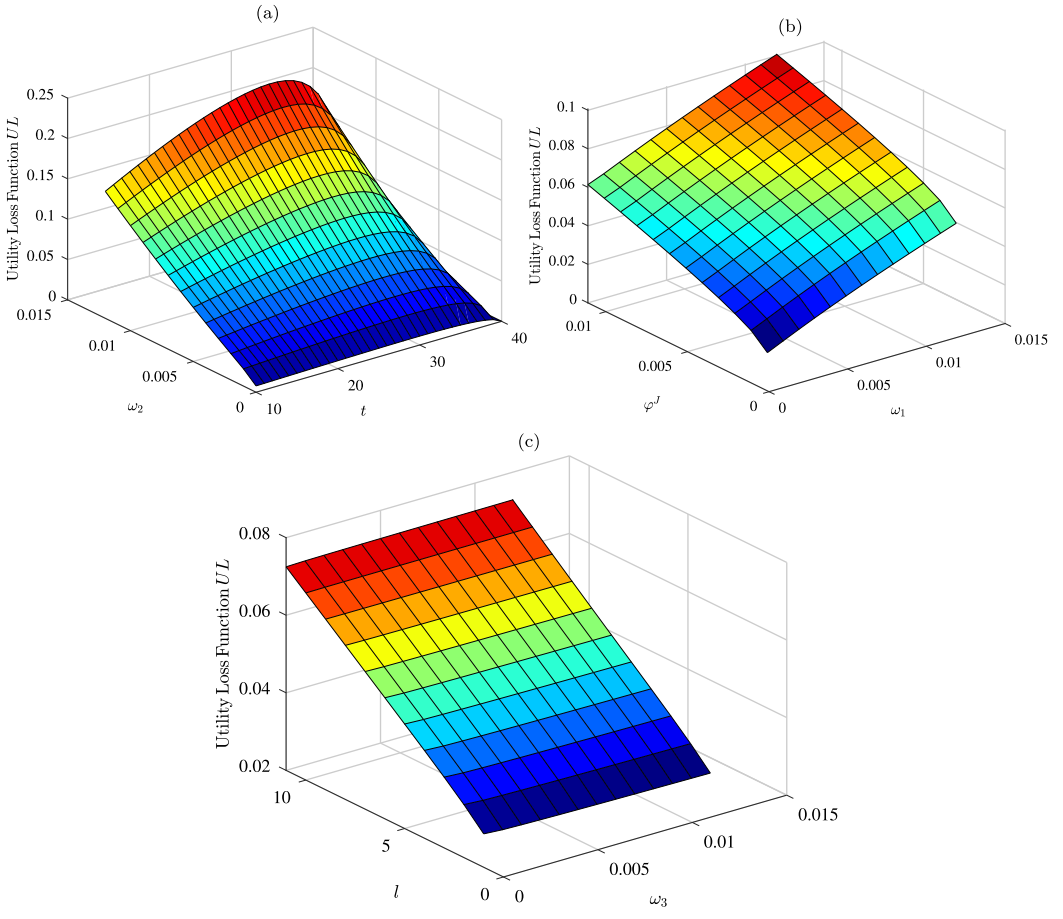


Figure 5. Effects of parameters t, $\omega_1, \omega_2, \omega_3$, l, and φ^J on UL. We set $Y^{\mathbf{u}}(t) = Y^{\mathbf{u}}(10)$ for all $t \in [0, 40]$ and $t = 20$ in Panels (c) and (d). The remaining parameters are from Table 1.

indicates that the utility loss increases with ω_1, ω_2, l and φ^J while being insensitive to ω_3. We also find that utility loss can be even larger, going way above 40% if we let $Y^{\mathbf{u}^*}$ vary with t. To save space, we do not report this here. In sum, the results indicate that the AAM can incur significant utility loss from ignoring ambiguity.

5. Conclusion

This paper investigates a mean-variance portfolio problem of an ambiguity averse member of a DC pension plan in a setting with stochastic opportunity set and with jump in stock returns and contribution rate processes. The member is ambiguous about both diffusive, and jump size and jump intensity parameters of stock returns. Given the long horizon of DC pension plans, spanning several decades, we concentrate on strategies that are time consistent. This, together with our specification of rectangular set of priors, gives rise to strategies that are robust to model misspecification and dynamically consistent.

Since the original problem is not self financing due to the continuous injection of contribution from the member, we follow previous literature and transform it into a self financing one. After solving the self financing problem, we invert it in order to obtain the original robust equilibrium strategy. Finally, we provide some comparative statics to illustrate the implication of our theoretical results.

