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Measuring the Value-at-Risk of a stocks and bonds portfolio: A Comparative Analysis of VaR Models

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Master in Finance

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BUSINESS
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Department of Finance

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Resumo

Neste estudo será efetuada a medição do Value-at-Risk (VaR) para uma carteira de investimento diversificada, composta por ações e obrigações. Através de um backtesting, vai ser avaliado o desempenho de diferentes modelos de VaR.

Esta investigação contribui para o campo da gestão de risco, fornecendo uma análise de como diferentes modelos de VaR podem ser selecionados e aplicados de forma ideal para gerir carteiras de investimento. Identificando os 4 modelos mais eficazes, o estudo procede ao cálculo do VaR utilizando estes modelos durante um período de um ano. No final desse período, estes modelos serão analisados em conjunto, incluindo o número de vezes que a estimativa de VaR é excedida bem como o Return on Risk-Adjusted Capital (RORAC). Esta análise comparativa oferece *insights* valiosos sobre o desempenho de cada modelo e a fiabilidade destes 4 modelos na gestão de carteiras no mundo real, destacando a sua eficácia em equilibrar risco e retorno.

Palavras Chave: Value-at-Risk, Backtest, Return on Risk-Adjusted Capital

Classificação JEL: C10, G32

Abstract

In this work it will be explored the measurement of Value at Risk (VaR) for a diversified investment portfolio consisting of equities and bonds. The performance of different VaR models will be studied to determine the 4 most accurate models for estimating the portfolio's risk. These models are evaluated based on their ability to predict potential losses accurately, through a backtesting process.

This work contributes to the field of risk management by providing an analysis of how different VaR models can be optimally selected and applied to manage investment portfolios. By identifying the 4 most effective models, the study proceeds to compute the VaR using these models for one year time. At the end of this period, will be compared the advantages and disadvantage of each of these models, including the number of times that the VaR estimate is exceeded and the Return on Risk-Adjusted Capital (RORAC). This comparative analysis offers valuable insights into the relative performance and reliability of these 4 models in real world portfolio management, highlighting their effectiveness in balancing risk and return.

Keywords: Value-at-Risk, Backtest, Return on Risk-Adjusted Capital

JEL Classification: C10, G32

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List of Abbreviations

BCP - Berkowitz, Christoffersen and Pelletier

EC - Economic Capital

EUR - Euro

EWMA - Exponentially Weighted Moving Average

HKD - Hong Kong Dollar

JPY - Japanese Yen

PV - Present Value

PV01 - Present Value of a Basis Point

QR - Quantile Regression

RM - RiskMetrics

RORAC - Return on Risk-Adjusted Capital

S&P500 - Standard & Poor's 500

SGSt - Skewed Generalized Student-t

U.S. - United States

UC - Unconditional Coverage

USD - United States Dollar

VaR - Value at Risk

CHAPTER 1

INTRODUCTION

In a modern competitive and volatile market environment, risk management plays a critical role in financial institutions in ensuring their stability, sustainability, and success. All financial institutions must be very efficient in identifying, measuring and mitigate risks, as it enables them to safeguard their assets and keep investor's confidence.

Risk management is also a legal obligation for financial institutions, reinforced by regulatory frameworks such as the Basel Accords, particularly Basel II (2004) and Basel III (2010). These regulations require financial institutions to have sufficient capital reserves and implement practices to minimize systemic risks to ensure financial stability.

One of the most widely used metrics in risk management is Value-at-Risk (VaR) which measures the maximum potential losses of a portfolio, over a specific time horizon, that we are confident will not be exceeded for a given significance level. In other words, the VaR measures the capital at risk that derives from financial institutions' financial activities – Economic Capital (EC). This concept represents the amount of capital that a financial institution needs to set aside to cover eventual losses from its risk exposure. So, numerically, EC is the same as the VaR, as both metrics tend to quantify the maximum potential loss (Jorion, 2007), and if a pre-defined maximum value of the EC is reached, financial institutions can establish a benchmark where they can implement their risk management strategies.

The importance of effective risk management has been underscored by several financial crises, which have revealed the vulnerabilities of financial institutions to unexpected market movements. For example, the global financial crisis of 2007-2008 revealed significant deficiencies in risk management practices, particularly in the areas of credit and market risk. According to Stulz (2008), the crisis exposed the limitations of traditional risk models and showed how the complexities of the financial markets require far more advanced techniques. This has driven the development of advanced risk management methodologies, including the refinement of VaR models. Despite its widespread adoption, the accuracy and reliability of VaR are heavily dependent on the models used to estimate it, thus, a lot of research and debate has been done on what methods really work the best.

