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Caps, floors and collars on continuous flows and investment decisions

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In memory of my parents

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Resumo

O objetivo desta tese é avaliar *caps* e *floors* e utilizar as fórmulas propostas para analisar decisões de investimento no contexto da teoria das opções reais em três artigos separados, mas relacionados.

O primeiro artigo apresenta novas fórmulas analíticas para *caps* e *floors* com maturidade finita que foram derivadas diretamente sem utilizar a técnica de decomposição temporal de Shackleton e Wojakowski (2007). Usando relações de não arbitragem, apresenta-se um novo método para calcular *optimal triggers of investment* que é mais simples e fácil de implementar do que os normalmente utilizados na literatura. O segundo artigo oferece novas soluções analíticas para avaliar *caps* e *floors* (perpétuos e finitos) em fluxos contínuos, estendendo o contexto do clássico *geometric Brownian motion (GBM)* para o modelo *constant elasticity of variance (CEV)*. Além disso, discute-se o impacto das soluções de maturidade finita no desempenho ótimo de uma empresa, em relação ao caso perpétuo. O terceiro artigo fornece uma representação analítica para avaliar decisões de investimento ótimas associadas a um *feed-in tariff (FIT) contract* com uma garantia de preço mínimo sob o modelo CEV, considerando garantias perpétuas e finitas.

A tese demonstra a importância de usar modelos realistas que incorporem os chamados *volatility smile effects*. De facto, a especificação incorreta do processo estocástico que modela a variável aleatória pode impactar significativamente nas decisões de investimento. Consequentemente, as diferenças entre os *optimal triggers of investment* podem levar ao pagamento de rendas excessivas à custa dos contribuintes. As soluções analíticas propostas abrem imediatamente uma ampla variedade de aplicações para outros tipos de contratos no campo das decisões de investimento em ativos reais.

Classificação JEL: G12; G13

Palavras chave: Opções Reais; Investimentos ótimos; Caps e Floors; Collars; GBM; modelo CEV

Abstract

The objective of this research is to evaluate caps and floors and use the proposed valuation formulas to address investment decisions within the real options theory in three separate but related articles.

The first paper presents novel analytic representations for finite maturity profit caps and floors that can be analyzed directly without using the time decomposition technique of Shackleton and Wojakowski (2007). Using some arbitrage-free relations, we provide a new method for calculating the optimal triggers of investment projects containing price floors and price collars clauses that is much simpler and easier to implement than the ones usually used in the literature. The second paper offers novel analytical solutions for evaluating (perpetual and finite) caps and floors on continuous flows extending the context from the classic geometric Brownian motion (GBM) process to the constant elasticity of variance (CEV) model. Furthermore, we discuss the impact of the finite-lived solutions on the optimal behavior of a firm, relative to the perpetual case. The third paper provides an analytical representation for evaluating optimal investment decisions associated with a feed-in tariff (FIT) contract with a minimum price guarantee under the CEV model, considering both perpetual and finite-lived guarantees.

The thesis shows the importance of using realistic models that incorporate leverage and volatility smile effects. In fact, the incorrect specification of the underlying state variable process for the valuation of caps and floors can significantly impact executive management decisions. Consequently, the differences between trigger points of investment can lead to excessive rents being paid at the expense of taxpayers. The proposed analytical solutions immediately open a wide variety of applications for other contractual agreements in the investment decisions field.

JEL Classification: G12; G13

Keywords: Real Options; Optimal investments; Caps and Floors; Collars; GBM; CEV model

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1. Introduction

This thesis addresses the problem of evaluating projects and investment opportunities by applying the real options analysis (ROA). As we know, the traditional net present value (NPV) approach cannot incorporate the interaction between irreversibility, uncertainty and flexibility on which investment decisions depend. Instead, ROA not only embodies those features, but also gives the company the value of waiting time to invest. Myers (1977) proposed the concept of seeing investment opportunities as growth options and attached the label "real options" to it, but Brennan and Schwartz (1985) and McDonald and Siegel (1985) were the first economists to propose the idea of incorporating option valuation techniques into the investment decision-making process. Since then many studies have been published about real options and their applications¹.

Most of the real options literature assumes the underlying uncertainty follows a random walk, specifically the geometric Brownian motion (GBM). However, this process adopts a constant volatility assumption and can diverge over time because it is unbounded from above. The choice of the stochastic process that models the project's state variable has a great impact on the valuation of a real investment project and also on the rule for determining the optimal timing of investment, as highlighted in Dias and Nunes (2011), for example. Being the output price of the project the source of uncertainty, it is of crucial importance to choose a stochastic process that best fits the pricing dynamics to overcome problems about project mispricing. The use of the GBM as a model for the output price P is convenient in terms of analytical tractability, but it may not be realistic in most cases. There is plenty of evidence that highlights the greater suitability of the constant elasticity of variance (CEV) process in capturing the leverage effect often observed in commodity markets. For instance, according to an empirical study conducted by Geman and Shih (2009), the spot price of crude oil, copper, and gold started to behave like a CEV process after 2000.

Therefore, we are going to derive the value of the project and examine the investment oppor-

¹A systematic treatment and comprehensive review on the real options approach to capital investment decisions of firms can be found in Dixit and Pindyck (1994). For an in-depth literature review of how the real options approach is a valuable problem-formulation and problem-solving tool for improving managerial decision-making, see Trigeorgis and Tsekrekos (2018). See also Schwartz and Trigeorgis (2001) for a neatly collection of novel research on real options and capital investment under uncertainty and Ipsmiller et al. (2019) for a review on the empirical real options research in top management journals over the past 25 years.

tunity rule by assuming that P follows a CEV process, which nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) as a special case.

This thesis contains three separate but interrelated papers.

The first paper presents novel analytic representations for caps, floors and collars on continuous flows. Firstly, we produce valuation formulae for finite maturity profit caps and profit floors that are contingent on continuous flows by solving the finite time integral of caplets through a relatively simple integration by parts method, that is, without the need of using the time decomposition technique of Shackleton and Wojakowski (2007). Secondly, we use some arbitrage-free relations to easily obtain the related price caps, price floors and price collars from the proposed analytic representations of profit caps and profit floors. Finally, we provide a new method for calculating the optimal triggers of investment projects containing price floors and price collars clauses that is much simpler and easier to implement than the ones usually used in the literature.

The second paper is related to the computation of perpetual and finite maturity caps and floors on continuous flows under the CEV model. Among the various types of financial derivatives, caps (floors) are the ones that best fit the investment concept under uncertainty and flexibility, especially when we deal with investments that can generate continuous flows, because they can be described as a continuum of caplets (floorlets), each of infinitesimal size as pointed out in McDonald and Siegel (1985) and Shackleton and Wojakowski (2007). The main advantage of caps is that the buyer limits his potential loss to the premium paid, but retains the right to benefit from favorable movements of the underlying asset (the output price of a project, for example). A floor helps an investor to protect himself against a fall in the output price of the project and maintains the ability to participate in a rise in it. Furthermore, caps and floors are flexible. The strike can be positioned to reflect the level of protection investor seeks. The maturity of the cap (floor) is also flexible and does not have to match the term of the underlying investment. In addition, as a cap (floor) is not part of the underlying investment, the protection afforded can apply to any investment with similar commercial terms. Thus, paper 2 extends the previous literature on caps and floors arrangements by providing new analytical formulae for valuing perpetual and finite maturity caps and floors that are contingent on continuous flows. It also discusses the impact of the finite-lived solutions on the optimal behavior of a firm, relative to the perpetual case. Finally, it shows the implications of the correct specification of the underlying state variable process for the valuation of caps and floors by comparing the CEV results with the ones obtained when assuming a lognormal diffusion.

Based on the formulae presented in Paper 2, the third paper offers an analytical representation for evaluating optimal investment decisions associated to a feed-in tariff (FIT) contract with a minimum price guarantee (i.e., a price floor regime) under the CEV model. We demon-

strate that the optimal price-floor level triggering immediate investment in the presence of a perpetual guarantee is independent of the elasticity parameter of the CEV model. However, such independence is not valid any more in the case of FIT contracts with a finite maturity guarantee. Our results provide evidence that care must be taken when a policymaker aims to design a given instrument to induce investment decisions with FIT contracts because the differences between trigger points under alternative modeling assumptions are quite significant and the excessive rents are usually paid at the expense of tax payers.

Finally, it should be noted that the second paper was already presented at

- *INFORMS Annual Meeting*, Indianapolis, Indiana, USA (2022);
- *19th EUROPT Workshop on Advances in Continuous Optimization, Session on Optimal Control and Optimization in Finance and Insurance*, Costa da Caparica, Portugal (2022);

and the third paper is going to be presented at the upcoming

- *26th International Conference on Real Options*, Durham, UK (2023);
- *12th International Conference of the Portuguese Finance Network*, Funchal, Madeira, Portugal (2023).

The thesis proceeds as follows. Chapters 2, 3 and 4 present the first, second and third articles, respectively. Finally, Chapter 5 concludes.

2. Novel analytic representations for caps, floors and collars on continuous flows, arbitrage-free relations and optimal investments

2.1. Introduction

Shackleton and Wojakowski (2007) provide analytic formulae for valuing finite maturity (profit) caps and floors that are contingent on continuous flows following a lognormal distribution. To accomplish this purpose, they use perpetual methods inspired in the real options literature and evaluate the finite-lived case using the risk-neutral expectation of a forward start perpetuity. More specifically, and considering for instance the case of a finite-lived profit cap expiring in T years, its value is decomposed (or replicated) by a portfolio that includes a long position in a perpetual profit cap and a short position in a forward start perpetual profit cap that begins after T years. The use of this replicating portfolio is feasible because the individual caplets contained within each integral are independent. This time decomposition technique has emerged in the literature as the stepping-stone to many other recent applications of real options models with related caps, floors and collars—e.g., Barbosa et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022) and Paxson et al. (2022).

The main goal of this paper is to produce novel analytic representations for valuing finite-lived profit caps and profit floors that are contingent on continuous flows, but without the need of using the replicating portfolio approach offered by Shackleton and Wojakowski (2007), i.e., that do not require the computation of forward starting perpetuities. By doing so, we are able to directly tackle the valuation of finite-lived profit caps and profit floors. Moreover, we show that the usual put-call duality in the spirit of McDonald and Schroder (1998) is still valid for a sum of finite or infinite continuum of European-style caplets and floorlets, which implies that any finite maturity (resp., perpetual) profit cap solution can be used to evaluate the corresponding finite-lived (resp., perpetual) profit floor (or vice versa), with a simple and appropriate switching of model parameters.

We recall that different, but related, contractual price arrangements with caps, floors and collars clauses are often used by policy makers as incentive vehicles for promoting investment decisions of private firms—see, for example, Barbosa et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022), Paxson et al. (2022) and the references contained therein. For instance, a price floor is a contract in which there is a floor level (or a minimum price guarantee) that protects the investor against the downside risk of adverse cash flows. By contrast, a price cap arrangement includes a cap level (or a maximum price guarantee) aiming that the investor foregoes abnormally favorable cash flows. A price collar arrangement contains both cap and floor clauses.

Even though price caps, price floors and price collars can be evaluated analytically using also a replicating portfolio approach, their direct link with the profit caps and profits floors studied in Shackleton and Wojakowski (2007) has not been properly acknowledged in the literature. To fill this gap, we show that such price caps, price floors and price collars are easily obtained from the proposed analytic representations of profit caps and profit floors using some arbitrage-free relations. If any of these relationships is not obeyed, then an arbitrage opportunity emerges instantaneously. Moreover, we also show how to derive the Greeks of all these finite-lived contractual agreements.

Finally, our novel definitions allow us to calculate the optimal triggers of investment in a way that is much simpler than the ones offered in Barbosa et al. (2018) and Adkins et al. (2019) for the cases of projects with floor-style and collar-style arrangements, respectively.

The remainder of the paper is organized in the following way. Section 2.2 produces the novel analytic representations for valuing profit caps and profit floors and highlights the possibility of applying profit cap-profit floor dualities on continuous flows. Section 2.3 shows how to evaluate price caps and price floors using some arbitrage-free relations with profit caps and profit floors. Section 2.4 extends these arbitrage-free relations for valuing price collars arrangements. The derivation of Greeks of all these finite-lived contractual agreements is shown in Section 2.5. Section 2.6 provides a very simple approach for calculating the optimal thresholds of investment projects with price floors and price collars regimes. Finally, Section 2.7 provides the main conclusions.

2.2. Profit caps and profit floors

The main goal of this section is to produce novel analytical formulae to evaluate (profit) caps and floors that are contingent on continuous flows following a lognormal distribution. Similarly to Shackleton and Wojakowski (2007), we consider a continuous price or revenue process $\{P_t \in \mathbb{R}^+ : t \geq 0\}$ that generates cash at an instantaneous rate of flow $P_t dt$ and that is governed by the risk-neutral dynamics

$$dP_t = (r - q) P_t dt + \sigma P_t dW_t^{\mathbb{Q}}, \quad (2.1)$$

where r , q and σ are the (positive and constant) risk free interest rate, dividend yield (or rate of return shortfall) and volatility, respectively, while $\{W_t^{\mathbb{Q}} \in \mathbb{R} : t \geq 0\}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.

Let $V(P_t, X, T)$ be the time- t value of a finite-lived profit cap. Its value can be derived from a cash flow of an instantaneous maximum flow rate $\Pi(P_t) dt$, with $\Pi(P_t) = (P_t - X)^+$, over stochastic revenue P_t and fixed cost X . Under the risk-neutral measure, its (current) time-0 value is given by

$$\begin{aligned} V(P_0, X, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [\Pi(P_t)] dt \\ &= \int_0^T c_0(P_0, X, t) dt, \end{aligned} \quad (2.2)$$

where $c_0(P_0, X, t)$ is interpreted as the time-0 price of the European-style call option (or caplet) offered in Black and Scholes (1973) and Merton (1973) on an underlying asset with spot price P_0 (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed cost rate) and with expiry date at time t (≥ 0).

Similarly, the value of a finite-lived floor can be obtained as

$$\begin{aligned} F(P_0, X, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [\Pi(P_t)] dt \\ &= \int_0^T p_0(P_0, X, t) dt, \end{aligned} \quad (2.3)$$

but now with $\Pi(P_t) = (X - P_t)^+$, while $p_0(P_0, X, t)$ is interpreted as the time-0 price of a European-style put option (or floorlet) given in Black and Scholes (1973) and Merton (1973) on an underlying asset with spot price P_0 (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed proceeds rate) and with expiry date at time t (≥ 0).

Next proposition produces novel formulae for valuing the contingent claims on continuous flows (2.2) and (2.3) that do not require the use of the time decomposition technique considered in Shackleton and Wojakowski (2007). This is accomplished by simply using integration by parts and, hence, it has the advantage that no risk-neutral expectations of forward start perpetuities are required.

In summary, the proposed novel approach allows us to directly tackle the finite maturity case—which is the most important one in the context of executive management decisions—without the need of using the respective perpetual solutions. Nevertheless, the corresponding perpetual cap and floor claims, denoted by $V(P_0, X, \infty)$ and $F(P_0, X, \infty)$, respectively, arise immediately by taking $T \rightarrow \infty$.

Proposition 2.1. Assume the lognormal process (2.1) with positive r , q and σ .

(i) The time-0 value of a finite-lived cap, $V(P_0, X, T)$, is given by

$$V(P_0, X, T) = -\frac{P_0}{q}I(0, T, P_0, X, 1, q) + \frac{X}{r}I(0, T, P_0, X, -1, r), \quad (2.4)$$

where

$$\begin{aligned} & I(0, T, P_0, X, \phi, v) \\ = & e^{-vT} N\left(\frac{a}{\sqrt{T}} + b_\phi \sqrt{T}\right) - N(d(0)) \\ & - \frac{1}{2} \left(\frac{b_\phi}{c_v} + 1\right) e^{a(c_v - b_\phi)} \left[N\left(\frac{a}{\sqrt{T}} + c_v \sqrt{T}\right) - N(d(0)) \right] \\ & - \frac{1}{2} \left(\frac{b_\phi}{c_v} - 1\right) e^{-a(c_v + b_\phi)} \left[N\left(-\frac{a}{\sqrt{T}} + c_v \sqrt{T}\right) - 1 + N(d(0)) \right], \end{aligned} \quad (2.5)$$

with $N(\cdot)$ representing the cumulative density function of the standard univariate normal distribution,

$$a := \frac{\ln(P_0/X)}{\sigma}, \quad (2.6)$$

$$b_\phi := \frac{r - q + \phi\sigma^2/2}{\sigma}, \quad (2.7)$$

$$c_v := \sqrt{b_\phi^2 + 2v}, \quad (2.8)$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}^+$, and¹

$$N(d(0)) := 1 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 0 \times \mathbb{1}_{\{P_0 < X\}}. \quad (2.9)$$

(ii) The time-0 value of a finite-lived floor, $F(P_0, X, T)$, is given by

$$F(P_0, X, T) = V(P_0, X, T) + \frac{X}{r} (1 - e^{-rT}) - \frac{P_0}{q} (1 - e^{-qT}). \quad (2.10)$$

(iii) The time-0 value of a perpetual cap, $V(P_0, X, \infty)$, is given by

$$V(P_0, X, \infty) = -\frac{P_0}{q}I(0, \infty, P_0, X, 1, q) + \frac{X}{r}I(0, \infty, P_0, X, -1, r), \quad (2.11)$$

¹Even though the parameter defined in equation (2.8) could be formally stated as $c_{\phi, v}$, we only define it by c_v to lighten notation and because c_q and c_r are always associated with $\phi = 1$ and $\phi = -1$, respectively.

where

$$I(0, \infty, P_0, X, \phi, v) = -N(d(0)) - \frac{1}{2} \left(\frac{b_\phi}{c_v} + 1 \right) e^{a(c_v - b_\phi)} [1 - N(d(0))] - \frac{1}{2} \left(\frac{b_\phi}{c_v} - 1 \right) e^{-a(c_v + b_\phi)} N(d(0)). \quad (2.12)$$

(iv) The time-0 value of a perpetual floor, $F(P_0, X, \infty)$, is given by

$$F(P_0, X, \infty) = V(P_0, X, \infty) + \frac{X}{r} - \frac{P_0}{q}. \quad (2.13)$$

Proof. Please see Appendix 2A. ■

Although the closed-form solutions (2.10) and (2.13) have been derived without explicitly invoking the caplet-floorlet parity, such arbitrage-free relation is, as expected, observed in both cases. Another interesting arbitrage-free relation for profit caps and profit floors on continuous flows can be also stated using the insights of McDonald and Schroder (1998). In particular, they show that a put-call duality holds whenever the price of a put option can be recovered from the price of a call option (and vice versa) through a suitable change in its arguments—more specifically, by changing the spot price by the strike and the interest rate by the dividend yield.

Taking the plain-vanilla call (or caplet) and put (or floorlet) solutions offered in Black and Scholes (1973) and Merton (1973) explicitly defined as functions of all of their input parameters, it follows that

$$c_0(P_0, X, T, \sigma, r, q) = p_0(X, P_0, T, \sigma, q, r). \quad (2.14)$$

Since this is true for any expiry date T , then it should be valid also for a sum of finite or infinite continuum of European-style caplets and floorlets. Therefore, the following two profit cap-profit floor dualities on continuous flows will be also true:

$$V(P_0, X, T, \sigma, r, q) = F(X, P_0, T, \sigma, q, r) \quad (2.15)$$

and

$$V(P_0, X, \infty, \sigma, r, q) = F(X, P_0, \infty, \sigma, q, r). \quad (2.16)$$

The intuition for equation (2.14) rests on the duality between the underlying asset and cash. A call option gives the right (but not the obligation) to exchange cash for the asset, while a put option gives the right (but not the obligation) to exchange the asset for cash. This duality result implies that any finite maturity or perpetual profit cap solution can be used to evaluate the corresponding finite-lived or perpetual profit floor (or the other way around) with a simple

substitution of parameters, as shown in equations (2.15) and (2.16).

2.3. Price caps and price floors

The goal now is to analyze the price caps and price floors in the spirit of Barbosa et al. (2018), Adkins and Paxson (2019), Barbosa et al. (2020), Barbosa et al. (2022) and Paxson et al. (2022) that are expressed also in the form of perpetual real options and, therefore, require the use of the time decomposition technique considered in Shackleton and Wojakowski (2007). Using the novel results of Proposition 2.1, we show that the finite-lived price caps and price floors can be evaluated without the need of computing risk-neutral expectations of forward start perpetuities. Again, the corresponding perpetual claims arise immediately by taking $T \rightarrow \infty$ in the finite maturity solutions.

2.3.1. Price caps

A price cap provides an instantaneous flow rate $\Pi(P_t) = \min(P_t, H)$ and can be thought as the value of an active project containing a cap level H that provides a ceiling to a given market price. Considering for instance a project associated to a given energy generating facility, the ceiling level H prevents governments to pay excessive rents to producers of energy and shields final customers against escalating prices.

Noting that $\min(P_t, H) = H - \max(H - P_t, 0)$, it is straightforward to conclude that the finite-lived price cap can be represented by

$$\begin{aligned} V_c(P_0, H, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [H - (H - P_t)^+] dt \\ &= H \int_0^T e^{-rt} dt - \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(H - P_t)^+] dt \\ &= \frac{H}{r} (1 - e^{-rT}) - F(P_0, H, T), \end{aligned} \quad (2.17)$$

with $F(P_0, H, T)$ being the finite maturity profit floor (2.10) with a strike $X = H$. This originates an arbitrage-free relation between a finite-lived price cap and a finite maturity profit floor.

Alternatively, since $\min(P_t, H) = P_t - \max(P_t - H, 0)$, it is possible to show that the finite-lived price cap can be also calculated as

$$\begin{aligned} V_c(P_0, H, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [P_t - (P_t - H)^+] dt \\ &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [P_t] dt - \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(P_t - H)^+] dt \\ &= \frac{P_0}{q} (1 - e^{-qT}) - V(P_0, H, T), \end{aligned} \quad (2.18)$$

with $V(P_0, H, T)$ being the finite maturity profit cap (2.4) with a strike $X = H$. In this case,

this originates an arbitrage-free relation between a finite-lived price cap and a finite maturity profit cap.

The corresponding perpetual price cap can be obtained either via a perpetual profit floor or through a perpetual profit cap by taking the limit $T \rightarrow \infty$ of the analytic representations (2.17) and (2.18), respectively, thus resulting in

$$\begin{aligned} V_c(P_0, H, \infty) &= \lim_{T \rightarrow \infty} V_c(P_0, H, T) \\ &= \frac{H}{r} - F(P_0, H, \infty) \end{aligned} \quad (2.19)$$

$$= \frac{P_0}{q} - V(P_0, H, \infty). \quad (2.20)$$

Clearly, equating the solutions (2.17) and (2.18) yields the finite-lived caplet-floorlet parity relation (2.10), whereas the combination of the perpetual solutions (2.19) and (2.20) obeys the infinite caplet-floorlet parity relation (2.13).

Remark 2.1. *The arbitrage-free relations (2.19) and (2.20) constitute viable alternatives for computing the value of an active project of a firm in the spirit of Adkins and Paxson (2019, equation 14).*

2.3.2. Price floors

A price floor generates an instantaneous flow rate $\Pi(P_t) = \max(P_t, L)$ and can be understood as the value of an active project containing a floor level L that guarantees a minimum to the prevailing market price in the face of adverse circumstances. Considering for example a project associated to a given energy generating facility, the floor level L aims to prevent the downside risk to producers of energy.

Noting that $\max(P_t, L) = \max(P_t - L, 0) + L$, it is straightforward to conclude that the finite-lived price floor can be represented by

$$\begin{aligned} V_f(P_0, L, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(P_t - L)^+ + L] dt \\ &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(P_t - L)^+] dt + L \int_0^T e^{-rt} dt \\ &= V(P_0, L, T) + \frac{L}{r} (1 - e^{-rT}), \end{aligned} \quad (2.21)$$

with $V(P_0, L, T)$ being the finite maturity profit cap (2.4) with a strike $X = L$. This originates an arbitrage-free relation between a finite-lived price floor and a finite maturity profit cap. In particular, definition (2.21) will be used for calculating the optimal thresholds of investment in the presence of a price floor regime that is covered in Section 2.6.

Alternatively, since $\max(P_t, L) = \max(L - P_t, 0) + P_t$, it is possible to show that the

finite-lived price floor can be also computed as

$$\begin{aligned}
V_f(P_0, L, T) &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(L - P_t)^+ + P_t] dt \\
&= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(L - P_t)^+] dt + \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [P_t] dt \\
&= F(P_0, L, T) + \frac{P_0}{q} (1 - e^{-qT}), \tag{2.22}
\end{aligned}$$

with $F(P_0, L, T)$ being the finite maturity profit floor (2.10) with a strike $X = L$. In this case, this originates an arbitrage-free relation between a finite-lived price floor and a finite maturity profit floor.

Remark 2.2. *The arbitrage-free relations (2.21) and (2.22) can be understood as viable alternatives for evaluating the value of an active project of a firm in the spirit of Barbosa et al. (2018, equation 19).*

The corresponding perpetual price floor can be obtained either via a perpetual profit cap or through a perpetual profit floor by taking the limit $T \rightarrow \infty$ of the analytic representations (2.21) and (2.22), respectively, thus yielding

$$\begin{aligned}
V_f(P_0, L, \infty) &= \lim_{T \rightarrow \infty} V_f(P_0, L, T) \\
&= V(P_0, L, \infty) + \frac{L}{r} \tag{2.23}
\end{aligned}$$

$$= F(P_0, L, \infty) + \frac{P_0}{q}. \tag{2.24}$$

Clearly, equalizing the solutions (2.21) and (2.22) yields the finite-lived caplet-floorlet parity relation (2.10), whereas the combination of the perpetual solutions (2.23) and (2.24) obeys the infinite caplet-floorlet parity relation (2.13).

Remark 2.3. *The arbitrage-free relations (2.23) and (2.24) can be understood as viable alternatives for evaluating the value of an active project of a firm in the spirit of Barbosa et al. (2018, equation 8) and Adkins and Paxson (2019, equation 12).*

2.3.3. Price caps, price floors and vertical bullish spreads

A portfolio composed by a long price cap and a long price floor can be defined in terms of a vertical bullish spread. Combining equations (2.18) and (2.21) as well as equations (2.17)

and (2.22) yields, respectively,

$$\begin{aligned} & V_c(P_0, H, T) + V_f(P_0, L, T) \\ &= \frac{L}{r} (1 - e^{-rT}) + \frac{P_0}{q} (1 - e^{-qT}) + V(P_0, L, T) - V(P_0, H, T) \end{aligned} \quad (2.25)$$

$$= \frac{H}{r} (1 - e^{-rT}) + \frac{P_0}{q} (1 - e^{-qT}) + F(P_0, L, T) - F(P_0, H, T), \quad (2.26)$$

where $V(P_0, L, T) - V(P_0, H, T)$ is interpreted as a bull profit cap spread (i.e., a long profit cap with lower strike L and a short profit cap with higher strike H), whereas $F(P_0, L, T) - F(P_0, H, T)$ is understood as a bull profit floor spread (i.e., a long profit floor with lower strike L and a short profit floor with higher strike H).

The special case with $L = H := X$ results in

$$V_c(P_0, X, T) + V_f(P_0, X, T) = \frac{X}{r} (1 - e^{-rT}) + \frac{P_0}{q} (1 - e^{-qT}), \quad (2.27)$$

which corresponds to a portfolio (without restrictions in the flow P_t) paying a cash flow rate $(X + P_t) dt$. Again, the relations (2.25), (2.26) and (2.27) must be satisfied in order to avoid the existence of arbitrage opportunities.

2.4. Price collars

Let us now consider the price collars studied in Adkins and Paxson (2019), Barbosa et al. (2020), Barbosa et al. (2022) and Paxson et al. (2022), which have been represented in the form of perpetual methods valued through the portfolio decomposition approach of Shackleton and Wojakowski (2007).

In a price collar arrangement the price floats freely subject to a price floor level L and a price cap level H (with $H \geq L$), so that the investor receives L if $P_t < L$, receives the unit price P_t if $L \leq P_t < H$ and receives H if $P_t \geq H$. Its instantaneous flow rate is defined as $\Pi(P_t) = \min(\max(L, P_t), H)$, that is

$$\Pi(P_t) = L \mathbb{1}_{\{P_t < L\}} + P_t \mathbb{1}_{\{L \leq P_t < H\}} + H \mathbb{1}_{\{P_t \geq H\}}. \quad (2.28)$$

Clearly, this payoff structure can be analyzed under different perspectives. For instance, the payoff (2.28) can be understood as a portfolio of path-independent binary options in the sense of Rubinstein and Reiner (1991) with the following components: (i) a European-style cash-or-nothing put on continuous flows with a strike price equal to the predetermined fixed cash amount L ; (ii) a European-style range asset on the flow rate P_t , with lower strike L and upper strike H ; and (iii) a European-style cash-or-nothing call on continuous flows with a strike price equal to the predetermined fixed cash amount H .

Apart from this interpretation, the payoff (2.28) can also be expressed in terms of profit

caps, profit floors, price caps or price floors. Such arbitrage-free relations will be presented next.

2.4.1. Finite-lived price collars expressed in terms of profit caps

It is straightforward to show that the payoff (2.28) is equivalent to

$$\Pi(P_t) = L + (P_t - L)\mathbb{1}_{\{P_t \geq L\}} - (P_t - H)\mathbb{1}_{\{P_t \geq H\}}. \quad (2.29)$$

Therefore, a finite-lived price collar can be expressed as

$$\begin{aligned} C(P_0, L, H, T) &= L \int_0^T e^{-rt} dt + \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(P_t - L)^+] dt - \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(P_t - H)^+] dt \\ &= \frac{L}{r} (1 - e^{-rT}) + V(P_0, L, T) - V(P_0, H, T) \end{aligned} \quad (2.30)$$

$$= V_f(P_0, L, T) - V(P_0, H, T), \quad (2.31)$$

with $V(P_0, X, T)$, for $X \in \{L, H\}$, and $V_f(P_0, L, T)$ being the finite maturity profit cap (2.4) and the finite maturity price floor (2.21), respectively.

This implies that the finite-lived price collar (2.30) can be interpreted as a portfolio composed by: (i) a cash amount with a present value equal to $\frac{L}{r} (1 - e^{-rT})$ that can be understood as the sum of long forward contracts with instantaneous delivery price L ; (ii) a long position in a finite-lived profit cap with fixed cost L ; and (iii) a short position in a finite-lived profit cap with fixed cost H . In particular, definition (2.30) will be used for determining the optimal triggers of investment in the presence of a price collar regime that is covered in Section 2.6. Moreover, the equivalent finite-lived price collar (2.31) can be understood as a portfolio composed by: (i) a long position in a finite-lived price floor with a fixed guarantee L ; and (ii) a short position in a finite-lived profit cap with fixed cost H .

2.4.2. Finite-lived price collars expressed in terms of profit floors

It can be shown that the payoff (2.28) is equivalent to

$$\Pi(P_t) = H + (L - P_t)\mathbb{1}_{\{P_t < L\}} - (H - P_t)\mathbb{1}_{\{P_t < H\}}. \quad (2.32)$$

Therefore, a finite-lived price collar can be expressed as

$$\begin{aligned} C(P_0, L, H, T) &= H \int_0^T e^{-rt} dt + \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(L - P_t)^+] dt - \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(H - P_t)^+] dt \\ &= \frac{H}{r} (1 - e^{-rT}) + F(P_0, L, T) - F(P_0, H, T) \end{aligned} \quad (2.33)$$

$$= V_c(P_0, H, T) + F(P_0, L, T), \quad (2.34)$$

with $F(P_0, X, T)$, for $X \in \{L, H\}$, and $V_c(P_0, H, T)$ being the finite maturity profit floor (2.10) and the finite maturity price cap (2.17), respectively.

This implies that the finite-lived price collar (2.33) can be interpreted as a portfolio composed by: (i) a cash amount with a present value equal to $\frac{H}{r} (1 - e^{-rT})$ that can be understood as the sum of long forward contracts with instantaneous delivery price H ; (ii) a long position in a finite-lived profit floor with a fixed proceeds rate L ; and (iii) a short position in a finite-lived profit floor with a fixed proceeds rate H . Alternatively, the equivalent finite-lived price collar (2.34) can be understood as a portfolio composed by: (i) a long position in a finite-lived price cap capped at H ; and (ii) a long position in a finite-lived profit floor with a fixed proceeds rate L .

2.4.3. Finite-lived price collars expressed in terms of portfolios of profit caps and profit floors

It is also straightforward to show that the payoff (2.28) is equivalent to

$$\Pi(P_t) = P_t - (P_t - H)\mathbb{1}_{\{P_t \geq H\}} + (L - P_t)\mathbb{1}_{\{P_t < L\}}. \quad (2.35)$$

Therefore, a finite-lived price collar can be expressed as

$$\begin{aligned} & C(P_0, L, H, T) \\ &= \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [P_t] dt - \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(P_t - H)^+] dt + \int_0^T e^{-rt} \mathbb{E}_{\mathbb{Q}} [(L - P_t)^+] dt \\ &= \frac{P_0}{q} (1 - e^{-qT}) - V(P_0, H, T) + F(P_0, L, T) \end{aligned} \quad (2.36)$$

$$= V_c(P_0, H, T) + F(P_0, L, T) \quad (2.37)$$

$$= V_f(P_0, L, T) - V(P_0, H, T), \quad (2.38)$$

with $V(P_0, H, T)$, $F(P_0, L, T)$, $V_c(P_0, H, T)$ and $V_f(P_0, L, T)$ being the finite maturity profit cap (2.4), the finite maturity profit floor (2.10), the finite maturity price cap (2.18) and the finite maturity price floor (2.22), respectively.

This implies that the finite-lived price collar (2.36) can be interpreted as a portfolio composed by: (i) the value of a plain active project with a duration equal to the time to maturity of the finite collar and having a present value equal to $\frac{P_0}{q} (1 - e^{-qT})$; (ii) a short position in a finite-lived profit cap with a fixed cost rate H ; and (iii) a long position in a finite-lived profit floor with a fixed proceeds rate L .

Remark 2.4. *The portfolio decomposition (2.36) is equivalent to the one that is offered in Adkins et al. (2019, equation 22 and Table 2), but only considering a finite concession with a duration equal to the maturity of the finite-lived collar, i.e., ignoring the value in operating the project after the end of the collar arrangement that is given by $\frac{P_0}{q}e^{-qT}$. We recall, however, that in our case it is not necessary to apply the replicating portfolio approach of Shackleton and Wojakowski (2007) to obtain the price collar (2.36). Moreover, the equivalent arbitrage-free relations (2.30) and (2.33) are expressed in terms of vertical bullish spreads of profit caps and profit floors, respectively, and offer an alternative economic intuition for the finite-lived price collar arrangement.*

Remark 2.5. *The portfolio decompositions (2.37) and (2.38) provide the rationale for two special cases of price collars that are often mentioned in the literature: (i) if $H \rightarrow \infty$, then $\lim_{H \rightarrow \infty} V(P_0, H, T) = 0$ because the price cap would never be exercised with an infinite strike price. This explains why a price collar arrangement with $H \rightarrow \infty$ is equivalent to a price floor after using the arbitrage-free relation (2.38); and (ii) if $L \rightarrow 0$, then $\lim_{L \rightarrow 0} F(P_0, L, T) = 0$ because the price floor would never be exercised with an infinitesimal strike price. Again, this offers the intuition why a price collar arrangement with $L \rightarrow 0$ is equivalent to a price cap after using the arbitrage-free relation (2.37).*

2.4.4. Finite-lived price collars expressed in terms of price floors or price caps

The finite-lived price collar (2.31) can be rearranged so that it is expressed in terms of price floors, that is

$$\begin{aligned} C(P_0, L, H, T) &= V_f(P_0, L, T) - V(P_0, H, T) - \frac{H}{r} (1 - e^{-rT}) + \frac{H}{r} (1 - e^{-rT}) \\ &= V_f(P_0, L, T) - V_f(P_0, H, T) + \frac{H}{r} (1 - e^{-rT}), \end{aligned} \quad (2.39)$$

after using the arbitrage-free relation (2.21).

Similarly, the finite-lived price collar (2.34) can be rewritten in terms of price caps, that is

$$\begin{aligned} C(P_0, L, H, T) &= V_c(P_0, H, T) + F(P_0, L, T) - \frac{L}{r} (1 - e^{-rT}) + \frac{L}{r} (1 - e^{-rT}) \\ &= \frac{L}{r} (1 - e^{-rT}) + V_c(P_0, H, T) - V_c(P_0, L, T), \end{aligned} \quad (2.40)$$

after using the arbitrage-free relation (2.17).

2.4.5. Perpetual price collars

The corresponding perpetual price collars can be obtained by taking the limit $T \rightarrow \infty$ of any of the analytic representations presented above. Using, for example, the arbitrage-free relations (2.30), (2.33) and (2.36), then

$$\begin{aligned} C(P_0, L, H, \infty) &= \lim_{T \rightarrow \infty} C(P_0, L, H, T) \\ &= \frac{L}{r} + V(P_0, L, \infty) - V(P_0, H, \infty) \end{aligned} \quad (2.41)$$

$$= \frac{H}{r} + F(P_0, L, \infty) - F(P_0, H, \infty) \quad (2.42)$$

$$= \frac{P_0}{q} - V(P_0, H, \infty) + F(P_0, L, \infty), \quad (2.43)$$

respectively.

Remark 2.6. *The arbitrage-free relations (2.41), (2.42) and (2.43) constitute viable alternatives for computing the value of an active project of a firm in the spirit of Adkins and Paxson (2019, equation 6) and Adkins et al. (2019, equation 9).*

2.5. Greeks

The so-called Greeks, also known as options sensitivity measures, are important tools for traders when implementing hedging and risk management strategies, especially in the case of naked short options positions. The computation of such Greek measures involves the partial derivative of the option price with respect to one of its input parameters. The most important Greeks are the delta, gamma and theta which correspond to the first and second derivatives with respect to the underlying asset price as well as time. These partial derivatives are especially relevant because they appear within the usual partial differential equation satisfied by the option price. Next proposition provides such Greek measures for a finite-lived cap.

Proposition 2.2. *Assume the lognormal process (2.1) with positive r , q and σ , as well as the finite-lived cap solution (2.4).*

(i) *The delta of a finite-lived cap, $\Delta_V(P_0, X, T) := \partial V(P_0, X, T) / \partial P_0$, is given by*

$$\begin{aligned} &\Delta_V(P_0, X, T) \\ &= -\frac{1}{q} [I(0, T, P_0, X, 1, q) + P_0 I_{P_0}(0, T, P_0, X, 1, q)] + \frac{X}{r} I_{P_0}(0, T, P_0, X, -1, r), \end{aligned} \quad (2.44)$$

where

$$\begin{aligned}
& I_{P_0}(0, T, P_0, X, \phi, v) \\
& =: \frac{\partial I(0, T, P_0, X, \phi, v)}{\partial P_0} \\
& = -\frac{1}{2\sigma P_0} \left(\frac{b_\phi}{c_v} + 1 \right) (c_v - b_\phi) e^{a(c_v - b_\phi)} \left[N \left(\frac{a}{\sqrt{T}} + c_v \sqrt{T} \right) - N(d(0)) \right] \\
& \quad + \frac{1}{2\sigma P_0} \left(\frac{b_\phi}{c_v} - 1 \right) (c_v + b_\phi) e^{-a(c_v + b_\phi)} \left[N \left(-\frac{a}{\sqrt{T}} + c_v \sqrt{T} \right) - 1 + N(d(0)) \right].
\end{aligned} \tag{2.45}$$

(ii) The gamma of a finite-lived cap, $\Gamma_V(P_0, X, T) := \partial^2 V(P_0, X, T) / \partial P_0^2$, is given by

$$\begin{aligned}
& \Gamma_V(P_0, X, T) \\
& = -\frac{1}{q} \left[2 I_{P_0}(0, T, P_0, X, 1, q) + P_0 I_{P_0^2}(0, T, P_0, X, 1, q) \right] + \frac{X}{r} I_{P_0^2}(0, T, P_0, X, -1, r),
\end{aligned} \tag{2.46}$$

where

$$\begin{aligned}
& I_{P_0^2}(0, T, P_0, X, \phi, v) \\
& =: \frac{\partial^2 I(0, T, P_0, X, \phi, v)}{\partial P_0^2} \\
& = \left(1 - \frac{c_v - b_\phi}{\sigma} \right) \frac{1}{2\sigma P_0^2} \left(\frac{b_\phi}{c_v} + 1 \right) (c_v - b_\phi) e^{a(c_v - b_\phi)} \left[N \left(\frac{a}{\sqrt{T}} + c_v \sqrt{T} \right) - N(d(0)) \right] \\
& \quad - \left(1 + \frac{c_v + b_\phi}{\sigma} \right) \frac{1}{2\sigma P_0^2} \left(\frac{b_\phi}{c_v} - 1 \right) (c_v + b_\phi) e^{-a(c_v + b_\phi)} \\
& \quad \times \left[N \left(-\frac{a}{\sqrt{T}} + c_v \sqrt{T} \right) - 1 + N(d(0)) \right].
\end{aligned} \tag{2.47}$$

(iii) The theta of a finite-lived cap, $\theta_V(P_0, X, T) := \partial V(P_0, X, T) / \partial T$, is given by

$$\theta_V(P_0, X, T) = c_0(P_0, X, T), \tag{2.48}$$

where $c_0(P_0, X, t)$ is the time-0 price of a European-style call option (or caplet) given in Black and Scholes (1973) and Merton (1973).

Proof. These derivatives arise immediately after differentiating the finite-lived cap solution (2.4) with respect to the underlying asset price and time and by applying the chain rule to the cumulative normal distribution. Full details of the proof are available upon request. ■

In summary, the analytic representation of the Greeks presented in Proposition 2.2 constitute a viable alternative to the calculation of such partial derivatives obtained via the replicating portfolio approach of Shackleton and Wojakowski (2007). Even though Proposition 2.2 focuses only on the Greeks of finite-lived caps, the corresponding sensitivity measures of the other

contractual arrangements discussed in this paper arise immediately using the aforementioned arbitrage-free relations.

2.6. Optimal investment decisions with price floors and price collars regimes

In this section, we analyze investment decisions with price floors and price collars regimes in the spirit of Barbosa et al. (2018) and Adkins et al. (2019) and we produce new and simpler formulae for calculating the optimal thresholds of investment.

2.6.1. Investment opportunity with a price floor regime

The goal now is to calculate the value of a (perpetual) option to invest in a renewable energy project to produce and sell energy with a price floor regime—also known as a feed-in-tariff (hereafter, FIT) contract—and the corresponding optimal investment rule.

2.6.1.1. Finite-lived case

Following Barbosa et al. (2018), we consider that after the expiry date of the finite maturity guarantee the producer of energy receives a cash flow with a present value equal to $\frac{P_0}{q}e^{-qT}$, that is understood as the value of selling the energy to the market after the maturity date of the FIT contract. Therefore, the total (or full) value of the project that includes the period of the FIT contract and the (perpetual) period thereafter is given by

$$\begin{aligned} V_f^F(P_0, L, T) &= V_f(P_0, L, T) + \frac{P_0}{q}e^{-qT} \\ &= V(P_0, L, T) + \frac{L}{r}(1 - e^{-rT}) + \frac{P_0}{q}e^{-qT}, \end{aligned} \quad (2.49)$$

after using equation (2.21) that expresses the finite-lived price floor in terms of the profit cap (2.4). We recall that in the FIT contract under analysis the producer (or investor) is entitled to receive the price floor L for every unit of energy produced when the market price P_t is below the price floor; otherwise (i.e., when $P_t \geq L$), the producer receives the market price. Notice also that our definition (2.21) is equivalent to Barbosa et al. (2018, equation 19), though without the need of having different solutions for the regions $P_0 < L$ and $P_0 \geq L$.

It is well known that the value of the (perpetual) option to invest in a project with a total value as given in equation (2.49), $F_f(P_0, \bar{P}, L, T, K)$, with $\bar{P}$ being interpreted as the optimal price trigger level that induces the producer to invest in a project requiring an investment amount equal to K , is obtained as the solution of an ordinary differential equation subject to the usual boundary condition $\lim_{P_0 \rightarrow 0} F_f(P_0, \bar{P}, L, T, K) = 0$, thus yielding

$$F_f(P_0, \bar{P}, L, T, K) = AP_0^{\beta_1}, \quad (2.50)$$

where A is to be determined from appropriate value-matching and smooth-pasting conditions,

while the constant parameter $\beta_1(> 1)$ is given by

$$\begin{aligned}\beta_1 &= \frac{1}{2} - \frac{r-q}{\sigma^2} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \\ &= \frac{c_r - b_{-1}}{\sigma}.\end{aligned}\quad (2.51)$$

Applying the usual value-matching and smooth-pasting conditions to the option solution (2.50) at the optimal trigger $\bar{P}$, yields

$$\begin{cases} A\bar{P}^{\beta_1} &= V_f^F(\bar{P}, L, T) - K \\ \beta_1 A\bar{P}^{\beta_1-1} &= \frac{\partial}{\partial \bar{P}} V_f^F(\bar{P}, L, T) \end{cases}, \quad (2.52)$$

which is equivalent to

$$\begin{cases} \beta_1 A\bar{P}^{\beta_1} &= \beta_1 V_f^F(\bar{P}, L, T) - \beta_1 K \\ \beta_1 A\bar{P}^{\beta_1} &= \bar{P} \frac{\partial}{\partial \bar{P}} V_f^F(\bar{P}, L, T) \end{cases}, \quad (2.53)$$

after multiplying both sides of the value-matching and smooth-pasting conditions (2.52) by β_1 and $\bar{P}$, respectively.

Equating both branches of the system (2.53) yields

$$\beta_1 V_f^F(\bar{P}, L, T) - \beta_1 K = \bar{P} \frac{\partial}{\partial \bar{P}} V_f^F(\bar{P}, L, T), \quad (2.54)$$

which can be rewritten as

$$\begin{aligned}& \beta_1 \left(V(\bar{P}, L, T) + \frac{L}{r} (1 - e^{-rT}) + \frac{\bar{P}}{q} e^{-qT} - K \right) \\ &= \bar{P} \frac{\partial}{\partial \bar{P}} \left(V(\bar{P}, L, T) + \frac{L}{r} (1 - e^{-rT}) + \frac{\bar{P}}{q} e^{-qT} \right),\end{aligned}\quad (2.55)$$

by combining equations (2.49) and (2.54).

Finally, using equations (2.4) and (2.44) and applying straightforward calculus, the optimal trigger $\bar{P}$ can be determined by numerically solving the following nonlinear equation

$$\begin{aligned}& (\beta_1 - 1) \frac{\bar{P}}{q} [e^{-qT} - I(0, T, \bar{P}, L, 1, q)] + \beta_1 \left(\frac{L}{r} [1 - e^{-rT} + I(0, T, \bar{P}, L, -1, r)] - K \right) \\ &+ \bar{P} \left[\frac{\bar{P}}{q} I_{\bar{P}}(0, T, \bar{P}, L, 1, q) - \frac{L}{r} I_{\bar{P}}(0, T, \bar{P}, L, -1, r) \right] = 0.\end{aligned}\quad (2.56)$$

Once the optimal trigger $\bar{P}$ is known we can immediately determine A using the value-matching condition (2.52) and replace it in equation (2.50), thus allowing the calculation of

the value of the (perpetual) option to invest in the project as

$$F_f(P_0, \bar{P}, L, T, K) = \begin{cases} (V_f^F(\bar{P}, L, T) - K) \left(\frac{P_0}{\bar{P}}\right)^{\beta_1} & \Leftarrow P_0 < \bar{P} \\ V_f^F(P_0, L, T) - K & \Leftarrow P_0 \geq \bar{P} \end{cases}. \quad (2.57)$$

Notice that equation (2.56) is equivalent to Barbosa et al. (2018, equation 26), but with the great advantage that it is not necessary to know in advance in which region the threshold is, i.e., whether $\bar{P} < L$ or $\bar{P} \geq L$. In our case, we immediately know in which region the optimal trigger $\bar{P}$ is once equation (2.56) is solved. By contrast, the use of Barbosa et al. (2018, equation 26) for determining the optimal threshold requires that first we need to find the investment cost level K_L^* that separates the regions $\bar{P} < L$ and $\bar{P} \geq L$. This is accomplished by solving Barbosa et al. (2018, equation 30) with respect to such investment cost level $K \equiv K_L^*$ and then compare it with the initial investment cost K . If $K_L^* \leq K$, the optimal threshold $\bar{P}$ is computed in the region $\bar{P} \geq L$ using the second branch of Barbosa et al. (2018, equation 26). Otherwise, i.e., if $K_L^* > K$, the optimal trigger $\bar{P}$ is calculated in the region $\bar{P} < L$ using the first branch of Barbosa et al. (2018, equation 26). Clearly, our approach for analyzing the optimal investment rule of a project with a price floor regime is much simpler than the one proposed by Barbosa et al. (2018).²

Nevertheless, and for the sake of completeness, we note that it is easy to determine analytically the investment cost level K_L^* that makes the current minimum price guarantee L the optimal trigger of investment. This is accomplished by replacing $\bar{P}$ by L in equation (2.56), that is

$$\begin{aligned} & (\beta_1 - 1) \frac{L}{q} [e^{-qT} - I(0, T, L, L, 1, q)] + \beta_1 \left(\frac{L}{r} [1 - e^{-rT} + I(0, T, L, L, -1, r)] - K \right) \\ & + L \left[\frac{L}{q} I_{\bar{P}=L}(0, T, L, L, 1, q) - \frac{L}{r} I_{\bar{P}=L}(0, T, L, L, -1, r) \right] = 0. \end{aligned} \quad (2.58)$$

For $\bar{P} = L$, it follows immediately that $a = 0$ and $N(d(0)) = 1/2$ after using equations (2.6) and (2.9), respectively. Therefore, equations (2.5) and (2.45) can be simplified to

$$I(0, T, L, L, \phi, v) = e^{-vT} N(b_\phi \sqrt{T}) - \frac{1}{2} - \frac{b_\phi}{c_v} \left[N(c_v \sqrt{T}) - \frac{1}{2} \right] \quad (2.59)$$

and

$$I_{\bar{P}=L}(0, T, L, L, \phi, v) = -\frac{2v}{\sigma L c_v} \left[N(c_v \sqrt{T}) - \frac{1}{2} \right], \quad (2.60)$$

²Furthermore, we only need to apply one value-matching condition and one smooth-pasting condition to derive the investment threshold of a renewable energy project with a finite guarantee, whereas Barbosa et al. (2018, Appendix C) show that the calculation of the optimal trigger in each region must be derived separately. By the way, and for the sake of clarity, we notice that there is a very small typo in Barbosa et al. (2018, equation C.3). The correct formula is the one given in Barbosa et al. (2018, equation 26).

respectively.