This paper can be extended by incorporating strategic interaction, similar to recent ambiguity-averse DB and DC game-theoretic models, allowing multiple DC pension members or fund managers to compete under jump-diffusion ambiguity and time-varying model uncertainty. Such an extension would unify robust mean-variance optimization with the robust stochastic differential game frameworks used in Guan et al. (2022), Zhang et al. (2025), and Josa-Fombellida and López-Casado (2023).

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Disclaimer

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Appendix. Proof of existence and uniqueness of optimal jump size and jump intensity perturbations

The optimal controls η^* are determined by minimizing the Lagrangian $\mathcal{L}^J(\eta, \Lambda^J)$, which incorporates the HJB jump term $\mathcal{H}$ and the penalty term $\Lambda^J \cdot \mathcal{R}(\eta)$:

$$\mathcal{L}^J(\eta, \Lambda^J) = \mathcal{H}(\eta, t) + \Lambda^J \cdot \mathcal{R}(\eta).$$

The FOCs define a system of three implicit, non-linear equations for $\eta^* = (a_l^*, \Delta\mu_S^*, s_\sigma^*)$ given the multiplier Λ^J:

$$\mathbf{f}(\eta^*, \Lambda^J) = \nabla_\eta \mathcal{L}^J(\eta^*, \Lambda^J) = 0.$$

Since the objective function $\mathcal{L}^J$ is strictly convex with respect to η^*, the Hessian matrix, $\mathbf{H} = \nabla_\eta \mathbf{f}$, is positive definite everywhere in the interior of the domain, which implies that its determinant is strictly positive: $\det(\mathbf{H}) \neq 0$. Because the Jacobian of the FOC system (which is the Hessian of the objective function) is non-singular, the inverse function theorem guarantees that locally around any point (η_0, Λ_0^J), there exists a unique solution η^* that is continuously differentiable with respect to Λ^J. This proves that η^* can be uniquely determined as an implicit function of Λ^J: $\eta^* = \mathbf{F}_{\text{Implicit}}(\Lambda^J)$.

We now prove the global solvability (monotonicity of Λ^J). We substitute the unique implicit solutions $\eta^*(\Lambda^J)$ back into the entropy constraint (Equation (77)), which becomes a function of Λ^J alone:

$$H(\Lambda^J) = lV(t)e^{a_l^*(\Lambda^J)}\left(a_l^*(\Lambda^J) + \frac{\Delta\mu_S^*(\Lambda^J)^2}{2\Sigma^2} + \frac{s_\sigma^*(\Lambda^J)}{2}\ln s_\sigma^*(\Lambda^J) - \frac{3}{2}\right) + lV(t) - \varphi^J = 0.$$

By the envelope theorem (or direct differentiation), the derivative of the constraint function $H(\Lambda^J)$ is the derivative of the instantaneous relative entropy rate with respect to the penalty:

$$\frac{\mathrm{d}H}{\mathrm{d}\Lambda^J} = \frac{\partial}{\partial\Lambda^J}\left[\mathcal{R}(\eta^*(\Lambda^J))\right].$$

Since the relative entropy growth rate $\mathcal{R}(\eta)$ is defined as a sum of positive, strictly convex terms (a penalty on the distance from the reference measure $\mathbb{P}$), the optimal controls $\eta^*(\Lambda^J)$ must adjust such that the overall entropy rate increases strictly as the Lagrange multiplier Λ^J increases.

This means:

$$\frac{\mathrm{d}H}{\mathrm{d}\Lambda^J} > 0,$$

i.e. $H(\Lambda^J)$ is a continuous, strictly increasing function. It has the following boundary conditions:

- As $\Lambda^J \to 0$ (no ambiguity penalty), the controls η^* adjust to minimize $\mathcal{H}$, and the resulting unconstrained entropy rate $\mathcal{R}(\eta^*)$ is typically greater than the fixed tolerance φ^J. Thus, $H(0) > 0$.
- As $\Lambda^J \to \infty$ (infinite ambiguity penalty), the controls η^* are forced towards the reference measure ($\eta^* \to \mathbf{0}$), making the entropy rate go to zero, $\mathcal{R}(\mathbf{0}) = 0$. Thus, $H(\infty) = -\varphi^J < 0$ (assuming $\varphi^J > 0$).

Existence and uniqueness: Since $H(\Lambda^J)$ is continuous and changes sign (from positive to negative), the intermediate value theorem guarantees the existence of at least one root. The strict monotonicity ($\frac{\mathrm{d}H}{\mathrm{d}\Lambda^J} > 0$) guarantees that this root is unique.

Solvability: A unique $\Lambda^J(t)$ is guaranteed to exist for all time $t \in [0, T]$, and by the implicit function theorem, this unique $\Lambda^J(t)$ uniquely determines the implicit functions $\eta^* = (a_l^*, \Delta\mu_S^*, s_\sigma^*)$, thus proving the global solvability and uniqueness of the entire system.