Finally, combining equations (2.58), (2.59) and (2.60) and rearranging the resulting terms we obtain a closed-form solution for determining the investment cost level $K \equiv K_L^*$ that makes the current minimum price guarantee L the optimal trigger of investment, that is

$$K_L^* = \frac{LY}{\beta_1}, \quad (2.61)$$

which is equivalent to Barbosa et al. (2018, equation 30), with

$$\begin{aligned} Y = & \frac{\beta_1 - 1}{q} \left(\frac{1}{2} + e^{-qT} \left[1 - N(b_1 \sqrt{T}) \right] + \frac{b_1}{c_q} \left[N(c_q \sqrt{T}) - \frac{1}{2} \right] \right) \\ & + \frac{\beta_1}{r} \left(\frac{1}{2} - e^{-rT} \left[1 - N(b_{-1} \sqrt{T}) \right] - \frac{b_{-1}}{c_r} \left[N(c_r \sqrt{T}) - \frac{1}{2} \right] \right) \\ & - \frac{2}{\sigma c_q} \left[N(c_q \sqrt{T}) - \frac{1}{2} \right] + \frac{2}{\sigma c_r} \left[N(c_r \sqrt{T}) - \frac{1}{2} \right]. \end{aligned} \quad (2.62)$$

There are other interesting points that are important from the perspective of the policy maker. For instance, taking now the initial investment cost K as given it is also possible to determine the L^* level that would be required to be offered by the government so that the firm starts the project immediately. This is accomplished by combining, again, equations (2.58), (2.59) and (2.60), but now solving the resulting expression with respect to $L \equiv L^*$, thus yielding the analytical solution

$$L^* = \frac{\beta_1 K}{Y}, \quad (2.63)$$

which, as expected, can be also derived from equation (2.61). If the policy maker sets the minimum price guarantee $L \geq L^*$, the investor starts immediately the project and receives a revenue from the guarantee instead of a revenue from the market price.

A further interesting point from the policymaker perspective is the price floor level L_0 that turns the net present value (henceforth, NPV) of the project equal to zero, because any value of L above this point generates a positive NPV independently of the market price P . Notice that if $L \geq L_0$ the investment will be deployed immediately because there is no waiting option. This point can be determined analytically by solving the equation $\text{NPV} := \lim_{P_0 \rightarrow 0^+} V_f^F(P_0, L_0, T) - K = 0$ with respect to L_0 and with $V_f^F(P_0, L_0, T)$ being defined as the project value given in equation (2.49), which yields

$$L_0 = \frac{rK}{1 - e^{-rT}} \quad (2.64)$$

after applying straightforward calculus, thus being consistent, as expected, with Barbosa et al. (2018, equation 31).

There is another interesting point that, to the best of our knowledge, has not been mentioned in the context of optimal investment rules of projects with minimum price guarantees. Let us now assume the minimum price guarantee L that is subject to the government policy. Even though the investment cost K is assumed to be fixed, it might be useful to know what would be the investment cost level that would make the current market price P_0 the trigger. In other words, what is the required investment cost, denoted hereafter by K^* , that induces the firm to invest immediately? We can answer this question if we replace $\bar{P}$ by P_0 in equation (2.56) and solve it with respect to the investment cost level $K \equiv K^*$, thus yielding the closed-form solution

$$K^* = \frac{\beta_1 - 1}{\beta_1} \frac{P_0}{q} [e^{-qT} - I(0, T, P_0, L, 1, q)] + \frac{L}{r} [1 - e^{-rT} + I(0, T, P_0, L, -1, r)] + \frac{P_0}{\beta_1} \left[\frac{P_0}{q} I_{P_0}(0, T, P_0, L, 1, q) - \frac{L}{r} I_{P_0}(0, T, P_0, L, -1, r) \right]. \quad (2.65)$$

As expected, if $K \leq K^*$ it would be better for the firm to invest immediately because $P_0 = \bar{P}$ at the K^* level. Hence, when $K > K^*$, if the firm is able to reduce the investment cost from K to K^* —using, for instance, a cheaper technology that might be available in the market—, the current market price P_0 would be enough for the investor to exercise the option to invest immediately. Alternatively, for $K > K^*$, the amount $K - K^*$ can be understood as a subsidy value on investment that might be supported by the government if the goal is to deploy the investment immediately without the need of changing the policy with respect to the minimum price guarantee L . That is, the government supports the firm with a one-time subsidy amount that is paid upfront, while maintaining the fixed price floor L until the end of the finite-lived FIT scheme.

2.6.1.2. Perpetual case

Even though our main goal is to calculate the optimal investment triggers of the finite-lived case, the corresponding optimal thresholds for the perpetual case can be also determined from a nonlinear equation that is derived by taking the limit of equation (2.56) as $T \rightarrow \infty$, that is

$$-(\beta_1 - 1) \frac{\bar{P}}{q} I(0, \infty, \bar{P}, L, 1, q) + \beta_1 \left(\frac{L}{r} [1 + I(0, \infty, \bar{P}, L, -1, r)] - K \right) + \bar{P} \left[\frac{\bar{P}}{q} I_{\bar{P}}(0, \infty, \bar{P}, L, 1, q) - \frac{L}{r} I_{\bar{P}}(0, \infty, \bar{P}, L, -1, r) \right] = 0, \quad (2.66)$$

where $I(0, \infty, \bar{P}, L, \phi, v)$ is calculated via equation (2.12) and $I_{\bar{P}}(0, \infty, \bar{P}, L, \phi, v)$ is obtained by taking the limit of equation (2.45) as $T \rightarrow \infty$, yielding

$$I_{\bar{P}}(0, \infty, \bar{P}, L, \phi, v) = -\frac{1}{2\sigma P_0} \left(\frac{b_\phi}{c_v} + 1 \right) (c_v - b_\phi) e^{a(c_v - b_\phi)} [1 - N(d(0))] \\ + \frac{1}{2\sigma P_0} \left(\frac{b_\phi}{c_v} - 1 \right) (c_v + b_\phi) e^{-a(c_v + b_\phi)} N(d(0)). \quad (2.67)$$

Notice that equation (2.66) is an alternative to Barbosa et al. (2018, equation 16) for computing the optimal triggers for a project with a perpetual minimum price guarantee L . As usual, equation (2.66) is valid only when $\bar{P} \geq L$, because the value-matching and smooth-pasting conditions are not accommodated in the region $\bar{P} < L$ for the perpetual case. Adopting a similar procedure to one employed in the finite-lived case, we can determine the value of the (perpetual) option to invest in the project once the optimal trigger $\bar{P}$ of the perpetual case is known, that is

$$F_f(P_0, \bar{P}, L, \infty, K) = \begin{cases} (V_f^F(\bar{P}, L, \infty) - K) \left(\frac{P_0}{\bar{P}} \right)^{\beta_1} & \Leftarrow P_0 < \bar{P} \\ V_f^F(P_0, L, \infty) - K & \Leftarrow P_0 \geq \bar{P} \end{cases}, \quad (2.68)$$

where $V_f^F(P, L, \infty)$ is computed as the limit of equation (2.49) as $T \rightarrow \infty$, which yields the perpetual price floor solution (2.23).

Moreover, it is also possible to calculate the interesting point where the trigger $\bar{P}$ is equal to the price floor $L \equiv L_\infty^*$ with the nonlinear equation (2.66) being rewritten as

$$-(\beta_1 - 1) \frac{L}{q} I(0, \infty, L, L, 1, q) + \beta_1 \left(\frac{L}{r} [1 + I(0, \infty, L, L, -1, r)] - K \right) \\ + L \left[\frac{L}{q} I_{\bar{P}=L}(0, \infty, L, L, 1, q) - \frac{L}{r} I_{\bar{P}=L}(0, \infty, L, L, -1, r) \right] = 0. \quad (2.69)$$

Taking the limits of equations (2.59) and (2.60) as $T \rightarrow \infty$, it follows that

$$I(0, \infty, L, L, \phi, v) = -\frac{1}{2} \left(1 + \frac{b_\phi}{c_v} \right) \quad (2.70)$$

and

$$I_{\bar{P}=L}(0, \infty, L, L, \phi, v) = -\frac{v}{\sigma L c_v}. \quad (2.71)$$

Finally, combining equations (2.69), (2.70) and (2.71) it is possible to show that

$$L_\infty^* = rK, \quad (2.72)$$

after applying straightforward calculus, which is equivalent to Barbosa et al. (2018, equation 18). This allows us to determine also the required investment cost that makes the fixed L the optimal trigger of investment, i.e., $K \equiv K_L^* = L/r$. In summary and assuming the given

initial investment cost K , if $K_L^* \leq K$, the optimal threshold $\bar{P}$ can be calculated since $\bar{P} \geq L$; otherwise, i.e., if $K_L^* > K$, the value-matching and smooth-pasting conditions are not met under the perpetual case and, hence, equation (2.66) is not valid.

Two additional results can be now provided combining some of the finite maturity and perpetual solutions. For instance, comparing equations (2.63) and (2.72) allows us to offer the asymptotic result $\lim_{T \rightarrow \infty} \beta_1/Y = r$, which can be easily proved analytically. Furthermore, combining equations (2.64) and (2.72) yields

$$L_0 (1 - e^{-rT}) = L_\infty^*, \quad (2.73)$$

which is consistent with the observation of Barbosa et al. (2018) that if the government sets a policy with a price floor $L \geq L_0$ in a finite maturity contract or a price floor $L \geq L_\infty^*$ in a perpetual contract, the investment will be made immediately generating a risk-free profit. Hence, we would have the counter-intuitive result of a project generating a positive NPV independently of the market price P in both FIT schemes.

2.6.2. Investment opportunity with a price collar regime

The purpose now is to show how to calculate the value of a (perpetual) option to invest in a project with a collar-style arrangement and the corresponding optimal investment rule.

2.6.2.1. Finite-lived case

Similarly to Adkins et al. (2019), we assume that after the expiry date of the collar-style arrangement the investor receives a cash flow with a present value equal to $\frac{P_0}{q}e^{-qT}$ that captures the value of operating the project perpetually without restrictions on the market price after the maturity date of the collar agreement. Hence, the total (or full) value of the project that includes the period of the collar arrangement and the (perpetual) period thereafter is given by

$$\begin{aligned} C^F(P_0, L, H, T) &= C(P_0, L, H, T) + \frac{P_0}{q}e^{-qT} \\ &= V(P_0, L, T) - V(P_0, H, T) + \frac{L}{r}(1 - e^{-rT}) + \frac{P_0}{q}e^{-qT}, \end{aligned} \quad (2.74)$$

after using equation (2.30) that expresses the finite-lived collar in terms of the profit cap (2.4). Notice that our definition (2.74) is equivalent to Adkins et al. (2019, equation 22), though without the need of having different solutions for the regions $P_0 < L$, $L \leq P_0 < H$ and $P_0 \geq H$.

The value of the (perpetual) option to invest in a project with a total value as given in equation (2.74), $F_C(P_0, \bar{P}, L, H, T, K)$, with $\bar{P}$ being interpreted again the optimal price

trigger level inducing the producer to invest in a project requiring an investment cost K , is also obtained as the solution of an ordinary differential equation subject to the boundary condition $\lim_{P_0 \rightarrow 0} F_C(P_0, \bar{P}, L, H, T, K) = 0$, which yields

$$F_C(P_0, \bar{P}, L, H, T, K) = AP_0^{\beta_1}, \quad (2.75)$$

where A needs to be determined from appropriate value-matching and smooth-pasting conditions, whereas the constant parameter $\beta_1(> 1)$ is still given by equation (2.51).

Applying the value-matching and smooth-pasting conditions to the option solution (2.75) at the optimal trigger $\bar{P}$, yields

$$\begin{cases} A\bar{P}^{\beta_1} &= C^F(\bar{P}, L, H, T) - K \\ \beta_1 A\bar{P}^{\beta_1-1} &= \frac{\partial}{\partial \bar{P}} C^F(\bar{P}, L, H, T) \end{cases}, \quad (2.76)$$

which is equivalent to

$$\begin{cases} \beta_1 A\bar{P}^{\beta_1} &= \beta_1 C^F(\bar{P}, L, H, T) - \beta_1 K \\ \beta_1 A\bar{P}^{\beta_1} &= \bar{P} \frac{\partial}{\partial \bar{P}} C^F(\bar{P}, L, H, T) \end{cases}, \quad (2.77)$$

after multiplying both sides of the value-matching and smooth-pasting conditions (2.76) by β_1 and $\bar{P}$, respectively.

Equating both branches of the system (2.77) yields

$$\beta_1 C^F(\bar{P}, L, H, T) - \beta_1 K = \bar{P} \frac{\partial}{\partial \bar{P}} C^F(\bar{P}, L, H, T), \quad (2.78)$$

which can be rewritten as

$$\begin{aligned} & \beta_1 \left(V(\bar{P}, L, T) - V(\bar{P}, H, T) + \frac{L}{r} (1 - e^{-rT}) + \frac{\bar{P}}{q} e^{-qT} - K \right) \\ &= \bar{P} \frac{\partial}{\partial \bar{P}} \left(V(\bar{P}, L, T) - V(\bar{P}, H, T) + \frac{L}{r} (1 - e^{-rT}) + \frac{\bar{P}}{q} e^{-qT} \right), \end{aligned} \quad (2.79)$$

by combining equations (2.74) and (2.78).

Finally, using equations (2.4) and (2.44) and applying straightforward calculus, the optimal trigger $\bar{P}$ can be determined by numerically solving the following nonlinear equation

$$\begin{aligned} & (\beta_1 - 1) \frac{\bar{P}}{q} \left[e^{-qT} + \tilde{I}(T, 1, q) \right] + \beta_1 \left[\frac{L}{r} (1 - e^{-rT}) - \frac{1}{r} \tilde{I}(T, -1, r) - K \right] \\ & - \frac{\bar{P}^2}{q} \tilde{I}_{\bar{P}}(T, 1, q) + \frac{\bar{P}}{r} \tilde{I}_{\bar{P}}(T, -1, r) = 0, \end{aligned} \quad (2.80)$$

where

$$\tilde{I}(T, 1, q) := I(0, T, \bar{P}, H, 1, q) - I(0, T, \bar{P}, L, 1, q), \quad (2.81)$$

$$\tilde{I}(T, -1, r) := H \times I(0, T, \bar{P}, H, -1, r) - L \times I(0, T, \bar{P}, L, -1, r), \quad (2.82)$$

$$\tilde{I}_{\bar{P}}(T, 1, q) := I_{\bar{P}}(0, T, \bar{P}, H, 1, q) - I_{\bar{P}}(0, T, \bar{P}, L, 1, q) \quad (2.83)$$

and

$$\tilde{I}_{\bar{P}}(T, -1, r) := H \times I_{\bar{P}}(0, T, \bar{P}, H, -1, r) - L \times I_{\bar{P}}(0, T, \bar{P}, L, -1, r). \quad (2.84)$$

We note that we obtain the required investment trigger for the case of finite-lived collars by simply solving equation (2.80). Hence, it is not necessary to know in advance in which region the threshold is, i.e., whether $\bar{P} > H$, $L \leq \bar{P} < H$ or $\bar{P} < L$. Similarly to the FIT case, we immediately know in which region the optimal trigger $\bar{P}$ is once equation (2.80) is solved. By contrast, the procedure proposed by Adkins et al. (2019, Appendix B) for determining the optimal threshold requires first the determination of the region in which the threshold $\bar{P}$ is. Clearly, our approach for analyzing the optimal investment rule of a project with a price collar regime is much simpler than the one offered by Adkins et al. (2019).³

Once the optimal trigger $\bar{P}$ is known we are able to determine A using the value-matching condition (2.76). Finally, replacing the obtained A in equation (2.75) allows us to calculate the value of the (perpetual) option to invest in the project as

$$F_C(P_0, \bar{P}, L, H, T, K) = \begin{cases} (C^F(\bar{P}, L, H, T) - K) \left(\frac{P_0}{\bar{P}}\right)^{\beta_1} & \Leftarrow P_0 < \bar{P} \\ C^F(P_0, L, H, T) - K & \Leftarrow P_0 \geq \bar{P} \end{cases}. \quad (2.85)$$

2.6.2.2. Perpetual case

Although the main purpose here is not to tackle the infinite-horizon problem, the optimal thresholds for the perpetual case can be also determined from a nonlinear equation that is derived by taking the limit of equation (2.80) as $T \rightarrow \infty$, that is

$$\begin{aligned} & (\beta_1 - 1) \frac{\bar{P}}{q} \tilde{I}(\infty, 1, q) + \beta_1 \left[\frac{L}{r} - \frac{1}{r} \tilde{I}(\infty, -1, r) - K \right] \\ & - \frac{\bar{P}^2}{q} \tilde{I}_{\bar{P}}(\infty, 1, q) + \frac{\bar{P}}{r} \tilde{I}_{\bar{P}}(\infty, -1, r) = 0, \end{aligned} \quad (2.86)$$

³Moreover, we only need to apply one value-matching condition and one smooth-pasting condition when deriving the investment threshold of a project with a finite-lived collar-style arrangement, whereas in Adkins et al. (2019) such boundary conditions must be applied individually three times, i.e., for each one of the three possible regions. By the way, and for the sake of clarity, we notice that there are some small typos in Adkins et al. (2019, equation 53). The correct formula is the one given in Adkins et al. (2019, equation 23).

where

$$\tilde{I}(\infty, 1, q) := I(0, \infty, \bar{P}, H, 1, q) - I(0, \infty, \bar{P}, L, 1, q), \quad (2.87)$$

$$\tilde{I}(\infty, -1, r) := H \times I(0, \infty, \bar{P}, H, -1, r) - L \times I(0, \infty, \bar{P}, L, -1, r), \quad (2.88)$$

$$\tilde{I}_{\bar{P}}(\infty, 1, q) := I_{\bar{P}}(0, \infty, \bar{P}, H, 1, q) - I_{\bar{P}}(0, \infty, \bar{P}, L, 1, q) \quad (2.89)$$

and

$$\tilde{I}_{\bar{P}}(\infty, -1, r) := H \times I_{\bar{P}}(0, \infty, \bar{P}, H, -1, r) - L \times I_{\bar{P}}(0, \infty, \bar{P}, L, -1, r). \quad (2.90)$$

Notice that our approach allows us to determine the optimal trigger $\bar{P}$ by simply solving equation (2.86), whereas the alternative that is available in the literature requires a more complex find-root scheme involving three equations as shown in Adkins et al. (2019, equations 18, 19 and 20), because they need to find first in which region the optimal threshold is placed.

Finally, using a similar procedure to one employed in the finite-lived case, we can determine the value of the (perpetual) option to invest in the project once the optimal trigger $\bar{P}$ of the perpetual case is known, that is

$$F_C(P_0, \bar{P}, L, H, \infty, K) = \begin{cases} (C^F(\bar{P}, L, H, \infty) - K) \left(\frac{P_0}{\bar{P}}\right)^{\beta_1} & \Leftarrow P_0 < \bar{P} \\ C^F(P_0, L, H, \infty) - K & \Leftarrow P_0 \geq \bar{P} \end{cases}, \quad (2.91)$$

where $C^F(P, L, H, \infty)$ is computed as the limit of equation (2.74) as $T \rightarrow \infty$, which yields the perpetual price collar solution (2.41).

To sum up, the novel solutions proposed in this section allow us to evaluate optimal decisions of investment projects with a minimum price guarantee (i.e., a price floor regime) and with a price collar regime that have been recently analyzed in Barbosa et al. (2018) and Adkins et al. (2019), respectively. In particular, the results reported by these authors (e.g., tables and figures) can now be immediately replicated using a viable alternative approach that is very simple and easier to implement in any software package.

2.7. Conclusions

The valuation of finite maturity caps, floors and collars on continuous flows has been typically tackled using perpetual methods inspired in the real options literature. Under this approach, the perpetual solution of the problem needs to be solved first and only afterwards the more interesting finite maturity case is deduced by subtracting the risk-neutral expectation of the forward start perpetual solution from the corresponding (current) perpetual solution.

By contrast, we propose a novel approach that does not require the use of the replicating portfolio scheme of Shackleton and Wojakowski (2007), since we are able to directly analyze

the finite maturity case by solving the finite time integral through a relatively simple integration by parts method. Whenever necessary, the corresponding perpetual claims arise immediately by taking $T \rightarrow \infty$ in the finite maturity solutions.

Several arbitrage-free relations between different, but related, contractual arrangements and the so-called Greeks are also derived in closed-form solution. Moreover, we provide a new method for calculating the optimal triggers of investment projects containing price floors and price collars clauses that is shown to be very simple and easy to implement.

Appendix

2A. Proof of Proposition 2.1.

This appendix is organized in four parts:

(i) Combining equation (2.2) with the closed-form solution of a plain-vanilla call (or caplet) offered in Black and Scholes (1973) and Merton (1973), the finite-lived cap can be rewritten as

$$\begin{aligned}
 V(P_0, X, T) &= \int_0^T c_0(P_0, X, t) dt \\
 &= \int_0^T [P_0 e^{-qt} N(d_1(t)) - X e^{-rt} N(d_2(t))] dt \\
 &= -\frac{P_0}{q} \int_0^T (-q) e^{-qt} N(d_1(t)) dt + \frac{X}{r} \int_0^T (-r) e^{-rt} N(d_2(t)) dt \\
 &= -\frac{P_0}{q} I_1 + \frac{X}{r} I_2
 \end{aligned} \tag{2A.1}$$

with

$$d_1(t) := \frac{\ln(P_0/X) + (r - q + \sigma^2/2)t}{\sigma\sqrt{t}}, \tag{2A.2}$$

$$d_2(t) := \frac{\ln(P_0/X) + (r - q - \sigma^2/2)t}{\sigma\sqrt{t}}, \tag{2A.3}$$

$$I_1 := \int_0^T (-q) e^{-qt} N(d_1(t)) dt, \tag{2A.4}$$

$$I_2 := \int_0^T (-r) e^{-rt} N(d_2(t)) dt \tag{2A.5}$$

and where $N(\cdot)$ represents the cumulative density function of the univariate standard normal distribution.

The integrals I_1 and I_2 are similar and can be both solved using integration by parts. Defining $a_1 := \ln(P_0/X)/\sigma$ and $b_1 := (r - q + \sigma^2/2)/\sigma$, equation (2A.2) can be rewritten as

$$d_1(t) = \frac{a_1}{\sqrt{t}} + b_1\sqrt{t}. \tag{2A.6}$$

Applying integration by parts to the first integral and recalling that the probability density

function of the standard normal distribution is given by $n(x) = (1/\sqrt{2\pi}) \exp(-x^2/2)$, then

$$\begin{aligned}
I_1 &= e^{-qt} N(a_1 t^{-1/2} + b_1 t^{1/2}) \Big|_{t=0}^{t=T} \\
&\quad - \int_0^T e^{-qt} n(a_1 t^{-1/2} + b_1 t^{1/2}) \left(\frac{-a_1 t^{-3/2} + b_1 t^{-1/2}}{2} \right) dt \\
&= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - e^0 N(d_1(0)) \\
&\quad - \int_0^T e^{-qt} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(a_1 t^{-1/2} + b_1 t^{1/2})^2}{2}\right) \left(\frac{-a_1 t^{-3/2} + b_1 t^{-1/2}}{2} \right) dt \\
&= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \int_0^T e^{-qt} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a_1^2 t^{-1} + 2a_1 b_1 + b_1^2 t}{2}\right) \left(\frac{-a_1 t^{-3/2} + b_1 t^{-1/2}}{2} \right) dt \\
&= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}((b_1^2 + 2q)t + a_1^2 t^{-1})} \left(\frac{b_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} \right) dt \\
&= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} \left(\frac{b_1}{c_1} \frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} \right) dt, \tag{2A.7}
\end{aligned}$$

with $c_1 := \sqrt{b_1^2 + 2q}$ and

$$\begin{aligned}
N(d_1(0)) &:= N\left(\lim_{t \rightarrow 0} (a_1 t^{-1/2} + b_1 t^{1/2})\right) \\
&= \begin{cases} N(+\infty) & \Leftarrow a_1 > 0 \\ N(0) & \Leftarrow a_1 = 0 \\ N(-\infty) & \Leftarrow a_1 < 0 \end{cases} . \\
&= \begin{cases} 1 & \Leftarrow a_1 > 0 \\ \frac{1}{2} & \Leftarrow a_1 = 0 \\ 0 & \Leftarrow a_1 < 0 \end{cases} . \\
&= 1 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 0 \times \mathbb{1}_{\{P_0 < X\}}. \tag{2A.8}
\end{aligned}$$

Noting that

$$\begin{aligned}
&\frac{b_1}{c_1} \frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} \\
&= \frac{b_1}{c_1} \frac{c_1}{2} t^{-1/2} + \frac{c_1}{2} t^{-1/2} - \frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} + \frac{b_1}{c_1} \frac{a_1}{2} t^{-3/2} - \frac{b_1}{c_1} \frac{a_1}{2} t^{-3/2} \\
&= \frac{1}{2} \left(\frac{b_1}{c_1} + 1 \right) \left(\frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} \right) + \frac{1}{2} \left(\frac{b_1}{c_1} - 1 \right) \left(\frac{c_1}{2} t^{-1/2} + \frac{a_1}{2} t^{-3/2} \right),
\end{aligned}$$

then the integral (2A.7) can be rewritten as

$$\begin{aligned}
I_1 &= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} \frac{1}{2} \left(\frac{b_1}{c_1} + 1 \right) \left(\frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} \right) dt \\
&\quad - \int_0^T \frac{e^{-a_1 b_1}}{\sqrt{2\pi}} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} \frac{1}{2} \left(\frac{b_1}{c_1} - 1 \right) \left(\frac{c_1}{2} t^{-1/2} + \frac{a_1}{2} t^{-3/2} \right) dt. \quad (2A.9)
\end{aligned}$$

Moreover,

$$\begin{aligned}
e^{-a_1 b_1} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} &= e^{-a_1 b_1} e^{-c_1 a_1} e^{c_1 a_1} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} \\
&= e^{-a_1 b_1} e^{c_1 a_1} e^{-\frac{1}{2}(c_1^2 t + 2c_1 a_1 + a_1^2 t^{-1})} \\
&= e^{a_1(c_1 - b_1)} e^{-\frac{1}{2}(c_1 t^{1/2} + a_1 t^{-1/2})^2}
\end{aligned}$$

and

$$\begin{aligned}
e^{-a_1 b_1} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} &= e^{-a_1 b_1} e^{-c_1 a_1} e^{c_1 a_1} e^{-\frac{1}{2}(c_1^2 t + a_1^2 t^{-1})} \\
&= e^{-a_1 b_1} e^{-c_1 a_1} e^{-\frac{1}{2}(c_1^2 t - 2c_1 a_1 + a_1^2 t^{-1})} \\
&= e^{-a_1(c_1 + b_1)} e^{-\frac{1}{2}(c_1 t^{1/2} - a_1 t^{-1/2})^2},
\end{aligned}$$

which implies that the integral (2A.9) can be rewritten as

$$\begin{aligned}
I_1 &= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} + 1 \right) e^{a_1(c_1 - b_1)} \int_0^T \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(c_1 t^{1/2} + a_1 t^{-1/2})^2} \left(\frac{c_1}{2} t^{-1/2} - \frac{a_1}{2} t^{-3/2} \right) dt \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} - 1 \right) e^{-a_1(c_1 + b_1)} \int_0^T \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(c_1 t^{1/2} - a_1 t^{-1/2})^2} \left(\frac{c_1}{2} t^{-1/2} + \frac{a_1}{2} t^{-3/2} \right) dt \\
&= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} + 1 \right) e^{a_1(c_1 - b_1)} \int_0^T \left[\frac{d}{dt} N(c_1 t^{1/2} + a_1 t^{-1/2}) \right] dt \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} - 1 \right) e^{-a_1(c_1 + b_1)} \int_0^T \left[\frac{d}{dt} N(c_1 t^{1/2} - a_1 t^{-1/2}) \right] dt \\
&= e^{-qT} N(a_1 T^{-1/2} + b_1 T^{1/2}) - N(d_1(0)) \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} + 1 \right) e^{a_1(c_1 - b_1)} N(c_1 t^{1/2} + a_1 t^{-1/2}) \Big|_{t=0}^{t=T} \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} - 1 \right) e^{-a_1(c_1 + b_1)} N(c_1 t^{1/2} - a_1 t^{-1/2}) \Big|_{t=0}^{t=T}
\end{aligned}$$

$$\begin{aligned}
&= e^{-qT} N\left(\frac{a_1}{\sqrt{T}} + b_1\sqrt{T}\right) - N(d_1(0)) \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} + 1\right) e^{a_1(c_1 - b_1)} \left[N\left(\frac{a_1}{\sqrt{T}} + c_1\sqrt{T}\right) - N(d_1(0)) \right] \\
&\quad - \frac{1}{2} \left(\frac{b_1}{c_1} - 1\right) e^{-a_1(c_1 + b_1)} \left[N\left(-\frac{a_1}{\sqrt{T}} + c_1\sqrt{T}\right) - 1 + N(d_1(0)) \right], \quad (2A.10)
\end{aligned}$$

because, following the insights used for calculating the distribution (2A.8), it is possible to show that

$$\begin{aligned}
N\left(\lim_{t \rightarrow 0} (c_1 t^{1/2} + a_1 t^{-1/2})\right) &= 1 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 0 \times \mathbb{1}_{\{P_0 < X\}} \\
&= N(d_1(0))
\end{aligned}$$

and

$$\begin{aligned}
N\left(\lim_{t \rightarrow 0} (c_1 t^{1/2} - a_1 t^{-1/2})\right) &= 0 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 1 \times \mathbb{1}_{\{P_0 < X\}} \\
&= 1 - N(d_1(0)).
\end{aligned}$$

Applying the same rational to the second integral with $a_2 := \ln(P_0/X)/\sigma$, $b_2 := (r - q - \sigma^2/2)/\sigma$ and $c_2 := \sqrt{b_2^2 + 2r}$, it can be shown that

$$\begin{aligned}
I_2 &= e^{-rT} N\left(\frac{a_2}{\sqrt{T}} + b_2\sqrt{T}\right) - N(d_2(0)) \\
&\quad - \frac{1}{2} \left(\frac{b_2}{c_2} + 1\right) e^{a_2(c_2 - b_2)} \left[N\left(\frac{a_2}{\sqrt{T}} + c_2\sqrt{T}\right) - N(d_2(0)) \right] \\
&\quad - \frac{1}{2} \left(\frac{b_2}{c_2} - 1\right) e^{-a_2(c_2 + b_2)} \left[N\left(-\frac{a_2}{\sqrt{T}} + c_2\sqrt{T}\right) - 1 + N(d_2(0)) \right], \quad (2A.11)
\end{aligned}$$

with $N(d_2(0)) = 1 \times \mathbb{1}_{\{P_0 > X\}} + \frac{1}{2} \times \mathbb{1}_{\{P_0 = X\}} + 0 \times \mathbb{1}_{\{P_0 < X\}} = N(d_1(0))$.

Finally, equation (2.4) arises immediately from equation (2A.1) since the integrals (2A.10) and (2A.11) can be expressed in a compact form as shown in equation (2.5).

(ii) Combining equation (2.3) with the closed-form solution of a plain-vanilla put (or floorlet) offered in Black and Scholes (1973) and Merton (1973), the finite-lived floors can be rewritten as

$$\begin{aligned}
&F(P_0, X, T) \\
&= \int_0^T p_0(P_0, X, t) dt \\
&= \int_0^T [X e^{-rt} N(-d_2(t)) - P_0 e^{-qt} N(-d_1(t))] dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^T X e^{-rt} dt - \int_0^T X e^{-rt} N(d_2(t)) dt - \int_0^T P_0 e^{-qt} dt + \int_0^T P_0 e^{-qt} N(d_1(t)) dt \\
&= -\frac{X}{r} \int_0^T (-r) e^{-rt} dt + \frac{X}{r} \int_0^T (-r) e^{-rt} N(d_2(t)) dt \\
&\quad + \frac{P_0}{q} \int_0^T (-q) e^{-qt} dt - \frac{P_0}{q} \int_0^T (-q) e^{-qt} N(d_1(t)) dt \\
&= -\frac{X}{r} e^{-rt} \Big|_{t=0}^{t=T} + \frac{X}{r} I_2 + \frac{P_0}{q} e^{-qt} \Big|_{t=0}^{t=T} - \frac{P_0}{q} I_1 \\
&= \frac{X}{r} (1 - e^{-rT}) - \frac{P_0}{q} (1 - e^{-qT}) + \frac{X}{r} I_2 - \frac{P_0}{q} I_1,
\end{aligned} \tag{2A.12}$$

which finally yields equation (2.10) after replacing the integrals (2A.10) and (2A.11) and equation (2.4) in equation (2A.12).

(iii) The perpetual cap is defined as

$$\begin{aligned}
V(P_0, X, \infty) &= \int_0^\infty c_0(P_0, X, t) dt \\
&= \lim_{T \rightarrow \infty} \int_0^T c_0(P_0, X, t) dt \\
&= \lim_{T \rightarrow \infty} V(P_0, X, T),
\end{aligned} \tag{2A.13}$$

with $V(P_0, X, T)$ being the finite-lived cap solution given in equation (2.4). The analytical representation of the perpetual cap can then be obtained after evaluating the integral

$$I(0, \infty, P_0, X, \phi, v) := \lim_{T \rightarrow \infty} I(0, T, P_0, X, \phi, v), \tag{2A.14}$$

with $I(0, T, P_0, X, \phi, v)$ being the integral defined in equation (2.5).

To accomplish this purpose three limits must be evaluated. Since r and q are both positive then $v > 0$ and, therefore,

$$\lim_{T \rightarrow \infty} e^{-vT} N\left(\frac{a}{\sqrt{T}} + b\sqrt{T}\right) = 0, \tag{2A.15}$$

because e^{-vT} is an infinitesimal, $N(\cdot)$ is a limited function in $[0, 1]$ and, hence, its product is an infinitesimal. Regarding the other two limits, and because $c > 0$,

$$\begin{aligned}
\lim_{T \rightarrow \infty} N\left(\frac{a}{\sqrt{T}} + c\sqrt{T}\right) &= N(+\infty) \\
&= 1
\end{aligned} \tag{2A.16}$$

and

$$\begin{aligned}\lim_{T \rightarrow \infty} N \left(-\frac{a}{\sqrt{T}} + c\sqrt{T} \right) &= N(+\infty) \\ &= 1.\end{aligned}\tag{2A.17}$$

The integral $I(0, \infty, P_0, X, \phi, v)$ shown in equation (2.12) is then obtained after combining the limits (2A.14), (2A.15), (2A.16) and (2A.17), which finally allows the computation of the perpetual cap shown in equation (2.11).

(iv) The perpetual floor is defined as

$$\begin{aligned}F(P_0, X, \infty) &= \int_0^\infty p_0(P_0, X, t) dt \\ &= \lim_{T \rightarrow \infty} \int_0^T p_0(P_0, X, t) dt \\ &= \lim_{T \rightarrow \infty} F(P_0, X, T),\end{aligned}\tag{2A.18}$$

with $F(P_0, X, T)$ being the finite-lived floor solution given in equation (2.10). Finally, equation (2.13) arises immediately after computing the limit (2A.18).

3. Finite maturity caps and floors on continuous flows under the constant elasticity of variance process

3.1. Introduction

This paper offers novel analytical formulae for evaluating (perpetual and finite) caps and floors on continuous flows under the constant elasticity of variance (hereafter, CEV) model that is known to nest the limiting geometric Brownian motion (henceforth, GBM) process as a special case. The proposed solutions are important for both academics and corporate managers since the CEV diffusion is known to be better able to accommodate several stylized-facts that are commonly reported in the literature, namely the so-called leverage and volatility smile effects.¹

In a nutshell, we offer four main contributions to the literature. First, our new perpetual cap solutions allow us to show that the inclusion of a perpetual bubble value in the Davydov and Linetsky (2001) formulae is required to avoid arbitrage opportunities in the case of the CEV process with upward-sloping volatility skews.

Second, we extend the perpetual and finite maturity cap and floor solutions offered by Shackleton and Wojakowski (2007) in the context of the GBM model to the CEV process. As usual, the computation of finite maturity caps (resp., floors) is obtained from the perpetual caps (resp., floors) by subtracting the risk-neutral expectation of the forward start perpetual caps (resp., floors). Even though the level of complexity for computing such forward start perpetuities under the CEV model is much higher than under the (nested) GBM process, we show that such risk-neutral expectations can be expressed in terms of the truncated moments of a noncentral χ^2 distribution law shown in Carr and Linetsky (2006) and Dias and Nunes (2018).

Since the approach of Shackleton and Wojakowski (2007) has been the basis to many other recent applications of caps and floors—e.g., Barbosa et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022) and Paxson et al. (2022),

¹Additional background on the CEV diffusion process can be consulted, for instance, in Cox (1975), Andersen and Andreasen (2000), Davydov and Linetsky (2001), Delbaen and Shirakawa (2002), Davydov and Linetsky (2003), Geman and Shih (2009), Nunes (2009), Dias and Nunes (2011), Larguinho et al. (2013), Ruas et al. (2013), Dias et al. (2015), Nunes et al. (2015) and Dias et al. (2020).

just to name a few—we contend that our paper will pave the way to a wide variety of other applications, for example in the context of executive management decisions or government policy interventions associated to renewable energy projects. As a specific example, and as our third contribution, we evaluate perpetual and finite maturity feed-in tariff (hereafter, FIT) contracts under the CEV model, thus extending Barbosa et al. (2018) from the GBM model to the CEV process. This should be important for decision makers given the interest of these support schemes in a wide variety of recent applications—e.g., Couture and Gagnon (2010), Ritzenhofen et al. (2016), Ritzenhofen and Spinler (2016), Pineda et al. (2018) and Li et al. (2023). Moreover, we also demonstrate an alternative and simpler solution to evaluate FIT contracts than the one prescribed by Barbosa et al. (2018) once the cap (or floor) solutions are known. In particular, our (model-free) alternative approach shows that a perpetual (resp., finite maturity) guarantee can be interpreted as a portfolio of a perpetual (resp., finite maturity) profit cap and a perpetual (resp., finite maturity) cash amount.

As our final contribution, we show that it is straightforward to extend the Keswani and Shackleton (2006) methodology to evaluate investment projects under the fully reversible assumption from a GBM context to a CEV process.

In general, our results allow us to conclude that a firm (or a government) that uses the standard lognormal assumption is exposed to significant errors of analysis when the underlying asset price is better described by a CEV process. In other words, exorbitant or insufficient profit flows might be paid/received in many practical applications arising within the context of managerial and government decisions. In summary, our numerical experiments strongly highlight the importance of moving beyond more simplistic real options analysis based on the GBM process to more realistic models incorporating leverage and volatility smile effects.

The remainder of the manuscript is organized as follows. Section 3.2 discusses the valuation of caps and floors under the perpetual case. Section 3.3 presents the analytical formulae for valuing finite maturity caps and floors. Section 3.4 deals with the valuation of FIT contracts with a minimum price guarantee. Section 3.5 shows how to evaluate fully reversible investment projects using the analytical solutions of caps. Section 3.6 provides some numerical examples and Section 3.7 presents some concluding remarks.

3.2. Perpetual case

Following (Dixit and Pindyck, 1994, pages 186-195) and Shackleton and Wojakowski (2007), we consider a continuous price or revenue process P that generates cash at an instantaneous rate of flow Pdt . However, instead of considering that P follows the classic (and more restrictive) GBM assumption, we assume that P is governed by a CEV process whose risk-neutral dynamics is described as

$$dP_t = (r - q) P_t dt + \sigma(P_t) P_t dW_t^{\mathbb{Q}}, \quad (3.1)$$

with the local volatility function—assumed to be continuous and strictly positive for all $P \in (0, \infty)$ —given by

$$\sigma(P_t) := \delta P_t^{\frac{\beta}{2}-1}, \quad (3.2)$$

for $\delta \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, and where r and q are the (positive and constant) interest rate and dividend yield (or rate of return shortfall), respectively, while $\{W_t^{\mathbb{Q}} \in \mathbb{R} : t \geq 0\}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.

We recall that the CEV model specification given by equations (3.1) and (3.2) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) (if $\beta = 2$) as well as the absolute diffusion (if $\beta = 0$) and the square-root diffusion (if $\beta = 1$) processes of Cox and Ross (1976) as special cases. We further notice that whenever $\beta < 2$ (resp., $\beta > 2$), the local volatility function (3.2) is a decreasing (resp., increasing) function of the asset price, thus being able to generate downward-sloping (resp., upward-sloping) volatility skews that are observed in the market. The model parameter δ is interpreted as the scale parameter fixing the initial instantaneous volatility at time $t = 0$, $\sigma_P \equiv \sigma(P_0) = \delta P_0^{\beta/2-1}$. This calibration procedure is standard in the literature in order to ensure that CEV models with different values of β have the same variance at inception, thus permitting valid comparisons between different CEV models.

3.2.1. Perpetual real options problem

Let $V(P_t, \infty)$ be the time- t value of an infinite horizon flexible project or profit cap. Its value can be derived from a perpetual cash flow of an instantaneous maximum flow rate $\Pi(P_t) dt$, with $\Pi(P_t) = (P_t - X)^+$, over stochastic revenue P_t and fixed cost X . Under the risk-neutral measure, this is given by

$$\begin{aligned} V(P_t, \infty) &= \int_t^\infty e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} [\Pi(P_u)] du \\ &= \int_t^\infty c_t(P_t, X, u) du, \end{aligned} \quad (3.3)$$

where $c_t(P_t, X, u)$ is interpreted as the time- t price of a European-style call option (or caplet) on an underlying asset with spot price P_t (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed cost rate) and with expiry date at time $u (\geq t)$ given, for instance, in Schroder (1989), for $\beta < 2$, and Heston et al. (2007), for $\beta > 2$, that is²

²The solution for the case with $\beta > 2$ corrects the misprint error of (Heston et al., 2007, page 367) highlighted in (Veestraeten, 2017, footnote 1) and in Dias et al. (2020, equation 10).

$$\begin{aligned}
& c_t(P_t, X, u) \tag{3.4} \\
= & \begin{cases} P_t e^{-q\tau} Q(2y; 2 + 2\nu, 2w_t) - X e^{-r\tau} [1 - Q(2w_t; 2\nu, 2y)] & \Leftarrow \beta < 2 \\ P_t e^{-q\tau} Q(2w_t; 2\nu, 2y) - X e^{-r\tau} [1 - Q(2y; 2 + 2\nu, 2w_t)] - G_t(P_t, u) & \Leftarrow \beta > 2 \end{cases},
\end{aligned}$$

with $\tau = T - t$, and where

$$G_t(P_t, u) = P_t e^{-q(u-t)} \frac{\Gamma(\nu, w_t)}{\Gamma(\nu)} \tag{3.5}$$

is interpreted as the time- t bubble value of the call option solution, $\frac{\Gamma(\nu, w)}{\Gamma(\nu)}$ is the incomplete gamma function ratio (or normalized function) shown in Olver et al. (2010, equation 8.2.4), with $\Gamma(\nu, w)$ and $\Gamma(\nu)$ being the upper incomplete gamma function and the Euler gamma function given in Olver et al. (2010, equations 8.2.2 and 5.2.1), respectively, for $\nu, w \in \mathbb{R}_+$, $Q(w; \nu, \lambda)$ is the complementary distribution function of a noncentral chi-square law with ν (≥ 0) degrees of freedom and noncentrality parameter λ (≥ 0),

$$w_t \equiv w(P_t) := \kappa_{t,u} P_t^{2-\beta} e^{(r-q)(2-\beta)(u-t)}, \tag{3.6}$$

$$y := y(X) = \kappa_{t,u} X^{2-\beta}, \tag{3.7}$$

$$\kappa_{t,u} := \begin{cases} \frac{2(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)(u-t)} - 1]} & \Leftarrow r \neq q \\ \frac{2}{\delta^2(2-\beta)^2(u-t)} & \Leftarrow r = q \end{cases} \tag{3.8}$$

and

$$\nu := \frac{1}{|\beta - 2|}. \tag{3.9}$$

Remark 3.1. We recall that if the bubble value $G_t(P_t, u)$ included in equation (3.4) for $\beta > 2$ is ignored, we obtain the well known call option pricing solution proposed by Emanuel and MacBeth (1982). Note that both option pricing formulas are nonnegative solutions of the CEV partial differential equation subject to the same boundary conditions. This implies that there are distinct strategies that exactly replicate identical option payoffs at different costs. Clearly, the risk-neutral expected discounted value of the payoff of a European-style call option should be interpreted as the minimum quantity needed to replicate the call option payoff at maturity, that is the cheapest solution of Heston et al. (2007). As shown recently in Dias et al. (2020), ignoring the presence of bubbles in the call and in the underlying spot price under the CEV model can produce significant errors of analysis for pricing and hedging purposes. The importance of the bubble value is even magnified in the case of the profit cap $V(P_t, \infty)$ under analysis. The perpetual bubble value is obtained by integrating the difference between the two plain-vanilla call option pricing

solutions, that is

$$G_t^\infty(P_t) = \int_t^\infty G_t(P_t, u) du. \quad (3.10)$$

As noticed by McDonald and Siegel (1985), expression (3.3) can be interpreted as a perpetual continuum of European-style calls or caplets, where each individual caplet is of infinitesimal size and the integral sums an infinite number of caplets. As expected, a positive dividend yield q is required when valuing perpetual calls or caplets, otherwise the model collapses. We recall that it is only the presence of a non-zero cost of waiting (i.e., $q > 0$) that makes these infinite horizon investment problems reasonable, because if waiting was costless perpetual calls or caplets would never be exercised. Even though it might be possible to tackle numerically the integral (3.3), it is preferable to follow the insights of Shackleton and Wojakowski (2007) and use perpetual real options solutions that better provide the economic rationale associated to net present valuation problems. To accomplish this purpose, we first derive the perpetual real options solution of this infinite horizon problem under the CEV process and then the corresponding forward start perpetuity will be derived in Section 3.3.

The perpetual real options problem under analysis needs to be specialized in each of the cases $P_t < X$ and $P_t \geq X$. For the range of prices $P_t \in [0, X)$, the (perpetual) claim $V(P_t, \infty)$ must satisfy the following ordinary differential equation (hereafter, ODE):

$$\frac{1}{2} \delta^2 P_t^\beta \frac{d^2 V}{dP_t^2} + (r - q) P_t \frac{dV}{dP_t} - rV = 0. \quad (3.11)$$

For the range of prices $P_t \in [X, \infty)$, the (perpetual) claim $V(P_t, \infty)$ must satisfy a similar ODE, but now containing the profit flow $\Pi(P_t)$, that is:

$$\frac{1}{2} \delta^2 P_t^\beta \frac{d^2 V}{dP_t^2} + (r - q) P_t \frac{dV}{dP_t} - rV + \Pi(P_t) = 0. \quad (3.12)$$

The solutions to the homogeneous part of (3.11) and (3.12)—i.e., with no profit flow in (3.12)—are known as the complementary functions of (3.11) and (3.12) and will represent the options components that need to be specialized for each region subject to appropriate value-matching and smooth-pasting conditions, whereas in the region $P_t \geq X$ the uncontingent time- t values of perpetual revenues and costs, P_t/q and X/r , should be included in the solution and are interpreted as the particular integral of (3.12).

Similarly, the value of the corresponding floor can be obtained as

$$\begin{aligned} F(P_t, \infty) &= \int_t^\infty e^{-r(u-t)} \mathbb{E}_\mathbb{Q} [\Pi(P_u)] du \\ &= \int_t^\infty p_t(P_t, X, u) du, \end{aligned} \quad (3.13)$$

but now with $\Pi(P_t) = (X - P_t)^+$, while $p_t(P_t, X, u)$ is interpreted as the time- t price of a European-style put option (or floorlet) on an underlying asset with spot price P_t (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed divestment proceeds rate) and with expiry date at time u ($\geq t$) given, for instance, in Larginho et al. (2013), that is

$$p_t(P_t, X, u) = \begin{cases} X e^{-r\tau} Q(2w_t; 2\nu, 2y) - P_t e^{-q\tau} [1 - Q(2y; 2 + 2\nu, 2w_t)] & \Leftarrow \beta < 2 \\ X e^{-r\tau} Q(2y; 2 + 2\nu, 2w_t) - P_t e^{-q\tau} [1 - Q(2w_t; 2\nu, 2y)] & \Leftarrow \beta > 2 \end{cases} \quad (3.14)$$

Remark 3.2. *As expected, the risk-neutral pricing solution (3.4) does not satisfy the put-call parity property for $\beta > 2$. As argued by Heston et al. (2007), it is necessary to choose between risk-neutral pricing and put-call parity because it is not possible to choose both. This implies that the caplet-floorlet parity is also not satisfied when $\beta > 2$.*

3.2.2. Perpetual caps

The solutions to the ordinary differential equations (3.11) and (3.12) need to be specialized for two different volatility skew patterns, $\beta < 2$ and $\beta > 2$, each one encompassing the cases with $r = q$ (this corresponds to a CEV process with no drift) and $r \neq q$ (i.e., the drift of the CEV process can be positive or negative). The next two propositions offer the novel analytical solutions for evaluating perpetual caps under the CEV process.

Proposition 3.1. *Assume the CEV process with $r = q$ and introduce the following notation:*

$$z_t \equiv z(P_t) := \frac{2\sqrt{2r}}{\delta|\beta - 2|} P_t^{1-\frac{\beta}{2}}. \quad (3.15)$$

The time- t value of the perpetual cap $V(P_t, \infty)$ is given by

$$\begin{aligned} & V(P_t, \infty) \quad (3.16) \\ = & \begin{cases} AP_t^{\frac{1}{2}} I_\nu(z_t) \mathbb{1}_{\{P_t < X\}} + \left(BP_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_t}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_t \geq X\}} & \Leftarrow \beta < 2 \\ BP_t^{\frac{1}{2}} K_\nu(z_t) \mathbb{1}_{\{P_t < X\}} + \left(AP_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_t}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_t \geq X\}} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases} \end{aligned}$$

where $I_\nu(z_t)$ and $K_\nu(z_t)$ are, respectively, the modified Bessel's functions of first and second kind with order ν , as given by Olver et al. (2010, equations 10.25.2 and 10.27.4), whereas A and B are constants that solve the following system of equations, for $\zeta := z(X)$ and for any $\beta \neq 2$:

$$\begin{cases} AI_\nu(\zeta) - BK_\nu(\zeta) = 0 \\ AI_{\nu+1}(\zeta) + BK_{\nu+1}(\zeta) = \frac{\delta}{q\sqrt{2r}} X^{\frac{\beta-1}{2}} \end{cases} \quad (3.17)$$

Proposition 3.2. *Assume the CEV process with $r \neq q$ and introduce the following notation:*

$$x_t \equiv x(P_t) := \frac{2|r-q|}{\delta^2|\beta-2|} P_t^{2-\beta}, \quad (3.18)$$

$$\epsilon := \text{sign}[(r-q)(\beta-2)], \quad (3.19)$$

$$k := \epsilon \left(\frac{1}{2} + \frac{1}{2(\beta-2)} \right) - \frac{r}{|(r-q)(\beta-2)|}, \quad (3.20)$$

$$m := \frac{1}{2|\beta-2|}, \quad (3.21)$$

$$a := \frac{1}{2} + m - k, \quad (3.22)$$

$$b := 1 + 2m \quad (3.23)$$

and

$$Y := \left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{\frac{b}{2}}. \quad (3.24)$$

The time- t value of the perpetual cap $V(P_t, \infty)$ is given by

$$V(P_t, \infty) = \begin{cases} AY e^{(\epsilon-1)\frac{x_t}{2}} P_t M(a, b, x_t) \mathbb{1}_{\{P_t < X\}} \\ + \left(BY e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a, b, x_t) + \frac{P_t}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_t \geq X\}} & \Leftarrow \beta < 2 \\ BY e^{(\epsilon-1)\frac{x_t}{2}} U(a, b, x_t) \mathbb{1}_{\{P_t < X\}} \\ + \left(AY e^{(\epsilon-1)\frac{x_t}{2}} M(a, b, x_t) + \frac{P_t}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_t \geq X\}} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \quad (3.25)$$

where $M(a, b, x_t)$ and $U(a, b, x_t)$ are, respectively, the Kummer's and Tricomi's confluent hypergeometric functions defined in Olver et al. (2010, equations 13.2.2 and 13.2.42), whereas A and B are constants that solve the following system of equations, for $\chi := x(X)$:

$$\begin{cases} AM(a, b, \chi) - BU(a, b, \chi) = \frac{r-q}{rq} Y^{-1} e^{(1-\epsilon)\frac{\chi}{2}} \\ A \frac{a}{b} M(a+1, b+1, \chi) + BaU(a+1, b+1, \chi) = e^{(1-\epsilon)\frac{\chi}{2}} \left(\frac{(1-\epsilon)(r-q)}{2rq} + \frac{\delta^2 X^{\beta-2}}{2q|r-q|} \right) Y^{-1} \end{cases}, \quad (3.26)$$

for $\beta < 2$, and

$$\begin{cases} AM(a, b, \chi) - BU(a, b, \chi) = \frac{q-r}{rq} XY^{-1} e^{(1-\epsilon)\frac{\chi}{2}} \\ A \frac{a}{b} M(a+1, b+1, \chi) + BaU(a+1, b+1, \chi) = e^{(1-\epsilon)\frac{\chi}{2}} \left(\frac{(1-\epsilon)(q-r)}{2rq} + \frac{\delta^2 X^{\beta-2}}{2q|r-q|} \right) XY^{-1} \end{cases}, \quad (3.27)$$

for $\beta > 2$.

Remark 3.3. Equations (3.16) and (3.25) are the CEV counterparts of the solution proposed by Shackleton and Wojakowski (2007, equation 12) for the GBM process, whose numerical results can be obtained by taking $\beta \rightarrow 2$ in our solutions.

Remark 3.4. Following the insights exposed in Remark 3.1, we notice that it is necessary to include a perpetual bubble value in the solutions of Davydov and Linetsky (2001, equations 37 and 38) when their elasticity of the local volatility is positive in order to obtain the cheapest solutions of the CEV ordinary differential equation and avoid arbitrage opportunities.

3.2.3. Perpetual floors

Let us first recall that the difference between the cap and the floor is a swap, a result that depends on the validity of the caplet-floorlet parity that is observed for the case $\beta < 2$. In this situation, a perpetual floor can be valued as the corresponding perpetual cap less a swap of the revenues associated with P_t against those with X over the infinite horizon. Since the caplet-floorlet parity is not satisfied when $\beta > 2$, then the bubble value (3.10) should not be included in the solutions for $\beta > 2$. Therefore,

$$F(P_t, \infty) = \begin{cases} V(P_t, \infty) - \frac{P_t}{q} + \frac{X}{r} & \Leftarrow \beta < 2 \\ V(P_t, \infty) + G_t^\infty(P_t) - \frac{P_t}{q} + \frac{X}{r} & \Leftarrow \beta > 2 \end{cases} . \quad (3.28)$$

Note that the (perpetual) bubble value (3.10) is summed in equation (3.28), in the case of $\beta > 2$, only to cancel out the bubble value appearing in equations (3.16) and (3.25) when $\beta > 2$. The economic rational here is that there are no bubble values in the case of puts as highlighted in equation (3.14). In other words, and as highlighted in Remark 3.2, we are choosing the put-call parity and not the risk-neutral pricing for $\beta > 2$. Notice that the perpetual cap, $V(P_t, \infty)$, should be understood as the cheapest cap price (i.e., the risk-neutral solution) in the spirit of Heston et al. (2007), whereas $V(P_t, \infty) + G_t^\infty(P_t)$ is interpreted also as a cap price but in the sense of Emanuel and MacBeth (1982). Thus, even though the caplet-floorlet parity is not satisfied when $\beta > 2$ if one uses the risk-neutral (perpetual) cap value $V(P_t, \infty)$ —i.e., $F(P_t, \infty) \neq V(P_t, \infty) - \frac{P_t}{q} + \frac{X}{r}$ for $\beta > 2$, as highlighted in Remark 3.2—, by using instead the (more expensive) cap solution $V(P_t, \infty) + G_t^\infty(P_t)$ it is possible to determine the risk-neutral perpetual floor price, $F(P_t, \infty)$, as shown in the second branch of equation (3.28).

Alternatively, the corresponding analytical solutions for evaluating perpetual floors under the CEV process can be obtained using the same rational considered in Propositions 3.1 and 3.2, by applying appropriate boundary conditions and value-matching and smooth-pasting conditions

to the problem. Hence, the time- t value of the perpetual floor $F(P_t, \infty)$ is given by

$$F(P_t, \infty) = \begin{cases} \left(AP_t^{\frac{1}{2}} I_\nu(z_t) + \frac{X}{r} - \frac{P_t}{q} \right) \mathbb{1}_{\{P_t < X\}} + BP_t^{\frac{1}{2}} K_\nu(z_t) \mathbb{1}_{\{P_t \geq X\}} & \Leftarrow \beta < 2 \\ \left(BP_t^{\frac{1}{2}} K_\nu(z_t) + \frac{X}{r} - \frac{P_t}{q} \right) \mathbb{1}_{\{P_t < X\}} + AP_t^{\frac{1}{2}} I_\nu(z_t) \mathbb{1}_{\{P_t \geq X\}} & \Leftarrow \beta > 2 \end{cases}, \quad (3.29)$$

for $r = q$, with the constants A and B being still determined from the system of equations (3.17), and

$$F(P_t, \infty) = \begin{cases} \left(AY e^{(\epsilon-1)\frac{x_t}{2}} P_t M(a, b, x_t) + \frac{X}{r} - \frac{P_t}{q} \right) \mathbb{1}_{\{P_t < X\}} \\ \quad + BY e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a, b, x_t) \mathbb{1}_{\{P_t \geq X\}} & \Leftarrow \beta < 2 \\ \left(BY e^{(\epsilon-1)\frac{x_t}{2}} U(a, b, x_t) + \frac{X}{r} - \frac{P_t}{q} \right) \mathbb{1}_{\{P_t < X\}} \\ \quad + AY e^{(\epsilon-1)\frac{x_t}{2}} M(a, b, x_t) \mathbb{1}_{\{P_t \geq X\}} & \Leftarrow \beta > 2 \end{cases}, \quad (3.30)$$

for $r \neq q$, with the constants A and B being still determined from the systems of equations (3.26) and (3.27) for $\beta < 2$ and $\beta > 2$, respectively.

3.3. Finite case

The goal now is to extend the valuation problem to cap and floor claims with finite maturity T , whose time- t values will be denoted by $V(P_t, T)$ and $F(P_t, T)$, respectively. The cap claim corresponds to an integral of European-style calls or caplets over a finite integration range, that is

$$V(P_t, T) = \int_t^T c_t(P_t, X, u) du. \quad (3.31)$$

As expected, the valuation problem under analysis needs also to be specialized in each of the cases $P_t < X$ and $P_t \geq X$. For the range of prices $P_t \in [0, X)$, the (finite) claim $V(P_t, T)$ must now satisfy the following partial differential equation (henceforth, PDE):

$$\frac{1}{2} \delta^2 P_t^\beta \frac{\partial^2 V}{\partial P_t^2} + (r - q) P_t \frac{\partial V}{\partial P_t} - \frac{\partial V}{\partial \tau} - rV = 0. \quad (3.32)$$

For the range of prices $P_t \in [X, \infty)$, the (finite) claim $V(P_t, T)$ must satisfy a similar PDE, but now containing the profit flow $\Pi(P_t) = (P_t - X)^+$, that is:

$$\frac{1}{2} \delta^2 P_t^\beta \frac{\partial^2 V}{\partial P_t^2} + (r - q) P_t \frac{\partial V}{\partial P_t} - \frac{\partial V}{\partial \tau} - rV + \Pi(P_t) = 0. \quad (3.33)$$

Note that now a time differential needs to be included in both PDEs (3.32) and (3.33) and, hence, the finite maturity cap is no longer a time homogeneous valuation problem.

Since the caplets contained within the integral (3.31) are of European-style, we can compute the finite cap integral from the perpetual cap by subtracting the risk-neutral expectation of

the forward start perpetual cap, that is

$$V(P_t, T) = V(P_t, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V(P_T, \infty) | \mathcal{F}_t], \quad (3.34)$$

where the risk-neutral expectation of the forward start perpetual cap can be expressed as

$$\begin{aligned} e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V(P_T, \infty) | \mathcal{F}_t] &= \int_T^\infty e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [c_T(P_T, X, u) | \mathcal{F}_t] du \\ &= \int_T^\infty c_t(P_t, X, u) du, \end{aligned} \quad (3.35)$$

using equation (3.3) and Fubini's theorem.

In particular, the value at the forward time T can be expressed by rewriting equations (3.16) and (3.25) with P_T , that is

$$\begin{aligned} &V(P_T, \infty) \quad (3.36) \\ = &\begin{cases} AP_T^{\frac{1}{2}} I_\nu(z_T) \mathbb{1}_{\{P_T < X\}} + \left(BP_T^{\frac{1}{2}} K_\nu(z_T) + \frac{P_T}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_T \geq X\}} & \Leftarrow \beta < 2 \\ BP_T^{\frac{1}{2}} K_\nu(z_T) \mathbb{1}_{\{P_T < X\}} + \left(AP_T^{\frac{1}{2}} I_\nu(z_T) + \frac{P_T}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_T \geq X\}} - G_T^\infty(P_T) & \Leftarrow \beta > 2 \end{cases}, \end{aligned}$$

for $r = q$ and $z_T := z(P_T)$, and

$$\begin{aligned} V(P_T, \infty) = &\begin{cases} AY e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \\ + \left(BY e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) + \frac{P_T}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_T \geq X\}} & \Leftarrow \beta < 2 \\ BY e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \\ + \left(AY e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) + \frac{P_T}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P_T \geq X\}} - G_T^\infty(P_T) & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (3.37)$$

for $r \neq q$ and $x_T := x(P_T)$.

In summary, in order to calculate the finite cap, we first need to compute the risk-neutral expectation of the forward start perpetuity $V(P_T, \infty)$, which is a function of the random variable P_T that needs to be specialized in each of the regions $P_T < X$ and $P_T \geq X$.

The finite floors can be treated similarly, but now with $\Pi(P_t) = (X - P_t)^+$, thus resulting in

$$F(P_t, T) = F(P_t, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [F(P_T, \infty) | \mathcal{F}_t], \quad (3.38)$$

where the risk-neutral expectation of the forward start perpetual floor can be expressed as

$$\begin{aligned} e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [F(P_T, \infty) | \mathcal{F}_t] &= \int_T^\infty e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [p_T(P_T, X, u) | \mathcal{F}_t] du \\ &= \int_T^\infty p_t(P_t, X, u) du, \end{aligned} \quad (3.39)$$

with

$$F(P_T, \infty) = \begin{cases} \left(AP_T^{\frac{1}{2}} I_\nu(z_T) + \frac{X}{r} - \frac{P_T}{q} \right) \mathbb{1}_{\{P_T < X\}} + BP_T^{\frac{1}{2}} K_\nu(z_T) \mathbb{1}_{\{P_T \geq X\}} & \Leftarrow \beta < 2 \\ \left(BP_T^{\frac{1}{2}} K_\nu(z_T) + \frac{X}{r} - \frac{P_T}{q} \right) \mathbb{1}_{\{P_T < X\}} + AP_T^{\frac{1}{2}} I_\nu(z_T) \mathbb{1}_{\{P_T \geq X\}} & \Leftarrow \beta > 2 \end{cases}, \quad (3.40)$$

for $r = q$, and

$$F(P_T, \infty) = \begin{cases} \left(AY e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) + \frac{X}{r} - \frac{P_T}{q} \right) \mathbb{1}_{\{P_T < X\}} \\ + BY e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} & \Leftarrow \beta < 2 \\ \left(BY e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) + \frac{X}{r} - \frac{P_T}{q} \right) \mathbb{1}_{\{P_T < X\}} \\ + AY e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} & \Leftarrow \beta > 2 \end{cases}, \quad (3.41)$$

for $r \neq q$, using equations (3.29) and (3.30)

3.3.1. Forward start perpetuities

Before attacking the problem of computing the risk-neutral expectation of the forward start perpetuities defined in equations (3.35) and (3.39), it is convenient to examine first one important building block that will be required later. This key result is presented in the next proposition.

Proposition 3.3. *Assume the CEV process and let $\Phi_\phi(p, x; v, \lambda)$ be the p -th lower tail (if $\phi = -1$) or the p -th upper tail (if $\phi = +1$) truncated moment of a noncentral χ^2 random variable with v degrees of freedom and noncentrality parameter λ , as defined in Carr and Linetsky (2006, equations 5.12 and 5.11) and Dias and Nunes (2018, equations 21 and 24). Introduce the notation*

$$\alpha := (r - q)|\beta - 2| \quad (3.42)$$

and assume a deterministic time change given by

$$f(\tau) := \begin{cases} \frac{\delta^2|\beta-2|}{4(r-q)} (1 - e^{-\alpha\tau}) & \Leftarrow r \neq q \\ \frac{\delta^2(\beta-2)^2}{4} \tau & \Leftarrow r = q \end{cases}, \quad (3.43)$$

with $\tau := T - t$.

(i) *The time- t expected value of the claim $P_T^{1+(2-\beta)j}$, $\forall j \in \mathbb{N}_0$, for $P_T \geq X$ or $P_T < X$, is equal to:*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \begin{cases} e^{(r-q+\alpha j)\tau} P_t (f(\tau))^j \Phi_{+\phi} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta < 2 \\ (f(\tau))^{j-\nu} \Phi_{-\phi} \left(j - \nu, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.44)$$

(ii) The time- t expected value of the claim $P_T^{(2-\beta)j}$, $\forall j \in \mathbb{N}_0$, for $P_T \geq X$ or $P_T < X$, is equal to:

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{\phi P_T \geq \phi X\}} \mid \mathcal{F}_t \right] \\ &= \begin{cases} e^{\alpha j \tau} P_t (f(\tau))^{j-\nu} \Phi_{+\phi} \left(j - \nu, \frac{e^{-\alpha \tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ + \frac{\Gamma\left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)}\right)}{\Gamma(\nu)} \mathbb{1}_{\{j=0, \phi=-1\}} \\ (f(\tau))^j \Phi_{-\phi} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha \tau} P_t^{2-\beta}}{f(\tau)} \right) \end{cases} \begin{aligned} & \Leftarrow \beta < 2 \\ & \Leftarrow \beta > 2 \end{aligned} \end{aligned} \quad (3.45)$$

Armed with this instrumental proposition we can now proceed with the calculation of the risk-neutral expectations of the value functions contained in equations (3.36), (3.37), (3.40) and (3.41). The easiest to calculate are the ones associated with the linear components P_T/q and X/r , which are conditional to the region $P_T \geq X$ (for caps) or $P_T < X$ (for floors). For $\phi = +1$ (resp, $\phi = -1$), the first component can be (financially) interpreted as a European-style asset-or-nothing call (resp., put) option on the payoff P_T/q and strike price X , whereas, for $\phi = +1$ (resp, $\phi = -1$), the second ingredient can be (financially) understood as a European-style cash-or-nothing call (resp, put) option with a predetermined fixed cash amount X/r and strike price X . Even though the corresponding solutions arise immediately from Proposition 3.3 (by imposing $j = 0$), we provide next the explicit solutions of these binary options for the sake of completeness and easier reading.

Proposition 3.4. *Under the CEV process,*

(i) The time- t expected value of the claim P_T/q , conditional on $P_T \geq X$ or $P_T < X$, is equal to:

$$\begin{aligned} & \frac{1}{q} \mathbb{E}_{\mathbb{Q}} \left[P_T \mathbb{1}_{\{\phi P_T \geq \phi X\}} \mid \mathcal{F}_t \right] \\ &= \begin{cases} e^{(r-q)\tau} \frac{P_t}{q} \Phi_{+\phi} \left(0, \frac{e^{-\alpha \tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta < 2 \\ \frac{1}{q} (f(\tau))^{-\nu} \Phi_{-\phi} \left(-\nu, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha \tau} P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta > 2 \end{cases} \\ &:= P_q(\phi). \end{aligned} \quad (3.46)$$

(ii) The time- t expected value of the claim X/r , conditional on $P_T \geq X$ or $P_T < X$, is

equal to:

$$\begin{aligned}
& \frac{X}{r} \mathbb{E}_{\mathbb{Q}} [\mathbb{1}_{\{\phi P_T \geq \phi X\}} | \mathcal{F}_t] \\
&= \begin{cases} \frac{X}{r} P_t(f(\tau))^{-\nu} \Phi_{+\phi} \left(-\nu, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) + \frac{X}{r} \frac{\Gamma\left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)}\right)}{\Gamma(\nu)} \mathbb{1}_{\{\phi=-1\}} & \Leftarrow \beta < 2 \\ \frac{X}{r} \Phi_{-\phi} \left(0, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta > 2 \end{cases} \\
&:= X_r(\phi).
\end{aligned} \tag{3.47}$$

The next proposition derives the risk-neutral expectations of the terms contained in equations (3.36) and (3.40) that involve the modified Bessel's functions of first and second kind with order ν , which can be straightforwardly specialized for the cases $\beta < 2$ and $\beta > 2$ from the compact notation shown below.

Proposition 3.5. Assume the CEV process with $r = q$, and take $\varphi := \mathbb{1}_{\{\beta < 2\}}$ as well as $\tilde{\phi} := \phi \mathbb{1}_{\{\beta < 2\}} - \phi \mathbb{1}_{\{\beta > 2\}}$.

(i) The time- t expected value of the claims contained in equations (3.36) and (3.40) involving the modified Bessel's functions of first kind is equal to:

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} I_{\nu}(z_T) \mathbb{1}_{\{\phi P_T \geq \phi X\}} | \mathcal{F}_t \right] \\
&= P_t^{\varphi} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{+\tilde{\phi}} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
&:= I(\phi).
\end{aligned} \tag{3.48}$$

(ii) The time- t expected value of the claims contained in equations (3.36) and (3.40) involving the modified Bessel's functions of second kind is equal to:

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} K_{\nu}(z_T) \mathbb{1}_{\{\phi P_T \geq \phi X\}} | \mathcal{F}_t \right] \\
&= \frac{\pi}{2 \sin(\nu\pi)} P_t^{\varphi} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j-\nu}}{j!} \frac{(f(\tau))^{j-\nu}}{\Gamma(j+1-\nu)} \Phi_{+\tilde{\phi}} \left(j - \nu, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad + \mathbb{1}_{\{\beta < 2, \phi=-1\}} \frac{\pi}{2 \sin(\nu\pi)} \frac{\Gamma\left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)}\right)}{\Gamma(\nu)} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{-\nu}}{\Gamma(1-\nu)} \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} P_t^{\varphi} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{+\tilde{\phi}} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
&:= K(\phi).
\end{aligned} \tag{3.49}$$

The computation of the risk-neutral expectations contained in equations (3.37) and (3.41)

involving the Kummer's and Tricomi's confluent hypergeometric functions needs also to be specialized for the cases $\beta < 2$ and $\beta > 2$, which is done in the next proposition.

Proposition 3.6. *Assume the CEV process with $r \neq q$.*

(i) *The time- t expected value of the claims contained in equations (3.37) and (3.41) involving the Kummer's functions when $\beta < 2$ is equal to:*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) \mathbb{1}_{\{\phi P_T \geq \phi X\}} \middle| \mathcal{F}_t \right] \\ &= e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \Phi_{+\phi} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ &:= M_P(\phi), \end{aligned} \quad (3.50)$$

where

$$c := a \mathbb{1}_{\{r < q, \beta < 2\}} + (b - a) \mathbb{1}_{\{r > q, \beta < 2\}} \quad (3.51)$$

and $(c)_j$ is the Pochhammer's symbol as defined in Olver et al. (2010, equations 5.2.4 and 5.2.5).

(ii) *The time- t expected value of the claims contained in equations (3.37) and (3.41) involving the Kummer's functions when $\beta > 2$ is equal to:*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) \mathbb{1}_{\{\phi P_T \geq \phi X\}} \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \Phi_{-\phi} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ &:= M(\phi), \end{aligned} \quad (3.52)$$

where

$$d := (b - a) \mathbb{1}_{\{r < q, \beta > 2\}} + a \mathbb{1}_{\{r > q, \beta > 2\}}. \quad (3.53)$$

(iii) *The time- t expected value of the claims contained in equations (3.37) and (3.41) involving the Tricomi's functions when $\beta < 2$ is equal to:*

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) \mathbb{1}_{\{\phi P_T \geq \phi X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(-\nu)}{\Gamma(a - \nu)} e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \Phi_{+\phi} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ &+ \frac{\Gamma(\nu)}{\Gamma(a)} e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c - \nu)_j}{(2 - b)_j} \frac{\xi^j}{j!} \left(\xi \frac{1 - e^{\alpha\tau}}{2} \right)^{j-\nu} \Phi_{+\phi} \left(j - \nu, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ &+ \mathbb{1}_{\{\phi = -1\}} \frac{\Gamma(\nu)}{\Gamma(a)} \frac{\Gamma\left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)}\right)}{\Gamma(\nu)} \left(\frac{2|r - q|}{\delta^2|\beta - 2|} \right)^{-\nu} \\ &:= U_P(\phi), \end{aligned} \quad (3.54)$$

where $\xi := \mathbb{1}_{\{r < q\}} - \mathbb{1}_{\{r > q\}}$ and c is given by equation (3.51).

(iv) The time- t expected value of the claims contained in equations (3.37) and (3.41) involving the Tricomi's functions when $\beta > 2$ is equal to:

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) \mathbb{1}_{\{\phi P_T \geq \phi X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(-\nu)}{\Gamma(a-\nu)} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1-e^{-\alpha\tau}}{2} \right)^j \Phi_{-\phi} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ &+ \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(d-\nu)_j}{(2-b)_j} \frac{\tilde{\xi}^j}{j!} \left(\tilde{\xi} \frac{1-e^{-\alpha\tau}}{2} \right)^{j-\nu} \Phi_{-\phi} \left(j-\nu, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ &:= U(\phi), \end{aligned} \tag{3.55}$$

with $\tilde{\xi} := \mathbb{1}_{\{r > q\}} - \mathbb{1}_{\{r < q\}} = -\xi$ and d is given by equation (3.53).

Next proposition derives the last required ingredient which is another novel result in the literature: the computation of the forward start perpetuity of a bubble under the CEV process.

Proposition 3.7. Assume the CEV process with $\beta > 2$ and introduce the notation

$$\tilde{f} := \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \tag{3.56}$$

and

$$A_{t,T,u} := \begin{cases} \frac{1-e^{-\alpha(T-t)}}{2(e^{\alpha(u-T)}-1)} & \Leftarrow r \neq q \\ \frac{T-t}{2(u-T)} & \Leftarrow r = q \end{cases}, \tag{3.57}$$

with $\tau = T - t$. Then, the time- t expected value of the perpetual bubble is equal to

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} [G_T^{\infty}(P_T) | \mathcal{F}_t] &= \frac{1}{\Gamma(\nu)} \int_T^{\infty} \left[\kappa_{T,u}^{\nu} e^{-r(u-T)} \int_1^{\infty} \frac{\exp\left(-\frac{s\tilde{f}A_{t,T,u}}{1+2sA_{t,T,u}}\right)}{(1+2sA_{t,T,u})^{1+\nu}} s^{\nu-1} ds \right] d\mathfrak{s} \\ &:= G^{\infty}(t, T). \end{aligned} \tag{3.58}$$

In summary, equipped with Propositions 3.4, 3.5, 3.6 and 3.7, coupled with the perpetual solutions offered in Section 3.2, we can now tackle the main goal of this paper: the valuation of finite maturity caps and floors on continuous flows under the CEV model.

3.3.2. Finite caps

Combining Propositions 3.4, 3.5 and 3.7, the (discounted) risk-neutral expectation of the forward start perpetual cap can be immediately determined for the no-drift case, that is

$$\begin{aligned} & e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V(P_T, \infty) | \mathcal{F}_t] \\ = & e^{-r(T-t)} \times \begin{cases} \{AI(-1) + BK(1) + P_q(1) - X_r(1)\} & \Leftarrow \beta < 2 \\ \{BK(-1) + AI(1) + P_q(1) - X_r(1) - G^\infty(t, T)\} & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.59)$$

Similarly, the case with $r \neq q$ can be tackled in an identical way by properly combining Propositions 3.4, 3.6 and 3.7, that is

$$\begin{aligned} & e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V(P_T, \infty) | \mathcal{F}_t] \\ = & e^{-r(T-t)} \times \begin{cases} \{AYM_P(-1) + BYU_P(1) + P_q(1) - X_r(1)\} & \Leftarrow \beta < 2 \\ \{BYU(-1) + AYM(1) + P_q(1) - X_r(1) - G^\infty(t, T)\} & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.60)$$

Finally, the time- t value of the finite maturity cap, $V(P_t, T)$, can be computed using the time decomposition (3.34) by combining equations (3.16) and (3.59), for $r = q$, or equations (3.25) and (3.60), for $r \neq q$.

3.3.3. Finite floors

The finite maturity floors can be calculated using an analogous approach: combining Propositions 3.4 and 3.5, the (discounted) risk-neutral expectation of the forward start perpetual floor can be immediately determined for the no-drift case, that is

$$\begin{aligned} & e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [F(P_T, \infty) | \mathcal{F}_t] \\ = & e^{-r(T-t)} \times \begin{cases} \{AI(-1) + X_r(-1) - P_q(-1) + BK(1)\} & \Leftarrow \beta < 2 \\ \{BK(-1) + X_r(-1) - P_q(-1) + AI(1)\} & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.61)$$

Similarly, the case with $r \neq q$ can be tackled in an identical way by properly combining Propositions 3.4 and 3.6, that is

$$\begin{aligned} & e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [F(P_T, \infty) | \mathcal{F}_t] \\ = & e^{-r(T-t)} \times \begin{cases} \{AYM_P(-1) + X_r(-1) - P_q(-1) + BYU_P(1)\} & \Leftarrow \beta < 2 \\ \{BYU(-1) + X_r(-1) - P_q(-1) + AYM(1)\} & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.62)$$

Finally, the time- t value of the finite maturity floor, $F(P_t, T)$, can be computed using the time decomposition (3.38) by combining equations (3.29) and (3.61), for $r = q$, and equations (3.30) and (3.62), for $r \neq q$.

Alternatively, the finite maturity floor can be computed as

$$F(P_t, T) = \begin{cases} V(P_t, T) - \frac{P_t}{q} (1 - e^{-q\tau}) + \frac{X}{r} (1 - e^{-r\tau}) & \Leftarrow \beta < 2 \\ V(P_t, T) + G_t^T(P_t) - \frac{P_t}{q} (1 - e^{-q\tau}) + \frac{X}{r} (1 - e^{-r\tau}) & \Leftarrow \beta > 2 \end{cases}, \quad (3.63)$$

where

$$\begin{aligned} G_t^T(P_t) &= G_t^\infty(P_t) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[G_T^\infty(P_T) | \mathcal{F}_t] \\ &= \int_t^T G_t(P_t, u) du. \end{aligned} \quad (3.64)$$

Notice that the (finite) bubble value (3.64) is summed in equation (3.63), in the case of $\beta > 2$, only to cancel out the bubble value included in the solution of the corresponding finite maturity cap. Again, the rational is that there are no bubble values in the case of puts as highlighted in equation (3.14). To better clarify this point, we recall that the finite maturity cap, $V(P_t, T)$, should be understood as the cheapest cap price (i.e., the risk-neutral solution) in the spirit of Heston et al. (2007), whereas $V(P_t, T) + G_t^T(P_t)$ is interpreted also as a cap price but in the sense of Emanuel and MacBeth (1982). Thus, even though the caplet-floorlet parity is not satisfied when $\beta > 2$ if one uses the risk-neutral (finite maturity) cap value $V(P_t, T)$ —i.e., $F(P_t, T) \neq V(P_t, T) - \frac{P_t}{q} (1 - e^{-q\tau}) + \frac{X}{r} (1 - e^{-r\tau})$ for $\beta > 2$, as highlighted in Remark 3.2—, by using instead the (more expensive) cap solution $V(P_t, T) + G_t^T(P_t)$ it is possible to determine the risk-neutral finite maturity floor price, $F(P_t, T)$, as shown in the second branch of equation (3.63).

Remark 3.5. Clearly, for $\beta > 2$, it is possible to price a finite maturity cap in the spirit of Emanuel and MacBeth (1982) using the time decomposition (3.34) by combining equations (3.16) and (3.59), for $r = q$, or equations (3.25) and (3.60), for $r \neq q$, but ignoring the bubble effects both in the perpetual cap solutions (3.16) and (3.25), and in the corresponding forward start perpetuities (3.59) and (3.60). Since the obtained value is equal to $V(P_t, T) + G_t^T(P_t)$, then the risk-neutral cap price, $V(P_t, T)$, is immediately obtained by subtracting the bubble value (3.64), with the integrand function $G_t(P_t, u)$ being defined by equation (3.5). This has the advantage of calculating only one integral and thus avoids the need of computing two integrals: the one attached to the perpetual cap solution that is obtained via equation (3.10) and the one included in the forward start perpetuity involving the (more complex) double integral contained in Proposition 3.7.

3.4. Feed-in tariffs with minimum price guarantees

The analytical solutions of perpetual and finite maturity caps and floors derived in the previous section can now be applied in a variety of contexts. As an example of application, we follow the insights of Barbosa et al. (2018, Section 3) by analyzing the case of a renewable energy

project from a single investor's perspective, but now considering the CEV model.

3.4.1. Perpetual guarantee

Let us start with a FIT contract with a perpetual guarantee. Notice that under this time-independent assumption the investor may receive payments from the guarantee forever. Although a FIT contract with a perpetual guarantee is not used in practice, it is important to analyze it first in order to be compared with the more realistic FIT contract with a finite guarantee.

The FIT contract under analysis is based on a price-floor regime, i.e., the producer (or investor) receives the price floor P_L for every unit of energy produced if the market price P is below the price floor; otherwise (if $P > P_L$), the producer receives the market price. Therefore, the perpetual profit flow of the renewable energy project with a FIT can be formally defined as $\Pi(P_t) = \max(P_t, P_L)$, with P_t being interpreted as the time- t revenue for one unit of energy produced and P_L being understood as the revenue from the guarantee for one unit of energy.

Assuming that the (stochastic) revenue P is governed by the CEV model defined in equations (3.1) and (3.2), then the value of the project $V^F(P_t, \infty)$ must satisfy an ODE similar to (3.12) but now with a profit flow equal to $\Pi(P_t) = \max(P_t, P_L)$, that is

$$\frac{1}{2}\delta^2 P_t^\beta \frac{d^2 V^F}{dP_t^2} + (r - q) P_t \frac{dV^F}{dP_t} - rV^F + \Pi(P_t) = 0, \quad (3.65)$$

whose solution arises using the same rational employed when deriving Propositions 3.1 and 3.2. In particular, apart from the usual two options components—i.e., the solutions of the homogeneous part of the ODE (3.65)—that need to be specialized for each region subject to appropriate value-matching and smooth-pasting conditions, the full solution of the ODE (3.65) depends also on the corresponding particular integral, that is: the region $P_t < P_L$ should include the uncontingent time- t value of perpetual (fixed) revenues P_L/r , whereas the region $P_t > P_L$ must contain the uncontingent time- t value of perpetual (stochastic) revenues P_t/q .

The next two propositions offer the novel analytical solutions for evaluating a FIT contract with a perpetual guarantee (and considering both volatility skew patterns of the CEV model), whose derivations follow straightforwardly from the proofs of Propositions 3.1 and 3.2 and, hence, are omitted.

Proposition 3.8. *Assume the CEV process with $r = q$ and the notation introduced in Proposition 3.1. The time- t value of the project $V^F(P_t, \infty)$ is given by*

$$\begin{aligned} & V^F(P_t, \infty) \\ = & \begin{cases} \left(AP_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_L}{r} \right) \mathbb{1}_{\{P_t < P_L\}} + \left(BP_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq P_L\}} & \Leftarrow \beta < 2 \\ \left(BP_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_L}{r} \right) \mathbb{1}_{\{P_t < P_L\}} + \left(AP_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq P_L\}} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (3.66)$$

where the constants A and B are the solutions of the system of equations (3.17) with $X = P_L$, $\zeta := z(P_L)$ and for any $\beta \neq 2$.

Proposition 3.9. Assume the CEV process with $r \neq q$ and the notation introduced in Proposition 3.2. The time- t value of the project $V^F(P_t, \infty)$ is given by

$$V^F(P_t, \infty) = \begin{cases} \left(AY e^{(\epsilon-1)\frac{x_t}{2}} P_t M(a, b, x_t) + \frac{P_t}{r} \right) \mathbb{1}_{\{P_t < P_L\}} \\ + \left(BY e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a, b, x_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq P_L\}} & \Leftarrow \beta < 2 \\ \left(BY e^{(\epsilon-1)\frac{x_t}{2}} U(a, b, x_t) + \frac{P_t}{r} \right) \mathbb{1}_{\{P_t < P_L\}} \\ + \left(AY e^{(\epsilon-1)\frac{x_t}{2}} M(a, b, x_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq P_L\}} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \quad (3.67)$$

where the constants A and B are the solutions of the system of equations (3.26) with $X = P_L$ and $\chi := x(P_L)$, for $\beta < 2$, or the solutions of the system of equations (3.27) with $X = P_L$ and $\chi := x(P_L)$, for $\beta > 2$.

3.4.2. Finite guarantee

The FIT contract with a finite guarantee can be evaluated using a similar modeling approach to the one considered in Section 3.3. In particular, the FIT contract must now satisfy the PDE

$$\frac{1}{2} \delta^2 P_t^\beta \frac{\partial^2 V^F}{\partial P_t^2} + (r - q) P_t \frac{\partial V^F}{\partial P_t} - \frac{\partial V^F}{\partial \tau} - r V^F + \Pi(P_t) = 0. \quad (3.68)$$

Using the same rational used for the case of finite maturity caps and floors, the value of the project with a finite guarantee, $V^F(P_t, T)$, can be expressed as the difference between the value of the project with a perpetual guarantee, $V^F(P_t, \infty)$, and the value of a forward start project with a perpetual guarantee, $V^F(P_T, \infty)$, that is

$$V^F(P_t, T) = V^F(P_t, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V^F(P_T, \infty) | \mathcal{F}_t]. \quad (3.69)$$

The value of the project at the forward time T can be expressed by rewriting equations (3.66) and (3.67) with P_T . Clearly, the computation of the required risk-neutral expectation of the forward start perpetuity $V^F(P_T, \infty)$ follows easily from the propositions presented in Subsection 3.3.1 by simply replacing X by P_L . Therefore, combining Propositions 3.4, 3.5 and 3.7, with $X = P_L$, the (discounted) risk-neutral expectation of the forward start perpetual FIT contract with a minimum price guarantee can be immediately determined for the no-drift case, that is

$$\begin{aligned} & e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V^F(P_T, \infty) | \mathcal{F}_t] \\ &= e^{-r(T-t)} \times \begin{cases} \{AI(-1) + X_r(-1) + BK(1) + P_q(1)\} & \Leftarrow \beta < 2 \\ \{BK(-1) + X_r(-1) + AI(1) + P_q(1) - G^\infty(t, T)\} & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.70)$$

Similarly, the case with $r \neq q$ can be tackled in an identical way by properly combining Propositions 3.4, 3.6 and 3.7, again with $X = P_L$, that is

$$\begin{aligned} & e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V^F(P_T, \infty) | \mathcal{F}_t] \\ = & e^{-r(T-t)} \times \begin{cases} \{AYM_P(-1) + X_r(-1) + BYU_P(1) + P_q(1)\} & \Leftarrow \beta < 2 \\ \{BYU(-1) + X_r(-1) + AYM(1) + P_q(1) - G^\infty(t, T)\} & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (3.71)$$

Finally, the time- t value of the finite maturity FIT contract, $V^F(P_t, T)$, can be computed using the time decomposition (3.69) by combining equations (3.66) and (3.70), for $r = q$, and equations (3.67) and (3.71), for $r \neq q$.

3.4.3. Alternative FIT formulas via caps

The pricing solutions of FIT contracts with perpetual and finite guarantees derived in the previous two subsections have the advantage of directly using the economic rationale associated to real options problems. However, once the analytical formulae of perpetual and finite maturity caps are available it is possible to express the FIT formulas via caps (or floors).

To accomplish this purpose, let us first recall that $\max(P_u, X) = \max(P_u - X, 0) + X$. Hence, it is straightforward to show that the time- t value of a FIT contract with a perpetual guarantee can be alternatively expressed as (for $X = P_L$)

$$\begin{aligned} V^F(P_t, \infty) &= \int_t^\infty e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} [(P_u - X)^+ + X] du \\ &= \int_t^\infty c_t(P_t, X, u) du + X \int_t^\infty e^{-r(u-t)} du \\ &= V(P_t, \infty) + \frac{X}{r}, \end{aligned} \quad (3.72)$$

i.e., the FIT contract with a perpetual guarantee can be interpreted as a portfolio of a perpetual profit cap and a perpetual cash amount with a present value equal to X/r .

Adopting a similar procedure for the case of a FIT contract with a finite guarantee, then its time- t value can be alternatively computed as

$$\begin{aligned} V^F(P_t, T) &= V^F(P_t, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V^F(P_T, \infty) | \mathcal{F}_t] \\ &= V(P_t, \infty) + \frac{X}{r} - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[\left(V(P_T, \infty) + \frac{X}{r} \right) \middle| \mathcal{F}_t \right] \\ &= V(P_t, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V(P_T, \infty) | \mathcal{F}_t] + \frac{X}{r} (1 - e^{-r(T-t)}) \\ &= V(P_t, T) + \frac{X}{r} (1 - e^{-r(T-t)}), \end{aligned} \quad (3.73)$$

after combining the time decomposition (3.69) with equations (3.72) and (3.34).

Remark 3.6. Notice that the valuation of FIT contracts with perpetual and finite guarantees under the (nested) GBM diffusion highlighted in Barbosa et al. (2018, Section 3) can alternatively be made by combining the closed-form solutions of caps offered by Shackleton and Wojakowski (2007) with our equations (3.72) and (3.73).

Remark 3.7. In addition, since $\max(P_u, X) = P_u + \max(X - P_u, 0)$ it is straightforward to show that the valuation of perpetual and finite FIT contracts can also be computed via floors (for both the CEV and GBM diffusions) as follows (further details are available upon request):

$$V^F(P_t, \infty) = F(P_t, \infty) + \frac{P_t}{q} - G_t^\infty(P_t) \mathbb{1}_{\{\beta > 2\}} \quad (3.74)$$

and

$$V^F(P_t, T) = F(P_t, T) + \frac{P_t}{q} (1 - e^{-q(T-t)}) - G_t^T(P_t) \mathbb{1}_{\{\beta > 2\}}. \quad (3.75)$$

3.5. Costless reversible NPV

The proposed novel solutions for caps on continuous flows can be straightforwardly adapted to analyze investment projects under the fully reversible assumption considered in Keswani and Shackleton (2006, Section 6). Both the perpetual and finite cases will be discussed next.

3.5.1. Perpetual case

Propositions 3.1 and 3.2 have an immediate and important application to the real options literature: the perpetual costless reversible NPV considered in Keswani and Shackleton (2006, Section 6.1) under a GBM diffusion can now be easily extended to the CEV model.

To accomplish this purpose, let us first assume in this subsection the existence of a single project whose value V evolves through a CEV diffusion and that a given firm needs to pay the investment cost $\overline{X}$ to switch from the idle state to the operating state. Notice that under this fully (or costless) reversible assumption the disinvestment proceeds from a project, $\underline{X}$, are equal to the investment costs of setting up the project, $\overline{X}$, that is $\underline{X} = \overline{X} = \overline{\underline{X}}$. Moreover, with no round trip costs of investment and disinvestment the thresholds at which investment and disinvestment will occur collapse to a common level $\overline{V}$, which implies that the usual hysteretic band observed in dynamic entry and exit problems vanishes.

In this situation, the optimal exercise strategy is to activate the project when the opportunity cost (or cash yield) on the project qV exceeds the opportunity cost on the required investment $r\overline{X}$ and to deactivate it when $r\overline{X} > qV$. The cash flows from being disinvested can be interpreted as a risk free return on $\overline{X}$, i.e., this amount is invested in a risk free account whilst disinvested. Notice that, in this case, the switching occurs at the optimal threshold $V = \overline{V}$ when the so-called Jorgenson (1963) user costs of capital are equal.

Thinking about the NPV in this way, it is clear that the NPV is a continuum of infinitesimal

(European-style) options which enables us to evaluate it as

$$\begin{aligned} NPV_t &= \int_t^\infty e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} \left[(qV_u - r\bar{X})^+ \right] du \\ &= \int_t^\infty c_t(qV_t, r\bar{X}, u) du, \end{aligned} \quad (3.76)$$

where $c_t(P_t, X, u)$ is interpreted as the time- t price of a European-style call option (or caplet) on an underlying asset with spot price P_t (i.e., the underlying stochastic revenue), strike price X (i.e., the fixed cost rate) and with expiry date at time u ($\geq t$), and is given by equation (3.4). Then, the perpetual costless reversible NPV under the CEV diffusion can be immediately obtained using Propositions 3.1 and 3.2.

3.5.2. Finite case

Armed with the finite caps solutions, the reversible analysis presented in Subsection 3.5.1 is also analytically tractable for finite maturities. In this case, the finite maturity costless reversible NPV can be interpreted as the sum of a finite integral of infinitesimal CEV flow options and one final European-style call option on terminal values at time T , that is

$$\begin{aligned} NPV_t &= \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} \left[(qV_u - r\bar{X})^+ \right] du + e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left[(V_T - \bar{X})^+ \right] \\ &= \int_t^T c_t(qV_t, r\bar{X}, u) du + c_t(V_t, \bar{X}, T), \end{aligned} \quad (3.77)$$

where the flow option component is computed using the time decomposition (3.34)—by combining equations (3.16) and (3.59), for $r = q$, or equations (3.25) and (3.60), for $r \neq q$ —, whereas the plain-vanilla call is calculated using equation (3.4).

In summary, equation (3.77) allows the computation of the finite costless reversible NPV under the CEV model, thus generalizing the fully reversible case considered in Keswani and Shackleton (2006, Section 6.2) but under a GBM diffusion.

3.6. Numerical applications

The main goal of this section is to provide some numerical experiments to analyze the implications of the correct specification of the underlying state variable process by comparing the CEV results computed via our formulas with the ones obtained when assuming a lognormal diffusion determined via the analytical solutions offered by Shackleton and Wojakowski (2007). Even though we are focusing our analysis on the valuation of caps and floors on continuous flows, the obtained results also have immediate implications, as expected, for the analysis of a FIT contract with a minimum price guarantee and for the value of the costless reversible NPV of a firm's project, mainly because their perpetual and finite maturity solutions essentially depend on the (correct) valuation of perpetual and finite maturity caps, as shown in Sections

3.4 and 3.5. In other words, using an incorrect specification for the underlying state variable process affects not only the prices of caps and floors but also the prices of FIT contracts and the values of the costless reversible NPVs.

3.6.1. Caps and floors on continuous flows

To analyze the effect of β on the values of perpetual caps, perpetual bubbles, finite caps, finite bubbles and finite floors, Table 3.1 adopts $\beta \in \{-6, -4, -2, 0, 1, 1.7, 2, 2.3, 5\}$, thus considering both downward-sloping—for $\beta < 2$ —and upward-sloping—for $\beta > 2$ —volatility skews. The initial asset price is $P_0 = 100$, the instantaneous volatility at this price level is $\sigma_0 = 0.25$ —i.e., the volatility scale parameter δ is selected so that $\sigma_0 \equiv \sigma(P_0) = \delta P_0^{\frac{\beta}{2}-1} = 0.25$ when $P_0 = 100$ —and the expiry date is $T = 1$. Different moneyness levels with $X \in \{80, 100, 120\}$ and different drift specifications—i.e., positive, zero and negative drifts—are considered. Therefore, a constellation of nine contracts is obtained for each contingent claim. The two utmost right columns give the maximum absolute value of the percentage difference between the GBM price and the CEV prices for $\beta < 2$ and $\beta > 2$, respectively, relative to the GBM price.

Our main interest is in the dependence of perpetual caps, perpetual bubbles, finite caps, finite bubbles and finite floors on the β parameter governing the CEV process. Let us first recall that while for the limiting GBM process (i.e., for $\beta = 2$) and for CEV models with $\beta > 2$ the process never hits zero, the CEV process with $\beta < 2$ can hit the zero default boundary with positive probability. Even though the probability of touching the zero level is generally quite small for downward-sloping volatility scenarios with β values close to 2, such killing boundary rises substantially as we move farther away from 2.

Starting our analysis with the perpetual and finite bubbles, we notice that, as expected, there are no bubbles for $\beta \leq 2$ since the martingale property is preserved in these cases. However, the loss of such property for $\beta > 2$ implies the existence of bubble values in these situations. As we move farther away to the right of $\beta = 2$ the effect of the bubble is stronger. In other words, the impact of the loss of the martingale property under upward-sloping volatility scenarios is much more pronounced for higher levels of β . Since the bubble value (3.5) is independent of X , the perpetual and finite bubble values are the same for any moneyness level. However, the level of the drift produces significant changes in the perpetual bubble values. The reason lies in the fact that as the drift of the stochastic differential equation (3.1) rises the underlying asset value P will grow at a higher rate, thus boosting the perpetual bubble values. A similar behavior associated to different drift specifications is observed also for the case of finite bubbles but now the values are much smaller since we are dealing with contracts with a (relatively) short maturity, i.e., $T = 1$.

The same drift effect on the prices of perpetual and finite caps (while keeping the remaining parameters fixed) is also found: higher drift rates imply a rise in the prices of perpetual and

finite caps since the underlying asset value P will grow at bigger rates. However, as expected, the opposite is true for the case of finite (and perpetual) floors: higher drift rates imply a fall in the prices of finite (and perpetual) floors due to the different nature of such contingent claims payoffs. We recall that while the flow rate associated to caps is equal to $(P_t - X)^+$, the payoff rate attached to floors is equal to $(X - P_t)^+$. The structure of these flow rates explains also why caps rise as X falls and floors fall as X falls (while maintaining, again, the remaining parameters fixed). This is an immediate consequence of the moneyness level and its impact on the price of these contingent claims.

Table 3.1.: CEV caps and floors for different betas, moneyness levels and drift specifications.

X	r	q	Beta					Maximum % difference					
			-6	-4	-2	0	1		1.7	2	2.3	5	
80	0.07	0.03	2358.481	2369.482	2377.112	2368.940	2350.132	2332.725	2326.349	1898.828	270.676	2.18	88.36
100	0.07	0.03	2120.478	2137.937	2155.104	2162.856	2156.621	2150.610	2149.829	1728.407	150.598	1.37	92.99
120	0.07	0.03	1902.184	1928.289	1958.255	1986.952	1996.871	2005.540	2012.082	1598.727	89.548	5.46	95.55
80	0.05	0.05	854.218	887.969	920.239	932.744	916.497	894.886	885.453	846.629	166.611	5.34	81.18
100	0.05	0.05	578.305	627.157	680.973	727.582	736.447	735.512	735.215	705.557	85.999	21.34	88.30
120	0.05	0.05	337.521	403.353	481.397	564.954	599.682	619.571	628.502	608.153	48.977	46.30	92.21
80	0.03	0.07	440.419	445.496	442.816	415.514	382.788	356.666	347.881	339.849	95.830	28.06	72.45
100	0.03	0.07	215.716	233.175	250.342	258.094	251.859	245.848	245.067	244.591	45.737	11.98	81.34
120	0.03	0.07	59.422	85.163	117.655	152.971	166.956	176.201	181.625	187.130	24.810	67.28	86.34
Perpetual bubble													
80	0.07	0.03	-	-	-	-	-	-	-	422.202	2010.718	-	-
100	0.07	0.03	-	-	-	-	-	-	-	422.202	2010.718	-	-
120	0.07	0.03	-	-	-	-	-	-	-	422.202	2010.718	-	-
80	0.05	0.05	-	-	-	-	-	-	-	29.955	620.912	-	-
100	0.05	0.05	-	-	-	-	-	-	-	29.955	620.912	-	-
120	0.05	0.05	-	-	-	-	-	-	-	29.955	620.912	-	-
80	0.03	0.07	-	-	-	-	-	-	-	1.256	210.817	-	-
100	0.03	0.07	-	-	-	-	-	-	-	1.256	210.817	-	-
120	0.03	0.07	-	-	-	-	-	-	-	1.256	210.817	-	-
Finite cap													
80	0.07	0.03	23.717	23.112	22.584	22.194	22.036	21.938	21.899	21.861	21.502	8.30	1.81
100	0.07	0.03	7.658	7.566	7.502	7.470	7.462	7.460	7.460	7.460	7.398	2.66	0.83
120	0.07	0.03	0.733	0.960	1.232	1.555	1.739	1.877	1.938	2.001	2.572	62.18	32.70
80	0.05	0.05	22.402	21.729	21.138	20.697	20.517	20.403	20.358	20.314	19.921	10.04	2.14
100	0.05	0.05	6.640	6.548	6.485	6.453	6.446	6.443	6.443	6.443	6.400	3.06	0.66
120	0.05	0.05	0.480	0.670	0.904	1.188	1.352	1.475	1.530	1.587	2.113	68.62	38.13
80	0.03	0.07	21.145	20.400	19.745	19.251	19.048	18.919	18.867	18.817	18.385	12.08	2.55
100	0.03	0.07	5.724	5.631	5.568	5.535	5.528	5.526	5.525	5.526	5.498	3.59	0.50
120	0.03	0.07	0.304	0.456	0.651	0.894	1.036	1.145	1.193	1.244	1.721	74.52	44.22
Finite bubble													
80	0.07	0.03	-	-	-	-	-	-	-	2.8E-149	0.085	-	-
100	0.07	0.03	-	-	-	-	-	-	-	2.8E-149	0.085	-	-
120	0.07	0.03	-	-	-	-	-	-	-	2.8E-149	0.085	-	-
80	0.05	0.05	-	-	-	-	-	-	-	3.3E-150	0.066	-	-
100	0.05	0.05	-	-	-	-	-	-	-	3.3E-150	0.066	-	-
120	0.05	0.05	-	-	-	-	-	-	-	3.3E-150	0.066	-	-
80	0.03	0.07	-	-	-	-	-	-	-	3.9E-151	0.051	-	-
100	0.03	0.07	-	-	-	-	-	-	-	3.9E-151	0.051	-	-
120	0.03	0.07	-	-	-	-	-	-	-	3.9E-151	0.051	-	-
Finite floor													
80	0.07	0.03	2.467	1.861	1.333	0.943	0.786	0.687	0.648	0.611	0.337	280.53	48.08
100	0.07	0.03	5.724	5.631	5.568	5.535	5.528	5.525	5.525	5.526	5.548	3.59	0.42
120	0.07	0.03	18.114	18.341	18.614	18.937	19.120	19.258	19.320	19.383	20.038	6.24	3.72
80	0.05	0.05	2.894	2.220	1.630	1.188	1.008	0.895	0.849	0.806	0.479	240.68	43.63
100	0.05	0.05	6.640	6.548	6.485	6.453	6.446	6.443	6.443	6.466	6.466	3.06	0.36
120	0.05	0.05	19.988	20.178	20.412	20.697	20.860	20.983	21.038	21.095	21.688	4.99	3.09
80	0.03	0.07	3.377	2.632	1.976	1.483	1.280	1.151	1.099	1.049	0.668	207.36	39.23
100	0.03	0.07	7.658	7.566	7.502	7.470	7.462	7.460	7.460	7.483	7.483	2.66	0.31
120	0.03	0.07	21.942	22.093	22.288	22.532	22.674	22.782	22.831	22.881	23.409	3.89	2.53

This table values perpetual caps, perpetual bubbles, finite caps, finite bubbles and finite floors under the CEV process with $\beta \in \{-6, -4, -2, 0, 1, 1.7, 2, 2.3, 5\}$. The two utmost right columns give the maximum absolute value of the percentage difference between the GBM price and the CEV prices for $\beta < 2$ and $\beta > 2$, respectively, relative to the GBM price. The strike price X , riskless interest rate r and dividend yield q vary as indicated in the three left columns. Parameters used in the calculation: $P_0 = 100$, $\sigma(P_0) = 0.25$ and $T = 1$.

Moving now to the effect of β on the prices of perpetual caps, finite caps and finite floors, Table 3.1 reports that the prices under CEV processes with $\beta < 2$ can be significantly different from the ones computed via the limiting GBM model. For example, considering a perpetual cap with $X = 120$, $r = 0.03$ and $q = 0.07$ the price under the GBM setup is 181.625; the corresponding price under a CEV process with $\beta = -6$ is only 59.422, which produces a difference of 67.28%. Similar highly pronounced effects are also observed for other combinations of parameters. Moreover, the results show even more noticeable differences when comparing the cases with $\beta = 2$ and $\beta = 5$. This is a direct consequence of the strong effect of the bubble when $\beta = 5$. Considering the aforementioned combination of parameters, the perpetual cap when $\beta = 5$ is $V(P_t, \infty) = 24.810$, thus originating a difference of 86.34%. Notice that the corresponding bubble value is $G_t^\infty(P_t) = 210.817$. Thus, the bubble value is much higher than the perpetual cap value. We recall that $V(P_t, \infty)$ should be understood as the cheapest cap price (i.e., the risk-neutral solution) in the spirit of Heston et al. (2007), whereas $V(P_t, \infty) + G_t^\infty(P_t)$ can be interpreted also as a cap price but in the sense of Emanuel and MacBeth (1982). The same line of reasoning can be used with the finite maturity components $V(P_t, T)$ and $V(P_t, T) + G_t^T(P_t)$.

The qualitative and economic insights to be drawn from the results of finite caps and finite floors are essentially similar to the ones already discussed for the perpetual caps. Using again the same combination of parameters, the finite cap value under the GBM model is 1.193, whereas the corresponding finite cap under a CEV process with $\beta = -6$ is only 0.304; this produces a difference of 74.52%. Similar conclusions can be taken from the finite floors results. Notice that since all the contracts (with different β values while maintaining the remaining parameters fixed) were calibrated so that they possess the same initial volatility, all the differences found in the numerical results highlighted in Table 3.1 stem purely from the effect of the relationship between volatility and price levels, which is captured only by the CEV volatility specification (3.2).

Table 3.2 adopts the same parameters configuration of Table 3.1, but now considering a driftless case—i.e., $r = q = 0.05$ —and different expiry dates with $T \in \{1, 5, 10\}$, thus admitting longer maturity contracts for each contingent claim. Notice that perpetual caps are independent of T and perpetual bubbles are not influenced by both X and T . Therefore, the corresponding results are not included in Table 3.2 since they are shown and discussed already in the context of Table 3.1. Although the results of finite caps, finite bubbles and finite floors for $T = 1$ are already included in Table 3.1, they are also displayed in Table 3.2 to facilitate comparisons. The salient feature of these results is that, as expected, the values of finite caps, finite bubbles and finite floors rise as the time to maturity of the contracts augments. Moreover, similar significant differences across different β values are also observed for the whole set of expiry dates configuration.

To sum up, the extensive numerical analysis that was performed suggests that corporate managers and policy-makers should pay more attention to the possibility of moving beyond real options models based on the classic GBM process to more realistic models incorporating leverage and volatility smile effects.

Table 3.2.: CEV caps and floors for different betas, moneyness levels and expiry dates.

X	T	Beta									Maximum	
		-6	-4	-2	0	1	1.7	2	2.3	5	% difference	
Finite cap												
80	1	22.402	21.729	21.138	20.697	20.517	20.403	20.358	20.314	19.921	10.04	2.14
100	1	6.640	6.548	6.485	6.453	6.446	6.443	6.443	6.443	6.400	3.06	0.66
120	1	0.480	0.670	0.904	1.188	1.352	1.475	1.530	1.587	2.113	68.62	38.13
80	5	135.397	132.720	127.151	119.573	116.368	114.407	113.620	112.862	82.491	19.17	27.40
100	5	65.941	66.390	65.567	64.121	63.753	63.650	63.639	63.650	40.225	4.32	36.79
120	5	19.557	23.336	27.113	31.098	33.406	35.180	35.985	36.820	21.743	45.65	39.58
80	10	272.235	272.249	266.368	250.728	241.791	236.509	234.444	232.478	120.370	16.13	48.66
100	10	152.411	157.599	159.783	156.969	155.352	154.893	154.849	154.893	61.237	3.19	60.45
120	10	61.304	72.606	83.779	93.658	98.995	103.340	105.371	107.515	34.353	41.82	67.40
Finite bubble												
80	1	-	-	-	-	-	-	-	3.3E-150	0.066	-	-
100	1	-	-	-	-	-	-	-	3.3E-150	0.066	-	-
120	1	-	-	-	-	-	-	-	3.3E-150	0.066	-	-
80	5	-	-	-	-	-	-	-	5.63E-27	24.564	-	-
100	5	-	-	-	-	-	-	-	5.63E-27	24.564	-	-
120	5	-	-	-	-	-	-	-	5.63E-27	24.564	-	-
80	10	-	-	-	-	-	-	-	1.01E-11	97.691	-	-
100	10	-	-	-	-	-	-	-	1.01E-11	97.691	-	-
120	10	-	-	-	-	-	-	-	1.01E-11	97.691	-	-
Finite floor												
80	1	2.894	2.220	1.630	1.188	1.008	0.895	0.849	0.806	0.479	240.68	43.63
100	1	6.640	6.548	6.485	6.453	6.446	6.443	6.443	6.443	6.466	3.06	0.36
120	1	19.988	20.178	20.412	20.697	20.860	20.983	21.038	21.095	21.688	4.99	3.09
80	5	46.917	44.241	38.671	31.094	27.888	25.927	25.141	24.382	18.575	86.62	26.11
100	5	65.941	66.390	65.567	64.121	63.753	63.650	63.639	63.650	64.789	4.32	1.81
120	5	108.036	111.816	115.592	119.578	121.886	123.660	124.465	125.300	134.787	13.20	8.29
80	10	114.847	114.861	108.980	93.341	84.403	79.122	77.056	75.090	60.673	49.06	21.26
100	10	152.411	157.599	159.783	156.969	155.352	154.893	154.849	154.893	158.928	3.19	2.63
120	10	218.691	229.994	241.166	251.046	256.383	260.728	262.759	264.903	289.432	16.77	10.15

This table values perpetual caps, perpetual bubbles, finite caps, finite bubbles and finite floors under the CEV process with $\beta \in \{-6, -4, -2, 0, 1, 1.7, 2, 2.3, 5\}$. The two utmost right columns give the maximum absolute value of the percentage difference between the GBM price and the CEV prices for $\beta < 2$ and $\beta > 2$, respectively, relative to the GBM price. The strike price X and expiry date T vary as indicated in the two left columns. Parameters used in the calculation: $P_0 = 100$, $\sigma(P_0) = 0.25$, $r = 0.05$ and $q = 0.05$.

3.6.2. Feed-in tariffs with minimum price guarantees

Equations (3.72) and (3.73) show that the differences observed between FIT contracts under alternative CEV models are only due to the differences obtained between the attached cap solutions of the corresponding portfolios. Even though we have already highlighted the potential huge disparities between CEV models when valuing caps and floors—which, ultimately, will be transposed to the corresponding FIT contracts—, Figures 3.1 and 3.2 implement the perpetual and finite maturity FIT formulas, respectively, as a function of the price floor P_L and adopting different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 5\}$, $P_0 = 100$, $\sigma(P_0) = 0.25$, $r = 0.05$ and $q = 0.05$. For the finite maturity case an even longer maturity contract with $T = 15$ is considered. As expected, the pattern of the differences is similar to the ones reported already in Tables 3.1 and 3.2.

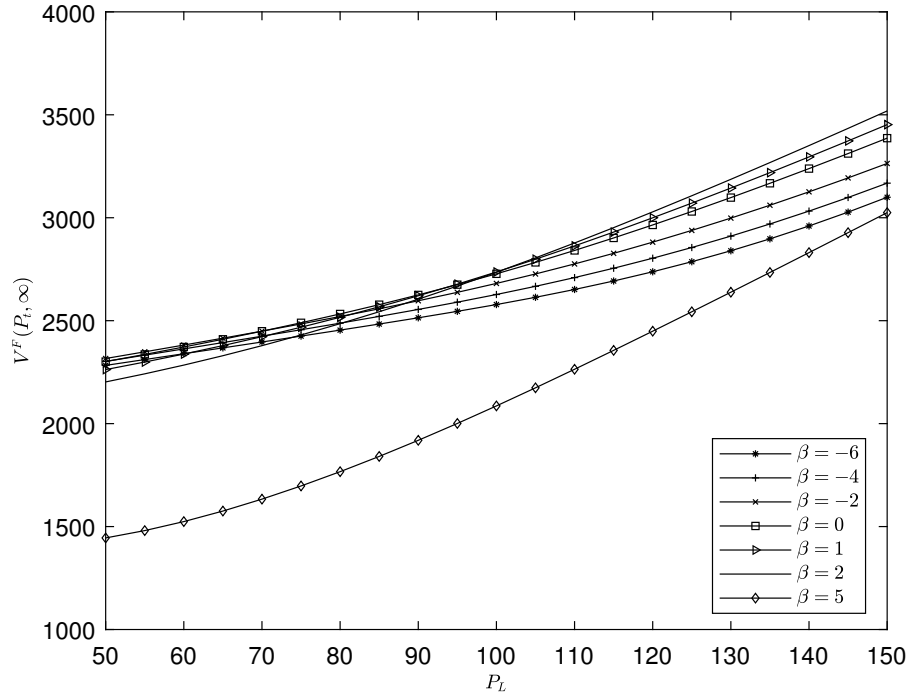


Figure 3.1.: Perpetual FIT contracts.

Contract values as a function of the price floor P_L and adopting different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 5\}$, $P_0 = 100$, $\sigma(P_0) = 0.25$, $r = 0.05$ and $q = 0.05$.

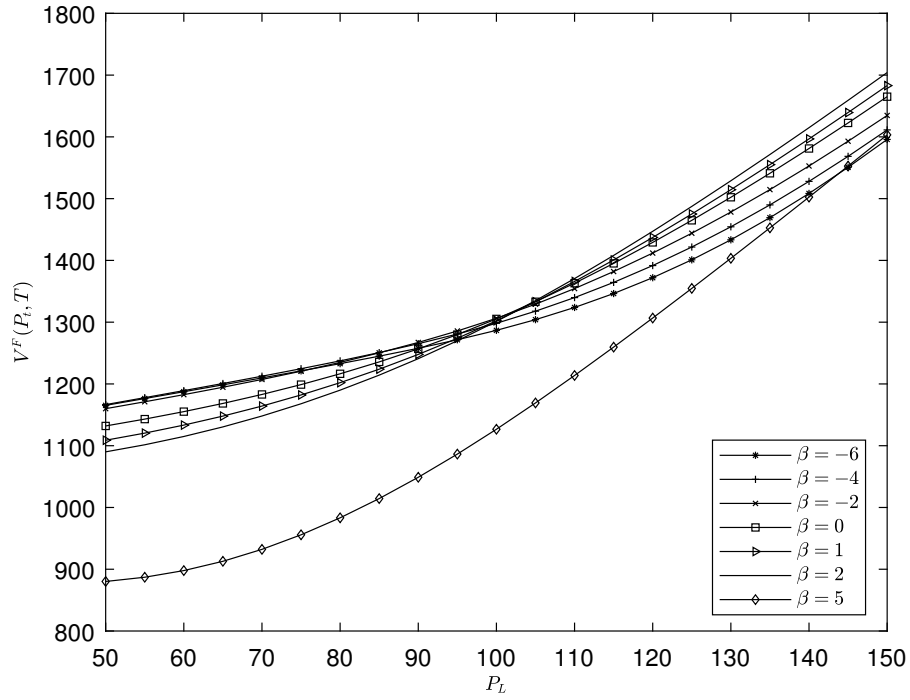


Figure 3.2.: Finite maturity FIT contracts.

Contract values as a function of the price floor P_L and adopting different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 5\}$, $P_0 = 100$, $T = 15$, $\sigma(P_0) = 0.25$, $r = 0.05$ and $q = 0.05$.

We recall that when $P_t > P_L$ the producer receives a revenue from the market price instead of the revenue from the guarantee. By contrast, the producer receives the price floor P_L if $P_t \leq P_L$. From a policy-maker perspective, Figures 3.1 and 3.2 reveal that a given government might be paying a too high or a too low profit flow if the classic GBM assumption is considered when the true generating process for modeling the uncertainty associated with the output price of energy is the CEV model. Therefore, these issues should be carefully analyzed by governments when defining policies to encourage new renewable energy projects aiming to achieve specific energy policy goals.

3.7. Conclusions

This paper offers an analytic representation for evaluating (perpetual and finite) caps and floors on continuous flows under the CEV model that nests the classic GBM process as a special case. Therefore, we have been able to extend the real options literature in several directions. The paper shows that the incorrect specification of the underlying state variable process for the valuation of caps and floors can have a significant impact in many executive management decisions. The proposed analytical solutions of caps and floors immediately opens a wide variety of applications on other contractual agreements with cap, floor and collar clauses that have been considered in the many price mechanisms and on several government policies implemented in the context of energy problems.

Appendices

3A. Proof of Proposition 3.1

Similarly to Davydov and Linetsky (2001, page 963), we look for a solution of the form $V(P) = P^{\frac{1}{2}}h(z)$ for some unknown function $h(z)$ with z defined in equation (3.15). Substituting this functional form into equation (3.11) with $r = q$, we obtain the ordinary differential equation (hereafter, ODE) for h

$$z^2 h''(z) + zh'(z) - (z^2 + \nu^2) h(z) = 0, \quad (3A.1)$$

with ν being given by equation (3.9). This is the modified Bessel's differential equation shown in Olver et al. (2010, equation 10.25.1) that has two linearly independent solutions, $I_\nu(z)$ and $K_\nu(z)$, that are known as the modified Bessel's functions of first and second kind, respectively, as defined in Olver et al. (2010, equations 10.25.2 and 10.27.4).

The general solution of the modified Bessel's equation (3A.1) is $h(z) = A_0 I_\nu(z) + B_0 K_\nu(z)$, with A_0 and B_0 being two constants to be determined from boundary conditions. Since the trial solution is $V(P) = P^{\frac{1}{2}}h(z)$, we must multiply $h(z)$ by $P^{\frac{1}{2}}$ to get the general solution for equation (3.11):

$$V(P) = A_0 P^{\frac{1}{2}} I_\nu(z) + B_0 P^{\frac{1}{2}} K_\nu(z). \quad (3A.2)$$

We notice that equation (3A.2) is also the solution of the homogeneous part of equation (3.12). In the region $P < X$ there is no profit flow and only the homogeneous part matters. However, in the region $P > X$ the profit flow $\Pi(P_t)$ must be considered. This means that we need to take another linear combination of the solutions for the homogeneous part and add any particular solution of the full equation. The particular solution of equation (3.12), for $P > X$, is given by $V(P) = \frac{P}{q} - \frac{X}{r}$. Therefore, the general solution of equation (3.12), for $P > X$, is

$$V(P) = A_1 P^{\frac{1}{2}} I_\nu(z) + B_1 P^{\frac{1}{2}} K_\nu(z) + \frac{P}{q} - \frac{X}{r}, \quad (3A.3)$$

with A_1 and B_1 being two constants to be determined from boundary conditions.

3A.1. Boundary conditions

Solving equations (3.11) and (3.12) subject to appropriate boundary conditions yields the value functions for the idle (i.e., $P < X$) and active (i.e., $P > X$) states of the real options problem denoted by $V_0(P)$ and $V_1(P)$, respectively. Note that as P becomes small, the possibility that the state variable P rises above the strike X becomes unlikely. Hence, the value of the project in the idle state should go to zero as P tends to zero; in other words, the option to invest is worthless. This means that equation (3.11) must be solved subject to the boundary condition

$$\lim_{P \rightarrow 0^+} V_0(P) = 0. \quad (3A.4)$$

Therefore, the solution of the ODE (3.11) subject to the boundary condition (3A.4) is given by $V_0(P) = A_0 P^{\frac{1}{2}} I_\nu(z)$, for $\beta < 2$. Note that $z \rightarrow 0$ as $P \rightarrow 0^+$ and, hence, using Olver et al. (2010, equations 10.30.1 and 10.30.2), we observe that $I_\nu(z)$ and $K_\nu(z)$ approach 0 and $+\infty$, respectively. Consequently, the constant B_0 in equation (3A.2) should be zero. Moreover, $V_0(P) = B_0 P^{\frac{1}{2}} K_\nu(z)$, for $\beta > 2$. In this case, $z \rightarrow +\infty$ as $P \rightarrow 0^+$ and, using Olver et al. (2010, equations 10.30.4 and 10.25.3), we notice that $I_\nu(z)$ and $K_\nu(z)$ tend to $+\infty$ and 0, respectively, as z tends to $+\infty$. Hence, the constant A_0 in equation (3A.2) should be zero. The imposition of these boundary conditions to both cases ensures that the options components terms that diverge as P goes to 0^+ are ruled out.

When the project is in the active state, the suspension option is unlikely to be exercised as P becomes larger. Thus, its value should be worthless. Moreover, to rule out explosive solutions it is necessary to use the boundary condition

$$\lim_{P \rightarrow +\infty} V_1(P) = \frac{P}{q} - \frac{X}{r} \quad (3A.5)$$

to solve equation (3.12). Therefore, the solution of the ODE (3.12) subject to the boundary condition (3A.5) is given by $V_1(P) = B_1 P^{\frac{1}{2}} K_\nu(z) + \frac{P}{q} - \frac{X}{r}$, for $\beta < 2$. Notice that $z \rightarrow +\infty$ as $P \rightarrow +\infty$ and, using Olver et al. (2010, equations 10.30.4 and 10.25.3), we note that $I_\nu(z)$ and $K_\nu(z)$ tend to $+\infty$ and 0, respectively, as z tends to $+\infty$. Consequently, the constant A_1 in equation (3A.3) should be zero. Furthermore, $V_1(P) = A_1 P^{\frac{1}{2}} I_\nu(z) + \frac{P}{q} - \frac{X}{r}$, for $\beta > 2$. In this case, $z \rightarrow 0$ as $P \rightarrow +\infty$ and, hence, using Olver et al. (2010, equations 10.30.1 and 10.30.2), we notice that $I_\nu(z)$ and $K_\nu(z)$ approach 0 and $+\infty$, respectively. Therefore, the constant B_1 in equation (3A.3) should be zero. The imposition of these boundary conditions to both cases ensures that the options components terms that diverge as P goes to $+\infty$ are ruled out.

In summary, the solutions for the case $r = q$ are given by

$$V(P) = \begin{cases} A_0 P^{\frac{1}{2}} I_\nu(z) \mathbb{1}_{\{P < X\}} + \left(B_1 P^{\frac{1}{2}} K_\nu(z) + \frac{P}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P \geq X\}} & \Leftarrow \beta < 2 \\ B_0 P^{\frac{1}{2}} K_\nu(z) \mathbb{1}_{\{P < X\}} + \left(A_1 P^{\frac{1}{2}} I_\nu(z) + \frac{P}{q} - \frac{X}{r} \right) \mathbb{1}_{\{P \geq X\}} & \Leftarrow \beta > 2 \end{cases} \quad (3A.6)$$

3A.2. Value-matching and smooth-pasting conditions

Using the recurrence relations $\frac{\partial I_\nu(z)}{\partial z} = \frac{\nu}{z}I_\nu(z) + I_{\nu+1}(z)$ and $\frac{\partial K_\nu(z)}{\partial z} = \frac{\nu}{z}K_\nu(z) - K_{\nu+1}(z)$ borrowed from Olver et al. (2010, equation 10.29.2) for computing the derivatives of the modified Bessel's functions wrt z and applying the value-matching and smooth-pasting conditions at $P = X$, and with $\zeta := z(X)$, given by

$$\begin{cases} V_0(X) = V_1(X) \\ V'_0(X) = V'_1(X) \end{cases}, \quad (3A.7)$$

we obtain, after some straightforward calculus, the system of equations (3.17) that is valid for both cases $\beta < 2$ and $\beta > 2$ and where the letters A and B denote the constants that appear in the solution (3A.6) in order to simplify notation and provide a more compact formula.

Finally, equation (3.16) is obtained by subtracting the perpetual bubble value (3.10) from the solution (3A.6) for the case with $\beta > 2$.

3B. Proof of Proposition 3.2

Similarly to Davydov and Linetsky (2001, page 963), we look for a solution of the form $V(P) = P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} h(x)$ for some unknown function $h(x)$ with x and ϵ defined in equations (3.18) and (3.19), respectively. Substituting this functional form into equation (3.11) with $r \neq q$, we obtain the ODE for h

$$h''(x) + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2} \right) h(x) = 0, \quad x \in (0, +\infty), \quad (3B.1)$$

with k and m being defined by equations (3.20) and (3.21), respectively. This is the Whittaker's differential equation shown in Olver et al. (2010, equation 13.14.1) that has two linearly independent solutions, $M_{k,m}(x)$ and $W_{k,m}(x)$, that are known as the Whittaker's functions defined in Olver et al. (2010, equations 13.14.2 and 13.14.3), respectively, and are expressed in terms of the Kummer's confluent hypergeometric function $M(a, b, x)$ and the Tricomi's confluent hypergeometric function (also known as the Kummer's confluent hypergeometric function of the second kind) $U(a, b, x)$ presented in Olver et al. (2010, equations 13.2.2 and 13.2.42), respectively.

The general solution of the Whittaker's equation (3B.1) is $h(z) = A_0 M_{k,m}(x) + B_0 W_{k,m}(x)$, with A_0 and B_0 being two constants to be determined from boundary conditions. Since the trial solution is $V(P) = P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} h(x)$, we must multiply $h(x)$ by $P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x}$ to get the general solution for equation (3.11):

$$V(P) = A_0 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} M_{k,m}(x) + B_0 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} W_{k,m}(x). \quad (3B.2)$$

We notice that equation (3B.2) is also the solution of the homogeneous part of equation (3.12). In the region $P < X$ there is no profit flow and only the homogeneous part matters. However, in the region $P > X$ the profit flow $\Pi(P_t)$ must be considered. This means that we need to take another linear combination of the solutions for the homogeneous part and add any particular solution of the full equation. The particular solution of equation (3.12), for $P > X$, is given by $V(P) = \frac{P}{q} - \frac{X}{r}$. Therefore, the general solution of equation (3.12), for $P > X$, is

$$V(P) = A_1 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} M_{k,m}(x) + B_1 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} W_{k,m}(x) + \frac{P}{q} - \frac{X}{r}, \quad (3B.3)$$

with A_1 and B_1 being two constants to be determined from boundary conditions.

3B.1. Boundary conditions

The solution of the ODE (3.11) subject to the boundary condition (3A.4) is given by $V_0(P) = A_0 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} M_{k,m}(x)$, for $\beta < 2$. In this case, $x \rightarrow 0$ as $P \rightarrow 0^+$ and $W_{k,m}(x) \rightarrow +\infty$ as $x \rightarrow 0$ using the limiting forms in Olver et al. (2010, page 335). Consequently, the constant

B_0 in equation (3B.2) should be zero. Moreover, $V_0(P) = B_0 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} W_{k,m}(x)$, for $\beta > 2$. In this case, $x \rightarrow +\infty$ as $P \rightarrow 0^+$ and $M_{k,m}(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ using the asymptotic properties in Olver et al. (2010, page 335). Hence, the constant A_0 in equation (3B.2) should be zero. Again, the imposition of these boundary conditions to both cases ensures that the options components terms that diverge as P goes to 0^+ are ruled out.

The solution of the ODE (3.12) subject to the boundary condition (3A.5) is given by $V_1(P) = B_1 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} W_{k,m}(x) + \frac{P}{q} - \frac{X}{r}$, for $\beta < 2$. In this case, $x \rightarrow +\infty$ as $P \rightarrow +\infty$ and $M_{k,m}(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ using the limiting forms in Olver et al. (2010, page 335). Consequently, the constant A_1 in equation (3B.3) should be zero. Furthermore, $V_1(P) = A_1 P^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x} M_{k,m}(x) + \frac{P}{q} - \frac{X}{r}$, for $\beta > 2$. In this case, $x \rightarrow 0$ as $P \rightarrow +\infty$ and $W_{k,m}(x) \rightarrow +\infty$ as $x \rightarrow 0$ using the asymptotic properties in Olver et al. (2010, page 335). Therefore, the constant B_1 in equation (3B.3) should be zero. As before, the imposition of these boundary conditions to both cases ensures that the options components terms that diverge as P goes to $+\infty$ are ruled out.

Finally, using Olver et al. (2010, equations 13.14.2 and 13.14.3), the solutions for the case $r \neq q$ are given by

$$V(P) = \begin{cases} A_0 e^{(\epsilon-1)\frac{x}{2}} YPM(a, b, x) \mathbb{1}_{\{P < X\}} \\ + \left[B_1 e^{(\epsilon-1)\frac{x}{2}} YPU(a, b, x) + \frac{P}{q} - \frac{X}{r} \right] \mathbb{1}_{\{P \geq X\}} & \Leftarrow \beta < 2 \\ B_0 e^{(\epsilon-1)\frac{x}{2}} YU(a, b, x) \mathbb{1}_{\{P < X\}} \\ + \left[A_1 e^{(\epsilon-1)\frac{x}{2}} YM(a, b, x) + \frac{P}{q} - \frac{X}{r} \right] \mathbb{1}_{\{P \geq X\}} & \Leftarrow \beta > 2 \end{cases}, \quad (3B.4)$$

with a , b and Y being defined by equations (3.22), (3.23) and (3.24), respectively.

3B.2. Value-matching and smooth-pasting conditions

Using the differentiation formulas for the Kummer's confluent hypergeometric functions $M(a, b, x)$ and $U(a, b, x)$ wrt x borrowed from Olver et al. (2010, equations 13.3.15 and 13.3.22) and applying the value-matching and smooth-pasting conditions (3A.7) at $P = X$, and with $\chi := x(X)$, we obtain, after some straightforward calculus, the systems of equations (3.26) and (3.27) for the cases $\beta < 2$ and $\beta > 2$, respectively, and where the letters A and B denote the constants that appear in the solution (3B.4) in order to simplify notation.

Finally, equation (3.25) is obtained by subtracting the perpetual bubble value (3.10) from the solution (3B.4) for the case with $\beta > 2$.

3C. Proof of Proposition 3.3

This appendix is organized in two parts: (i) for $\beta < 2$ the CEV process will be restated as a (time-changed) squared Bessel's process; and (ii) for $\beta > 2$ the CEV process will be converted into an equivalent square root process.

3C.1. The case with $\beta < 2$

Let us first recall that the CEV process with $\beta < 2$ can hit zero with positive probability, as highlighted, for instance, in Andersen and Andreasen (2000), Davydov and Linetsky (2001), Delbaen and Shirakawa (2002), Dias et al. (2015) and Dias et al. (2020). The goal now is to restate the CEV process $\{P_u, u \geq 0\}$ as a (time-changed) squared Bessel's process of order γ , $\{y_u^{(\gamma)}, u \geq 0\}$, whose first passage time through zero is denoted by $\tau_0^{(\gamma)} := \inf \{u > 0 : y_u^{(\gamma)} = 0\}$.

Following Delbaen and Shirakawa (2002, Theorem 2.2), and assuming that the boundary $y_u^{(\gamma)} = 0$ is absorbing for $\gamma < 2$, we have

$$P_u = e^{(r-q)(u-t)} \left(y_{f(u-t) \wedge \tau_0^{(\gamma)}}^{(\gamma)} \right)^{1-\frac{\gamma}{2}}, \quad (3C.1)$$

with

$$\gamma = \frac{2-2\beta}{2-\beta}, \quad (3C.2)$$

and where the deterministic time change is defined as

$$f(t) := \frac{\delta^2}{2(r-q)(2-\gamma)} \left(1 - e^{-\frac{2(r-q)t}{2-\gamma}} \right) = \begin{cases} \frac{\delta^2(2-\beta)}{4(r-q)} (1 - e^{-(r-q)(2-\beta)t}) & \Leftarrow r \neq q \\ \frac{\delta^2(2-\beta)^2}{4} t & \Leftarrow r = q \end{cases}, \quad (3C.3)$$

which is equivalent to equation (3.43) for $\beta < 2$ (i.e., for $\gamma < 2$).

At time $u = T$, equation (3C.1) becomes

$$P_T = e^{(r-q)\tau} \left(y_{f(\tau) \wedge \tau_0^{(\gamma)}}^{(\gamma)} \right)^{1-\frac{\gamma}{2}}, \quad (3C.4)$$

with $\tau = T - t$ and $1 - \frac{\gamma}{2} = \frac{1}{2-\beta}$.

Denoting by $\tau_0^P := \inf \{t > 0 : P_t = 0\}$ the first passage time through zero of the CEV process, the expected value on the left-hand side of equation (3.44), for $\beta < 2$ and $\phi = -1$,

can be written as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{\tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right], \tag{3C.5}
\end{aligned}$$

where the second equality follows since $\mathbb{1}_{\{P_T < X, \tau_0^P \leq T\}} = \mathbb{1}_{\{\tau_0^P \leq T\}} - \mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}}$ and because $\mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}} = 0$, for $X > 0$, while the last equality arises because $P_T = 0$ if $\tau_0^P \leq T$. Using equation (3C.4), the first term on the right-hand side of equation (3C.5), which should be interpreted as a risk-neutral expectation conditional on no default, can be expressed as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] \\
&= e^{(r-q)\tau + (r-q)\tau(2-\beta)j} \mathbb{E}_{\mathbb{Q}} \left[\left(y_{f(\tau) \wedge \tau_0^{(\gamma)}}^{(\gamma)} \right)^{\frac{1}{2-\beta} + j} \mathbb{1}_{\left\{ e^{(r-q)\tau} \left(y_{f(\tau) \wedge \tau_0^{(\gamma)}}^{(\gamma)} \right)^{\frac{1}{2-\beta}} < X, \tau_0^{(\gamma)} > f(\tau) \right\}} \middle| \mathcal{F}_t \right] \\
&= e^{(r-q)\tau + (r-q)\tau(2-\beta)j} \mathbb{E}_{\mathbb{Q}} \left[\left(y_{f(\tau)}^{(\gamma)} \right)^{\frac{1}{2-\beta} + j} \mathbb{1}_{\{y_{f(\tau)}^{(\gamma)} < e^{-(r-q)\tau(2-\beta)} X^{2-\beta}, \tau_0^{(\gamma)} > f(\tau)\}} \middle| \mathcal{F}_t \right] \\
&= e^{(r-q)\tau + (r-q)\tau(2-\beta)j} \int_0^L y^{\frac{1}{2-\beta} + j} p_{\gamma} \left(y, f(\tau); y_{f(0)}^{(\gamma)} \right) dy, \tag{3C.6}
\end{aligned}$$

where $L \equiv e^{-(r-q)\tau(2-\beta)} X^{2-\beta} = e^{-\alpha\tau} X^{2-\beta}$, with α being defined by equation (3.42), and $p_{\gamma}(y_u, u; y_t)$ is the transition density function of the squared Bessel's process $\{y_u^{(\gamma)}, u \geq t\}$.

The use of the symmetry relationship

$$y_u^{1-\frac{\gamma}{2}} p_{\gamma}(y_u, u; y_t) = y_t^{1-\frac{\gamma}{2}} p_{\gamma}(y_t, u; y_u), \tag{3C.7}$$

allows equation (3C.6) to be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] \\
&= e^{(r-q+\alpha j)\tau} \int_0^L y^{\frac{1}{2-\beta} + j} y^{-1+\frac{\gamma}{2}} \left(y_{f(0)}^{(\gamma)} \right)^{1-\frac{\gamma}{2}} p_{\gamma} \left(y_{f(0)}^{(\gamma)}, f(\tau); y \right) dy \\
&= e^{(r-q+\alpha j)\tau} \left(y_{f(0)}^{(\gamma)} \right)^{1-\frac{\gamma}{2}} \int_0^L y^j p_{\gamma} \left(y_{f(0)}^{(\gamma)}, f(\tau); y \right) dy. \tag{3C.8}
\end{aligned}$$

It is well known that the transition density function of the squared Bessel's process of order γ is related to the density of a noncentral χ^2 law as

$$p_\gamma(y_u, u; y_t) = \frac{1}{u} f_{\chi^2(4-\gamma, \frac{y_u}{u})} \left(\frac{y_t}{u} \right), \quad (3C.9)$$

where $f_{\chi^2(v, \lambda)}(\cdot)$ denotes the density function of a noncentral χ^2 law with v degrees of freedom and noncentrality parameter λ . Therefore, and since $4 - \gamma = 2 + 2\nu$, with ν being defined by equation (3.9), equation (3C.8) can be rewritten as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] \\ &= e^{(r-q+\alpha j)\tau} \left(y_{f(0)}^{(\gamma)} \right)^{\frac{1}{2-\beta}} \int_0^L y^j \frac{1}{f(\tau)} f_{\chi^2\left(2+2\nu, \frac{y_{f(0)}^{(\gamma)}}{f(\tau)}\right)} \left(\frac{y}{f(\tau)} \right) dy \\ &= e^{(r-q+\alpha j)\tau} \left(y_{f(0)}^{(\gamma)} \right)^{\frac{1}{2-\beta}} \int_0^{\frac{L}{f(\tau)}} (f(\tau)z)^j \frac{1}{f(\tau)} f_{\chi^2\left(2+2\nu, \frac{y_{f(0)}^{(\gamma)}}{f(\tau)}\right)}(z) f(\tau) dz \\ &= e^{(r-q+\alpha j)\tau} \left(P_t^{2-\beta} \right)^{\frac{1}{2-\beta}} (f(\tau))^j \int_0^{\frac{L}{f(\tau)}} z^j f_{\chi^2\left(2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right)}(z) dz \\ &= e^{(r-q+\alpha j)\tau} P_t (f(\tau))^j \Phi_{-1} \left(j, \frac{L}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (3C.10)$$

where the last line follows from Carr and Linetsky (2006, Lemma 5.1) and $\Phi_{-1}(p, x; v, \lambda)$ is the p -th lower tail truncated moment (for an upper integration limit x) of a noncentral χ^2 random variable with v degrees of freedom and noncentrality parameter λ , as defined in Carr and Linetsky (2006, equation 5.12) and Dias and Nunes (2018, equation 21).

Finally, combining equations (3C.5) and (3C.10) yields equation (3.44) for the case with $\beta < 2$ and $\phi = -1$.

The expected value on the left-hand side of equation (3.44), but now for $\beta < 2$ and $\phi = +1$, can be expressed as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right], \end{aligned} \quad (3C.11)$$

because $\mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}} = 0$, for $X > 0$. Following the same steps to the ones used above, equation (3.44) for the case with $\beta < 2$ and $\phi = +1$ arises immediately, but now involving the

use of p -th upper tail truncated moment $\Phi_{+1}(p, x; v, \lambda)$ for a lower integration limit x and of a noncentral χ^2 random variable with v degrees of freedom and noncentrality parameter λ , as defined in Carr and Linetsky (2006, equation 5.11) and Dias and Nunes (2018, equation 24). Additional details of the full derivation are available upon request.

Tackling now the expected value on the left-hand side of equation (3.45), for $\beta < 2$ and $\phi = -1$, we note that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{\tau_0^P \leq T\}} \middle| \mathcal{F}_t \right], \end{aligned} \quad (3C.12)$$

because $\mathbb{1}_{\{P_T < X, \tau_0^P \leq T\}} = \mathbb{1}_{\{\tau_0^P \leq T\}} - \mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}}$ and $\mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}} = 0$, for $X > 0$.

Following a similar procedure to the one used for deriving equation (3.44), it is straightforward to show that the first term on the right-hand side of equation (3C.12) is equal to

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] \\ &= e^{\alpha j \tau} P_t(f(\tau))^{j-\nu} \Phi_{-1} \left(j - \nu, \frac{L}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (3C.13)$$

that should be interpreted as the conditional on no default component.

The second expectation on the right-hand side of equation (3C.12) needs to be solved carefully due to the value of j . For $j \in \mathbb{N}$ it is equal to zero because the asset price would be stuck forever at zero due to the default event. However, for $j = 0$ the second term on the right-hand side of equation (3C.12) is interpreted as the risk-neutral probability of default given by, for instance, Cox (1975) or Carr and Linetsky (2006, page 321):

$$\mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{\tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] \mathbb{1}_{\{j=0\}} = \mathbb{E}_{\mathbb{Q}} \left[\mathbb{1}_{\{\tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] = \frac{\Gamma \left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)} \right)}{\Gamma(\nu)}. \quad (3C.14)$$

Finally, combining equations (3C.12), (3C.13) and (3C.14), equation (3.45) for the case with $\beta < 2$ and $\phi = -1$ arises immediately.

The expected value on the left-hand side of equation (3.45), but now for $\beta < 2$ and $\phi = +1$, is obtained using a similar approach. Note that now only the conditional on no

default component matters since

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right] + \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X, \tau_0^P > T\}} \middle| \mathcal{F}_t \right], \tag{3C.15}
\end{aligned}$$

because $\mathbb{1}_{\{P_T \geq X, \tau_0^P \leq T\}} = 0$, for $X > 0$. Additional details of the full derivation are available upon request.

3C.2. The case with $\beta > 2$

Even though $\tau_0^P < \infty$ for $\beta < 2$, the probability that the CEV process hits zero is null for $\beta > 2$. Moreover, the CEV process will now be converted into an equivalent square root process through the transformation $y_u = P_u^{2-\beta}$. Applying Itô's lemma, it is possible to show that the process $\{y_u, u \geq t\}$ satisfies the “square-root” stochastic differential equation

$$dy_t = \alpha (\theta - y_t) dt + \sigma \sqrt{y_t} dW_t^{\mathbb{Q}}, \tag{3C.16}$$

with α being defined by equation (3.42) and where

$$\theta := \frac{\frac{\delta^2}{2} (2 - \beta) (1 - \beta)}{\alpha} \tag{3C.17}$$

and

$$\sigma := (2 - \beta) \delta. \tag{3C.18}$$

Furthermore, it is well known that, conditional on the sigma-algebra $\mathcal{F}_t$, the scaled random variable $\frac{y_T}{f(\tau)}$ possesses a noncentral χ^2 distribution with $\frac{4\alpha\theta}{\sigma^2}$ degrees of freedom and a noncentrality parameter equal to $\frac{4\alpha y_t}{\sigma^2(e^{\alpha\tau} - 1)}$, where

$$f(\tau) := \frac{\sigma^2 (1 - e^{-\alpha\tau})}{4\alpha} = \begin{cases} \frac{\delta^2(\beta-2)}{4(r-q)} (1 - e^{-(r-q)(\beta-2)\tau}) & \Leftarrow r \neq q \\ \frac{\delta^2(\beta-2)^2}{4} \tau & \Leftarrow r = q \end{cases} \tag{3C.19}$$

is equivalent to equation (3.43) for $\beta > 2$. Therefore, Carr and Linetsky (2006, Lemma 5.1) implies that

$$\mathbb{E}_{\mathbb{Q}} \left[\left(\frac{y_T}{f(\tau)} \right)^j \mathbb{1}_{\left\{ \frac{y_T}{f(\tau)} < \frac{X^{2-\beta}}{f(\tau)} \right\}} \middle| \mathcal{F}_t \right] = \Phi_{-1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; \frac{4\alpha\theta}{\sigma^2}, \frac{4\alpha y_t}{\sigma^2 [e^{\alpha\tau} - 1]} \right) \tag{3C.20}$$

and

$$\mathbb{E}_{\mathbb{Q}} \left[\left(\frac{y_T}{f(\tau)} \right)^j \mathbb{1}_{\left\{ \frac{y_T}{f(\tau)} > \frac{X^{2-\beta}}{f(\tau)} \right\}} \middle| \mathcal{F}_t \right] = \Phi_{+1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; \frac{4\alpha\theta}{\sigma^2}, \frac{4\alpha y_t}{\sigma^2 [e^{\alpha\tau} - 1]} \right). \quad (3C.21)$$

Using equation (3C.21), equation (3.44), for $\beta > 2$ and $\phi = -1$, arises because

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] &= \mathbb{E}_{\mathbb{Q}} \left[\left(y_T^{\frac{1}{2-\beta}} \right)^{1+(2-\beta)j} \mathbb{1}_{\left\{ y_T^{\frac{1}{2-\beta}} < X \right\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[y_T^{\frac{1}{2-\beta}+j} \mathbb{1}_{\{y_T > X^{2-\beta}\}} \middle| \mathcal{F}_t \right] \\ &= (f(\tau))^{j+\frac{1}{2-\beta}} \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{y_T}{f(\tau)} \right)^{j+\frac{1}{2-\beta}} \mathbb{1}_{\left\{ \frac{y_T}{f(\tau)} > \frac{X^{2-\beta}}{f(\tau)} \right\}} \middle| \mathcal{F}_t \right] \\ &= (f(\tau))^{j-\nu} \Phi_{+1} \left(j-\nu, \frac{X^{2-\beta}}{f(\tau)}; \frac{4\alpha\theta}{\sigma^2}, \frac{4\alpha y_t}{\sigma^2 [e^{\alpha\tau} - 1]} \right), \end{aligned} \quad (3C.22)$$

and since $\frac{4\alpha\theta}{\sigma^2} = 2 + 2\nu$ and $\frac{4\alpha y_t}{\sigma^2 [e^{\alpha\tau} - 1]} = \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}$.

The expected value on the left-hand side of equation (3.44), for $\beta > 2$ and $\phi = +1$, as well as the risk-neutral expectation on the left-hand side of equation (3.45), for the case $\beta > 2$ and $\phi = -1$ and for the case $\beta > 2$ and $\phi = +1$, are obtained using a similar approach. Additional details of the full derivation are available upon request.

3D. Proof of Proposition 3.5

This appendix is organized in two parts: (i) derivation of the risk-neutral expectations contained in equations (3.36) and (3.40) involving the modified Bessel's functions of first kind with order ν ; and (ii) derivation of the risk-neutral expectations contained in equations (3.36) and (3.40) involving the modified Bessel's functions of second kind with order ν . Each part will be divided in two subparts to accommodate the cases $\beta < 2$ and $\beta > 2$.

3D.1. The case involving the modified Bessel's functions of first kind

The case with $\beta < 2$

Starting with the case conditional on $P_T < X$, recalling that z_T is defined through equation (3.15) and using the solution of the modified Bessel's function of first kind with order ν defined by

$$I_\nu(z) := \sum_{j=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2j+\nu}}{j!\Gamma(j+1+\nu)}, \quad (3D.1)$$

as shown in (Olver et al., 2010, equation 10.25.2), then

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} I_\nu(z_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} P_T^{1-\frac{\beta}{2}} \right)^{2j+\nu}}{j!\Gamma(j+1+\nu)} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j!\Gamma(j+1+\nu)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right], \end{aligned} \quad (3D.2)$$

since

$$P_T^{\frac{1}{2}} \left(P_T^{1-\frac{\beta}{2}} \right)^{2j+\nu} = P_T^{1+(2-\beta)j}, \quad (3D.3)$$

for $\beta < 2$.

Using equation (3.44) with $\phi = -1$, $\beta < 2$, $r = q$ and $\alpha = 0$, equation (3D.2) can be rewritten as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} I_\nu(z_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= P_t \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j!\Gamma(j+1+\nu)} (f(\tau))^j \Phi_{-1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (3D.4)$$

which results in equation (3.48).

Even though the case conditional on $P_T \geq X$ with $\beta < 2$ is not required in equations (3.36) and (3.40), we notice that the corresponding expected value can be obtained similarly via equation (3.48), though being dependent on the upper tail truncated moment.

The case with $\beta > 2$

Moving now to the case conditional on $P_T \geq X$,

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} I_{\nu}(z_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} P_T^{1-\frac{\beta}{2}} \right)^{2j+\nu}}{j! \Gamma(j+1+\nu)} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+1+\nu)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right], \quad (3D.5)
\end{aligned}$$

since

$$P_T^{\frac{1}{2}} \left(P_T^{1-\frac{\beta}{2}} \right)^{2j+\nu} = P_T^{(2-\beta)j}, \quad (3D.6)$$

for $\beta > 2$.

Using equation (3.45) with $\phi = 1$, $\beta > 2$, $r = q$ and $\alpha = 0$, equation (3D.5) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} I_{\nu}(z_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+1+\nu)} (f(\tau))^j \Phi_{-1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \quad (3D.7)
\end{aligned}$$

thus resulting, again, in equation (3.48), because $\varphi = \mathbb{1}_{\{\beta < 2\}}$ is equal to zero for $\beta > 2$.

Even though the case conditional on $P_T < X$ with $\beta > 2$ is not required in equations (3.36) and (3.40), we notice that the corresponding expected value can be obtained similarly via equation (3.48), though being dependent on the upper tail truncated moment.

3D.2. The case involving the modified Bessel's functions of second kind

The case with $\beta < 2$

Starting with the case conditional on $P_T \geq X$ and using the solution of the modified Bessel's function of second kind with order ν defined by

$$K_{\nu}(z) := \left(\frac{\pi}{2} \right) \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin(\nu\pi)}, \quad (3D.8)$$

as shown in Olver et al. (2010, equation 10.27.4), as well as equations (3.15) and (3D.1), it follows that

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} K_{\nu}(z_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\pi}{2 \sin(\nu\pi)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} P_T^{1-\frac{\beta}{2}} \right)^{2j-\nu}}{j! \Gamma(j+1-\nu)} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} P_T^{1-\frac{\beta}{2}} \right)^{2j+\nu}}{j! \Gamma(j+1+\nu)} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j-\nu} \frac{1}{j! \Gamma(j+1-\nu)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+1+\nu)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right],
\end{aligned} \tag{3D.9}$$

since

$$P_T^{\frac{1}{2}} \left(P_T^{1-\frac{\beta}{2}} \right)^{2j-\nu} = P_T^{(2-\beta)j}, \tag{3D.10}$$

for $\beta < 2$, and using equation (3D.3).

Using equations (3.44) and (3.45) with $\phi = 1$, $\beta < 2$, $r = q$ and $\alpha = 0$, equation (3D.9) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} K_{\nu}(z_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\pi}{2 \sin(\nu\pi)} P_t \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j-\nu} \frac{1}{j! \Gamma(j+1-\nu)} (f(\tau))^{j-\nu} \Phi_{+1} \left(j - \nu, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} P_t \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+1+\nu)} (f(\tau))^j \Phi_{+1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right),
\end{aligned} \tag{3D.11}$$

which results in equation (3.49).

Even though the case conditional on $P_T < X$ with $\beta < 2$ is not required in equations (3.36) and (3.40), we notice that the corresponding expected value can be obtained similarly via equation (3.49), though being dependent on the lower tail truncated moment. In this case, an extra component must be included to tackle the possibility of default, as shown in the second line of equation (3.49), but that it is used only for $j = 0$ and $\phi = -1$.

The case with $\beta > 2$

Moving now to the case conditional on $P_T < X$, while using equations (3.15), (3D.1) and (3D.8), then

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} K_{\nu}(z_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\pi}{2 \sin(\nu\pi)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} P_T^{1-\frac{\beta}{2}} \right)^{2j-\nu}}{j! \Gamma(j+1-\nu)} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} P_T^{1-\frac{\beta}{2}} \right)^{2j+\nu}}{j! \Gamma(j+1+\nu)} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j-\nu} \frac{1}{j! \Gamma(j+1-\nu)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+1+\nu)} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right], \tag{3D.12}
\end{aligned}$$

since

$$P_T^{\frac{1}{2}} \left(P_T^{1-\frac{\beta}{2}} \right)^{2j-\nu} = P_T^{1+(2-\beta)j}, \tag{3D.13}$$

for $\beta > 2$, and applying, once more, identity (3D.6).

Using equations (3.44) and (3.45) with $\phi = -1$, $\beta > 2$, $r = q$ and $\alpha = 0$, equation (3D.12) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[P_T^{\frac{1}{2}} K_{\nu}(z_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \tag{3D.14} \\
&= \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j-\nu} \frac{1}{j! \Gamma(j+1-\nu)} (f(\tau))^{j-\nu} \Phi_{+1} \left(j - \nu, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu} \frac{1}{j! \Gamma(j+1+\nu)} (f(\tau))^j \Phi_{+1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right),
\end{aligned}$$

thus resulting, again, in equation (3.49), because $\varphi = \mathbb{1}_{\{\beta < 2\}}$ is equal to zero for $\beta > 2$.

Even though the case conditional on $P_T \geq X$ with $\beta > 2$ is not required in equations (3.36) and (3.40), we notice that the corresponding expected value can be obtained similarly via equation (3.49), though being dependent on the lower tail truncated moment.

3E. Proof of Proposition 3.6

This appendix is organized in two parts: (i) derivation of the risk-neutral expectations contained in equations (3.37) and (3.41) involving the Kummer's confluent hypergeometric functions $M(a, b, x)$; and (ii) derivation of the risk-neutral expectations contained in equations (3.37) and (3.41) involving the Tricomi's confluent hypergeometric functions $U(a, b, x)$. Each part will be divided in two subparts to accommodate the cases $\beta < 2$ and $\beta > 2$.

3E.1. The cases involving the Kummer's confluent hypergeometric functions

The case with $\beta < 2$

Starting with the Kummer's function associated to the case conditional on $P_T < X$ in the presence of downward-sloping volatility skews—i.e., for $\beta < 2$ —, recalling definition (3.18), since $e^{(\epsilon-1)\frac{x}{2}} = \mathbb{1}_{\{r < q, \beta < 2\}} + e^{-x} \mathbb{1}_{\{r > q, \beta < 2\}}$ and using the solution of the Kummer's confluent hypergeometric function

$$M(a, b, x) = \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{(x)^j}{j!}, \quad (3E.1)$$

as defined in Olver et al. (2010, equation 13.2.2), then, for $r < q$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right]. \end{aligned} \quad (3E.2)$$

Using equation (3.44) with $\phi = -1$ and $\beta < 2$, equation (3E.2) can be rewritten as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} e^{(r-q+\alpha j)\tau} P_t (f(\tau))^j \Phi_{-1} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \end{aligned} \quad (3E.3)$$

which results in equation (3.50), since combining equations (3.42) and (3.43) it follows that

$$\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j e^{\alpha j \tau} (f(\tau))^j = \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j, \quad (3E.4)$$

for $r < q$ and $\beta < 2$.

For the case with $r > q$, it is necessary to use not only the definition (3E.1) of the Kummer's

function but also the Kummer's transformation

$$M(a, b, x) = e^x M(b - a, b, -x), \quad (3E.5)$$

as defined in Olver et al. (2010, equation 13.2.39). Hence,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T M(b - a, b, -x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[P_T \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} \frac{\left(-\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right]. \end{aligned} \quad (3E.6)$$

Using equation (3.44) with $\phi = -1$ and $\beta < 2$, equation (3E.6) can be rewritten as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T M(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} e^{(r-q+\alpha j)\tau} P_t(f(\tau))^j \Phi_{-1} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (3E.7)$$

which results in equation (3.50), since combining equations (3.42) and (3.43) it follows that

$$(-1)^j \left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j e^{\alpha j \tau} (f(\tau))^j = \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j, \quad (3E.8)$$

for $r > q$ and $\beta < 2$.

Even though the case conditional on $P_T \geq X$ with $\beta < 2$ is not required in equations (3.37) and (3.41), we notice that the corresponding expected values can be obtained similarly via equation (3.50), though being dependent on the upper tail truncated moment.

The case with $\beta > 2$

Moving now to the other Kummer's function that is associated to the case conditional on $P_T \geq X$ in the presence of upward-sloping volatility skews—i.e., for $\beta > 2$ —, but now with $e^{(\epsilon-1)\frac{x}{2}} = e^{-x} \mathbb{1}_{\{r < q, \beta > 2\}} + \mathbb{1}_{\{r > q, \beta > 2\}}$, then, for $r < q$, and using, again, identities (3.18), (3E.1) and (3E.5),

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[M(b-a, b, -x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} \frac{\left(-\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right]. \tag{3E.9}
\end{aligned}$$

Using equation (3.45) with $\phi = 1$ and $\beta > 2$, equation (3E.9) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} (f(\tau))^j \Phi_{-1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right), \tag{3E.10}
\end{aligned}$$

which results in equation (3.52), since combining equations (3.42) and (3.43) it follows that

$$(-1)^j \left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j (f(\tau))^j = \left(\frac{1-e^{-\alpha\tau}}{2} \right)^j, \tag{3E.11}$$

for $r < q$ and $\beta > 2$.

For $r > q$, definitions (3.18) and (3E.1) implies tat

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right]. \tag{3E.12}
\end{aligned}$$

Using equation (3.45) with $\phi = 1$ and $\beta > 2$, equation (3E.12) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} (f(\tau))^j \Phi_{-1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right), \tag{3E.13}
\end{aligned}$$

which results in equation (3.52), since combining equations (3.42) and (3.43) it follows that

$$\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j (f(\tau))^j = \left(\frac{1-e^{-\alpha\tau}}{2} \right)^j, \quad (3E.14)$$

for $r > q$ and $\beta > 2$.

Even though the case conditional on $P_T < X$ with $\beta > 2$ is not required in equations (3.37) and (3.41), we notice that the corresponding expected values can be obtained similarly via equation (3.52), though being dependent on the upper tail truncated moment.

3E.2. The cases involving the Tricomi's confluent hypergeometric functions

The case with $\beta < 2$

Starting with the Tricomi's function associated to the case conditional on $P_T \geq X$ in the presence of downward-sloping volatility skews—i.e., for $\beta < 2$ —, recalling again that $e^{(\epsilon-1)\frac{x_T}{2}} = \mathbb{1}_{\{r < q, \beta < 2\}} + e^{-x} \mathbb{1}_{\{r > q, \beta < 2\}}$ and using the Kummer's transformation

$$U(a, b, x) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a, b, x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} M(1+a-b, 2-b, x), \quad (3E.15)$$

as defined in Olver et al. (2010, equation 13.2.42), as well as definitions (3.18) and (3E.1), then, for $r < q$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[P_T M(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ & \quad + \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[P_T x_T^{1-b} M(1+a-b, 2-b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[P_T \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ & \quad + \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[P_T \left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^{1-b} \sum_{j=0}^{\infty} \frac{(1+a-b)_j}{(2-b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ & \quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1+a-b)_j}{(2-b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j+1-b}}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right], \end{aligned} \quad (3E.16)$$

since $1 - b = -\nu$ and

$$P_T \left(P_T^{2-\beta} \right)^{1-b} \left(P_T^{2-\beta} \right)^j = P_T^{(2-\beta)j}, \quad (3E.17)$$

for $\beta < 2$.

Using equations (3.44) and (3.45) with $\phi = 1$ and $\beta < 2$, equation (3E.16) can be rewritten as

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} e^{(r-q+\alpha j)\tau} P_t (f(\tau))^j \Phi_{+1} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ &+ \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1+a-b)_j}{(2-b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j+1-b}}{j!} \\ &\times e^{\alpha j\tau} P_t (f(\tau))^{j-\nu} \Phi_{+1} \left(j-\nu, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right). \end{aligned} \quad (3E.18)$$

Finally, using equation (3E.4), combining equations (3.42) and (3.43), noting that $1 - b = -\nu$ and because

$$\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j-\nu} e^{\alpha j\tau} (f(\tau))^{j-\nu} = \left(\frac{1-e^{\alpha\tau}}{2} \right)^{j-\nu} e^{(r-q)\tau}, \quad (3E.19)$$

for $r < q$ and $\beta < 2$, then equation (3.54) is immediately obtained.

For $r > q$, and using again identities (3.18), (3E.1), (3E.5), (3E.15) and (3E.17),

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[P_T M(b-a, b, -x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &+ \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[P_T x_T^{1-b} M(1-a, 2-b, -x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[P_T \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} \frac{\left(-\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\ &+ \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[P_T \left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^{1-b} \sum_{j=0}^{\infty} \frac{(1-a)_j}{(2-b)_j} \frac{\left(-\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|}\right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1-a)_j}{(2-b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|}\right)^{j+1-b}}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right].
\end{aligned} \tag{3E.20}$$

Using equations (3.44) and (3.45) with $\phi = 1$ and $\beta < 2$, equation (3E.20) can be rewritten as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} P_T U(a, b, x_T) \mathbb{1}_{\{P_T \geq X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|}\right)^j}{j!} \\
&\quad \times e^{(r-q+\alpha j)\tau} P_t(f(\tau))^j \Phi_{+1} \left(j, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1-a)_j}{(2-b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|}\right)^{j+1-b}}{j!} \\
&\quad \times e^{\alpha j \tau} P_t(f(\tau))^{j-\nu} \Phi_{+1} \left(j-\nu, \frac{e^{-\alpha\tau} X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right).
\end{aligned} \tag{3E.21}$$

Finally, using equation (3E.8), combining equations (3.42) and (3.43) and because

$$(-1)^j \left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j-\nu} e^{\alpha j \tau} (f(\tau))^{j-\nu} = (-1)^j \left(-\frac{1-e^{\alpha\tau}}{2} \right)^{j-\nu} e^{(r-q)\tau}, \tag{3E.22}$$

for $r > q$ and $\beta < 2$, then equation (3.54) is immediately obtained.

Even though the case conditional on $P_T < X$ with $\beta < 2$ is not required in equations (3.37) and (3.41), we notice that the corresponding expected values can be obtained similarly via equation (3.54), though being dependent on the lower tail truncated moment. In this case, an extra component must be included to tackle the possibility of default, as shown in the third line of equation (3.54), but that is used only for $j = 0$ and $\phi = -1$.

The case with $\beta > 2$

Moving now to the other Tricomi's function that is associated to the case conditional on $P_T < X$ in the presence of upward-sloping volatility skews—i.e., for $\beta > 2$ —, but now with $e^{(\epsilon-1)\frac{x}{2}} = e^{-x} \mathbb{1}_{\{r < q, \beta > 2\}} + \mathbb{1}_{\{r > q, \beta > 2\}}$, then, for $r < q$, and using again identities (3.18),

(3E.1), (3E.5) and (3E.15),

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[M(b-a, b, -x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[x_T^{1-b} M(1-a, 2-b, -x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} \frac{\left(-\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^{1-b} \sum_{j=0}^{\infty} \frac{(1-a)_j}{(2-b)_j} \frac{\left(-\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1-a)_j}{(2-b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j+1-b}}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right], \quad (3E.23)
\end{aligned}$$

since $1-b = -\nu$ and

$$\left(P_T^{2-\beta} \right)^{1-b} \left(P_T^{2-\beta} \right)^j = P_T^{1+(2-\beta)j}, \quad (3E.24)$$

for $\beta > 2$.

Using equations (3.44) and (3.45) with $\phi = -1$ and $\beta > 2$, equation (3E.23) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \quad (3E.25) \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(b-a)_j}{(b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} (f(\tau))^j \Phi_{+1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1-a)_j}{(2-b)_j} (-1)^j \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j-\nu}}{j!} (f(\tau))^{j-\nu} \Phi_{+1} \left(j-\nu, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right).
\end{aligned}$$

Finally, using equation (3E.11), combining equations (3.42) and (3.43) and because

$$(-1)^j \left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j-\nu} (f(\tau))^{j-\nu} = (-1)^j \left(-\frac{1-e^{-\alpha\tau}}{2} \right)^{j-\nu}, \quad (3E.26)$$

for $r < q$ and $\beta > 2$, then equation (3.55) is immediately obtained.

For $r > q$, and using again identities (3.18), (3E.1), (3E.15) and (3E.24),

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[M(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[x_T^{1-b} M(1+a-b, 2-b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \mathbb{E}_{\mathbb{Q}} \left[\sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^{1-b} \sum_{j=0}^{\infty} \frac{(1+a-b)_j}{(2-b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} P_T^{2-\beta} \right)^j}{j!} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1+a-b)_j}{(2-b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j+1-b}}{j!} \mathbb{E}_{\mathbb{Q}} \left[P_T^{1+(2-\beta)j} \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right]. \quad (3E.27)
\end{aligned}$$

Using equations (3.44) and (3.45) with $\phi = -1$ and $\beta > 2$, equation (3E.27) can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{\mathbb{Q}} \left[e^{(\epsilon-1)\frac{x_T}{2}} U(a, b, x_T) \mathbb{1}_{\{P_T < X\}} \middle| \mathcal{F}_t \right] \quad (3E.28) \\
&= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \sum_{j=0}^{\infty} \frac{(a)_j}{(b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^j}{j!} (f(\tau))^j \Phi_{+1} \left(j, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad + \frac{\Gamma(b-1)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(1+a-b)_j}{(2-b)_j} \frac{\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j-\nu}}{j!} (f(\tau))^{j-\nu} \Phi_{+1} \left(j-\nu, \frac{X^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right).
\end{aligned}$$

Finally, using equation (3E.14), combining equations (3.42) and (3.43) and because

$$\left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{j-\nu} (f(\tau))^{j-\nu} = \left(\frac{1-e^{-\alpha\tau}}{2} \right)^{j-\nu}, \quad (3E.29)$$

for $r > q$ and $\beta > 2$, then equation (3.55) is immediately obtained.

Even though the case conditional on $P_T \geq X$ with $\beta > 2$ is not required in equations (3.37) and (3.41), we notice that the corresponding expected values can be obtained similarly via equation (3.55), though being dependent on the lower tail truncated moment.

3F. Proof of Proposition 3.7

Using equation (3.10), the time- t expected value of the perpetual bubble can be expressed as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[G_T^{\infty}(P_T, u) | \mathcal{F}_t] &= \mathbb{E}_{\mathbb{Q}}\left[\int_T^{\infty} G_T(P_T, u) du \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\int_T^{\infty} P_T e^{-q(u-T)} \frac{\Gamma(\nu, w_T)}{\Gamma(\nu)} du \middle| \mathcal{F}_t\right] \\ &= \frac{1}{\Gamma(\nu)} \int_T^{\infty} e^{-q(u-T)} \mathbb{E}_{\mathbb{Q}}\left[P_T \int_{w_T}^{\infty} e^{-z} z^{\nu-1} dz \middle| \mathcal{F}_t\right] du, \quad (3F.1)\end{aligned}$$

where the second line follows from equation (3.5), whereas the third line follows from Fubini's theorem and Olver et al. (2010, equation 8.2.2).

Making the change of variables $s = \frac{z}{w_T}$, the risk-neutral expectation contained on the right-hand side of equation (3F.1) can be rewritten as

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[P_T \int_{w_T}^{\infty} e^{-z} z^{\nu-1} dz \middle| \mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[P_T \int_1^{\infty} e^{-w_T s} (w_T s)^{\nu-1} w_T ds \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[P_T w_T^{\nu} \int_1^{\infty} e^{-w_T s} s^{\nu-1} ds \middle| \mathcal{F}_t\right]. \quad (3F.2)\end{aligned}$$

Using equations (3.6) and (3.9), it follows that

$$\begin{aligned}P_T w_T^{\nu} &= P_T \left[\kappa_{T,u} P_T^{2-\beta} e^{(r-q)(2-\beta)(u-T)}\right]^{\nu} \\ &= \kappa_{T,u}^{\nu} e^{-(r-q)(u-T)}, \quad (3F.3)\end{aligned}$$

and, therefore, equation (3F.2) becomes

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[P_T \int_{w_T}^{\infty} e^{-z} z^{\nu-1} dz \middle| \mathcal{F}_t\right] &= \kappa_{T,u}^{\nu} e^{-(r-q)(u-T)} \mathbb{E}_{\mathbb{Q}}\left[\int_1^{\infty} e^{-w_T s} s^{\nu-1} ds \middle| \mathcal{F}_t\right] \\ &= \kappa_{T,u}^{\nu} e^{-(r-q)(u-T)} \int_1^{\infty} \mathbb{E}_{\mathbb{Q}}[e^{-w_T s} | \mathcal{F}_t] s^{\nu-1} ds, \quad (3F.4)\end{aligned}$$

where the second line follows from Fubini's theorem.

Since

$$-w_T s = -P_T^{2-\beta} \kappa_{T,u} e^{(r-q)(2-\beta)(u-T)} s \quad (3F.5)$$

and $\{y_t, t \geq 0\} \equiv \{P_t^{2-\beta}, t \geq 0\}$ follows the square-root process (3C.16), then the risk-neutral expectation $\mathbb{E}_{\mathbb{Q}}[e^{-w_T s} | \mathcal{F}_t]$ can be understood as the Laplace's transform of the square-root

process $P_T^{2-\beta} \equiv y_T$ with parameter $\kappa_{T,u} e^{(r-q)(2-\beta)(u-T)} s \equiv \lambda$ that is equal to

$$\mathbb{E}_{\mathbb{Q}} [\exp(-\lambda y_T) | \mathcal{F}_t] = \frac{\exp\left(-\frac{\lambda \tilde{L} \tilde{f}}{1+2\lambda \tilde{L}}\right)}{\left(1+2\lambda \tilde{L}\right)^{\frac{2\alpha\theta}{\sigma^2}}}, \quad (3F.6)$$

where

$$\tilde{L} := \frac{\sigma^2 (1 - e^{-\alpha\tau})}{4\alpha} = f(\tau), \quad (3F.7)$$

and

$$\tilde{f} := \frac{4\alpha y_t}{\sigma^2 [e^{\alpha\tau} - 1]} = \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}, \quad (3F.8)$$

with $\tau = T - t$.

Combining equations (3F.4) and (3F.6), then

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[P_T \int_{w_T}^{\infty} e^{-z} z^{\nu-1} dz \middle| \mathcal{F}_t \right] \\ &= \kappa_{T,u}^{\nu} e^{-(r-q)(u-T)} \int_1^{\infty} \frac{\exp\left(-\frac{\kappa_{T,u} e^{-\alpha(u-T)} s f(\tau) \tilde{f}}{1+2\kappa_{T,u} e^{-\alpha(u-T)} s f(\tau)}\right)}{(1+2\kappa_{T,u} e^{-\alpha(u-T)} s f(\tau))^{\frac{2\alpha\theta}{\sigma^2}}} s^{\nu-1} ds. \end{aligned} \quad (3F.9)$$

Finally, combining equations (3F.1) and (3F.9), noting that $\frac{2\alpha\theta}{\sigma^2} = 1 + \nu$ and because

$$\kappa_{T,u} e^{-\alpha(u-T)} f(\tau) = \frac{1 - e^{-\alpha(T-t)}}{2(e^{\alpha(u-T)} - 1)}, \quad (3F.10)$$

then equation (3.58) arises immediately.

4. Optimal investment decisions with minimum price guarantees under the constant elasticity of variance process

4.1. Introduction

This paper proposes an analytical representation for analyzing optimal investment decisions associated to a feed-in tariff (henceforth, FIT) contract with a minimum price guarantee (i.e., a price-floor regime) under the constant elasticity of variance (hereafter, CEV) model, that is known to nest the limiting geometric Brownian motion (henceforward, GBM) process as a special case. Our analytic solutions are suitable to design FIT contractual schemes with both perpetual and finite-lived guarantees. Even though the project value of a FIT contract under the CEV process arises straightforwardly from the cap and floor solutions offered recently by Dias et al. (2023), the analytic formulae proposed in the present paper for studying investment decisions with minimum price guarantees constitutes a novel real options framework that should be useful for investors and policymakers in a wide variety of management decisions involving renewable energy projects.

The use of such support schemes has been considered as an important tool to design incentive policies related with renewable energy projects that might be relevant for mitigating many environmental problems and, hence, an abundant amount of research has emerged in the real options literature related with this topic, e.g., Shackleton and Wojakowski (2007), Couture and Gagnon (2010), Ritzenhofen et al. (2016), Ritzenhofen and Spinler (2016), Barbosa et al. (2018), Pineda et al. (2018), Adkins and Paxson (2019), Adkins et al. (2019), Barbosa et al. (2020), Barbosa et al. (2022), Paxson et al. (2022) and Li et al. (2023), just to name a few.

Building upon the work of Dias et al. (2023) and similarly to the case analyzed in Barbosa et al. (2018) under the GBM assumption, our real options model under the more general CEV process includes managerial flexibilities and calculates the optimal trigger level that induces a firm to invest in a FIT project with a minimum price guarantee. Within this policy, the producer receives a fixed amount if the market price is below the price floor or the market

price, otherwise. Although the use of a simplistic real options approach based on a GBM diffusion is easier to be tackled analytically and numerically, it is well known that the CEV diffusion is better able to accommodate several stylized-facts that are commonly reported in the literature (including in a real options context), namely the so-called leverage and volatility smile effects.¹

Our analytical formulae and the corresponding numerical analysis provide several interesting results that should be relevant for academics and researchers when studying such incentive policies. In particular, we show that the argument of Barbosa et al. (2018) that a perpetual guarantee only induces investment for prices below the price floor when offering a risk-free investment opportunity is still valid under the CEV process. This indicates that a perpetual guarantee is not economically sound even when considering a non-constant volatility setup such as the one considered in a CEV local volatility model. We also demonstrate that the optimal price-floor level L^* triggering immediate investment in the presence of a perpetual guarantee is independent of the elasticity parameter of the CEV model, which implies that it is equal to the one derived by Barbosa et al. (2018) under the GBM case. Nevertheless, for optimal threshold levels $\bar{P}$ above the optimal price-floor level L^* we observe that the leverage effect produces significant differences amongst models with different elasticity parameters.

By contrast, we show that such independence of L^* is not valid any more in the case of FIT contracts with a finite maturity guarantee. Using the European onshore wind farm with 25 wind turbines considered in Barbosa et al. (2018), we conclude that an annual difference higher than 1 million Euros of supporting rents (per farm) is required if the regulator negotiated the contract based on a GBM assumption when the price process was better described by an alternative CEV process. Hence, care must be taken when a policymaker aims to design a given instrument to induce investment decisions with FIT contracts because the differences between trigger points under alternative modeling assumptions are quite significant and the excessive rents are usually paid at the expense of tax payers.

The remainder of the manuscript is organized as follows. Section 4.2 briefly summarizes the CEV model setup adopted in this paper. Section 4.3 presents the analytical formulae for evaluating optimal investment decisions associated to FIT contracts with a minimum price guarantee under the CEV model and considering both the cases with perpetual and finite maturity guarantees. Section 4.4 provides some numerical examples and Section 4.5 presents some concluding remarks.

¹For additional theoretical background on the CEV process and applications in financial and real options contexts see, for instance, Cox (1975), Davydov and Linetsky (2001), Davydov and Linetsky (2003), Geman and Shih (2009), Nunes (2009), Dias and Nunes (2011), Larguinho et al. (2013), Ruas et al. (2013), Dias et al. (2015), Nunes et al. (2015) and Dias et al. (2020).

4.2. Model setup

Hereafter, we follow Dias et al. (2023) by assuming that the output price P is described via a CEV process represented by the stochastic differential equation

$$dP_t = (r - q) P_t dt + \sigma(P_t) P_t dW_t^{\mathbb{Q}}, \quad (4.1)$$

with the local volatility function—assumed to be continuous and strictly positive for all $P \in (0, \infty)$ —given by

$$\sigma(P_t) := \delta P_t^{\frac{\beta}{2}-1}, \quad (4.2)$$

for $\delta \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$, and where r and q are the (positive and constant) interest rate and dividend yield (or rate of return shortfall), respectively, while $\{W_t^{\mathbb{Q}} \in \mathbb{R} : t \geq 0\}$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$, initialized at zero and generating the augmented, right continuous and complete filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$.

The modeling specification given by equations (4.1) and (4.2) nests the lognormal assumption of Black and Scholes (1973) and Merton (1973) (if $\beta = 2$) as well as the absolute diffusion (if $\beta = 0$) and the square-root diffusion (if $\beta = 1$) processes of Cox and Ross (1976) as special cases. We recall also that if $\beta < 2$ (resp., $\beta > 2$), the local volatility function (4.2) is a decreasing (resp., increasing) function of the asset price, thus being able to generate downward-sloping (resp., upward-sloping) volatility skews that are observed in the market. As usual, the model parameter δ is interpreted as the scale parameter fixing the initial instantaneous volatility at time $t = 0$, $\sigma_P \equiv \sigma(P_0) = \delta P_0^{\beta/2-1}$. This calibration procedure is standard in the option pricing literature in order to ensure that CEV models with different values of β have the same variance at inception, thus permitting valid comparisons between different CEV diffusion processes.

4.3. Investment decisions with minimum price guarantees

This section considers a FIT contractual agreement with a minimum price guarantee—also known as a contract based on a price-floor regime—such as the one considered in Barbosa et al. (2018, Section 3) for the case of a renewable energy project and from a single investor's perspective, but now under the CEV modeling setup prescribed in Dias et al. (2023). Although the value of the project with a minimum price guarantee under the CEV process has been already derived in Dias et al. (2023, Section 4), the corresponding options to invest represent novel results to the real options literature.

4.3.1. Perpetual case

Let us start with a contractual scheme with a perpetual guarantee. This time-independent assumption implies that the investor may receive payments from the guarantee forever. Even

though such perpetual guarantee is not used in practice, it is important to analyze it first in order to be compared with the more realistic case of a finite guarantee.

4.3.1.1. Value of the project

In the FIT contract under analysis the producer (or investor) is entitled to receive the price floor L for every unit of energy produced when the market price P is below the price floor; otherwise (i.e., when $P \geq L$), the producer receives the market price. Hence, the perpetual profit flow of such renewable energy project can be formally defined as $\Pi(P_t) = \max(P_t, L)$, with P_t being understood as the time- t revenue for one unit of energy produced and L being interpreted as the revenue from the guarantee for one unit of energy.

Assuming that the output price P follows the CEV process defined in equations (4.1) and (4.2), then the value of the (infinite horizon) project, $V_f(P_t, L, \infty)$, with a perpetual profit flow $\Pi(P_t)$ must satisfy the following ordinary differential equation (hereafter, ODE)

$$\frac{1}{2}\delta^2 P_t^\beta \frac{d^2 V_f}{dP_t^2} + (r - q) P_t \frac{dV_f}{dP_t} - rV_f + \Pi(P_t) = 0, \quad (4.3)$$

whose analytical solution is dependent on the drift specification (i.e., whether $r = q$ or $r \neq q$) and on the elasticity parameter β , as already shown in Dias et al. (2023). For the sake of completeness and to introduce notation and explicit expressions that are required later, such results are highlighted in the next two propositions.

Proposition 4.1. *Assume the CEV process with $r = q$ and introduce the following notation:*

$$z_t \equiv z(P_t) := \frac{2\sqrt{2r}}{\delta|\beta - 2|} P_t^{1-\frac{\beta}{2}} \quad (4.4)$$

and

$$\nu := \frac{1}{|\beta - 2|}. \quad (4.5)$$

The time- t value of the project with a minimum price guarantee, $V_f(P_t, L, \infty)$, is given by

$$V_f(P_t, L, \infty) = \begin{cases} \left(A_0 P_t^{\frac{1}{2}} I_\nu(z_t) + \frac{L}{r} \right) \mathbb{1}_{\{P_t < L\}} \\ + \left(B_0 P_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq L\}} & \Leftarrow \beta < 2 \\ \left(B_0 P_t^{\frac{1}{2}} K_\nu(z_t) + \frac{L}{r} \right) \mathbb{1}_{\{P_t < L\}} \\ + \left(A_0 P_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq L\}} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \quad (4.6)$$

where $G_t^\infty(P_t)$ is the perpetual bubble value defined in (Dias et al., 2023, equation 10), $I_\nu(z_t)$ and $K_\nu(z_t)$ are, respectively, the modified Bessel's functions of first and second kind with order ν , as given by Olver et al. (2010, equations 10.25.2 and 10.27.4), whereas $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ are constants (associated to the value of the project) that

solve the following system of equations, for $\zeta := z(L)$ and for any $\beta \neq 2$:

$$\begin{cases} A_0 I_\nu(\zeta) - B_0 K_\nu(\zeta) = 0 \\ A_0 I_{\nu+1}(\zeta) + B_0 K_{\nu+1}(\zeta) = \frac{\delta}{q\sqrt{2r}} L^{\frac{\beta-1}{2}} \end{cases} . \quad (4.7)$$

Proof. Please see Dias et al. (2023, Proposition 8). ■

Remark 4.1. Even though the constants A_0 and B_0 are easily determined by solving (numerically) the system of equations (4.7), it is still possible to calculate each one via a very simple closed-form solution after using some calculus and the Wronskian shown in Olver et al. (2010, equation 10.28.2), that is

$$A_0 \equiv A_0(L) = \frac{2}{q|\beta-2|} L^{\frac{1}{2}} K_\nu(z(L)) \quad (4.8)$$

and

$$B_0 \equiv B_0(L) = \frac{2}{q|\beta-2|} L^{\frac{1}{2}} I_\nu(z(L)). \quad (4.9)$$

Proposition 4.2. Assume the CEV process with $r \neq q$ and introduce the following notation:

$$x_t \equiv x(P_t) := \frac{2|r-q|}{\delta^2|\beta-2|} P_t^{2-\beta}, \quad (4.10)$$

$$\epsilon := \text{sign}[(r-q)(\beta-2)], \quad (4.11)$$

$$k := \epsilon \left(\frac{1}{2} + \frac{1}{2(\beta-2)} \right) - \frac{r}{|(r-q)(\beta-2)|}, \quad (4.12)$$

$$m := \frac{1}{2|\beta-2|}, \quad (4.13)$$

$$a := \frac{1}{2} + m - k, \quad (4.14)$$

$$b := 1 + 2m \quad (4.15)$$

and

$$Y := \left(\frac{2|r-q|}{\delta^2|\beta-2|} \right)^{\frac{b}{2}}. \quad (4.16)$$

The time- t value of the project with a minimum price guarantee, $V_f(P_t, L, \infty)$, is given by

$$V_f(P_t, L, \infty) = \begin{cases} \left(A_0 Y e^{(\epsilon-1)\frac{x_t}{2}} P_t M(a, b, x_t) + \frac{L}{r} \right) \mathbb{1}_{\{P_t < L\}} \\ + \left(B_0 Y e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a, b, x_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq L\}} & \Leftarrow \beta < 2 \\ \left(B_0 Y e^{(\epsilon-1)\frac{x_t}{2}} U(a, b, x_t) + \frac{L}{r} \right) \mathbb{1}_{\{P_t < L\}} \\ + \left(A_0 Y e^{(\epsilon-1)\frac{x_t}{2}} M(a, b, x_t) + \frac{P_t}{q} \right) \mathbb{1}_{\{P_t \geq L\}} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \quad (4.17)$$

where $M(a, b, x_t)$ and $U(a, b, x_t)$ are, respectively, the Kummer's and Tricomi's confluent hypergeometric functions defined in Olver et al. (2010, equations 13.2.2 and 13.2.42), whereas $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ are constants (associated to the value of the project) that solve the following system of equations, for $\chi := x(L)$:

$$\begin{cases} A_0 M(a, b, \chi) - B_0 U(a, b, \chi) = \frac{r-q}{rq} Y^{-1} e^{(1-\epsilon)\frac{\chi}{2}} \\ A_0 \frac{a}{b} M(a+1, b+1, \chi) + B_0 a U(a+1, b+1, \chi) = e^{(1-\epsilon)\frac{\chi}{2}} \left(\frac{(1-\epsilon)(r-q)}{2rq} + \frac{\delta^2 L^{\beta-2}}{2r|r-q|} \right) Y^{-1} \end{cases}, \quad (4.18)$$

for $\beta < 2$, and

$$\begin{cases} A_0 M(a, b, \chi) - B_0 U(a, b, \chi) = \frac{q-r}{rq} LY^{-1} e^{(1-\epsilon)\frac{\chi}{2}} \\ A_0 \frac{a}{b} M(a+1, b+1, \chi) + B_0 a U(a+1, b+1, \chi) = e^{(1-\epsilon)\frac{\chi}{2}} \left(\frac{(1-\epsilon)(q-r)}{2rq} + \frac{\delta^2 L^{\beta-2}}{2q|r-q|} \right) LY^{-1} \end{cases}, \quad (4.19)$$

for $\beta > 2$.

Proof. Please see Dias et al. (2023, Proposition 9). ■

Remark 4.2. Again, although the constants A_0 and B_0 can be easily determined by solving (numerically) the systems of equations (4.18) and (4.19) for $\beta < 2$ and $\beta > 2$, respectively, it is still possible to calculate each one via an analytic solution after using some calculus and combining the standard solution and the Wronskian presented in Olver et al. (2010, equations 13.2.4 and 13.2.34), that is

$$\begin{aligned} A_0 \equiv A_0(L) &= \left[\frac{\delta^2 L^{\beta-2}}{2r|r-q|} + \frac{r-q}{rq} \left(\frac{1-\epsilon}{2} + a \frac{U(a+1, b+1, x(L))}{U(a, b, x(L))} \right) \right] \\ &\times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y} \frac{\Gamma(a)}{\Gamma(b)} (x(L))^b U(a, b, x(L)) \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} B_0 \equiv B_0(L) &= \left[\frac{\delta^2 L^{\beta-2}}{2r|r-q|} + \frac{r-q}{rq} \left(\frac{1-\epsilon}{2} - \frac{a}{b} \frac{M(a+1, b+1, x(L))}{M(a, b, x(L))} \right) \right] \\ &\times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y} \frac{\Gamma(a)}{\Gamma(b)} (x(L))^b M(a, b, x(L)), \end{aligned} \quad (4.21)$$

for $\beta < 2$, and

$$\begin{aligned} A_0 \equiv A_0(L) &= \left[\frac{\delta^2 L^{\beta-2}}{2q|r-q|} + \frac{q-r}{rq} \left(\frac{1-\epsilon}{2} + a \frac{U(a+1, b+1, x(L))}{U(a, b, x(L))} \right) \right] \\ &\times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y} L \frac{\Gamma(a)}{\Gamma(b)} (x(L))^b U(a, b, x(L)) \end{aligned} \quad (4.22)$$

and

$$B_0 \equiv B_0(L) = \left[\frac{\delta^2 L^{\beta-2}}{2q|r-q|} + \frac{q-r}{rq} \left(\frac{1-\epsilon}{2} - \frac{a}{b} \frac{M(a+1, b+1, x(L))}{M(a, b, x(L))} \right) \right] \times \frac{e^{-(1+\epsilon)\frac{x(L)}{2}}}{Y} L \frac{\Gamma(a)}{\Gamma(b)} (x(L))^b M(a, b, x(L)), \quad (4.23)$$

for $\beta > 2$.

4.3.1.2. Value of the investment opportunity

Assuming that an investor has a perpetual investment opportunity to invest in a project with a sunk cost I , we can (simultaneously) determine the value of the (perpetual) option to invest in a project with a perpetual guarantee, $F_f(P_t, \bar{P}, \infty)$, and the optimal investment rule, with $\bar{P}$ being interpreted as the optimal output price trigger level that will induce the investment. Assuming that the output price P follows the CEV process defined in equations (4.1) and (4.2), then the value of the option to invest in a perpetual project must satisfy another ODE but with no profit flow, that is

$$\frac{1}{2} \delta^2 P_t^\beta \frac{d^2 F_f}{dP_t^2} + (r-q) P_t \frac{dF_f}{dP_t} - r F_f = 0, \quad (4.24)$$

whose analytical solutions depend, again, on the drift level and β parameter and are determined subject to appropriate boundary conditions. The next two propositions offer such novel results for evaluating a perpetual option to invest in a project with an infinite-lived guarantee under the CEV process. We start, however, with a remark clarifying in which region the value-matching and smooth-pasting conditions can be applied.

Remark 4.3. *Similarly to what happens in the limiting GBM case considered in Barbosa et al. (2018), the value-matching and smooth-pasting conditions are not met for the first branch of equations (4.6) and (4.17) (i.e., when $\bar{P} < L$). Therefore, investment never occurs in this region, except when $\frac{L}{r} > I$, which would imply a counter-intuitive result of a project generating a positive net present value (hereafter, NPV) that is independent of P . Moreover, such risk-free payoff would be the same for any $\beta \in \mathbb{R}$, because the simultaneous presence of bubbles in the project value and in the option component would cancel out any bubble effect for $\beta > 2$. In summary, the optimal investment threshold for a FIT contract with a perpetual guarantee is restricted to the range $\bar{P} \geq L$.²*

Proposition 4.3. *Assume the CEV process with $r = q$ and the notation introduced in Proposition 4.1. The time- t value of the (perpetual American-style) option to invest in a*

²A formal proof of these arguments can be found in 4A.

project with a perpetual guarantee is given by

$$\begin{aligned}
& F_f(P_t, \bar{P}, \infty) \\
&= \begin{cases} [V_{f,op}(\bar{P}, L, \infty) + G_t^\infty(\bar{P}) \mathbb{1}_{\{\beta > 2\}} - I] S_f(P_t, \bar{P}, \infty) - G_t^\infty(P_t) \mathbb{1}_{\{\beta > 2\}} & \Leftarrow P_t < \bar{P} \\ V_{f,op}(P_t, L, \infty) - I & \Leftarrow P_t \geq \bar{P} \end{cases},
\end{aligned} \tag{4.25}$$

with $V_{f,op}(P_t, L, \infty)$ being given by equation (4.6) but conditional on $P_t \geq L$, that is

$$\begin{aligned}
V_{f,op}(P_t, L, \infty) &:= V_f(P_t, L, \infty) \mathbb{1}_{\{P_t \geq L\}} \\
&= \begin{cases} B_0 P_t^{\frac{1}{2}} K_\nu(z_t) + \frac{P_t}{q} & \Leftarrow \beta < 2 \\ A_0 P_t^{\frac{1}{2}} I_\nu(z_t) + \frac{P_t}{q} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases},
\end{aligned} \tag{4.26}$$

and where the stochastic discount factor of the perpetual option to invest is given by

$$S_f(P_t, \bar{P}, \infty) = \begin{cases} \sqrt{\frac{P_t}{\bar{P}}} \frac{I_\nu(z(P_t))}{I_\nu(z(\bar{P}))} & \Leftarrow \beta < 2 \\ \sqrt{\frac{P_t}{\bar{P}}} \frac{K_\nu(z(P_t))}{K_\nu(z(\bar{P}))} & \Leftarrow \beta > 2 \end{cases}, \tag{4.27}$$

which can be financially interpreted as the time- t price of a perpetual claim that pays \$1 the first time P reaches $\bar{P}$ from below, with the optimal threshold $\bar{P}$ ($\geq L$) and the constants $A_1 \equiv A_1(\bar{P})$ and $B_1 \equiv B_1(\bar{P})$ (associated to the value of the option to invest) being obtained as the solution of the following system of equations:

$$\begin{cases} A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) - B_0 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) = \frac{\bar{P}}{q} - I \\ A_1 I_{\nu+1}(z(\bar{P})) + B_0 K_{\nu+1}(z(\bar{P})) = \frac{\delta I}{\sqrt{2r}} \bar{P}^{\frac{\beta-3}{2}} \end{cases}, \tag{4.28}$$

for $\beta < 2$, and

$$\begin{cases} B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - A_0 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = \frac{\bar{P}}{q} - I \\ B_1 K_{\nu+1}(z(\bar{P})) + A_0 I_{\nu+1}(z(\bar{P})) = \frac{\delta}{q\sqrt{2r}} \bar{P}^{\frac{\beta-1}{2}} \end{cases}, \tag{4.29}$$

for $\beta > 2$, with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (4.7) for any $\beta \neq 2$.

Proof. Please see Appendix 4B. ■

Proposition 4.4. Assume the CEV process with $r \neq q$ and the notation introduced in Proposition 4.2. The time- t value of the (perpetual American-style) option to invest in a project with a perpetual guarantee is still given by equation (4.25), but now with

$V_{f,op}(P_t, L, \infty)$ being given by equation (4.17) conditional on $P_t \geq L$, that is

$$\begin{aligned} V_{f,op}(P_t, L, \infty) &:= V_f(P_t, L, \infty) \mathbb{1}_{\{P_t \geq L\}} \\ &= \begin{cases} B_0 Y e^{(\epsilon-1)\frac{x_t}{2}} P_t U(a, b, x_t) + \frac{P_t}{q} & \Leftarrow \beta < 2 \\ A_0 Y e^{(\epsilon-1)\frac{x_t}{2}} M(a, b, x_t) + \frac{P_t}{q} - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (4.30)$$

and where the stochastic discount factor of the perpetual option to invest being given by

$$S_f(P_t, \bar{P}, \infty) = \begin{cases} e^{(\epsilon-1)\frac{x(P_t)-x(\bar{P})}{2}} \frac{P_t}{\bar{P}} \frac{M(a, b, x(P_t))}{M(a, b, x(\bar{P}))} & \Leftarrow \beta < 2 \\ e^{(\epsilon-1)\frac{x(P_t)-x(\bar{P})}{2}} \frac{U(a, b, x(P_t))}{U(a, b, x(\bar{P}))} & \Leftarrow \beta > 2 \end{cases}, \quad (4.31)$$

with the optimal threshold $\bar{P}$ ($\geq L$) and the constants $A_1 \equiv A_1(\bar{P})$ and $B_1 \equiv B_1(\bar{P})$ (associated to the value of the option to invest) being obtained as the solution of the following system of equations:

$$\begin{cases} A_1 M(a, b, x(\bar{P})) - B_0 U(a, b, x(\bar{P})) = \left(\frac{\bar{P}}{q} - I\right) Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \bar{P}^{-1} \\ A_1 \frac{a}{b} M(a+1, b+1, x(\bar{P})) + B_0 a U(a+1, b+1, x(\bar{P})) \\ = \left[\frac{1}{q} - \left(\frac{\bar{P}}{q} - I\right) \left(\frac{\epsilon-1}{\delta^2} |r-q| \bar{P}^{1-\beta} + \bar{P}^{-1}\right)\right] Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \frac{\delta^2}{2|r-q|} \bar{P}^{\beta-2} \end{cases}, \quad (4.32)$$

for $\beta < 2$, and

$$\begin{cases} B_1 U(a, b, x(\bar{P})) - A_0 M(a, b, x(\bar{P})) = \left(\frac{\bar{P}}{q} - I\right) Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \\ B_1 a U(a+1, b+1, x(\bar{P})) + A_0 \frac{a}{b} M(a+1, b+1, x(\bar{P})) \\ = \left[\frac{1}{q} + \left(\frac{\bar{P}}{q} - I\right) \frac{\epsilon-1}{\delta^2} |r-q| \bar{P}^{1-\beta}\right] Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \frac{\delta^2}{2|r-q|} \bar{P}^{\beta-1} \end{cases}, \quad (4.33)$$

for $\beta > 2$, with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (4.18) and (4.19) for $\beta < 2$ and $\beta > 2$, respectively.

Proof. Please see Appendix 4C. ■

In summary, the optimal decision to invest in the case of a FIT contract with a perpetual guarantee can be analyzed via Propositions 4.3 and 4.4: exercise the (perpetual) option to invest only if $P_t \geq \bar{P}$; otherwise, i.e., if $P_t < \bar{P}$, it is better to wait and avoid loosing the value of waiting in the spirit of McDonald and Siegel (1986).

As highlighted in Remark 4.3, the perpetual case requires that $\bar{P} \geq L$ so that the problem is well behaved. In order to ensure that this condition is met, we determine the investment cost level I_L^* that separates the regions $\bar{P} \geq L$ and $\bar{P} < L$. This is accomplished by replacing $\bar{P}$ by the fixed price floor L (previously defined by the government) in the systems of equations

(4.28), (4.29), (4.32) and (4.33) and solving each one with respect to $I \equiv I_L^*$. The obtained I_L^* value can be economically interpreted as the required investment cost that makes the fixed L the optimal trigger of investment. Notice that the constants A_0 and B_0 should be taken as functions of L and, hence, they are known in advance using the systems of equations (4.7), (4.18) or (4.19). Alternatively, it is also possible to use the novel analytic solutions offered in Remarks 4.1 and 4.2 to compute the value of such ingredients. Armed with the values of $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$, the I_L^* level is then easily computed using the systems of equations (4.28), (4.29), (4.32) or (4.33) for the specific required case. If $I_L^* \leq I$, the optimal threshold $\bar{P}$ can be calculated since $\bar{P} \geq L$; otherwise, i.e., if $I_L^* > I$, the value-matching and smooth-pasting conditions are not met under the perpetual case and, hence, the investment problem falls in the counter-intuitive case exposed in Remark 4.3.

Several other interesting points can be also analyzed. For instance, for a given time- t price P_t (i.e., the current price) and a fixed price floor L (that is subject to the government policy), Barbosa et al. (2018, equation 16) allow us to determine the investment cost I^* that would induce the firm to exercise the option to invest immediately under the GBM process. In other words, this I^* value indicates the required level of investment that would make the time- t price P_t the trigger value to invest. In our case, this I^* value can be obtained by solving the systems of equations (4.28), (4.29), (4.32) or (4.33) with respect to $I \equiv I^*$ (and with $\bar{P}$ replaced by $P_t \geq L$) for the specific required case, which can then be compared with the initial investment cost I of the project. Again, the constants A_0 and B_0 are functions of L and, therefore, they are already known using the systems of equations (4.7), (4.18) and (4.19), or the analytical formulas shown in Remarks 4.1 and 4.2. As expected, if $I \leq I^*$ it would be better for the firm to invest immediately because $P_t = \bar{P}$ at the I^* level.

Furthermore, taking now the initial investment cost I of the project as given and substituting L for $\bar{P}$, Barbosa et al. (2018, equation 18) show how to compute (under the GBM process) the interesting point where the optimal threshold value $\bar{P}$ is equal to the price floor $L \equiv L^*$. The obtained L^* value can be economically interpreted as the required price floor level to be offered by the government to induce the firm to invest immediately (i.e., so that $\bar{P} = L^*$) for the fixed initial investment cost. In our case, this L^* value can be obtained by solving systems of four equations with the required constants being now functions of L^* . In particular, the following procedure is adopted to find the L^* value: (i) if $r = q$ and $\beta < 2$, use the systems of equations (4.7) with $L \equiv L^*$ and (4.28) with $\bar{P} \equiv L^*$; (ii) if $r = q$ and $\beta > 2$, use the systems of equations (4.7) with $L \equiv L^*$ and (4.29) with $\bar{P} \equiv L^*$; (iii) if $r \neq q$ and $\beta < 2$, use the systems of equations (4.18) with $L \equiv L^*$ and (4.32) with $\bar{P} \equiv L^*$; and (iv) if $r \neq q$ and $\beta > 2$, use the systems of equations (4.19) with $L \equiv L^*$ and (4.33) with $\bar{P} \equiv L^*$. The obtained L^* value can then be compared with the minimum price guarantee L that is offered by the government: if $L \geq L^*$ it would be better for the firm to invest right away.

Even though the levels of I_L^* and L^* can be calculated numerically using the aforementioned numerical procedures, it is possible to show analytically that $I_L^* = \frac{L}{r}$. This is accomplished using relatively simple calculus by appropriately combining the solutions offered in Remarks 4.1 and 4.2 with the systems of equations (4.28), (4.29), (4.32) and (4.33) and solving each particular case with respect to $I \equiv I_L^*$.³ This also implies that $L^* = rI$. Therefore, I_L^* and L^* are both independent of β under the perpetual case and, hence, are the same triggers offered by Barbosa et al. (2018, equation 18) for the GBM case. The independence of I_L^* and L^* on the parameter β seems to be surprising at first glance, but is explained by the perpetual nature of the guarantee; as we will discuss in the next subsection, such independence is not valid any more in the case of FIT contracts with a finite maturity guarantee.

Nevertheless, such independence of the β parameter is only observed at the level L^* , because for the range of price floors $L \in [0, L^*[$ the optimal trigger value $\bar{P}$ decreases as we move further away to the left of $\beta < 2$ and increases as β rises for the case with $\beta > 2$. Only at $\bar{P} \equiv L^*$ the optimal threshold of investment is the same. Since alternative CEV models imply the existence of different optimal triggers or, equivalently, distinct optimal stopping times, a government assuming a GBM process to define a FIT policy might be paying a too high or a too low price floor to stimulate a firm to exercise the option to invest if the specific investment problem is better described by a different CEV model.

4.3.2. Finite case

Let us now analyze the FIT contractual agreement with a finite maturity guarantee. Similarly to the perpetual case, both the value of the project and the value of the option to invest will be analyzed.

4.3.2.1. Value of the project

Following Dias et al. (2023, Section 4.2), the value of the project with a finite guarantee, $V_f(P_t, L, T)$, must now satisfy the partial differential equation

$$\frac{1}{2}\delta^2 P_t^\beta \frac{\partial^2 V_f}{\partial P_t^2} + (r - q) P_t \frac{\partial V_f}{\partial P_t} - \frac{\partial V_f}{\partial \tau} - rV_f + \Pi(P_t) = 0, \quad (4.34)$$

with $\tau := T - t$, and whose solution can be expressed as the difference between the value of the project with a perpetual guarantee, $V_f(P_t, L, \infty)$, and the value of a forward start project with a perpetual guarantee, $V_f(P_T, L, \infty)$, that is

$$V_f(P_t, L, T) = V_f(P_t, L, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t], \quad (4.35)$$

³The full analytical proofs are available upon request.

with $V_f(P_t, L, \infty)$ being computed by equations (4.6) and (4.17) for $r = q$ and $r \neq q$, respectively, whereas the risk-neutral expectation of the forward start perpetual FIT contract with a minimum price guarantee, $\mathbb{E}_{\mathbb{Q}}[V_f(P_T, L, \infty)|\mathcal{F}_t]$, is calculated using Dias et al. (2023, equations 70 and 71). These results are reported below to introduce notation and expressions explicitly dependent on the underlying spot P_t and price floor L that are required later.

For the no-drift case (i.e., for $r = q$),

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}}[V_f(P_T, L, \infty)|\mathcal{F}_t] \\ = & \begin{cases} A_0 I(-1, P_t, L) + X_r(-1, P_t, L) + B_0 K(1, P_t, L) + P_q(1, P_t, L) & \Leftarrow \beta < 2 \\ B_0 K(-1, P_t, L) + X_r(-1, P_t, L) + A_0 I(1, P_t, L) + P_q(1, P_t, L) & \\ -G^\infty(t, T, P_t) & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (4.36)$$

with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (4.7) for any $\beta \neq 2$, the functions

$$P_q(1, P_t, L) := \begin{cases} e^{(r-q)\tau} \frac{P_t}{q} \Phi_{+1} \left(0, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta < 2 \\ \frac{1}{q} (f(\tau))^{-\nu} \Phi_{-1} \left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta > 2 \end{cases} \quad (4.37)$$

and

$$\begin{aligned} & X_r(-1, P_t, L) \\ := & \begin{cases} \frac{L}{r} P_t (f(\tau))^{-\nu} \Phi_{-1} \left(-\nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) + \frac{L}{r} \frac{\Gamma\left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)}\right)}{\Gamma(\nu)} & \Leftarrow \beta < 2 \\ \frac{L}{r} \Phi_{+1} \left(0, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) & \Leftarrow \beta > 2 \end{cases} \end{aligned} \quad (4.38)$$

are borrowed from Dias et al. (2023, Proposition 4), with $\tau := T - t$,

$$\alpha := (r - q)|\beta - 2|, \quad (4.39)$$

$$f(\tau) := \begin{cases} \frac{\delta^2 |\beta - 2|}{4(r - q)} (1 - e^{-\alpha\tau}) & \Leftarrow r \neq q \\ \frac{\delta^2 (\beta - 2)^2}{4} \tau & \Leftarrow r = q \end{cases}, \quad (4.40)$$

and where $\Phi_\phi(p, x; \nu, \lambda)$ is interpreted as the p -th lower tail (if $\phi = -1$) or the p -th upper tail (if $\phi = +1$) truncated moment of a noncentral χ^2 random variable with ν degrees of freedom and noncentrality parameter λ , as defined in Carr and Linetsky (2006, equations 5.12 and 5.11) and Dias and Nunes (2018, equations 21 and 24), $\frac{\Gamma(\nu, w)}{\Gamma(\nu)}$ is the incomplete gamma function ratio (or normalized function) shown in Olver et al. (2010, equation 8.2.4), with $\Gamma(\nu, w)$ and $\Gamma(\nu)$ being the upper incomplete gamma function and the Euler gamma function

given in Olver et al. (2010, equations 8.2.2 and 5.2.1), respectively, for $\nu, w \in \mathbb{R}^+$, whereas the functions

$$I(\phi, P_t, L) := P_t^\varphi \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{+\tilde{\phi}} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \quad (4.41)$$

and

$$\begin{aligned} K(\phi, P_t, L) & \quad (4.42) \\ := & \frac{\pi}{2 \sin(\nu\pi)} P_t^\varphi \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j-\nu}}{j!} \frac{(f(\tau))^{j-\nu}}{\Gamma(j+1-\nu)} \Phi_{+\tilde{\phi}} \left(j-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ & + \mathbb{1}_{\{\beta < 2, \phi = -1\}} \frac{1}{2} \Gamma \left(\nu, \frac{P_t^{2-\beta}}{2f(\tau)} \right) \left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{-\nu} \\ & - \frac{\pi}{2 \sin(\nu\pi)} P_t^\varphi \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{+\tilde{\phi}} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \end{aligned}$$

with $\varphi := \mathbb{1}_{\{\beta < 2\}}$ and $\tilde{\phi} := \phi \mathbb{1}_{\{\beta < 2\}} - \phi \mathbb{1}_{\{\beta > 2\}}$, are taken from Dias et al. (2023, Proposition 5).⁴ Finally, the time- t expected value of the perpetual bubble, $G^\infty(t, T, P_t)$, is obtained from Dias et al. (2023, Proposition 7), that is

$$G^\infty(t, T, P_t) = \frac{1}{\Gamma(\nu)} \int_T^\infty \left[\kappa_{T,u}^\nu e^{-r(u-T)} \int_1^\infty \frac{\exp\left(-\frac{s\tilde{f}A_{t,T,u}}{1+2sA_{t,T,u}}\right)}{(1+2sA_{t,T,u})^{1+\nu}} s^{\nu-1} ds \right] du, \quad (4.43)$$

with

$$\kappa_{T,u} := \begin{cases} \frac{2(r-q)}{\delta^2(2-\beta)[e^{(r-q)(2-\beta)(u-T)}-1]} & \Leftarrow r \neq q \\ \frac{2}{\delta^2(2-\beta)^2(u-T)} & \Leftarrow r = q \end{cases}, \quad (4.44)$$

$$\tilde{f} \equiv \tilde{f}(P_t) := \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \quad (4.45)$$

and

$$A_{t,T,u} := \begin{cases} \frac{1-e^{-\alpha(T-t)}}{2(e^{\alpha(u-T)}-1)} & \Leftarrow r \neq q \\ \frac{T-t}{2(u-T)} & \Leftarrow r = q \end{cases}. \quad (4.46)$$

⁴Notice that the second line of Dias et al. (2023, equation 49) can be further simplified using the relation $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin(\nu\pi)$, for $\nu \neq 0, \pm 1, \dots$, borrowed from Olver et al. (2010, equation 5.5.3), thus resulting in the second line of equation (4.42) of the present paper.

For $r \neq q$,

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] \\ = & \begin{cases} A_0 Y M_P(-1, P_t, L) + X_r(-1, P_t, L) + B_0 Y U_P(1, P_t, L) + P_q(1, P_t, L) & \Leftarrow \beta < 2 \\ B_0 Y U_N(-1, P_t, L) + X_r(-1, P_t, L) + A_0 Y M_N(1, P_t, L) + P_q(1, P_t, L) & \\ -G^\infty(t, T, P_t) & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (4.47)$$

with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (4.18) and (4.19) for $\beta < 2$ and $\beta > 2$, respectively, and where the remaining required functions are borrowed from Dias et al. (2023, Proposition 6), that is

$$M_P(-1, P_t, L) := e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \Phi_{-1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \quad (4.48)$$

with $c := a \mathbb{1}_{\{r < q, \beta < 2\}} + (b - a) \mathbb{1}_{\{r > q, \beta < 2\}}$ and $(c)_j$ is the Pochhammer's symbol as defined in Olver et al. (2010, equations 5.2.4 and 5.2.5),

$$M_N(1, P_t, L) := \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \Phi_{-1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right), \quad (4.49)$$

with $d := (b - a) \mathbb{1}_{\{r < q, \beta > 2\}} + a \mathbb{1}_{\{r > q, \beta > 2\}}$,

$$\begin{aligned} & U_P(1, P_t, L) \\ = & \frac{\Gamma(-\nu)}{\Gamma(a - \nu)} e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \Phi_{+1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ & + \frac{\Gamma(\nu)}{\Gamma(a)} e^{(r-q)\tau} P_t \sum_{j=0}^{\infty} \frac{(c - \nu)_j}{(2 - b)_j} \frac{\xi^j}{j!} \left(\xi \frac{1 - e^{\alpha\tau}}{2} \right)^{j-\nu} \Phi_{+1} \left(j - \nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (4.50)$$

with $\xi := \mathbb{1}_{\{r < q\}} - \mathbb{1}_{\{r > q\}}$, and

$$\begin{aligned} & U_N(-1, P_t, L) \\ = & \frac{\Gamma(-\nu)}{\Gamma(a - \nu)} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \Phi_{+1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ & + \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(d - \nu)_j}{(2 - b)_j} \frac{\tilde{\xi}^j}{j!} \left(\tilde{\xi} \frac{1 - e^{-\alpha\tau}}{2} \right)^{j-\nu} \Phi_{+1} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (4.51)$$

with $\tilde{\xi} := \mathbb{1}_{\{r > q\}} - \mathbb{1}_{\{r < q\}} = -\xi$.

We recall that equation (4.35) allow us to compute the (time- t) value of the project until

the maturity date of the FIT contract, i.e., only while the producer benefits from a contractual agreement guaranteeing a minimum selling price of L . In practice, however, it might be reasonable to assume that the producer still runs the project after time T . Therefore, after the expiry date of the finite maturity FIT contract the producer is entitled to receive an unprotected cash flow equal to $e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T | \mathcal{F}_t] - e^{-r\tau} \mathbb{E}_{\mathbb{Q}} [G_T^{\infty}(P_T) | \mathcal{F}_t] \mathbb{1}_{\{\beta > 2\}}$ that should be economically interpreted as the cash amount that is obtained by selling energy at the prevailing market price for the remaining (perpetual) life of the project and with $G_T^{\infty}(P_T)$ being understood as a forward start perpetual bubble value in the spirit of Dias et al. (2023, equation 10).⁵ Hence, the (time- t) full value of the project, $V_p(P_t, L, T)$, that contains both the protected and unprotected sources of value, is equal to⁶

$$V_p(P_t, L, T) \tag{4.52}$$

$$= \begin{cases} V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} & \Leftarrow \beta < 2 \\ V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} \frac{\gamma(\nu, \tilde{f}/2)}{\Gamma(\nu)} - e^{-r(T-t)} G^{\infty}(t, T, P_t) & \Leftarrow \beta > 2 \end{cases},$$

where $\frac{\gamma(\nu, w)}{\Gamma(\nu)}$ is the incomplete gamma function ratio (or normalized function) shown in Olver et al. (2010, equation 8.2.4), with $\gamma(\nu, w)$ being the lower incomplete gamma function given in Olver et al. (2010, equations 8.2.1), for $\nu, w \in \mathbb{R}^+$, and with $G^{\infty}(t, T, P_t)$ being computed via equation (4.43).

Remark 4.4. We notice that equation (4.52) nests immediately two special cases of interest: (i) if there is no guarantee—i.e., if $L = 0$ or, equivalently, if $T \rightarrow t$ so that $\tau := T - t \rightarrow 0$ —, the present value of the project is simply equal to $\lim_{T \rightarrow t} V_p(P_t, L, T) = \frac{P_t}{q} - G_t^{\infty}(P_t) \mathbb{1}_{\{\beta > 2\}}$, thus demonstrating that the value of the project is equal to a cash flow without a guarantee; (ii) in the case of a perpetual guarantee, $\lim_{T \rightarrow \infty} V_p(P_t, L, T) = V_f(P_t, L, \infty)$ —i.e., the value of the project is obtained, as expected, via equations (4.6) and (4.17) for $r = q$ and $r \neq q$, respectively.⁷

4.3.2.2. Value of the investment opportunity

Let us now consider that the investor has a perpetual investment opportunity to invest in a project with a sunk cost I . The goal is still to (simultaneously) determine the value of the (perpetual) option to invest in a project but now with a finite guarantee, $F_f(P_t, \bar{P}, T)$, and the optimal investment rule, with $\bar{P}$ being still interpreted as the optimal output price trigger level that will induce the investment.

⁵This implies that we are implicitly assuming a perpetual concession with a finite maturity FIT. Nevertheless, the model can be also applied to the case of finite concessions having the same duration of the finite maturity FIT contract by simply ignoring the unprotected cash flow component.

⁶The proof of the present value of the unprotected cash flow component is presented in 4D.

⁷The proof of such asymptotic analysis is presented in 4E.

Notice that we still have an infinite-horizon optimal stopping problem such as the one considered in Section 4.3.1.2. The only difference is that now once the (perpetual American-style) option to invest is exercised the owner is entitled to receive a project with a finite-lived guarantee (coupled with an unprotected cash flow component that is received after the expiry date of the finite maturity FIT contract), whereas in Section 4.3.1.2 the producer receives a project with an infinite-lived guarantee (hence, with no additional unprotected cash flow component).

Note also that, in contrast to the perpetual case, the value-matching and smooth-pasting conditions are now met in both regions $P_t < L$ and $P_t \geq L$ and, therefore, it is necessary to derive a system of nonlinear equations being dependent on the optimal threshold $\bar{P}$ in both regions and for $\beta < 2$ and $\beta > 2$. The next two propositions show how to evaluate a perpetual option to invest in a project with a finite-lived guarantee under the CEV process.

Proposition 4.5. *Assume the CEV process with $r = q$ and the notation introduced in Proposition 4.1. The time- t value of the (perpetual American-style) option to invest in a project with a finite-lived guarantee is given by*

$$\begin{aligned} & F_f(P_t, \bar{P}, T) \\ = & \begin{cases} [V_p(\bar{P}, L, T) + G_t^\infty(\bar{P}) \mathbf{1}_{\{\beta > 2\}} - I] S_f(P_t, \bar{P}, \infty) - G_t^\infty(P_t) \mathbf{1}_{\{\beta > 2\}} & \Leftarrow P_t < \bar{P} \\ V_p(P_t, L, T) - I & \Leftarrow P_t \geq \bar{P} \end{cases}, \end{aligned} \quad (4.53)$$

with the stochastic discount factor of the perpetual option to invest being still given by equation (4.27). Two cases must now be considered:

(i) For $\bar{P} < L$, the optimal threshold $\bar{P}$ is obtained as the solution of the single nonlinear equation

$$\begin{aligned} & \left(\frac{L}{r} - e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + \frac{\bar{P}}{q} e^{-q(T-t)} - I \right) \left(\bar{P}^{-1} + z'(\bar{P}) \frac{I_{\nu+1}(z(\bar{P}))}{I_\nu(z(\bar{P}))} \right) \\ & + e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) - \frac{1}{q} e^{-q(T-t)} = 0, \end{aligned} \quad (4.54)$$

for $\beta < 2$, and

$$\begin{aligned} & \left(\frac{L}{r} - e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + g(\bar{P}) - I \right) \left(-z'(\bar{P}) \frac{K_{\nu+1}(z(\bar{P}))}{K_\nu(z(\bar{P}))} \right) \\ & + e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) - g'(\bar{P}) = 0, \end{aligned} \quad (4.55)$$

for $\beta > 2$, with

$$\begin{aligned} & V_{f,nb}(\bar{P}, L, \infty) \\ := & \begin{cases} [A_0 I(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + B_0 K(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta < 2 \\ [B_0 K(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + A_0 I(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta > 2 \end{cases} \end{aligned} \quad (4.56)$$

representing the time- t risk-neutral expectation (4.36) but excluding the bubble effect,

$$z'(\bar{P}) = \left(1 - \frac{\beta}{2}\right) \frac{2\sqrt{2r}}{\delta|\beta - 2|} \bar{P}^{-\frac{\beta}{2}}, \quad (4.57)$$

$$g(\bar{P}) = \frac{\bar{P}}{q} e^{-q(T-t)} \frac{\gamma\left(\nu, \tilde{f}(\bar{P})/2\right)}{\Gamma(\nu)} \quad (4.58)$$

and

$$g'(\bar{P}) = \frac{1}{q} e^{-q(T-t)} \left[\frac{\gamma\left(\nu, \tilde{f}(\bar{P})/2\right)}{\Gamma(\nu)} + \frac{2-\beta}{\Gamma(\nu)} \left[\tilde{f}(\bar{P})/2\right]^\nu e^{-\tilde{f}(\bar{P})/2} \right], \quad (4.59)$$

after employing Olver et al. (2010, equation 8.8.13) for tackling the derivative of the term involving the incomplete gamma function $\gamma(\nu, \tilde{f}(\bar{P})/2)$.

(ii) For $\bar{P} \geq L$, the optimal threshold $\bar{P}$ and the constants $A_1 \equiv A_1(\bar{P})$ and $B_1 \equiv B_1(\bar{P})$ (associated to the value of the option to invest) are obtained as the solution of the system of equations

$$\begin{cases} A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) - B_0 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - \frac{\bar{P}}{q} (1 + e^{-q(T-t)}) \\ \quad + e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + I = 0 \\ A_1 I_{\nu+1}(z(\bar{P})) + B_0 K_{\nu+1}(z(\bar{P})) \\ \quad - e^{-r(T-t)} \frac{\delta}{\sqrt{2r}} \bar{P}^{\frac{\beta-3}{2}} [V_{f,nb}(\bar{P}, L, \infty) - \bar{P} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty)] - \frac{\delta I}{\sqrt{2r}} \bar{P}^{\frac{\beta-3}{2}} = 0 \end{cases}, \quad (4.60)$$

for $\beta < 2$, and

$$\begin{cases} B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - A_0 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) - \frac{\bar{P}}{q} - g(\bar{P}) \\ \quad + e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + I = 0 \\ B_1 K_{\nu+1}(z(\bar{P})) + A_0 I_{\nu+1}(z(\bar{P})) - \frac{\delta}{\sqrt{2r}} \bar{P}^{\frac{\beta-1}{2}} \left[\frac{1}{q} + g'(\bar{P}) \right] \\ \quad + e^{-r(T-t)} \frac{\delta}{\sqrt{2r}} \bar{P}^{\frac{\beta-1}{2}} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) = 0 \end{cases}, \quad (4.61)$$

for $\beta > 2$, with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (4.7) for any $\beta \neq 2$.

Proof. Please see Appendix 4F. ■

Proposition 4.6. Assume the CEV process with $r \neq q$ and the notation introduced in Proposition 4.2. The time- t value of the (perpetual American-style) option to invest in a project with a finite-lived guarantee is still given by equation (4.53), but with the stochastic discount factor of the perpetual option to invest being given by equation (4.31). Again,

two cases must be considered:

(i) For $\bar{P} < L$, the optimal threshold $\bar{P}$ is obtained as the solution of the single nonlinear equation

$$\begin{aligned} & \left(\frac{L}{r} - e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + \frac{\bar{P}}{q} e^{-q(T-t)} - I \right) \left(\bar{P}^{-1} + \left[\frac{\epsilon-1}{2} + \frac{\frac{a}{b} M(a+1, b+1, x(\bar{P}))}{M(a, b, x(\bar{P}))} \right] x'(\bar{P}) \right) \\ & + e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) - \frac{1}{q} e^{-q(T-t)} = 0, \end{aligned} \quad (4.62)$$

for $\beta < 2$, and

$$\begin{aligned} & \left(\frac{L}{r} - e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + g(\bar{P}) - I \right) \left(\frac{\epsilon-1}{2} - \frac{aU(a+1, b+1, x(\bar{P}))}{U(a, b, x(\bar{P}))} \right) x'(\bar{P}) \\ & + e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) - g'(\bar{P}) = 0, \end{aligned} \quad (4.63)$$

for $\beta > 2$, with

$$\begin{aligned} & V_{f,nb}(\bar{P}, L, \infty) \quad (4.64) \\ := & \begin{cases} [A_0 Y M_P(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + B_0 Y U_P(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta < 2 \\ [B_0 Y U_N(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + A_0 Y M_N(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta > 2 \end{cases} \end{aligned}$$

representing the time- t risk-neutral expectation (4.47) but excluding the bubble effect and

$$x'(\bar{P}) = (2 - \beta) \frac{2|r - q|}{\delta^2 |\beta - 2|} \bar{P}^{1-\beta}. \quad (4.65)$$

(ii) For $\bar{P} \geq L$, the optimal threshold $\bar{P}$ and the constants $A_1 \equiv A_1(\bar{P})$ and $B_1 \equiv B_1(\bar{P})$ (associated to the value of the option to invest) are obtained as the solution of the system of equations

$$\begin{cases} A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) - B_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} U(a, b, x(\bar{P})) \\ \quad - \frac{\bar{P}}{q} (1 + e^{-q(T-t)}) + e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + I = 0 \\ A_1 \frac{a}{b} M(a+1, b+1, x(\bar{P})) + B_0 a U(a+1, b+1, x(\bar{P})) \\ \quad - \left[I + e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) - \frac{\bar{P}}{q} (1 + e^{-q(T-t)}) \right] \\ \quad \times Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \bar{P}^{-1} \left(\frac{\epsilon-1}{2} + \frac{\delta^2}{2|r-q|} \bar{P}^{\beta-2} \right) \\ \quad - Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \frac{\delta^2}{2|r-q|} \bar{P}^{\beta-2} \left[\frac{1}{q} (1 + e^{-q(T-t)}) - e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) \right] = 0 \end{cases} \quad (4.66)$$

for $\beta < 2$, and

$$\begin{cases} B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) - A_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y M(a, b, x(\bar{P})) \\ - \frac{\bar{P}}{q} - g(\bar{P}) + e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) + I = 0 \\ B_1 a U(a+1, b+1, x(\bar{P})) + A_0 \frac{a}{b} M(a+1, b+1, x(\bar{P})) \\ - \left[\frac{\bar{P}}{q} + g(\bar{P}) - e^{-r(T-t)} V_{f,nb}(\bar{P}, L, \infty) - I \right] \frac{\epsilon-1}{2} Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \\ - Y^{-1} e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \frac{\delta^2}{2|r-q|} \bar{P}^{\beta-1} \left[\frac{1}{q} + g'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) \right] = 0 \end{cases}, \quad (4.67)$$

for $\beta > 2$, with the (already known) constants $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$ (associated to the value of the project) being obtained as the solution of the system of equations (4.18) and (4.19) for $\beta < 2$ and $\beta > 2$, respectively.

Proof. Please see Appendix 4G. ■

To sum up, Propositions 4.5 and 4.6 can be used to analyze the optimal decision to invest in the case of a FIT contract with a finite maturity guarantee: exercise the (perpetual) option to invest only if $P_t \geq \bar{P}$; otherwise, i.e., if $P_t < \bar{P}$, it is better to wait and keep the option alive.

Notice that Propositions 4.5 and 4.6 are the CEV counterparts of the solution offered in Barbosa et al. (2018, Section 3.2) for the GBM process, whose numerical results can be obtained by taking $\beta \rightarrow 2$ in our solutions.⁸ Fortunately, the optimal thresholds of the investment problem under analysis are independent of any bubble effect because both the perpetual option to invest and the value of the active project possess the same bubble. Therefore, equations (4.56) and (4.64) do not include any bubble effect for computing the trigger levels $\bar{P}$ in Propositions 4.5 and 4.6.

Note also that the value-matching and smooth-pasting conditions are now met in both regions $\bar{P} \geq L$ and $\bar{P} < L$. The case $\bar{P} < L$ in Propositions 4.5 and 4.6 requires only the use of a single nonlinear equation to determine the corresponding optimal trigger level $\bar{P}$ (in particular, equations (4.54) and (4.62) for $\beta < 2$ and equations (4.55) and (4.63) for $\beta > 2$). As expected, the associated constants of the perpetual option to invest ($A_1 \equiv A_1(\bar{P})$ for $\beta < 2$ and $B_1 \equiv B_1(\bar{P})$ for $\beta > 2$) can be calculated using equations (4F.4) and (4F.6), for $r = q$, and equations (4G.3) and (4G.5), for $r \neq q$, respectively.

To determine which region should be used to calculate the optimal trigger $\bar{P}$ it is necessary to find first the investment cost level I_L^* that separates the regions $\bar{P} \geq L$ and $\bar{P} < L$. This is accomplished by replacing $\bar{P}$ by the fixed price floor L (previously defined by the government) in the systems of equations (4.60), (4.61), (4.66) and (4.67) and solving each one with respect to $I \equiv I_L^*$. Notice that these are the equations that should be used since are the ones for which

⁸By the way, and for the sake of clarity, we notice that there is a very small typo in Barbosa et al. (2018, equation C.3). The correct formula is the one given in Barbosa et al. (2018, equation 26).

the condition $\bar{P} = L$ is valid. Similarly to the perpetual case, I_L^* is economically interpreted as the required investment cost that makes the current L the optimal trigger of investment, but now for the finite maturity case. Again, the constants A_0 and B_0 are functions of L and, hence, they are known in advance using the systems of equations (4.7), (4.18) and (4.19) or, alternatively, the analytic solutions offered in Remarks 4.1 and 4.2. Armed with the values of $A_0 \equiv A_0(L)$ and $B_0 \equiv B_0(L)$, the I_L^* level is then easily computed using the systems of equations (4.60) or (4.61), for the case $r = q$, and (4.66) or (4.67), for the case $r \neq q$. If $I_L^* \leq I$, the optimal threshold $\bar{P}$ is calculated using the region $\bar{P} \geq L$, i.e., $\bar{P}$ is obtained by solving the systems of nonlinear equations (4.60) or (4.61), for $r = q$, and (4.66) or (4.67), for $r \neq q$. If $I_L^* > I$, the optimal trigger $\bar{P}$ is computed in the region $\bar{P} < L$, i.e., $\bar{P}$ is calculated by solving the single nonlinear equations (4.54) or (4.55), for $r = q$, and (4.62) or (4.63), for $r \neq q$.

The knowledge of I_L^* under the finite guarantee case allow us to find an additional interesting point that is obtained in the appropriate region $\bar{P} < L$ or $\bar{P} \geq L$, depending on $I_L^* > I$ or $I_L^* \leq I$, respectively. For instance, suppose that $I_L^* \leq I$ for the case with $\beta < 2$ and $r = q$. By solving the system of equations (4.60) with respect to $I \equiv I^*$, and with $\bar{P}$ replaced by the given time- t price P_t and assuming a fixed price floor L (that is subject to the government policy), it is possible to find the required level of investment I^* that would induce the firm to exercise the option to invest immediately, which can then be compared with the initial investment cost I of the project. As before, this I^* value indicates the required level of investment that would make the time- t price P_t the trigger value to invest. Hence, it would be preferable for the firm to invest immediately if $I \leq I^*$ because $P_t = \bar{P}$ at the I^* level. A similar rational can be applied to the case of other combinations of β parameters and drift specifications.

Several other interesting points from the perspective of a policymaker can also be determined. For instance, taking now the initial investment cost I of the project as given and using the previous example, we can determine the point where the optimal threshold value $\bar{P}$ is equal to the price floor L , by solving the system of equations (4.60) with respect to $L \equiv L^*$, which can then be compared with the minimum price guarantee L that is offered by the government. Note that if the policymaker sets the minimum price guarantee $L \geq L^*$, the investor starts immediately the project and receives a revenue from the guarantee instead of a revenue from the market price. Again, a similar rational can be used in the case of other combinations of β parameters and drift specifications.

Even though the levels of I_L^* and L^* under the perpetual case have been shown to be independent of β , such independence is not valid in the case of FIT contracts with a finite maturity guarantee. In this case, in the range of price floors $L \in [0, L^*]$ both the price floor L^* and the optimal trigger value $\bar{P}$ decrease as we move further away to the left of $\beta < 2$ and increase as β rises for the case with $\beta > 2$. This contrasting behavior when compared with the

perpetual case is explained by the finite nature of the guarantee, i.e., now the producer is not entitled to receive a fixed guarantee forever and, hence, the effect of β on I_L^* and L^* do not disappear. Again, since different CEV models imply the existence of different optimal triggers (including now distinct L^* levels), a policy scheme based on a GBM assumption might imply a too high or a too low price floor to encourage a producer to exercise the option to invest if the specific investment problem is better described with an alternative CEV specification.

A further interesting point from the policymaker perspective is the price floor level L_0 that turns the NPV of the project equal to zero, because any value of L above this point generates a positive NPV independently of the market price P . Notice that if $L \geq L_0$ the investment will be deployed immediately because there is no waiting option. Since under the CEV process with $\beta < 2$ the origin is attainable (and assumed to be absorbing), this point can be determined analytically by solving the equation $\text{NPV} := V_p(P_t, L_0, T) - I = 0$ with respect to L_0 when $P_t = 0$ and with $V_p(0, L_0, T)$ being defined as the project value given in equation (4.52), which yields⁹

$$L_0 = \frac{Ir}{1 - e^{-r(T-t)}}. \quad (4.68)$$

However, since under the CEV process with $\beta > 2$ the origin is not attainable, it is not possible to obtain a closed-form solution as given in equation (4.68). Nevertheless, as shown in 4H, the level L_0 can still be determined numerically by solving the equation

$$\begin{aligned} & \lim_{P_t \rightarrow 0^+} V_p(P_t, L_0, T) - I = 0, \quad (4.69) \\ \text{i.e., } & \begin{cases} \lim_{P_t \rightarrow 0^+} [B_0(L_0)K(-1, P_t, L_0)] + \frac{L_0}{r} (1 - e^{-r(T-t)}) - I = 0 & \Leftarrow r = q \\ \lim_{P_t \rightarrow 0^+} [B_0(L_0)YU_N(-1, P_t, L_0)] + \frac{L_0}{r} (1 - e^{-r(T-t)}) - I = 0 & \Leftarrow r \neq q \end{cases} \end{aligned}$$

with respect to L_0 when $P_t \rightarrow 0^+$ and with $B_0(L_0)$ being computed by equations (4.9) and (4.23) for $r = q$ and $r \neq q$, respectively. The differences found between equations (4.68) and (4.69) are clearly expected and can be explained by the fact that the process with $\beta > 2$ is not trapped at the origin at time $t = 0$ and, therefore, the forward start perpetuity that is included in the project value (4.52) is influenced by a nonzero forward value P_T . By contrast, if $\beta < 2$ the process is absorbed at the origin and, hence, the forward start perpetuity that is included in the project value (4.52) does not depend on the forward value P_T because $P_T = 0$ (i.e., the process remains in a cemetery state forever).

Therefore, if the government sets a policy with a price floor $L \geq L_0$, the investment will be made immediately generating a risk-free profit. Moreover, by setting L in the range $L^* < L < L_0$, the policymaker gives the investor a waiting option and the project starts with

⁹The proof of such asymptotic analysis for both $\beta < 2$ and $\beta > 2$ is presented in 4H using results borrowed from Sankaran (1963), Temme (1993), Johnson et al. (1995, Chapter 29) and Ruas et al. (2013).

a revenue from the guarantee.

Even though equation (4.68) that has been obtained using the CEV model with $\beta < 2$ is exactly the same to the one derived in Barbosa et al. (2018, equation 31) under the GBM assumption¹⁰—thus, implying that L_0 is, as expected, independent of β (for $\beta < 2$)—, the range of values $]L^*, L_0[$ is not the same across models with different skew patterns due to the influence of the β parameter on L^* . Furthermore, the limit of (4.68) when $T \rightarrow \infty$ is consistent with the observations that have been made in Remark 4.3, which also implies that the range $]L^*, L_0[$ vanishes as $T \rightarrow \infty$. This means that, under the perpetual case, there is no range of values in which the investor has an option to wait and a revenue from the guarantee when the project starts.

4.3.3. Derivatives of the forward start perpetuity components

The implementation of Proposition 4.5 still requires the computation of the derivative of the risk-neutral expectation (4.56) with respect to $\bar{P}$, that is

$$\begin{aligned} & \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) \\ = & \frac{\partial}{\partial \bar{P}} \begin{cases} [A_0 I(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + B_0 K(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta < 2 \\ [B_0 K(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + A_0 I(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (4.70)$$

with the (already known) constants A_0 and B_0 (associated to the value of the project) being obtained as the solution of the system of equations (4.7) for any $\beta \neq 2$, whereas the computation of Proposition 4.6 requires the derivative of the risk-neutral expectation (4.64) with respect to $\bar{P}$, that is

$$\begin{aligned} & \frac{\partial}{\partial \bar{P}} V_{f,nb}(\bar{P}, L, \infty) \\ = & \frac{\partial}{\partial \bar{P}} \begin{cases} [A_0 Y M_P(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + B_0 Y U_P(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta < 2 \\ [B_0 Y U_N(-1, \bar{P}, L) + X_r(-1, \bar{P}, L) + A_0 Y M_N(1, \bar{P}, L) + P_q(1, \bar{P}, L)] & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (4.71)$$

with the (already known) constants A_0 and B_0 (associated to the value of the project) being obtained as the solution of the system of equations (4.18) and (4.19) for $\beta < 2$ and $\beta > 2$, respectively.

¹⁰A tiny technical detail arises here that deserves to be mentioned for the sake of completeness. Although Barbosa et al. (2018, equation 31) has been derived by simply imposing $P_t = 0$ in Barbosa et al. (2018, equation 19), this should not be formally made because, as argued, for instance, in Carr and Linetsky (2006, Remark 5.2) or Nunes et al. (2015, page 352), the limiting process of GBM never hits zero (i.e., zero is a natural boundary and not an exit boundary). Nevertheless, Barbosa et al. (2018, equation 31) is still obtained if we compute the limit of Barbosa et al. (2018, equation 19) when $P_t \rightarrow 0^+$.

Even though the derivatives (4.70) and (4.71) can be easily computed through a finite difference scheme, we prefer to use the analytical expressions shown in 4I that have been derived following the insights of Carr and Linetsky (2006), Ruas et al. (2013) and Dias and Nunes (2018).

Remark 4.5. *Clearly, the delta of a finite maturity FIT contract, i.e., $\partial V_f(P_t, L, T)/\partial P_t$, is straightforward to be evaluated via the appropriate combination of our analytical results. Moreover, Dias et al. (2023, equation 73) implies that the delta of a finite maturity FIT is equal to the delta of a finite-lived cap. Hence, we are able to extend the computation of the delta shown in Shackleton and Wojakowski (2007, equation 28) under the GBM diffusion to the more general CEV process. This should be helpful for the potential design of dynamic hedging strategies of such contracts.*

4.4. Numerical results

The goal of this section is to present some numerical experiments of the novel theoretical results proposed in this paper. To accomplish this purpose, we adopt the base case parameters configuration of Barbosa et al. (2018) that is associated to a typical European onshore wind farm. Even though we are adopting the case of an onshore wind farm in our numerical examples, our analytical formulae is suitable to be used in any other technology with FIT contracts.

Following Barbosa et al. (2018), we assume a riskless interest rate $r = 5\%$, a dividend yield (or rate of return shortfall) $q = 5\%$, a volatility $\sigma = 19\%$, a price floor $L = 25$ EUR/MWh and a finite duration of $T = 15$ years. It is also assumed that each wind turbine has a capacity of 2 MW and requires a total investment cost of 1.5 million EUR/MW, thus leading to a total investment cost per wind turbine of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000/(0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$ for the case of a power plant with an assumed capacity factor of 30%. Although we analyze the investment threshold for a single wind turbine, the results can be easily calculated for a wind park with N turbines.

Figure 4.1 presents the optimal thresholds $\bar{P}$ under the case of a perpetual guarantee as a function of the price floor L for different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 3, 3.5\}$. The dashed line displays the price floor level L that separates the two regions in the graph, i.e., $\bar{P} \geq L$ and $\bar{P} < L$. Note that only the region of trigger values above or equal to the dashed line L represents the market conditions for which $\bar{P} \geq L$. Following the insights of Remark 4.3, trigger values below the dashed line L are not admissible for a FIT scheme with a perpetual guarantee since the value-matching and smooth-pasting conditions are not met and, hence, the investment problem is not well behaved in this case. The case with $\beta \neq 2$ has been computed through Proposition 4.3, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 16).

We first notice that the triggers of all tested β parameters decrease as the price floor rises, thus inducing earlier investment decisions for higher price floor values. Moreover, for the range of price floors $L \in [0, L^*[$ the optimal trigger value $\bar{P}$ decreases as we move further away to the left of $\beta < 2$ and increases as β rises for the case with $\beta > 2$. This is explained by the different behavior of the leverage effects attached to CEV models for $\beta < 2$ and $\beta > 2$. Note also that the aforementioned independence of the β parameter is observed only at the optimal trigger level $\bar{P} \equiv L^* = 28.54$, which is exactly the L^* value that is obtained under the nested GBM assumption considered in Barbosa et al. (2018, equation 18). The perpetual nature of the guarantee makes the effect of β on L^* (and I_L^*) to disappear thus leading to this interesting finding.

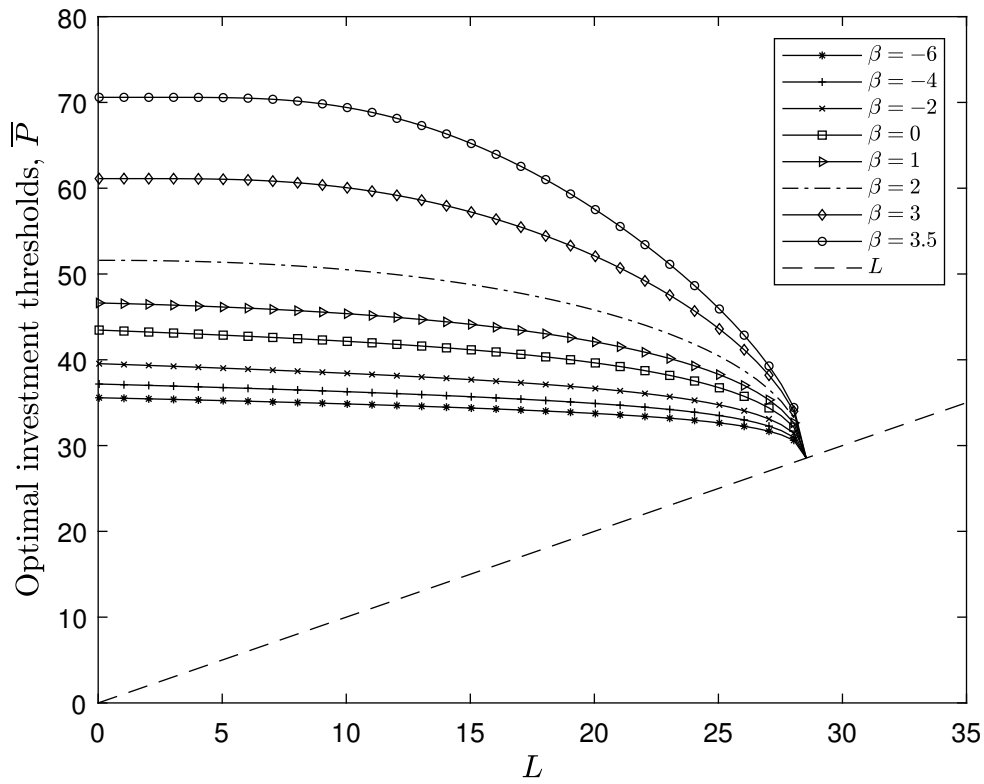


Figure 4.1.: Perpetual optimal investment thresholds, $\bar{P}$, as a function of the price floor L for different β parameters with $P_0 = 25$.

Other parameters borrowed from Barbosa et al. (2018): $\sigma(P_0) = 0.19$, $r = 0.05$, $q = 0.05$ and total investment cost of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000/(0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$. The case with $\beta \neq 2$ has been computed through Proposition 4.3, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 16).

In the case of a FIT contract with a perpetual guarantee under the CEV model, the value of $L^* = 28.54$ can be obtained numerically by solving the systems of four equations referred in Subsection 4.3.1.2 for each β value—with the relevant constants being functions of L^* , i.e., $A_0 \equiv A_0(L^*)$, $B_0 \equiv B_0(L^*)$, $A_1 \equiv A_1(L^*)$ and $B_1 \equiv B_1(L^*)$ —or, alternatively, via the closed-form solution $L^* = rI$, which directly highlights the independence of L^* with respect

to β . Nevertheless, different optimal triggers for distinct β values are obtained in the range $L \in [0, 28.54[$, which implies that a government that (wrongly) assumes a GBM process to define a FIT policy might be paying a too high or a too low price floor to stimulate a firm to exercise the option to invest if the specific investment problem is better described by a different CEV specification.

Figure 4.2 displays the optimal thresholds $\bar{P}$ under the case of a finite maturity guarantee as a function of the price floor L for different CEV models with $\beta \in \{-6, -4, -2, 0, 1, 2, 2.7, 4\}$. Again, the dashed line displays the price floor level L that separates the two regions in the graph, i.e., $\bar{P} \geq L$ and $\bar{P} < L$. As mentioned before, both regions are now valid because the value-matching and smooth-pasting conditions are always met in the presence of a finite maturity guarantee. The case with $\beta \neq 2$ has been computed through Proposition 4.5, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 26).

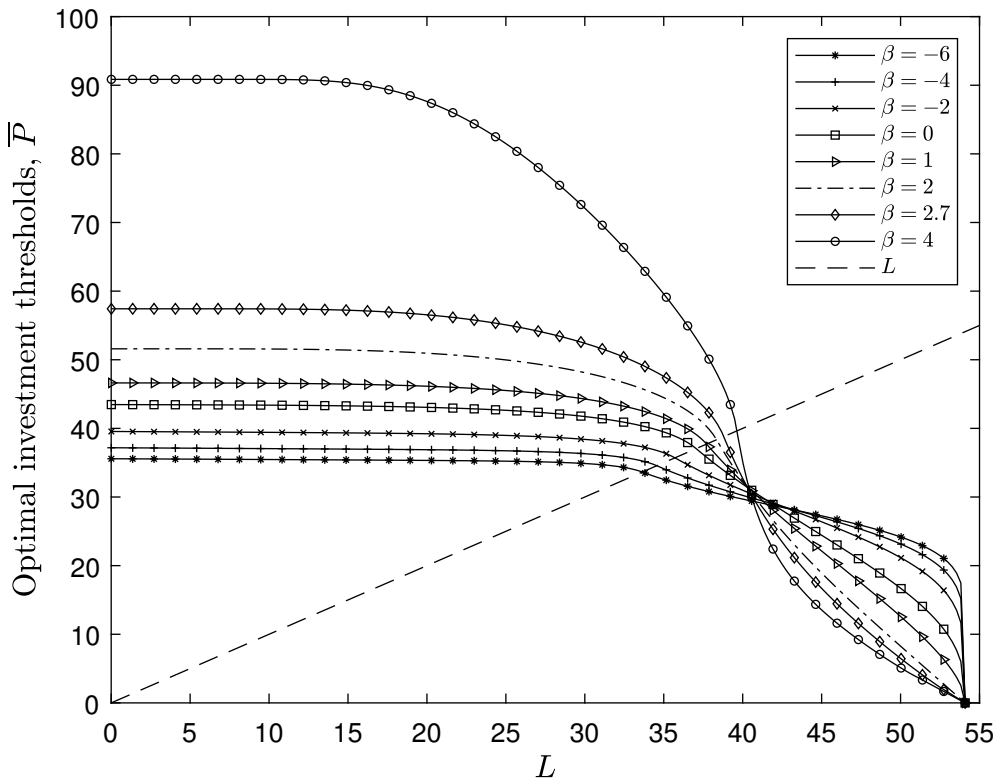


Figure 4.2.: Finite maturity optimal investment thresholds, $\bar{P}$, as a function of the price floor L for different β parameters with $P_0 = 25$.

Other parameters borrowed from Barbosa et al. (2018): $T = 15$, $\sigma(P_0) = 0.19$, $r = 0.05$, $q = 0.05$ and total investment cost of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000 / (0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$. The case with $\beta \neq 2$ has been computed through Proposition 4.5, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 26).

Similarly to the perpetual case, the region $\bar{P} \geq L$ under the finite maturity case originates triggers that decrease as the price floor increases for the whole set of β parameters. As expected, this induces earlier investment decisions for higher price floor values. Furthermore,

for the range of price floors $L \in [0, L^*]$ the optimal trigger value $\bar{P}$ still decreases as we move further away to the left of $\beta < 2$ and increases as β rises for the case with $\beta > 2$. The rational is again explained by the different behavior of the leverage effects associated to CEV models with $\beta < 2$ and $\beta > 2$. From a policymaker perspective, the region $\bar{P} \geq L$ creates scenarios for which the project starts receiving a revenue from the market price rather than a revenue from the guarantee. It is also possible to observe that for a wide range of price floors $L \in [0, L^*]$ the policy scheme is useless in practical terms because the lines representing the optimal thresholds are almost flat, at least for lower values of L and, in particular, for the case with $\beta < 2$.

In the case of a FIT contract with a finite maturity guarantee, however, the trigger point $\bar{P} \equiv L^*$ depends on the β parameter and needs to be determined numerically. Table 4.1 shows the optimal trigger points $\bar{P}$ that are equal to the price floor $L \equiv L^*$ of the finite maturity case displayed in Figure 4.2. We first note that the trigger point $L^* = 28.54$ determined for the perpetual guarantee case is always lower than the ones reported in Table 4.1 for the finite maturity case. This is expected since an investor requires a higher threshold to invest in a project that only ensures a finite guarantee instead of a perpetual one. We also conclude that the range $[0, L^*]$ widens as the β parameter rises. This should have important implications for a policymaking perspective. For instance, comparing the cases with $\beta = -6$ and $\beta = 2$, we observe that $(38.35 - 33.64) = 4.71$ EUR/MWh (per turbine) thus leading to an annual amount of $(38.35 - 33.64) \times 24 \text{ h} \times 365 \text{ days} = 41,259.60$ EUR per turbine. For a typical European onshore wind farm with 25 wind turbines this would lead to a difference of 1,031,490 Euros. This would represent an amount that tax payers would be paying in excess (only for a single wind farm) if the regulator negotiated the contract based on a GBM assumption when the price process was better described by a CEV process with $\beta = -6$.

Table 4.1.: Optimal investment thresholds $\bar{P}$ that are equal to the price floor $L \equiv L^*$ of the finite maturity case displayed in Figure 4.2.

β	-6	-4	-2	0	1	2	2.7	4
L^*	33.64	34.56	35.71	37.05	37.71	38.35	38.80	39.64

This table shows the optimal trigger points $\bar{P}$ that are equal to the price floor $L \equiv L^*$ of the finite maturity case displayed in Figure 4.2. For $r = q$ and $\beta < 2$, L^* is calculated by solving the systems of equations (4.7) with $L \equiv L^*$ and (4.60) with $\bar{P} \equiv L^*$. For $r = q$ and $\beta > 2$, L^* is computed by solving the systems of equations (4.7) with $L \equiv L^*$ and (4.61) with $\bar{P} \equiv L^*$. Notice that the relevant constants should be taken as functions of L^* , i.e., $A_0 \equiv A_0(L^*)$, $B_0 \equiv B_0(L^*)$, $A_1 \equiv A_1(L^*)$ and $B_1 \equiv B_1(L^*)$. Finally, for $\beta = 2$, L^* is determined via Barbosa et al. (2018, equation 30).

In contrast to the perpetual case, the region $\bar{P} < L$ allows the existence of threshold values that range in the interval $L \in [L^*, L_0]$, with $L_0 = 54.09$ EUR/MWh being determined through equations (4.68) and (4.69) for $\beta < 2$ and $\beta > 2$, respectively. Interestingly, this is the value that is obtained for the GBM process calculated via Barbosa et al. (2018, equation 31). This means that if the government sets a policy with a price floor $L \geq L_0$, then a

positive NPV arises and the firm can undertake the project immediately. Finally, by setting the price floor in the range $L \in]L^*, L_0[$, the policymaker gives the investor a waiting option and the project starts with a revenue from the finite maturity guarantee. Notice also that in this region there is a price floor level L for which the optimal thresholds are the same for pairs of β parameters. Moreover, after this point lower β parameters lead to higher threshold values.

Even though the optimization problem involving the implementation of Propositions 4.3, 4.4, 4.5 and 4.6 is easily tackled in any computer language, the calculation and the convergence of the required optimal threshold values $\bar{P}$ depend on the combination of the used parameters. Nevertheless, for the cases where it might be hard to obtain convergent solutions it is possible to apply a linear interpolation scheme. For example, a simple linear interpolation between the values reported in Table 4.1 for $\beta = 1$ and $\beta = 2.7$ (both calculated through Proposition 4.5) allows us to find an interpolated value for the case with $\beta = 2$, which can then be compared with the one obtained via Barbosa et al. (2018, equation 30), that is: $37.71 + (38.80 - 37.71) \times (2 - 1) / (2.7 - 1) = 38.35$, which is exactly the value that is obtained under the GBM process. Clearly, this is a further simple confirmation of the robustness of our theoretical results.

Table 4.2.: Optimal investment thresholds of perpetual and finite maturity FIT contracts.

#	L	r	q						Maximum	
				-4	-2	0	2	4	% difference	
Panel A: Optimal investment thresholds under the perpetual case										
1	10	0.07	0.03	41.03	42.63	46.16	54.88	106.44	25.24	93.93
2	15	0.07	0.03	41.03	42.61	46.10	54.75	105.99	25.06	93.61
3	20	0.07	0.03	41.03	42.59	46.01	54.41	102.64	24.59	88.66
4	10	0.05	0.05	36.28	38.42	42.16	50.51	89.16	28.16	76.53
5	15	0.05	0.05	35.70	37.69	41.17	48.78	81.15	26.82	66.36
6	20	0.05	0.05	34.91	36.66	39.65	45.80	67.03	23.77	46.34
7	10	0.03	0.07	32.44	33.88	36.79	43.59	61.56	25.58	41.22
8	15	0.03	0.07	27.13	27.37	28.40	31.14	34.37	12.88	10.36
9	20	0.03	0.07	-	-	-	-	-	-	-

Panel B: Optimal investment thresholds under the finite case with $T = 5$ years

1	15	0.05	0.05	37.17	39.55	43.47	51.59	90.85	27.95	76.10
2	20	0.05	0.05	37.17	39.54	43.46	51.58	90.77	27.93	76.00
3	25	0.05	0.05	37.16	39.52	43.42	51.50	90.26	27.85	75.28
4	30	0.05	0.05	37.09	39.44	43.30	51.25	88.81	27.62	73.29
5	35	0.05	0.05	36.70	39.10	42.93	50.67	86.08	27.56	69.89
6	40	0.05	0.05	35.31	37.59	41.76	49.44	81.90	28.59	65.64
7	15	0.03	0.04	33.66	35.25	37.49	40.63	47.98	17.14	18.09
8	20	0.03	0.04	33.64	35.22	37.44	40.55	47.80	17.03	17.89
9	25	0.03	0.04	33.57	35.13	37.30	40.25	46.98	16.59	16.71
10	30	0.03	0.04	33.33	34.84	36.90	39.49	45.02	15.58	14.00
11	35	0.03	0.04	32.16	33.82	35.78	37.74	41.31	14.80	9.46
12	40	0.03	0.04	31.82	32.82	30.83	34.54	32.16	10.74	6.90

This table shows the values of optimal investment thresholds of perpetual and finite maturity FIT contracts under the CEV process with $\beta \in \{-4, -2, 0, 2, 4\}$ and $P_0 = 25$. The price floor L , the riskless interest rate r and the dividend yield (or rate of return shortfall) q vary as indicated in the second, third and fourth columns, respectively. Other parameters borrowed from Barbosa et al. (2018): $\sigma(P_0) = 0.19$ and total investment cost of 3 million Euros, which implies an investment cost per MWh of $I = 3,000,000 / (0.3 \times 2 \text{ MWh} \times 24 \text{ h} \times 365 \text{ days})$. The two utmost right columns give the maximum absolute value of the percentage difference between the GBM trigger value and the CEV trigger values for $\beta < 2$ and $\beta > 2$, respectively, relative to the GBM trigger value. For Panel A, the case with $r = q$ and $\beta \neq 2$ has been computed through Proposition 4.3, the case with $r \neq q$ and $\beta \neq 2$ has been calculated using Proposition 4.4, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 16). For Panel B, the case with $r = q$ and $\beta \neq 2$ has been computed through Proposition 4.5, the case with $r \neq q$ and $\beta \neq 2$ has been calculated using Proposition 4.6, whereas the case with $\beta = 2$ has been implemented via Barbosa et al. (2018, equation 26).

Finally, even though we have adopted the base case parameters configuration of Barbosa et al. (2018), for the sake of completeness and future reproducibility of our results by other interested academics and practitioners, Table 4.2 provides investment thresholds for a wide variety of β parameters and different drift specifications. These results confirm the previous observations that care must be taken when a policymaker aims to design a given policy to induce investment decisions with the help of FIT contracts since the differences between trigger points under alternative modeling assumptions are quite significant and, in most of the cases, the excessive rents are paid at the expense of tax payers. Notice that contract #9 of Panel A is intended only to show an example of a contract where the value-matching and smooth-pasting conditions are not accommodated since it refers to the region $\bar{P} < L$ of the perpetual case.

4.5. Conclusions

This paper presents analytical formulae to analyze optimal investment decisions of FIT contracts with a minimum price guarantee under the CEV model considering both perpetual and finite-lived guarantees. Overall, we document that the argument that a perpetual guarantee only induces investment for prices below the price floor when offering a risk-free investment opportunity is still valid under the CEV process. We also show that the optimal price-floor level triggering immediate investment in the presence of a perpetual guarantee is independent of the elasticity parameter of the CEV model. Moreover, we show that such independence is not valid in the case of FIT contracts with a finite maturity guarantee. In summary, our

results strongly highlight the importance of policymakers to move beyond the more simplistic real options analysis based on the GBM process to more realistic models incorporating leverage and volatility smile effects because the differences between trigger points originate excessive rents that are commonly paid at the expense of tax payers.

Appendices

4A. Proof that the value-matching and smooth-pasting conditions are not met under the perpetual case conditional on $P_t < L$

4A.1. The case with $r = q$

Let us start with the case $\beta < 2$. The application of the value-matching and smooth-pasting conditions using equation (4.6) conditional on $P_t < L$ yields

$$\begin{aligned} & \begin{cases} F_f(\bar{P}, \bar{P}, \infty) = V_f(\bar{P}, L, \infty) \mathbb{1}_{\{P_t < L\}} - I \\ \left. \frac{\partial}{\partial P_t} F_f(P_t, \bar{P}, \infty) \right|_{P_t = \bar{P}} = \left. \frac{\partial}{\partial P_t} V_f(P_t, L, \infty) \mathbb{1}_{\{P_t < L\}} \right|_{P_t = \bar{P}} \end{cases} \\ \Leftrightarrow & \begin{cases} A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = A_0 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) + \frac{L}{r} - I \\ A_1 \frac{\partial}{\partial \bar{P}} \left[\bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) \right] = A_0 \frac{\partial}{\partial \bar{P}} \left[\bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) \right] \end{cases} . \end{aligned} \quad (4A.1)$$

Clearly, the smooth-pasting condition in the system of equations (4A.1) implies that $A_1 = A_0$ and, hence, the use of the corresponding value-matching condition would lead to $\frac{L}{r} = I$, thus resulting in the observation highlighted in Remark 4.3.

Moving now to the case $\beta > 2$, we would have

$$\begin{aligned} & \begin{cases} F_f(\bar{P}, \bar{P}, \infty) = V_f(\bar{P}, L, \infty) \mathbb{1}_{\{P_t < L\}} - I \\ \left. \frac{\partial}{\partial P_t} F_f(P_t, \bar{P}, \infty) \right|_{P_t = \bar{P}} = \left. \frac{\partial}{\partial P_t} V_f(P_t, L, \infty) \mathbb{1}_{\{P_t < L\}} \right|_{P_t = \bar{P}} \end{cases} \\ \Leftrightarrow & \begin{cases} B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) = B_0 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) + \frac{L}{r} - G_t^\infty(\bar{P}) - I \\ B_1 \frac{\partial}{\partial \bar{P}} \left[\bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) \right] = B_0 \frac{\partial}{\partial \bar{P}} \left[\bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) \right] \end{cases} . \end{aligned} \quad (4A.2)$$

In this case, $B_1 = B_0$ and, therefore, we obtain, again, $\frac{L}{r} = I$, thus leading to the arguments pointed out in Remark 4.3.

In summary, the value-matching and smooth-pasting conditions are not met under the perpetual case conditional on $P_t < L$.

4A.2. The case with $r \neq q$

This case is obtained similarly with the only difference being the requirement of using confluent hypergeometric functions instead of modified Bessel functions and, hence, its full proof is omitted here. Nevertheless, we still obtain $\frac{L}{r} = I$ in both cases $\beta < 2$ and $\beta > 2$, which shows, again, that the value-matching and smooth-pasting conditions are not accommodated under the perpetual case conditional on $P_t < L$.

4B. Proof of Proposition 4.3

Following Davydov and Linetsky (2001, page 963) and Dias et al. (2023, Section EC.1), it is possible to show that the general solution to ODE (4.24) with $r = q$ is of the form

$$F_f(P_t, \bar{P}, \infty) = A_1 P_t^{\frac{1}{2}} I_\nu(z(P_t)) + B_1 P_t^{\frac{1}{2}} K_\nu(z(P_t)), \quad (4B.1)$$

where $I_\nu(z(P_t))$ and $K_\nu(z(P_t))$ are, respectively, the modified Bessel's functions of first and second kind with order ν , as given by Olver et al. (2010, equations 10.25.2 and 10.27.4), $z(P_t)$ is defined in equation (4.4), and A_1 and B_1 are two constants (associated to the value of the option to invest) to be determined from appropriate boundary conditions.

4B.1. Boundary conditions

We know that as P_t becomes small the (perpetual) option to invest is worthless, that is

$$\lim_{P_t \rightarrow 0^+} F_f(P_t, \bar{P}, \infty) = 0. \quad (4B.2)$$

Therefore, the solution of the ODE (4.24) with $r = q$ and subject to the boundary condition (4B.2) is given by $F_f(P_t, \bar{P}, \infty) = A_1 P_t^{\frac{1}{2}} I_\nu(z(P_t))$, for $\beta < 2$. Note that $z(P_t) \rightarrow 0$ as $P_t \rightarrow 0^+$ and, hence, using Olver et al. (2010, equations 10.30.1 and 10.30.2), we observe that $I_\nu(z(P_t))$ and $K_\nu(z(P_t))$ approach 0 and $+\infty$, respectively. Consequently, the constant B_1 in equation (4B.1) should be zero. Moreover, $F_f(P_t, \bar{P}, \infty) = B_1 P_t^{\frac{1}{2}} K_\nu(z(P_t))$, for $\beta > 2$. In this case, $z(P_t) \rightarrow +\infty$ as $P_t \rightarrow 0^+$ and, using Olver et al. (2010, equations 10.30.4 and 10.25.3), we notice that $I_\nu(z(P_t))$ and $K_\nu(z(P_t))$ tend to $+\infty$ and 0, respectively, as $z(P_t)$ tends to $+\infty$. Hence, the constant A_1 in equation (4B.1) should be zero. The imposition of these boundary conditions to both cases ensures that the options components terms that diverge as P_t goes to 0^+ are ruled out.

In summary, the fundamental (increasing) solutions for the case $r = q$ are given by

$$F_f(P_t, \bar{P}, \infty) = \begin{cases} A_1 P_t^{\frac{1}{2}} I_\nu(z(P_t)) & \Leftarrow \beta < 2 \\ B_1 P_t^{\frac{1}{2}} K_\nu(z(P_t)) - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}. \quad (4B.3)$$

Notice that the case with $\beta > 2$ requires the subtraction of the perpetual bubble value to avoid the existence of arbitrage opportunities—see, for example, Dias et al. (2023, Remark 4).

4B.2. Value-matching and smooth-pasting conditions

To determine the constants A_1 and B_1 as well as the optimal trigger level $\bar{P}$ it is necessary to apply first the value-matching condition that essentially states that at the optimal investment threshold $\bar{P}$ the value of the (perpetual) option to invest is equal to the value of the project

(with an infinite-lived guarantee) minus the investment cost I , that is

$$F_f(\bar{P}, \bar{P}, \infty) = V_f(\bar{P}, L, \infty) - I. \quad (4B.4)$$

This means that the investor is indifferent between holding the option to invest or deploying the project at the optimal trigger level $\bar{P}$. We recall that since the value-matching and smooth-pasting conditions are not met for the branch of the project given in equation (4.6) conditional on $P_t < L$ —see Remark 4.3 and 4A for additional details explaining why investment never occurs in this region—, such conditions should be imposed using equation (4.26) rather than equation (4.6) for both cases $\beta < 2$ and $\beta > 2$, which implies that the value-matching condition should be formally defined as

$$F_f(\bar{P}, \bar{P}, \infty) = V_{f,op}(\bar{P}, L, \infty) - I. \quad (4B.5)$$

For $\beta < 2$, the combination of equations (4B.3) and (4B.5) results in

$$A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = V_{f,op}(\bar{P}, L, \infty) - I, \quad (4B.6)$$

which allow us to find an explicit expression for the constant A_1 as

$$A_1 = [V_{f,op}(\bar{P}, L, \infty) - I] \frac{1}{\bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P}))}. \quad (4B.7)$$

Finally, equations (4.25) and (4.27) are immediately obtained, for $\beta < 2$, after substituting equation (4B.7) in equation (4B.3).

The case $\beta > 2$ is treated in a similar way, though now we need to consider the bubble effect. Combining equations (4B.3) and (4B.5) leads to

$$B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) = V_{f,op}(\bar{P}, L, \infty) - I, \quad (4B.8)$$

which allow us to find an explicit expression for the constant B_1 as

$$B_1 = [V_{f,op}(\bar{P}, L, \infty) + G_t^\infty(\bar{P}) - I] \frac{1}{\bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P}))}. \quad (4B.9)$$

Finally, equations (4.25) and (4.27) are immediately obtained, for $\beta > 2$, after replacing equation (4B.9) in equation (4B.3).

Even though equations (4.25) and (4.27) provide already an explicit solution for the value of the option, it is still necessary to determine the critical point $\bar{P}$, which requires the use of the smooth-pasting condition so that the value of the option and the value of the project

meet tangentially (i.e., both have the same slope) at the optimal investment point $\bar{P}$. This is accomplished by solving a system of equations involving both the value-matching and smooth-pasting conditions (with $\bar{P} \geq L$), that is

$$\begin{cases} F_f(\bar{P}, \bar{P}, \infty) = V_{f,op}(\bar{P}, L, \infty) - I \\ \frac{\partial}{\partial P_t} F_f(P_t, \bar{P}, \infty) |_{P_t=\bar{P}} = \frac{\partial}{\partial P_t} V_{f,op}(P_t, L, \infty) |_{P_t=\bar{P}} \end{cases} \quad (4B.10)$$

Applying the value-matching and smooth-pasting conditions (4B.10) and noting that, for $\beta > 2$, the same bubble exists on the option and on the underlying asset price—see, for instance, Heston et al. (2007, page 367) and Dias et al. (2020, Remark 2)—, we obtain

$$\begin{cases} A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = B_0 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) + \frac{\bar{P}}{q} - I \\ \frac{A_1}{2} \bar{P}^{-\frac{1}{2}} I_\nu(z(\bar{P})) + A_1 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [I_\nu(z(\bar{P}))] z'(\bar{P}) = \frac{B_0}{2} \bar{P}^{-\frac{1}{2}} K_\nu(z(\bar{P})) \\ + B_0 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [K_\nu(z(\bar{P}))] z'(\bar{P}) + \frac{1}{q} \end{cases} ,$$

for $\beta < 2$, and

$$\begin{cases} B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) = A_0 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) + \frac{\bar{P}}{q} - I \\ \frac{B_1}{2} \bar{P}^{-\frac{1}{2}} K_\nu(z(\bar{P})) + B_1 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [K_\nu(z(\bar{P}))] z'(\bar{P}) = \frac{A_0}{2} \bar{P}^{-\frac{1}{2}} I_\nu(z(\bar{P})) \\ + A_0 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [I_\nu(z(\bar{P}))] z'(\bar{P}) + \frac{1}{q} \end{cases} ,$$

for $\beta > 2$, which finally results in the systems of equations (4.28) and (4.29), respectively, after some straightforward calculus and using the recurrence relations $\frac{\partial I_\nu(z)}{\partial z} = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z)$ and $\frac{\partial K_\nu(z)}{\partial z} = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z)$ borrowed from Olver et al. (2010, equation 10.29.2) for computing the derivatives of the modified Bessel's functions with respect to z . Full details of these calculations are available upon request.

4C. Proof of Proposition 4.4

Following Davydov and Linetsky (2001, page 963) and Dias et al. (2023, Section EC.2), it is possible to show that the general solution to ODE (4.24) with $r \neq q$ is of the form

$$F_f(P_t, \bar{P}, \infty) = A_1 P_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x(P_t)} M_{k,m}(x(P_t)) + B_1 P_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x(P_t)} W_{k,m}(x(P_t)), \quad (4C.1)$$

where $M_{k,m}(x(P_t))$ and $W_{k,m}(x(P_t))$ are known as the Whittaker's functions defined in Olver et al. (2010, equations 13.14.2 and 13.14.3), respectively, and are expressed in terms of the Kummer's confluent hypergeometric function $M(a, b, x(P_t))$ and the Tricomi's confluent hypergeometric function $U(a, b, x(P_t))$ —also known as the Kummer's confluent hypergeometric function of the second kind—presented in Olver et al. (2010, equations 13.2.2 and 13.2.42), respectively, with $x(P_t)$, ϵ , k , m , a and b being defined by equations (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15), respectively, and A_1 and B_1 are two constants (associated to the value of the option to invest) to be determined from appropriate boundary conditions.

4C.1. Boundary conditions

The solution of the ODE (4.24) with $r \neq q$ and subject to the boundary condition (4B.2) is given by $F_f(P_t, \bar{P}, \infty) = A_1 P_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x(P_t)} M_{k,m}(x(P_t))$, for $\beta < 2$. In this case, $x(P_t) \rightarrow 0$ as $P_t \rightarrow 0^+$ and $W_{k,m}(x(P_t)) \rightarrow +\infty$ as $x(P_t) \rightarrow 0$, using the limiting forms in Olver et al. (2010, page 335). Consequently, the constant B_1 in equation (4C.1) should be zero. Moreover, $F_f(P_t, \bar{P}, \infty) = B_1 P_t^{\frac{\beta-1}{2}} e^{\frac{\epsilon}{2}x(P_t)} W_{k,m}(x(P_t))$, for $\beta > 2$. In this case, $x(P_t) \rightarrow +\infty$ as $P_t \rightarrow 0^+$ and $M_{k,m}(x(P_t)) \rightarrow +\infty$ as $x(P_t) \rightarrow +\infty$, using the asymptotic properties in Olver et al. (2010, page 335). Hence, the constant A_1 in equation (4C.1) should be zero. Again, the imposition of these boundary conditions to both cases ensures that the options components terms that diverge as P_t goes to 0^+ are ruled out.

Finally, using Olver et al. (2010, equations 13.14.2 and 13.14.3), the (increasing) solutions for the case $r \neq q$ are given by

$$F_f(P_t, \bar{P}, \infty) = \begin{cases} A_1 e^{(\epsilon-1)\frac{x(P_t)}{2}} Y P_t M(a, b, x(P_t)) & \Leftarrow \beta < 2 \\ B_1 e^{(\epsilon-1)\frac{x(P_t)}{2}} Y U(a, b, x(P_t)) - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}, \quad (4C.2)$$

with Y being defined by equation (4.16).

Notice that the case with $\beta > 2$ requires, again, the subtraction of the perpetual bubble value to avoid the existence of arbitrage opportunities.

4C.2. Value-matching and smooth-pasting conditions

As before, the determination of the constants A_1 and B_1 as well as the optimal trigger level $\bar{P}$ requires the use of the value-matching and smooth-pasting conditions (4B.10), but now for $r \neq q$. Moreover, since the value-matching and smooth-pasting conditions are not met for the

branch of the project given in equation (4.17) conditional on $P_t < L$ —see, again, Remark 4.3 and 4A for additional details explaining why investment never occurs in this region—, such conditions should be imposed using equation (4.30) instead of equation (4.17) for both cases $\beta < 2$ and $\beta > 2$.

For $\beta < 2$, the combination of equations (4C.2) and (4B.5) results in

$$A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) = V_{f,op}(\bar{P}, L, \infty) - I, \quad (4C.3)$$

which allow us to find an explicit expression for the constant A_1 as

$$A_1 = [V_{f,op}(\bar{P}, L, \infty) - I] \frac{1}{e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P}))}. \quad (4C.4)$$

Finally, equations (4.25) and (4.31) are immediately obtained, for $\beta < 2$, after substituting equation (4C.4) in equation (4C.2).

The case $\beta > 2$ is treated in a similar way, though now we need to consider the bubble effect. Combining equations (4C.2) and (4B.5) leads to

$$B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) - G_t^\infty(\bar{P}) = V_{f,op}(\bar{P}, L, \infty) - I, \quad (4C.5)$$

which allow us to find an explicit expression for the constant B_1 as

$$B_1 = [V_{f,op}(\bar{P}, L, \infty) + G_t^\infty(\bar{P}) - I] \frac{1}{e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P}))}. \quad (4C.6)$$

Finally, equations (4.25) and (4.31) are immediately obtained, for $\beta > 2$, after replacing equation (4C.6) in equation (4C.2).

Applying the value-matching and smooth-pasting conditions (4B.10) and noting that, for $\beta > 2$, the same bubble exists on the option and on the underlying asset price—see, for instance, Heston et al. (2007, page 367) and Dias et al. (2020, Remark 2)—, we obtain

$$\left\{ \begin{array}{l} A_1 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \bar{P} M(a, b, x(\bar{P})) = B_0 Y e^{(1-\epsilon)\frac{x(\bar{P})}{2}} \bar{P} U(a, b, x(\bar{P})) + \frac{\bar{P}}{q} - I \\ A_1 Y (\epsilon - 1) \frac{x'(\bar{P})}{2} e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \bar{P} M(a, b, x(\bar{P})) + A_1 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} M(a, b, x(\bar{P})) \\ \quad + A_1 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \bar{P} \frac{\partial}{\partial x(\bar{P})} [M(a, b, x(\bar{P}))] x'(\bar{P}) \\ = B_0 Y (\epsilon - 1) \frac{x'(\bar{P})}{2} e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \bar{P} U(a, b, x(\bar{P})) + B_0 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} U(a, b, x(\bar{P})) \\ \quad + B_0 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \bar{P} \frac{\partial}{\partial x(\bar{P})} [U(a, b, x(\bar{P}))] x'(\bar{P}) + \frac{1}{q} \end{array} \right. ,$$

for $\beta < 2$, and

$$\left\{ \begin{array}{l} B_1 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} U(a, b, x(\bar{P})) - G_t^\infty(\bar{P}) = A_0 Y e^{(1-\epsilon)\frac{x(\bar{P})}{2}} M(a, b, x(\bar{P})) - G_t^\infty(\bar{P}) + \frac{\bar{P}}{q} - I \\ B_1 Y (\epsilon - 1) \frac{x'(\bar{P})}{2} e^{(\epsilon-1)\frac{x(\bar{P})}{2}} U(a, b, x(\bar{P})) + B_1 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \frac{\partial}{\partial x(\bar{P})} [U(a, b, x(\bar{P}))] x'(\bar{P}) \\ = A_0 Y (\epsilon - 1) \frac{x'(\bar{P})}{2} e^{(\epsilon-1)\frac{x(\bar{P})}{2}} M(a, b, x(\bar{P})) \\ + A_0 Y e^{(\epsilon-1)\frac{x(\bar{P})}{2}} \frac{\partial}{\partial x(\bar{P})} [M(a, b, x(\bar{P}))] x'(\bar{P}) + \frac{1}{q} \end{array} \right. ,$$

for $\beta > 2$, which finally results in the systems of equations (4.32) and (4.33), respectively, after some straightforward calculus and using the differentiation formulas for the Kummer's confluent hypergeometric functions $M(a, b, x)$ and $U(a, b, x)$ with respect to x borrowed from Olver et al. (2010, equations 13.3.15 and 13.3.22). Full details of these calculations are available upon request.

4D. Present value of the unprotected cash flow in equation (4.52)

4D.1. The case with $\beta < 2$

For $\beta < 2$, the (forward) time- T value of the perpetual project is computed as

$$\begin{aligned} \int_T^\infty e^{-r(u-T)} \mathbb{E}_{\mathbb{Q}} [P_u | \mathcal{F}_T] du &= \int_T^\infty e^{-r(u-T)} e^{(r-q)(u-T)} P_T du \\ &= \int_T^\infty e^{-q(u-T)} P_T du \\ &= \frac{P_T}{q}, \end{aligned} \quad (4D.1)$$

since the time- T mean of the CEV process should be equal to $P_T e^{(r-q)(u-T)}$ given the risk-neutral dynamics (4.1).

Finally, using Dias et al. (2023, equation 46) with $\beta < 2$, the present value (i.e., the time- t value) of the unprotected cash flow is determined as (with $\tau := T - t$)

$$\begin{aligned} &e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T | \mathcal{F}_t] \\ &= e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T \mathbb{1}_{\{P_T < L\}} | \mathcal{F}_t] + e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T \mathbb{1}_{\{P_T \geq L\}} | \mathcal{F}_t] \\ &= e^{-q\tau} \frac{P_t}{q} \Phi_{-1} \left(0, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) + e^{-q\tau} \frac{P_t}{q} \Phi_{+1} \left(0, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ &= \frac{P_t}{q} e^{-q\tau} \left[F_{\chi^2 \left(2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right)} \left(\frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)} \right) + Q_{\chi^2 \left(2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right)} \left(\frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)} \right) \right] \\ &= \frac{P_t}{q} e^{-q\tau}, \end{aligned} \quad (4D.2)$$

since for $p = 0$, $\Phi_{-1}(0, x; \nu, \lambda)$ and $\Phi_{+1}(0, x; \nu, \lambda)$ reduce to the noncentral chi-square distribution function $F_{\chi^2(\nu, \lambda)}(x)$ and the complementary distribution function $Q_{\chi^2(\nu, \lambda)}(x)$, respectively. As expected, the present value of the unprotected cash flow in equation (4.52) can be computed in a simpler manner since $e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T | \mathcal{F}_t] = e^{-r\tau} \frac{1}{q} P_t e^{(r-q)\tau} = \frac{P_t}{q} e^{-q\tau}$.

4D.2. The case with $\beta > 2$

For $\beta > 2$, and following, for instance, Dias et al. (2020, Remark 2), the (forward) time- T value of the perpetual project is computed as

$$\begin{aligned} \int_T^\infty e^{-r(u-T)} \mathbb{E}_{\mathbb{Q}} [P_u | \mathcal{F}_T] du &= \int_T^\infty e^{-r(u-T)} e^{(r-q)(u-T)} P_T \left[1 - \frac{\Gamma(\nu, w_T)}{\Gamma(\nu)} \right] du \\ &= \int_T^\infty e^{-q(u-T)} P_T \left[1 - \frac{\Gamma(\nu, w_T)}{\Gamma(\nu)} \right] du \\ &= \int_T^\infty e^{-q(u-T)} P_T du - \int_T^\infty e^{-q(u-T)} P_T \frac{\Gamma(\nu, w_T)}{\Gamma(\nu)} du \end{aligned}$$

$$= \frac{P_T}{q} - G_T^\infty(P_T), \quad (4D.3)$$

with

$$w_T \equiv w(P_T) := \kappa_{T,u} P_T^{2-\beta} e^{(r-q)(2-\beta)(u-T)}. \quad (4D.4)$$

Using Dias et al. (2023, equation 46) with $\beta > 2$, the first component of the present value (i.e., the time- t value) of the unprotected cash flow is determined as (with $\tau := T - t$)

$$\begin{aligned} & e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T \mid \mathcal{F}_t] \\ &= e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T \mathbb{1}_{\{P_T < L\}} \mid \mathcal{F}_t] + e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T \mathbb{1}_{\{P_T \geq L\}} \mid \mathcal{F}_t] \\ &= e^{-r\tau} \frac{1}{q} (f(\tau))^{-\nu} \Phi_{+1} \left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ &\quad + e^{-r\tau} \frac{1}{q} (f(\tau))^{-\nu} \Phi_{-1} \left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ &= e^{-r\tau} \frac{1}{q} P_t e^{(r-q)\tau} \left[\frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right]^\nu \mathcal{M} \left(-\nu; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right), \end{aligned} \quad (4D.5)$$

since the sum of the two truncated moments $\Phi_{-1}(p, x; \nu, \lambda)$ and $\Phi_{+1}(p, x; \nu, \lambda)$ can be explicitly represented as the p -th raw moment $\mathcal{M}(p; \nu, \lambda)$, as shown in Carr and Linetsky (2006, equation 5.13) and Dias and Nunes (2018, equation 46), and because

$$(f(\tau))^{-\nu} = P_t e^{(r-q)\tau} \left[\frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right]^\nu. \quad (4D.6)$$

Moreover, using the risk-neutral survival probability definition for the CEV model borrowed from Carr and Linetsky (2006, equation 5.14 and page 321), equation (4D.5) can be rewritten as

$$e^{-r\tau} \frac{1}{q} \mathbb{E}_{\mathbb{Q}} [P_T \mid \mathcal{F}_t] = \frac{P_t}{q} e^{-q\tau} \frac{\gamma \left(\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{2f(\tau)} \right)}{\Gamma(\nu)}, \quad (4D.7)$$

thus resulting in the first component of the present value (i.e., the time- t value) of the unprotected cash flow contained in equation (4.52) for $\beta > 2$ after using equation (4.45).

Finally, the second component of the present value (i.e., the time- t value) of the unprotected cash flow included in equation (4.52) for $\beta > 2$ is computed via Dias et al. (2023, Proposition 7), that is

$$e^{-r\tau} \mathbb{E}_{\mathbb{Q}} [G_T^\infty(P_T) \mid \mathcal{F}_t] = e^{-r\tau} G^\infty(t, T, P_t), \quad (4D.8)$$

with $G^\infty(t, T, P_t)$ being computed via equation (4.43).

4E. Asymptotic analysis of the value of the project with a finite guarantee

The main goal here is to extend the asymptotic analysis of Barbosa et al. (2018, Appendix B) to the CEV model.

4E.1. Asymptotic analysis for the case with no guarantee

This case occurs when $T \rightarrow t$, so that $\tau := T - t \rightarrow 0$, which is economically equivalent to have $L = 0$. Using equation (4.52) with $\beta < 2$,

$$\begin{aligned} & \lim_{T \rightarrow t} V_p(P_t, L, T) \\ &= \lim_{T \rightarrow t} \left[V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} \right] \\ &= \lim_{T \rightarrow t} \left[V_f(P_t, L, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] + \frac{P_t}{q} e^{-q(T-t)} \right], \end{aligned} \quad (4E.1)$$

after using the time decomposition (4.35). Moreover, since V_f is a continuous function in $\mathbb{R}_0^+$, then equation (4E.1) can be rewritten as

$$\begin{aligned} \lim_{T \rightarrow t} V_p(P_t, L, T) &= V_f(P_t, L, \infty) - e^{-r(t-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_t, L, \infty) | \mathcal{F}_t] + \frac{P_t}{q} e^{-q(t-t)} \\ &= V_f(P_t, L, \infty) - e^0 V_f(P_t, L, \infty) + \frac{P_t}{q} e^0 \\ &= \frac{P_t}{q}. \end{aligned} \quad (4E.2)$$

We note that this is a simpler (model-free) approach that can be also used to derive the limits contained in Barbosa et al. (2018, equations B.8 and B.9) without the need of directly employing the normal distribution law.

Moving now to the case with $\beta > 2$ and using the second branch of equation (4.52),

$$\begin{aligned} & \lim_{T \rightarrow t} V_p(P_t, L, T) \\ &= \lim_{T \rightarrow t} \left[V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} \frac{\gamma \left(\nu, \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)} \right)}{\Gamma(\nu)} - e^{-r(T-t)} G^\infty(t, T, P_t) \right] \\ &= \lim_{T \rightarrow t} \left[V_f(P_t, L, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] \right. \\ & \quad \left. + \frac{P_t}{q} e^{-q(T-t)} \frac{\gamma \left(\nu, \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)} \right)}{\Gamma(\nu)} - e^{-r(T-t)} G^\infty(t, T, P_t) \right], \end{aligned} \quad (4E.3)$$

after using the time decomposition (4.35). Moreover, since V_f is a continuous function in $\mathbb{R}_0^+$,

then equation (4E.3) can be rewritten as

$$\begin{aligned}
\lim_{T \rightarrow t} V_p(P_t, L, T) &= V_f(P_t, L, \infty) - e^{-r(t-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_t, L, \infty) | \mathcal{F}_t] \\
&\quad + \frac{P_t}{q} e^{-q(t-t)} \frac{\gamma\left(\nu, \frac{e^{-\alpha(t-t)} P_t^{2-\beta}}{2f(t-t)}\right)}{\Gamma(\nu)} - e^{-r(t-t)} G^\infty(t, t, P_t) \\
&= V_f(P_t, L, \infty) - e^0 V_f(P_t, L, \infty) + \frac{P_t}{q} e^0 \frac{\gamma\left(\nu, \frac{e^0 P_t^{2-\beta}}{2f(0)}\right)}{\Gamma(\nu)} - e^0 G^\infty(t, t, P_t) \\
&= \frac{P_t}{q} - G_t^\infty(P_t), \tag{4E.4}
\end{aligned}$$

because equation (4.40) implies that $f(0) = 0$, $\frac{\gamma(\nu, \infty)}{\Gamma(\nu)} = 1$ and $G^\infty(t, t, P_t)$ represents the perpetual bubble $G_t^\infty(P_t)$.

Finally, combining equations (4E.2) and (4E.4) leads to the asymptotic value (i) shown in Remark 4.4.

4E.2. Asymptotic analysis for the case with a perpetual guarantee

This case occurs when $T \rightarrow \infty$. Using equation (4.52) with $\beta < 2$ (and because $q > 0$),

$$\begin{aligned}
\lim_{T \rightarrow \infty} V_p(P_t, L, T) &= \lim_{T \rightarrow \infty} \left[V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} \right] \\
&= V_f(P_t, L, \infty) + \frac{P_t}{q} \times 0, \\
&= V_f(P_t, L, \infty). \tag{4E.5}
\end{aligned}$$

Using again equation (4.52) but now with $\beta > 2$,

$$\begin{aligned}
&\lim_{T \rightarrow \infty} V_p(P_t, L, T) \tag{4E.6} \\
&= \lim_{T \rightarrow \infty} \left[V_f(P_t, L, T) + \frac{P_t}{q} e^{-q(T-t)} \frac{\gamma\left(\nu, \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)}\right)}{\Gamma(\nu)} - e^{-r(T-t)} G^\infty(t, T, P_t) \right].
\end{aligned}$$

Analyzing first the limit in equation (4E.6) involving the normalized function, which is a continuous function in $\mathbb{R}_0^+$, it follows that

$$\lim_{T \rightarrow \infty} \frac{\gamma\left(\nu, \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)}\right)}{\Gamma(\nu)} = \frac{\gamma\left(\nu, \lim_{T \rightarrow \infty} \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)}\right)}{\Gamma(\nu)}. \tag{4E.7}$$

Three different cases must now be considered: $r > q$, $r = q$ and $r < q$, which implies that

$\alpha > 0$, $\alpha = 0$ and $\alpha < 0$, respectively. Notice that

$$\lim_{T \rightarrow \infty} e^{-\alpha(T-t)} P_t^{2-\beta} = \begin{cases} 0 & \Leftarrow r > q \\ P_t^{2-\beta} & \Leftarrow r = q \\ \infty & \Leftarrow r < q \end{cases} . \quad (4E.8)$$

Moreover, using equation (4.40), it is easy to show that

$$\lim_{T \rightarrow \infty} f(T-t) = \begin{cases} \frac{\delta^2 |\beta-2|}{4(r-q)} & \Leftarrow r > q \\ \infty & \Leftarrow r = q \\ \infty & \Leftarrow r < q \end{cases} . \quad (4E.9)$$

It is possible to observe that the limit (4E.7) for the case $r < q$ has an indetermination that needs to be solved:

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)} &= \lim_{T \rightarrow \infty} \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{\frac{\delta^2 |\beta-2|}{2(r-q)} (1 - e^{-\alpha(T-t)})} \\ &= \lim_{T \rightarrow \infty} \frac{-\alpha e^{-\alpha(T-t)} P_t^{2-\beta}}{\frac{\delta^2 |\beta-2|}{2(r-q)} \alpha e^{-\alpha(T-t)}} \\ &= \lim_{T \rightarrow \infty} \frac{-P_t^{2-\beta}}{\frac{\delta^2 |\beta-2|}{2(r-q)}} \\ &= x, \end{aligned} \quad (4E.10)$$

with $x \in \mathbb{R}^+$ because $r < q$.

Hence, combining the limits (4E.8), (4E.9) and (4E.10) allows the limit (4E.7) to be expressed as

$$\lim_{T \rightarrow \infty} \frac{\gamma\left(\nu, \frac{e^{-\alpha(T-t)} P_t^{2-\beta}}{2f(T-t)}\right)}{\Gamma(\nu)} = \begin{cases} \frac{\gamma(\nu, 0)}{\Gamma(\nu)} = 0 & \Leftarrow r > q \\ \frac{\gamma(\nu, 0)}{\Gamma(\nu)} = 0 & \Leftarrow r = q \\ \frac{\gamma(\nu, x)}{\Gamma(\nu)} < \infty & \Leftarrow r < q \end{cases} . \quad (4E.11)$$

Moving now to the limit in equation (4E.6) involving the time- t expected value of the perpetual bubble (4.43), it follows that

$$\lim_{T \rightarrow \infty} e^{-r(T-t)} G^\infty(t, T, P_t) = 0, \quad (4E.12)$$

because equation (4.43) implies that $G^\infty(t, \infty, P_t) = 0$ and given the assumption of a positive interest rate r .

Finally, combining the limits (4E.6), (4E.11) and (4E.12), it follows that

$$\begin{aligned}\lim_{T \rightarrow \infty} V_p(P_t, L, T) &= V_f(P_t, L, \infty) + \frac{P_t}{q} \times 0 - 0 \\ &= V_f(P_t, L, \infty).\end{aligned}\tag{4E.13}$$

In summary, equations (4E.5) and (4E.13) correspond to the asymptotic value (ii) shown in Remark 4.4.

4F. Proof of Proposition 4.5

This appendix derives the value of the option to invest in a project with a finite maturity guarantee and presents the derivation of the optimal investment thresholds for the case $r = q$.

4F.1. Fundamental solutions accommodating the appropriate boundary conditions

Following the same rational used in 4B, it is possible to show that the fundamental (increasing) solutions for the case $r = q$ are given by

$$F_f(P_t, \bar{P}, T) = \begin{cases} A_1 P_t^{\frac{1}{2}} I_\nu(z(P_t)) & \Leftarrow \beta < 2 \\ B_1 P_t^{\frac{1}{2}} K_\nu(z(P_t)) - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}. \quad (4F.1)$$

4F.2. Analytic solution for the option to invest

As usual, the determination of the constants A_1 and B_1 as well as the optimal trigger level $\bar{P}$ requires the use of the value-matching condition, which essentially states that at the optimal investment threshold $\bar{P}$ the value of the option to invest is equal to the full value of the project with a finite-lived guarantee, $V_p(\bar{P}, L, T)$, minus the investment cost I , that is

$$F_f(\bar{P}, \bar{P}, T) = V_p(\bar{P}, L, T) - I. \quad (4F.2)$$

For $\beta < 2$, the combination of equations (4F.1) and (4F.2) results in

$$A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = V_p(\bar{P}, L, T) - I, \quad (4F.3)$$

which allow us to find an explicit expression for the constant A_1 as

$$A_1 = [V_p(\bar{P}, L, T) - I] \frac{1}{\bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P}))}. \quad (4F.4)$$

Finally, equations (4.53) and (4.27) are immediately obtained, for $\beta < 2$, after substituting equation (4F.4) in equation (4F.1).

The case $\beta > 2$ is treated in a similar way, though now we need to consider the bubble effect. Combining equations (4F.1) and (4F.2) leads to

$$B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) = V_p(\bar{P}, L, T) - I, \quad (4F.5)$$

which allow us to find an explicit expression for the constant B_1 as

$$B_1 = [V_p(\bar{P}, L, T) + G_t^\infty(\bar{P}) - I] \frac{1}{\bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P}))}. \quad (4F.6)$$

Finally, equations (4.53) and (4.27) are immediately obtained, for $\beta > 2$, after replacing equation (4F.6) in equation (4F.1).

Even though equation (4.53) provides already an explicit solution for the value of the option to invest in a project with a finite maturity guarantee, it is still necessary to determine the critical point $\bar{P}$ which requires the use of the smooth-pasting condition so that the value of the option and the value of the project meet tangentially (i.e., both have the same slope) at the optimal investment point $\bar{P}$. This is accomplished by solving a system of equations involving both the value-matching and smooth-pasting conditions, that is

$$\begin{cases} F_f(\bar{P}, \bar{P}, T) = V_p(\bar{P}, L, T) - I \\ \frac{\partial}{\partial P_t} F_f(P_t, \bar{P}, T) |_{P_t=\bar{P}} = \frac{\partial}{\partial P_t} V_p(P_t, L, T) |_{P_t=\bar{P}} \end{cases} \quad (4F.7)$$

Since in the finite maturity case the value-matching and smooth-pasting conditions are met in both regions $P_t < L$ and $P_t \geq L$, it is necessary to derive a system of nonlinear equations for each region and for $\beta < 2$ and $\beta > 2$, which results in four different systems of equations. The next two subsections present such derivations.

4F.3. Investment threshold for the region $P_t < L$

Combining equations (4F.1), (4F.7), (4.52), (4.35) and (4.6) yields

$$\begin{cases} A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = A_0 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) + \frac{L}{r} - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + \frac{\bar{P}}{q} e^{-q(T-t)} - I \\ \frac{A_1}{2} \bar{P}^{-\frac{1}{2}} I_\nu(z(\bar{P})) + A_1 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [I_\nu(z(\bar{P}))] z'(\bar{P}) = \frac{A_0}{2} \bar{P}^{-\frac{1}{2}} I_\nu(z(\bar{P})) \\ \quad + A_0 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [I_\nu(z(\bar{P}))] z'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + \frac{1}{q} e^{-q(T-t)} \end{cases},$$

for $\beta < 2$, and

$$\begin{cases} B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) = B_0 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) + \frac{L}{r} - G_t^\infty(\bar{P}) \\ \quad - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + g(\bar{P}) \\ \quad - e^{-r(T-t)} G^\infty(t, T, \bar{P}) - I \\ \frac{B_1}{2} \bar{P}^{-\frac{1}{2}} K_\nu(z(\bar{P})) + B_1 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [K_\nu(z(\bar{P}))] z'(\bar{P}) = \frac{B_0}{2} \bar{P}^{-\frac{1}{2}} K_\nu(z(\bar{P})) \\ \quad + B_0 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [K_\nu(z(\bar{P}))] z'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + g'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} G^\infty(t, T, \bar{P}) \end{cases},$$

for $\beta > 2$.

Finally, the above two systems of value-matching and smooth-pasting conditions result in the single nonlinear equations (4.54) and (4.55), respectively, after some straightforward calculus and using the recurrence relations $\frac{\partial I_\nu(z)}{\partial z} = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z)$ and $\frac{\partial K_\nu(z)}{\partial z} = \frac{\nu}{z} K_\nu(z) -$

$K_{\nu+1}(z)$ borrowed from Olver et al. (2010, equation 10.29.2) for computing the derivatives of the modified Bessel's functions with respect to z . Notice that the risk-neutral expectation $\mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}}$ contains the expected value of the perpetual bubble $G^\infty(t, T, \bar{P})$ —please see equation (4.36)—, which will cancel out the other bubble term appearing in the case $\beta > 2$. This implies that it is not necessary to determine the derivative of $G^\infty(t, T, \bar{P})$ with respect to $\bar{P}$. Full details of these calculations are available upon request.

4F.4. Investment threshold for the region $P_t \geq L$

Combining equations (4F.1), (4F.7), (4.52), (4.35) and (4.6) yields

$$\left\{ \begin{array}{l} A_1 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) = B_0 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) + \frac{\bar{P}}{q} - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + \frac{\bar{P}}{q} e^{-q(T-t)} - I \\ \frac{A_1}{2} \bar{P}^{-\frac{1}{2}} I_\nu(z(\bar{P})) + A_1 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [I_\nu(z(\bar{P}))] z'(\bar{P}) = \frac{B_0}{2} \bar{P}^{-\frac{1}{2}} K_\nu(z(\bar{P})) \\ \quad + B_0 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [K_\nu(z(\bar{P}))] z'(\bar{P}) + \frac{1}{q} - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + \frac{1}{q} e^{-q(T-t)} \end{array} \right. ,$$

for $\beta < 2$, and

$$\left\{ \begin{array}{l} B_1 \bar{P}^{\frac{1}{2}} K_\nu(z(\bar{P})) - G_t^\infty(\bar{P}) = A_0 \bar{P}^{\frac{1}{2}} I_\nu(z(\bar{P})) + \frac{\bar{P}}{q} - G_t^\infty(\bar{P}) \\ \quad - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + g(\bar{P}) \\ \quad - e^{-r(T-t)} G^\infty(t, T, \bar{P}) - I \\ \frac{B_1}{2} \bar{P}^{-\frac{1}{2}} K_\nu(z(\bar{P})) + B_1 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [K_\nu(z(\bar{P}))] z'(\bar{P}) = \frac{A_0}{2} \bar{P}^{-\frac{1}{2}} I_\nu(z(\bar{P})) \\ \quad + A_0 \bar{P}^{\frac{1}{2}} \frac{\partial}{\partial z(\bar{P})} [I_\nu(z(\bar{P}))] z'(\bar{P}) + \frac{1}{q} - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + g'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} G^\infty(t, T, \bar{P}) \end{array} \right. ,$$

for $\beta > 2$.

Finally, the above two systems of value-matching and smooth-pasting conditions result in the systems of equations (4.60) and (4.61), respectively, after some straightforward calculus and using the recurrence relations $\frac{\partial I_\nu(z)}{\partial z} = \frac{\nu}{z} I_\nu(z) + I_{\nu+1}(z)$ and $\frac{\partial K_\nu(z)}{\partial z} = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z)$ borrowed from Olver et al. (2010, equation 10.29.2) for computing the derivatives of the modified Bessel's functions with respect to z . Notice that the risk-neutral expectation $\mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}}$ contains the expected value of the perpetual bubble $G^\infty(t, T, \bar{P})$ —please see again equation (4.36)—, which will cancel out the other bubble term appearing in the case $\beta > 2$. This implies that it is not necessary to determine the derivative of $G^\infty(t, T, \bar{P})$ with respect to $\bar{P}$. Full details of these calculations are available upon request.

4G. Proof of Proposition 4.6

This appendix derives the value of the option to invest in a project with a finite maturity guarantee and presents the derivation of the optimal investment thresholds for the case $r \neq q$.

4G.1. Fundamental solutions accommodating the appropriate boundary conditions

Following the same rational used in 4C, it is possible to show that the fundamental (increasing) solutions for the case $r \neq q$ are given by

$$F_f(P_t, \bar{P}, T) = \begin{cases} A_1 e^{(\epsilon-1)\frac{x(P_t)}{2}} Y P_t M(a, b, x(P_t)) & \Leftarrow \beta < 2 \\ B_1 e^{(\epsilon-1)\frac{x(P_t)}{2}} Y U(a, b, x(P_t)) - G_t^\infty(P_t) & \Leftarrow \beta > 2 \end{cases}. \quad (4G.1)$$

4G.2. Analytic solution for the option to invest

As before, the determination of the constants A_1 and B_1 as well as the optimal trigger level $\bar{P}$ requires the use of the value-matching condition (4F.2).

For $\beta < 2$, the combination of equations (4G.1) and (4F.2) results in

$$A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) = V_p(\bar{P}, L, \infty) - I, \quad (4G.2)$$

which allow us to find an explicit expression for the constant A_1 as

$$A_1 = [V_p(\bar{P}, L, \infty) - I] \frac{1}{e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P}))}. \quad (4G.3)$$

Finally, equations (4.53) and (4.31) are immediately obtained, for $\beta < 2$, after substituting equation (4G.3) in equation (4G.1).

The case $\beta > 2$ is treated in a similar way, though now we need to consider the bubble effect. Combining equations (4G.1) and (4F.2) leads to

$$B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) - G_t^\infty(\bar{P}) = V_p(\bar{P}, L, \infty) - I, \quad (4G.4)$$

which allow us to find an explicit expression for the constant B_1 as

$$B_1 = [V_p(\bar{P}, L, \infty) + G_t^\infty(\bar{P}) - I] \frac{1}{e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P}))}. \quad (4G.5)$$

Finally, equations (4.53) and (4.31) are immediately obtained, for $\beta < 2$, after substituting equation (4G.5) in equation (4G.1).

Similarly to the finite maturity case with $r = q$, the value-matching and smooth-pasting conditions (4F.7) are met in both regions $P_t < L$ and $P_t \geq L$ and, hence, it is necessary to derive a system of nonlinear equations for each region and for $\beta < 2$ and $\beta > 2$, which

results in other four different systems of equations. The next two subsections present such derivations.

4G.3. Investment threshold for the region $P_t < L$

Combining equations (4G.1), (4F.7), (4.52), (4.35) and (4.17) yields

$$\left\{ \begin{array}{l} A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) = A_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) + \frac{L}{r} \\ \quad - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + \frac{\bar{P}}{q} e^{-q(T-t)} - I \\ A_1 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) + A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y M(a, b, x(\bar{P})) \\ \quad + A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} \frac{\partial}{\partial x(\bar{P})} [M(a, b, x(\bar{P}))] x'(\bar{P}) \\ = A_0 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) + A_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y M(a, b, x(\bar{P})) \\ \quad + A_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} \frac{\partial}{\partial x(\bar{P})} [M(a, b, x(\bar{P}))] x'(\bar{P}) \\ \quad - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + \frac{1}{q} e^{-q(T-t)} \end{array} \right. ,$$

for $\beta < 2$, and

$$\left\{ \begin{array}{l} B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) - G_t^\infty(\bar{P}) = B_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) + \frac{L}{r} - G_t^\infty(\bar{P}) \\ \quad - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + g(\bar{P}) \\ \quad - e^{-r(T-t)} G^\infty(t, T, \bar{P}) - I \\ B_1 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) + B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \frac{\partial}{\partial x(\bar{P})} [U(a, b, x(\bar{P}))] x'(\bar{P}) \\ = B_0 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) + B_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \frac{\partial}{\partial x(\bar{P})} [U(a, b, x(\bar{P}))] x'(\bar{P}) \\ \quad - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + g'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} G^\infty(t, T, \bar{P}) \end{array} \right. ,$$

for $\beta > 2$.

Finally, the above two systems of value-matching and smooth-pasting conditions result in the single nonlinear equations (4.62) and (4.63), respectively, after some straightforward calculus and using the differentiation formulas for the Kummer's confluent hypergeometric functions $M(a, b, x)$ and $U(a, b, x)$ with respect to x borrowed from Olver et al. (2010, equations 13.3.15 and 13.3.22). Notice that the risk-neutral expectation $\mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}}$ contains the expected value of the perpetual bubble $G^\infty(t, T, \bar{P})$ —please see equation (4.47)—, which will cancel out the other bubble term appearing in the case $\beta > 2$. This implies that it is not necessary to determine the derivative of $G^\infty(t, T, \bar{P})$ with respect to $\bar{P}$. Full details of these calculations are available upon request.

4G.4. Investment threshold for the region $P_t \geq L$

Combining equations (4G.1), (4F.7), (4.52), (4.35) and (4.17) yields

$$\left\{ \begin{array}{l} A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) = B_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} U(a, b, x(\bar{P})) + \frac{\bar{P}}{q} \\ \quad - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + \frac{\bar{P}}{q} e^{-q(T-t)} - I \\ A_1 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} M(a, b, x(\bar{P})) + A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y M(a, b, x(\bar{P})) \\ \quad + A_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} \frac{\partial}{\partial x(\bar{P})} [M(a, b, x(\bar{P}))] x'(\bar{P}) \\ = B_0 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} U(a, b, x(\bar{P})) + B_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) \\ \quad + B_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \bar{P} \frac{\partial}{\partial x(\bar{P})} [U(a, b, x(\bar{P}))] x'(\bar{P}) + \frac{1}{q} \\ \quad - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + \frac{1}{q} e^{-q(T-t)} \end{array} \right. ,$$

for $\beta < 2$, and

$$\left\{ \begin{array}{l} B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) - G_t^\infty(\bar{P}) = A_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y M(a, b, x(\bar{P})) + \frac{\bar{P}}{q} - G_t^\infty(\bar{P}) \\ \quad - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} + g(\bar{P}) \\ \quad - e^{-r(T-t)} G^\infty(t, T, \bar{P}) - I \\ B_1 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y U(a, b, x(\bar{P})) + B_1 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \frac{\partial}{\partial x(\bar{P})} [U(a, b, x(\bar{P}))] x'(\bar{P}) \\ = A_0 \frac{\epsilon-1}{2} x'(\bar{P}) e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y M(a, b, x(\bar{P})) + A_0 e^{(\epsilon-1)\frac{x(\bar{P})}{2}} Y \frac{\partial}{\partial x(\bar{P})} [M(a, b, x(\bar{P}))] x'(\bar{P}) \\ \quad + \frac{1}{q} - e^{-r(T-t)} \frac{\partial}{\partial P_t} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}} \\ \quad + g'(\bar{P}) - e^{-r(T-t)} \frac{\partial}{\partial \bar{P}} G^\infty(t, T, \bar{P}) \end{array} \right. ,$$

for $\beta > 2$.

Finally, the above two systems of value-matching and smooth-pasting conditions result in the systems of equations (4.66) and (4.67), respectively, after some straightforward calculus and using the differentiation formulas for the Kummer's confluent hypergeometric functions $M(a, b, x)$ and $U(a, b, x)$ with respect to x borrowed from Olver et al. (2010, equations 13.3.15 and 13.3.22). Notice that the risk-neutral expectation $\mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=\bar{P}}$ contains the expected value of the perpetual bubble $G^\infty(t, T, \bar{P})$ —please see again equation (4.47)—, which will cancel out the other bubble term appearing in the case $\beta > 2$. This implies that it is not necessary to determine the derivative of $G^\infty(t, T, \bar{P})$ with respect to $\bar{P}$. Full details of these calculations are available upon request.

4H. Proof of equations (4.68) and (4.69)

This appendix derives equations (4.68) and (4.69) for the cases $\beta < 2$ and $\beta > 2$, respectively. Each case must be tackled in a different way due to the contrasting boundary characterization of the CEV diffusion in each of the two regions—see, for instance, Dias et al. (2015, Remark 7), Dias et al. (2020, Section 2.2) and the references contained therein for additional details.

4H.1. The case $\beta < 2$

In this case, zero is an exit boundary for $1 \leq \beta < 2$ and, for $\beta < 1$, zero is a regular boundary point and is specified as a killing boundary by adjoining a killing boundary condition. Since $\tau_0 := \inf \{t > 0 : P_t = 0\} < \infty$, we can formally impose $P_t = 0$ when deriving equation (4.68) for $\beta < 2$ because the boundary $P_t = 0$ is attainable and the origin becomes the absorbing state for the underlying asset price $\{P_t, t \geq 0\}$.

It is easy to observe that, for $\beta < 2$, the use of $P_t = 0$ in the first branch of equation (4.52) implies that $V_p(0, L, T) = V_f(0, L, T)$ since the unprotected cash flow component vanishes. However, the remaining analysis needs to be split in the cases $r = q$ and $r \neq q$, as shown below.

The case $r = q$.

Combining equations (4.52), (4.35), (4.6) and (4.36), with $\beta < 2$ and $r = q$, yields

$$\begin{aligned}
 & V_p(0, L, T) \\
 = & V_f(0, L, T) + 0 \\
 = & V_f(0, L, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=0} \\
 = & A_0 \times 0^{\frac{1}{2}} \times I_{\nu}(z(0)) + \frac{L}{r} \\
 & - e^{-r(T-t)} [A_0 I(-1, 0, L) + X_r(-1, 0, L) + B_0 K(1, 0, L) + P_q(1, 0, L)]. \quad (4H.1)
 \end{aligned}$$

Equation (4.4) implies that $z(0) = 0$ and $I_{\nu}(0)$ is a real number for $\nu \in \mathbb{R}$, as documented, for example, in Olver et al. (2010, page 249). Moreover, for $\beta < 2$, and using equations (4.37), (4.38), (4.41) and (4.42), it is straightforward to show that

$$\begin{aligned}
 P_q(1, 0, L) &= e^{(r-q)\tau} \times \frac{0}{q} \times \Phi_{+1} \left(0, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) \\
 &= 0,
 \end{aligned}$$

since a truncated moment $\Phi_{+1}(0, x; \nu, 0)$ is interpreted in statistics and probability theory as the upper tail of a central χ^2 distribution with ν degrees of freedom, thus yielding a real value

in the range $[0, 1]$,

$$\begin{aligned} X_r(-1, 0, L) &= \frac{L}{r} \times 0 \times (f(\tau))^{-\nu} \Phi_{-1} \left(-\nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) + \frac{L}{r} \frac{\Gamma(\nu, 0)}{\Gamma(\nu)} \\ &= \frac{L}{r}, \end{aligned}$$

because a truncated moment $\Phi_{-1}(p, x; v, 0)$ is understood as the p -th lower tail truncated moment of a central χ^2 distribution with v degrees of freedom generating a finite value $\in \mathbb{R}_0^+$ and $\frac{\Gamma(\nu, 0)}{\Gamma(\nu)} = 1$ using Olver et al. (2010, equations 8.2.2 and 5.2.1),

$$\begin{aligned} I(-1, 0, L) &= 0 \times \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{-1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} &K(1, 0, L) \\ &= \frac{\pi}{2 \sin(\nu\pi)} \times 0 \times \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j-\nu}}{j!} \frac{(f(\tau))^{j-\nu}}{\Gamma(j+1-\nu)} \Phi_{+1} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) \\ &\quad - \frac{\pi}{2 \sin(\nu\pi)} \times 0 \times \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{+1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right), \\ &= 0, \end{aligned}$$

given that a truncated moment $\Phi_{+1}(p, x; v, 0)$ is interpreted as the p -th upper tail truncated moment of a central χ^2 distribution with v degrees of freedom yielding also a finite value $\in \mathbb{R}_0^+$, and because the series of the functions $I(-1, 0, L)$ and $K(1, 0, L)$ return a finite sum.

Combining the above auxiliary results allows equation (4H.1) to be rewritten as

$$V_p(0, L, T) = \frac{L}{r} (1 - e^{-r(T-t)}). \quad (4H.2)$$

Finally, solving the equation $\text{NPV} := V_p(0, L_0, T) - I = 0$ with respect to L_0 yields equation (4.68).

The case $r \neq q$.

Combining equations (4.52), (4.35), (4.17) and (4.47), with $\beta < 2$ and $r \neq q$, yields

$$\begin{aligned}
& V_p(0, L, T) \\
&= V_f(0, L, T) + 0 \\
&= V_f(0, L, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] |_{P_t=0} \\
&= A_0 Y e^{(\epsilon-1) \frac{x(0)}{2}} \times 0 \times M(a, b, x(0)) + \frac{L}{r} \\
&\quad - e^{-r(T-t)} [A_0 Y M_P(-1, 0, L) + X_r(-1, 0, L) + B_0 Y U_P(1, 0, L) + P_q(1, 0, L)].
\end{aligned} \tag{4H.3}$$

Equation (4.10) implies that $x(0) = 0$, while Olver et al. (2010, equation 13.2.13) entails that $M(a, b, 0) = 1$. Moreover, for $\beta < 2$, and using equations (4.48) and (4.50), it is straightforward to show that

$$\begin{aligned}
M_P(-1, 0, L) &= e^{(r-q)\tau} \times 0 \times \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \Phi_{-1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
& U_P(1, 0, L) \\
&= \frac{\Gamma(-\nu)}{\Gamma(a-\nu)} e^{(r-q)\tau} \times 0 \times \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \Phi_{+1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) \\
&\quad + \frac{\Gamma(\nu)}{\Gamma(a)} e^{(r-q)\tau} \times 0 \times \sum_{j=0}^{\infty} \frac{(c-\nu)_j}{(2-b)_j} \frac{\xi^j}{j!} \left(\xi \frac{1 - e^{\alpha\tau}}{2} \right)^{j-\nu} \Phi_{+1} \left(j - \nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, 0 \right) \\
&= 0,
\end{aligned}$$

because the series of the functions $M_P(-1, 0, L)$ and $U_P(1, 0, L)$ yield finite sums.

Combining the aforementioned auxiliary results allows equation (4H.3) to be rewritten as equation (4H.2), thus finally yielding, again, equation (4.68) after solving $\text{NPV} := V_p(0, L_0, T) - I = 0$ with respect to L_0 .

4H.2. The case $\beta > 2$

In this case, zero is a natural boundary and the probability that the underlying asset price hits the point 0 is zero (i.e., $\tau_0 = \infty$). Hence, the derivation of equation (4.68) for $\beta > 2$ requires the asymptotic analysis of $P_t \rightarrow 0^+$, since imposing $P_t = 0$ is not possible.

Using the second branch of equation (4.52), it follows that

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} V_p(P_t, L, T) \tag{4H.4} \\
&= \lim_{P_t \rightarrow 0^+} V_f(P_t, L, T) + \lim_{P_t \rightarrow 0^+} \left[\frac{P_t}{q} e^{-q(T-t)} \frac{\gamma\left(\nu, \tilde{f}(P_t)/2\right)}{\Gamma(\nu)} \right] - \lim_{P_t \rightarrow 0^+} \left[e^{-r(T-t)} G^\infty(t, T, P_t) \right] \\
&= \lim_{P_t \rightarrow 0^+} \left[V_f(P_t, L, \infty) - e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} [V_f(P_T, L, \infty) | \mathcal{F}_t] \right] - \lim_{P_t \rightarrow 0^+} \left[e^{-r(T-t)} G^\infty(t, T, P_t) \right],
\end{aligned}$$

after applying the time decomposition (4.35) and because, for $\beta > 2$, equation (4.45) yields

$$\begin{aligned}
\lim_{P_t \rightarrow 0^+} \tilde{f}(P_t) &= \lim_{P_t \rightarrow 0^+} \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \\
&= \infty,
\end{aligned}$$

for both $r = q$ and $r \neq q$, and $\frac{\gamma(\nu, \infty)}{\Gamma(\nu)} = 1$ using Olver et al. (2010, equations 8.2.1 and 5.2.1).

The remaining analysis needs to be split in the cases $r = q$ and $r \neq q$ as shown below.

The case $r = q$.

Using equations (4.36), (4.56) and (4.6), with $\beta > 2$ and $r = q$, the limit (4H.4) can be rewritten as

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} V_p(P_t, L, T) \tag{4H.5} \\
&= \lim_{P_t \rightarrow 0^+} \left[V_f(P_t, L, \infty) - e^{-r(T-t)} V_{f,nb}(P_t, L, \infty) + e^{-r(T-t)} G^\infty(t, T, P_t) \right] \\
&\quad - \lim_{P_t \rightarrow 0^+} \left[e^{-r(T-t)} G^\infty(t, T, P_t) \right] \\
&= \lim_{P_t \rightarrow 0^+} \left[B_0 P_t^{\frac{1}{2}} K_\nu(z(P_t)) + \frac{L}{r} - G_t^\infty(P_t) \right] \\
&\quad - e^{-r(T-t)} \lim_{P_t \rightarrow 0^+} [B_0 K(-1, P_t, L) + X_r(-1, P_t, L) + A_0 I(1, P_t, L) + P_q(1, P_t, L)],
\end{aligned}$$

since the limits associated with the time- t expected value of the perpetual bubble cancel out.

Given that

$$\begin{aligned}
\lim_{P_t \rightarrow 0^+} z(P_t) &= \lim_{P_t \rightarrow 0^+} \frac{2\sqrt{2r}}{\delta|\beta - 2|} P_t^{1-\frac{\beta}{2}} \\
&= \infty,
\end{aligned}$$

from equation (4.4) and for $\beta > 2$, then, using Olver et al. (2010, equation 10.25.3),

$$\begin{aligned}\lim_{P_t \rightarrow 0^+} K_\nu(z(P_t)) &= \lim_{z \rightarrow \infty} K_\nu(z) \\ &\sim \lim_{z \rightarrow \infty} \sqrt{\frac{\pi}{2z}} e^{-z} \\ &= 0\end{aligned}$$

and, therefore,

$$\lim_{P_t \rightarrow 0^+} B_0 P_t^{\frac{1}{2}} K_\nu(z(P_t)) = 0. \quad (4H.6)$$

Moreover, combining Dias et al. (2023, equations 10, 5 and 6), it can be shown that

$$\begin{aligned}\lim_{P_t \rightarrow 0^+} G_t^\infty(P_t) &= \lim_{P_t \rightarrow 0^+} \int_t^\infty G_t(P_t, u) du \\ &= \lim_{P_t \rightarrow 0^+} \int_t^\infty P_t e^{-q(u-t)} \frac{\Gamma(\nu, \kappa_{t,u} P_t^{2-\beta} e^{(r-q)(2-\beta)(u-t)})}{\Gamma(\nu)} du \\ &= \int_t^\infty 0 \times e^{-q(u-t)} \frac{\Gamma(\nu, \infty)}{\Gamma(\nu)} \\ &= 0,\end{aligned} \quad (4H.7)$$

since, for $\beta > 2$,

$$\lim_{P_t \rightarrow 0^+} \kappa_{t,u} P_t^{2-\beta} e^{(r-q)(2-\beta)(u-t)} = \infty,$$

with $\kappa_{t,u}$ being defined by equation (4.44), and because $\frac{\Gamma(\nu, \infty)}{\Gamma(\nu)} = 0$ using Olver et al. (2010, equations 8.2.2 and 5.2.1).

Substituting the limits (4H.6) and (4H.7) in the first limit contained in last equality of expression (4H.5) yields

$$\lim_{P_t \rightarrow 0^+} \left[B_0 P_t^{\frac{1}{2}} K_\nu(z(P_t)) + \frac{L}{r} - G_t^\infty(P_t) \right] = \frac{L}{r}. \quad (4H.8)$$

Moreover, for $\beta > 2$, it is straightforward to show that equation (4.38) yields

$$\begin{aligned}\lim_{P_t \rightarrow 0^+} X_r(-1, P_t, L) &= \lim_{P_t \rightarrow 0^+} \frac{L}{r} \Phi_{+1} \left(0, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\ &= \frac{L}{r} \Phi_{+1} \left(0, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \infty \right) \\ &= \frac{L}{r},\end{aligned} \quad (4H.9)$$

because $\Phi_{+1} \left(0, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \infty \right) = 1$.

To better clarify this point, we note that a truncated moment $\Phi_{+1}(0, x; \nu, \lambda)$, with $\lambda \rightarrow$

∞ , is interpreted in statistics and probability theory as the upper tail of a noncentral χ^2 distribution with v degrees of freedom and infinite noncentrality parameter λ . It is well-known, however, that when the noncentrality parameter λ is large the non-central χ^2 distribution can be approximated to a normal distribution—see, for instance, Johnson et al. (1995, Chapter 29).

More specifically, the aim of such analytical approximation is to map the argument x and the parameters v and λ to a new variable w that is approximately normally distributed, that is

$$\begin{aligned}\Phi_{+1}(0, x; v, \lambda) &= Q_{\chi^2(v, \lambda)}(x) \\ &\approx N(-w),\end{aligned}$$

with $Q_{\chi^2(v, \lambda)}(x)$ being the upper tail of a noncentral χ^2 distribution with v degrees of freedom and noncentrality parameter λ and where $N(\cdot)$ is the standard normal distribution.

Using, for example, the approximation method of Sankaran (1963),

$$w = -\frac{1 - hs(1 - h + (2 - h)ms/2) - (x/(v + \lambda))^h}{h\sqrt{2s(1 + ms)}}$$

where

$$\begin{aligned}h &= 1 - \frac{2(v + \lambda)(v + 3\lambda)}{3(v + 2\lambda)^2}, \\ s &= \frac{v + 2\lambda}{(v + \lambda)^2}\end{aligned}$$

and

$$m = (h - 1)(1 - 3h).$$

Clearly, $s \rightarrow 0$ and $h \rightarrow \frac{1}{2}$ as $\lambda \rightarrow \infty$. Moreover, taking $x = \frac{L^{2-\beta}}{f(\tau)}$ and $v = 2 + 2\nu$, it is easy to show that $\lim_{\lambda \rightarrow \infty}(-w) = \infty$ and it is well-known that $N(\infty) = 1$, which ultimately proves that $\Phi_{+1}\left(0, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \infty\right) = 1$. Alternatively, this can also be proven using the asymptotic expansions of the noncentral χ^2 distribution offered by Temme (1993).

We notice, however, that the functions $P_q(1, P_t, L)$, $I(1, P_t, L)$ and $K(-1, P_t, L)$ involve a parameter $p \neq 0$ and, therefore, we cannot apply the previous approach that is used for the function $X_r(-1, P_t, L)$. Fortunately, the results of Ruas et al. (2013, equation 52 and appendix D) offer useful asymptotic expansions for the truncated moments of a noncentral χ^2 law that will allow us to tackle the problem. In particular, and as $\lambda, x \rightarrow \infty$, they show that

$$\lambda^{-p}\Phi_{-1}(p, x; v, \lambda) \sim 1 - \lambda^{-p}\Phi_{+1}(p, x; v, \lambda), \quad (4H.10)$$

where

$$\begin{aligned}
& \lambda^{-p} \Phi_{+1}(p, x; v, \lambda) \\
& \sim \left[\mathbb{1}_{\{\rho < 1\}} + \frac{1}{2} \operatorname{sign}(\rho - 1) \operatorname{erfc} \left(\left| \sqrt{\frac{x}{2}} - \sqrt{\frac{\lambda}{2}} \right| \right) \right] \left[1 + \frac{\tau_2(\tau_2 - 1)}{2\lambda} - \frac{A_1(\frac{v}{2} - 1)}{\lambda} \right] \\
& + \frac{\exp \left[- \left(\sqrt{\frac{x}{2}} - \sqrt{\frac{\lambda}{2}} \right)^2 \right]}{\sqrt{2\pi} \sqrt{\lambda}} \left[\frac{\rho^{\tau_2} - 1}{\rho - 1} + \frac{((\tau_2 - 1)(\rho - 1) - 1) \rho^{\tau_2 - 1} + 1}{\lambda(\rho - 1)^3} \right. \\
& \left. - \frac{\tau_2(\tau_2 - 1)}{2\lambda(\rho - 1)} - \frac{A_1(\frac{v}{2} - 1)}{\lambda} \frac{\rho^{\tau_2 - 1} - 1}{\rho - 1} \right], \\
& \rho := \sqrt{\frac{x}{\lambda}},
\end{aligned} \tag{4H.11}$$

$\operatorname{sign}(\cdot)$ is the signum function, $\operatorname{erfc}(\cdot)$ is the complementary error function,

$$\tau_l := 2p + \frac{v}{l} - \frac{1}{2},$$

and the function $A_1(\cdot)$ is defined in Temme (1993, page 58).

In the case of the function $P_q(1, P_t, L)$ with $\beta > 2$, equation (4.37) yields

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} P_q(1, P_t, L) \\
& = \lim_{P_t \rightarrow 0^+} \frac{1}{q} (f(\tau))^{-\nu} \Phi_{-1} \left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
& = 0,
\end{aligned} \tag{4H.12}$$

because $x = \frac{L^{2-\beta}}{f(\tau)} < \infty$ and $\lambda \equiv \lambda(P_t) = \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \rightarrow \infty$ as $P_t \rightarrow 0^+$, which implies that $\rho \rightarrow 0$, $\left(\sqrt{\frac{x}{2}} - \sqrt{\frac{\lambda}{2}} \right) \rightarrow -\infty$ and $\operatorname{erfc}(\infty) = 0$ using Olver et al. (2010, equation 7.2.2). Combining these results with the asymptotic expansions (4H.10) and (4H.11), the limit (4H.12) is immediately obtained.

Adopting the same rational, but now with $\lambda \equiv \lambda(P_t) = \frac{P_t^{2-\beta}}{f(\tau)} \rightarrow \infty$ as $P_t \rightarrow 0^+$, it is possible to show analytically that equation (4.41) yields

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} I(1, P_t, L) \\
& = \lim_{P_t \rightarrow 0^+} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|} \right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{-1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\
& = 0.
\end{aligned} \tag{4H.13}$$

The limit of the function $K(-1, P_t, L)$ seems to be impossible to be evaluated analytically because it involves gamma functions and the upper tail truncated moment contained in the asymptotic expansion (4H.11), which originates an indetermination of the type $\infty - \infty$ that, to be best of our knowledge, cannot be solved. Nevertheless, using equation (4.42) and since $\lambda^{-p}\Phi_{+1}(p, x; v, \lambda) \sim 1$ as $\lambda \rightarrow \infty$, it can be concluded that

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} K(-1, P_t, L) \\
&= \lim_{P_t \rightarrow 0^+} \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j-\nu}}{j!} \frac{(f(\tau))^{j-\nu}}{\Gamma(j+1-\nu)} \Phi_{+1}\left(j-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \\
&\quad - \lim_{P_t \rightarrow 0^+} \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \Phi_{+1}\left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right), \\
&\sim \lim_{P_t \rightarrow 0^+} \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j-\nu}}{j!} \frac{(f(\tau))^{j-\nu}}{\Gamma(j+1-\nu)} \lambda^{j-\nu} \\
&\quad - \lim_{P_t \rightarrow 0^+} \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j+\nu}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \lambda^j \\
&= \frac{\pi}{2 \sin(\nu\pi)} \left(\frac{\sqrt{2r}f(\tau)\lambda}{\delta|\beta-2|}\right)^{-\nu} \lim_{P_t \rightarrow 0^+} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1-\nu)} \lambda^j \\
&\quad - \frac{\pi}{2 \sin(\nu\pi)} \left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{\nu} \lim_{P_t \rightarrow 0^+} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta|\beta-2|}\right)^{2j}}{j!} \frac{(f(\tau))^j}{\Gamma(j+1+\nu)} \lambda^j \\
&\neq 0, \tag{4H.14}
\end{aligned}$$

for $\lambda \equiv \lambda(P_t) = \frac{P_t^{2-\beta}}{f(\tau)}$, after conducting some numerical tests for economically reasonable parameters combinations, whose results are available upon request, and because $1/\Gamma(j+1-\nu) \neq 1/\Gamma(j+1+\nu)$ for $j \in \mathbb{N}_0$. Nevertheless, and even though this limit cannot be assumed to be (algebraically) zero, its value is generally almost null for the tested parameters combinations.

Therefore, substituting the limits (4H.9), (4H.12), (4H.13) and (4H.14) in the second limit contained in last equality of expression (4H.5) yields

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} [B_0 K(-1, P_t, L) + X_r(-1, P_t, L) + A_0 I(1, P_t, L) + P_q(1, P_t, L)] \\
&= \lim_{P_t \rightarrow 0^+} [B_0 K(-1, P_t, L)] + \frac{L}{r}. \tag{4H.15}
\end{aligned}$$

Finally, the limits (4H.8) and (4H.15) allow the limit (4H.5) to be rewritten as

$$\lim_{P_t \rightarrow 0^+} V_p(P_t, L, T) = \lim_{P_t \rightarrow 0^+} [B_0 K(-1, P_t, L)] + \frac{L}{r} (1 - e^{-r(T-t)}), \quad (4H.16)$$

thus yielding the first branch of equation (4.69) after solving NPV := $\lim_{P_t \rightarrow 0^+} V_p(P_t, L_0, T) - I = 0$ with respect to L_0 .

The case $r \neq q$.

Using equations (4.47), (4.64) and (4.17), with $\beta > 2$ and $r \neq q$, the limit (4H.4) can be rewritten as

$$\begin{aligned} & \lim_{P_t \rightarrow 0^+} V_p(P_t, L, T) \quad (4H.17) \\ &= \lim_{P_t \rightarrow 0^+} [V_f(P_t, L, \infty) - e^{-r(T-t)} V_{f,nb}(P_t, L, \infty) + e^{-r(T-t)} G^\infty(t, T, P_t)] \\ & \quad - \lim_{P_t \rightarrow 0^+} [e^{-r(T-t)} G^\infty(t, T, P_t)] \\ &= \lim_{P_t \rightarrow 0^+} \left[B_0 Y e^{(\epsilon-1)\frac{x(P_t)}{2}} U(a, b, x(P_t)) + \frac{L}{r} - G_t^\infty(P_t) \right] - e^{-r(T-t)} \\ & \quad \times \lim_{P_t \rightarrow 0^+} [B_0 Y U_N(-1, P_t, L) + X_r(-1, P_t, L) + A_0 Y M_N(1, P_t, L) + P_q(1, P_t, L)], \end{aligned}$$

since the limits associated with the time- t expected value of the perpetual bubble cancel out.

Given that

$$\begin{aligned} \lim_{P_t \rightarrow 0^+} x(P_t) &= \lim_{P_t \rightarrow 0^+} \frac{2|r-q|}{\delta^2|\beta-2|} P_t^{2-\beta} \\ &= \infty, \end{aligned}$$

from equation (4.10) and for $\beta > 2$, and $U(a, b, x) \sim x^{-a}$ for $x \rightarrow \infty$, using Olver et al. (2010, equation 13.2.6), which implies that

$$\begin{aligned} \lim_{P_t \rightarrow 0^+} e^{(\epsilon-1)\frac{x(P_t)}{2}} U(a, b, x(P_t)) &= \lim_{x \rightarrow \infty} e^{(\epsilon-1)\frac{x}{2}} U(a, b, x) \\ &\sim \lim_{x \rightarrow \infty} e^{(\epsilon-1)\frac{x}{2}} x^{-a} \\ &= 0, \end{aligned} \quad (4H.18)$$

because, for $\beta > 2$, $\epsilon = \mathbb{1}_{\{r>q\}} - \mathbb{1}_{\{r<q\}}$ and $a > 0$ for both cases $r < q$ and $r > q$, and, hence,

$$\lim_{P_t \rightarrow 0^+} \left[B_0 Y e^{(\epsilon-1)\frac{x(P_t)}{2}} U(a, b, x(P_t)) + \frac{L}{r} - G_t^\infty(P_t) \right] = \frac{L}{r}, \quad (4H.19)$$

after using the limits (4H.7) and (4H.18).

Adopting the same rational used before it is possible to show analytically that equation

(4.49) yields

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} M_N(1, P_t, L) \\
&= \lim_{P_t \rightarrow 0^+} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \Phi_{-1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
&= 0.
\end{aligned} \tag{4H.20}$$

Moreover, following the insights used for the function $K(-1, P_t, L)$ it is possible to conclude that equation (4.51) yields

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} U_N(-1, P_t, L) \\
&= \lim_{P_t \rightarrow 0^+} \frac{\Gamma(-\nu)}{\Gamma(a - \nu)} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \Phi_{+1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
&\quad + \lim_{P_t \rightarrow 0^+} \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(d - \nu)_j}{(2 - b)_j} \frac{\tilde{\xi}^j}{j!} \left(\frac{\tilde{\xi} (1 - e^{-\alpha\tau})}{2} \right)^{j-\nu} \Phi_{+1} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \\
&\sim \lim_{P_t \rightarrow 0^+} \frac{\Gamma(-\nu)}{\Gamma(a - \nu)} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \lambda^j \\
&\quad + \lim_{P_t \rightarrow 0^+} \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(d - \nu)_j}{(2 - b)_j} \frac{\tilde{\xi}^j}{j!} \left(\frac{\tilde{\xi} (1 - e^{-\alpha\tau})}{2} \right)^{j-\nu} \lambda^{j-\nu} \\
&\neq 0,
\end{aligned} \tag{4H.21}$$

for $\lambda \equiv \lambda(P_t) = \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}$, after conducting some numerical tests for economically reasonable parameters combinations, whose results are available upon request.

Therefore, substituting the limits (4H.9), (4H.12), (4H.20) and (4H.21) in the second limit contained in last equality of expression (4H.17) yields

$$\begin{aligned}
& \lim_{P_t \rightarrow 0^+} [B_0 Y U_N(-1, P_t, L) + X_r(-1, P_t, L) + A_0 Y M_N(1, P_t, L) + P_q(1, P_t, L)] \\
&= \lim_{P_t \rightarrow 0^+} [B_0 Y U_N(-1, P_t, L)] + \frac{L}{r}.
\end{aligned} \tag{4H.22}$$

Finally, the limits (4H.19) and (4H.22) can be combined in order to allow the limit (4H.17) to be rewritten as

$$\lim_{P_t \rightarrow 0^+} V_p(P_t, L, T) = \lim_{P_t \rightarrow 0^+} [B_0 Y U_N(-1, P_t, L)] + \frac{L}{r} (1 - e^{-r(T-t)}), \tag{4H.23}$$

thus yielding the second branch of equation (4.69) by solving $\text{NPV} := \lim_{P_t \rightarrow 0^+} V_p(P_t, L_0, T) - I = 0$ with respect to L_0 .

4I. Derivatives of the forward start perpetuity components

This appendix shows the solutions of the derivatives (4.70) and (4.71), which can be straightforwardly obtained following the results of Carr and Linetsky (2006), Ruas et al. (2013) and Dias and Nunes (2018).

4I.1. Auxiliary result

For a given function $\lambda(P) \in \mathbb{R}_0^+$, it is possible to show that

$$\frac{\partial}{\partial P} \Phi_\phi(p, x; v, \lambda(P)) = \frac{1}{2} \lambda'(P) [\Phi_\phi(p, x; v+2, \lambda(P)) - \Phi_\phi(p, x; v, \lambda(P))], \quad (4I.1)$$

with $\Phi_\phi(p, x; v, \lambda)$ being interpreted as the p -th lower tail (if $\phi = -1$) or the p -th upper tail (if $\phi = +1$) truncated moment of a noncentral χ^2 random variable with v degrees of freedom and noncentrality parameter $\lambda(P)$, as defined in Carr and Linetsky (2006, equations 5.12 and 5.11) and Dias and Nunes (2018, equations 21 and 24).

4I.2. Derivatives involving the linear components

The derivative of the component (4.37) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} P_q(1, P_t, L) \\ &= \begin{cases} \frac{1}{q} e^{(r-q)\tau} \left(1 - \frac{2-\beta}{2f(\tau)} P_t^{2-\beta}\right) \Phi_{+1}\left(0, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \\ + \frac{1}{q} e^{(r-q)\tau} \frac{2-\beta}{2f(\tau)} P_t^{2-\beta} \Phi_{+1}\left(0, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) & \Leftarrow \beta < 2 \\ \frac{1}{q} e^{-\alpha\tau} \frac{2-\beta}{2(f(\tau))^{1+\nu}} P_t^{1-\beta} \left[\Phi_{-1}\left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}\right) \right. \\ \left. - \Phi_{-1}\left(-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}\right) \right] & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (4I.2)$$

The derivative of the component (4.38) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} X_r(-1, P_t, L) \\ &= \begin{cases} \frac{L}{r} (f(\tau))^{-\nu} \left(1 - \frac{2-\beta}{2f(\tau)} P_t^{2-\beta}\right) \Phi_{-1}\left(-\nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \\ + \frac{L}{r} \frac{(2-\beta)}{2(f(\tau))^{1+\nu}} P_t^{2-\beta} \Phi_{-1}\left(-\nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \\ + \frac{L}{r} \frac{\beta-2}{\Gamma(\nu)} (2f(\tau))^{-\nu} e^{-\frac{P_t^{2-\beta}}{2f(\tau)}} & \Leftarrow \beta < 2, \\ \frac{L}{r} \frac{e^{-\alpha\tau} (2-\beta)}{2f(\tau)} P_t^{1-\beta} \left[\Phi_{+1}\left(0, \frac{L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}\right) \right. \\ \left. - \Phi_{+1}\left(0, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)}\right) \right] & \Leftarrow \beta > 2 \end{cases}, \end{aligned} \quad (4I.3)$$

after employing Olver et al. (2010, equation 8.8.13) for tackling the derivative of the term involving the incomplete gamma function $\Gamma(\nu, \cdot)$.

4I.3. Derivatives involving the modified Bessel functions

The derivative of the component (4.41) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} I(\phi, P_t, L) \\ = & \begin{cases} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta(2-\beta)}\right)^{2j+\nu} (f(\tau))^j}{j! \Gamma(j+1+\nu)} \left[\left(1 - \frac{2-\beta}{2f(\tau)} P_t^{2-\beta}\right) \Phi_{+\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right. \\ \left. + \frac{2-\beta}{2f(\tau)} P_t^{2-\beta} \Phi_{+\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right] & \Leftarrow \beta < 2 \\ \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta(\beta-2)}\right)^{2j+\nu} (f(\tau))^j}{j! \Gamma(j+1+\nu)} \frac{2-\beta}{2f(\tau)} P_t^{1-\beta} \left[\Phi_{-\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right. \\ \left. - \Phi_{-\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right] & \Leftarrow \beta > 2 \end{cases}. \end{aligned} \quad (4I.4)$$

The derivative of the component (4.42) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} K(\phi, P_t, L) \\ = & \begin{cases} \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta(2-\beta)}\right)^{2j-\nu} (f(\tau))^{j-\nu}}{j! \Gamma(j+1-\nu)} \left[\left(1 - \frac{2-\beta}{2f(\tau)} P_t^{2-\beta}\right) \right. \\ \times \Phi_{+\phi} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \\ \left. + \frac{2-\beta}{2f(\tau)} P_t^{2-\beta} \Phi_{+\phi} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right] \\ + \mathbb{1}_{\{\beta < 2, \phi = -1\}} \frac{\beta-2}{2} \left(\frac{2f(\tau)\sqrt{2r}}{\delta(2-\beta)}\right)^{-\nu} e^{-\frac{P_t^{2-\beta}}{2f(\tau)}} \\ - \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta(2-\beta)}\right)^{2j+\nu} (f(\tau))^j}{j! \Gamma(j+1+\nu)} \left[\left(1 - \frac{2-\beta}{2f(\tau)} P_t^{2-\beta}\right) \right. \\ \times \Phi_{+\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) + \frac{2-\beta}{2f(\tau)} P_t^{2-\beta} \Phi_{+\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \Big] & \Leftarrow \beta < 2 \\ \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta(\beta-2)}\right)^{2j-\nu} (f(\tau))^{j-\nu}}{j! \Gamma(j+1-\nu)} \frac{2-\beta}{2f(\tau)} P_t^{1-\beta} \\ \times \left[\Phi_{-\phi} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) - \Phi_{-\phi} \left(j - \nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right] \\ - \frac{\pi}{2 \sin(\nu\pi)} \sum_{j=0}^{\infty} \frac{\left(\frac{\sqrt{2r}}{\delta(\beta-2)}\right)^{2j+\nu} (f(\tau))^j}{j! \Gamma(j+1+\nu)} \frac{2-\beta}{2f(\tau)} P_t^{1-\beta} \left[\Phi_{-\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right. \\ \left. - \Phi_{-\phi} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{P_t^{2-\beta}}{f(\tau)}\right) \right] & \Leftarrow \beta > 2 \end{cases} \end{aligned} \quad (4I.5)$$

after employing Olver et al. (2010, equation 8.8.13) for tackling the derivative of the term involving the incomplete gamma function $\Gamma(\nu, \cdot)$.

4I.4. Derivatives involving the confluent hypergeometric functions

The derivative of the component (4.48) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} M_P(-1, P_t, L) \\ = & e^{(r-q)\tau} \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \left[\left(1 - \frac{2 - \beta}{2f(\tau)} P_t^{2-\beta} \right) \Phi_{-1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \right. \\ & \left. + \frac{2 - \beta}{2f(\tau)} P_t^{2-\beta} \Phi_{-1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \right]. \end{aligned} \quad (4I.6)$$

The derivative of the component (4.49) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} M_N(1, P_t, L) \\ = & \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{-\alpha\tau}}{2} \right)^j \frac{e^{-\alpha\tau} (2 - \beta)}{2f(\tau)} P_t^{1-\beta} \left[\Phi_{-1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \right. \\ & \left. - \Phi_{-1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \right]. \end{aligned} \quad (4I.7)$$

The derivative of the component (4.50) with respect to P_t is equal to

$$\begin{aligned} & \frac{\partial}{\partial P_t} U_P(1, P_t, L) \\ = & e^{(r-q)\tau} \frac{\Gamma(-\nu)}{\Gamma(a - \nu)} \sum_{j=0}^{\infty} \frac{(c)_j}{(b)_j} \frac{1}{j!} \left(\frac{1 - e^{\alpha\tau}}{2} \right)^j \left[\left(1 - \frac{2 - \beta}{2f(\tau)} P_t^{2-\beta} \right) \right. \\ & \times \Phi_{+1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) + \frac{2 - \beta}{2f(\tau)} P_t^{2-\beta} \Phi_{+1} \left(j, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \Big] \\ & + e^{(r-q)\tau} \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(c - \nu)_j}{(2 - b)_j} \frac{\xi^j}{j!} \left(\xi \frac{1 - e^{\alpha\tau}}{2} \right)^{j-\nu} \left[\left(1 - \frac{2 - \beta}{2f(\tau)} P_t^{2-\beta} \right) \right. \\ & \times \Phi_{+1} \left(j - \nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 2 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \\ & \left. + \frac{2 - \beta}{2f(\tau)} P_t^{2-\beta} \Phi_{+1} \left(j - \nu, \frac{e^{-\alpha\tau} L^{2-\beta}}{f(\tau)}; 4 + 2\nu, \frac{P_t^{2-\beta}}{f(\tau)} \right) \right]. \end{aligned} \quad (4I.8)$$

The derivative of the component (4.51) with respect to P_t is equal to

$$\begin{aligned}
& \frac{\partial}{\partial P_t} U_N(-1, P_t, L) \\
= & \frac{\Gamma(-\nu)}{\Gamma(a-\nu)} \sum_{j=0}^{\infty} \frac{(d)_j}{(b)_j} \frac{1}{j!} \left(\frac{1-e^{-\alpha\tau}}{2} \right)^j \frac{e^{-\alpha\tau}(2-\beta)}{2f(\tau)} P_t^{1-\beta} \\
& \times \left[\Phi_{+1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) - \Phi_{+1} \left(j, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \right] \\
& + \frac{\Gamma(\nu)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(d-\nu)_j}{(2-b)_j} \frac{\tilde{\xi}^j}{j!} \left(\tilde{\xi} \frac{1-e^{-\alpha\tau}}{2} \right)^{j-\nu} \frac{e^{-\alpha\tau}(2-\beta)}{2f(\tau)} P_t^{1-\beta} \\
& \times \left[\Phi_{+1} \left(j-\nu, \frac{L^{2-\beta}}{f(\tau)}; 4+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) - \Phi_{+1} \left(j-\nu, \frac{L^{2-\beta}}{f(\tau)}; 2+2\nu, \frac{e^{-\alpha\tau} P_t^{2-\beta}}{f(\tau)} \right) \right].
\end{aligned} \tag{4I.9}$$

5. Conclusion

Throughout three separate but interrelated articles, this thesis provides important results concerning the valuation of caps and floors and then uses those results to analyze feed-in tariff contracts based on the real options theory. First, under the GBM assumption we present novel analytic representations for caps, floors and collars on continuous flows using arbitrage-free relations based on the proposed analytic representations of profit caps and profit floors. Then, under the CEV diffusion we provide analytical formulas for perpetual and finite maturity caps and floors, which are used to derive the optimal investment rule for investments covered by FIT contracts. The contributions of this study are three-fold:

Firstly, the proposed new method for calculating the optimal triggers of investment projects containing price floors and price collars clauses is simpler and easier to implement than the usual approaches adopted in the existing literature that use the GBM process to model the uncertainty source dynamics.

Secondly, by extending the perpetual and finite maturity cap and floor solutions offered by Shackleton and Wojakowski (2007) in the context of the GBM model to the CEV process, we show that an incorrect specification of the underlying state variable process can have a significant impact in many executive management decisions. Moreover, we prove that the inclusion of a perpetual bubble value in the Davydov and Linetsky (2001) formulae is required to avoid arbitrage opportunities in the case of the CEV process with upward-sloping volatility skews.

Finally, we have evaluated perpetual and finite maturity FIT contracts under the CEV model and derived the corresponding optimal investment rule, thus extending Barbosa et al. (2018) from the GBM model to the CEV process. In that way, we show that for a policy maker offering a FIT contract, it would be better to consider alternative modeling assumptions to the standard GBM process when setting the price floor as a guarantee for the producer. Otherwise, significant and excessive rents can occur, which are usually paid at the expense of taxpayers. We also show an easy way to compute the value of the project covered by a FIT contract, since we have derived a formula to express its value as a function of caps.

In conclusion, the proposed analytical solutions of caps and floors immediately opens a wide variety of applications on other contractual agreements with cap, floor and collar clauses that have been considered in many price mechanisms and on several government policies

implemented in the context of energy problems. The model given here, in particular, can be expanded by examining a collar contract as a mechanism for governments to encourage private investment by providing a guarantee to the investor while also preventing excessive profits at expense of tax payers.

Further extensions of this work may also include modeling uncertainty through an alternative stochastic process, namely including jumps, or trying to re-evaluate the conceptual framework proposed here with a mean-reverting process, which is well known for better fitting assets like commodities. All of these aspects are left for future research.

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