In this context, selecting the appropriate VaR model is crucial for financial institutions aiming to manage risk effectively. Different models offer varying levels of accuracy and sensitivity to market conditions, and their performance can significantly impact their protection

against potential losses. As McNeil, Frey and Embrechts (2015) underscore, when selecting a model from a variety of available risk models it is necessary to establish the purposes, the assumptions, the advantages, and disadvantages of each of the models.

This work provides a comprehensive analysis of how different VaR models can be optimally selected and applied to manage a portfolio. By examining the performance of various models through a backtesting procedure so that the most effective models could be chosen, the study relies on analyze key metrics such as the number of exceedances and Return on Risk-Adjusted Capital (RORAC). Beyond that it will be studied the advantages and disadvantages of each model and how would the portfolio impact on each model.

The structure of this study is as follows: Chapter 2 reviews the existing literature relevant to the study; Chapter 3 outlines the dataset and composition of the portfolio; Chapter 4 details the methodological approach used in the analysis; Chapter 5 presents the results of the backtesting and model selection; Chapter 6 analyzes the selected Value-at-Risk models; Chapter 7 concludes with reflections on the models' effectiveness and their implications for portfolio risk management.

CHAPTER 2

LITERATURE REVIEW

The financial crises of the past few decades show that financial institutions are vulnerable to periods of financial distress. These events, such as the global financial crisis of 2008, exposed significant weaknesses in risk management practices and emphasized the need for policies to deal with financial instability. The failure to effectively manage risks that financial institutions were exposed to and the underestimation of the level of risk they were taking on, contributed to the collapse of major financial institutions which lead to widespread economic repercussions (Stulz, 2008 and Brunnermeier, 2009). To protect financial stability and to avoid similar crises, the introduction of the Basel Accords in 1988 have become increasingly important. With this introduction, institutions must set minimum capital requirements to ensure that they will be safe against potential losses. This implementation made the broader financial system more protected from systemic risks.

The need for standardized metrics to measure risk became important as financial regulators and international banks recognized the importance of better understanding of how to manage risk. In this effort, in 1994, J.P. Morgan developed the RiskMetrics (RM) VaR model. The introduction of this method allowed institutions to have a measure of the maximum expected loss over a specific period of time at a given confidence level, offering financial institutions a powerful tool for understanding and controlling market risk (Jorion, 2007). VaR's adoption as the official measure of market risk stated in the Basel II Accords marked a turning point toward using economic capital based metrics to determine minimum capital requirements, reinforcing the importance of risk management practices in ensure financial stability (Allen et al., 2004).

The RM model is categorized as a parametric model, specifically referred to as Parametric Normal VaR. This means that the model relies on certain assumptions such as a normal distribution of returns (Jorion, 2007). Since financial returns are normally distributed, this model assumes that most returns will fall within a predictable range, with a small probability of existing extreme events. However, extreme events, such as sharp declines or financial crashes occur more frequently than a normal distribution predicts (Pritsker, 2006). At lower significance levels such as 1% (99% confidence level), the non-normality of financial returns becomes highly relevant. Which means that the Parametric normal VaR model tends to underestimate the true risk, leading to a potential scenario of large and unexpected losses. We can conclude that the higher the confidence level, the more the VaR will underestimate the risk.

This shortcoming has led researchers to explore more flexible models capable of capturing heavy tails and skewness often observed in financial return distributions. Studies like those by Kuester, Mittnik, and Paolella (2006) have emphasized this issue, reinforcing the need for alternative VaR models that more accurately reflect the reality of financial market returns.

The Skewed Generalized Student-t (SGSt) distribution was studied by Azzalini and Capitanio (2003), to address the limitations of the Generalized Student-t distribution, which was extended to incorporate skewness, allowing for a more flexible modeling of data with both tails and asymmetry. Marco Folpmers (2021) explored how the SGSt model effectively captures fat tails and asymmetries in financial return distributions, compared to normal distribution models, such as the RM VaR. By applying the SGSt model to S&P 500 index returns from 2017-2021, the study demonstrates how the SGSt model is particularly well-suited for modeling tail risk during periods of high volatility, such as the COVID-19 crisis, outperforming the RM model in extreme risk scenarios. This study underscores that while the RM model may perform reasonably well for moderate risk levels, it is not the best model to accurately predict the VaR when there is market distress. The SGSt model, with its flexibility and ability to account for both kurtosis and skewness shows that it is a better model compared to the RM VaR, especially when applied to low significance levels.

There are VaR models that do not rely on normal distribution approach, like the Historical Simulation VaR. This model uses actual past returns, which allows to capture the real behavior of the market, including fat tails and skewness. However, the accuracy of the model depends on the relevance and the sample size of past returns (Jorion, 2007; Pritsker, 2006). If the market has changed in a way that makes historical data less representative of future risks, the model may underperform (Jorion, 2007). While a larger sample size can result in a diversity of outcomes, it may also reduce the model's sensitivity to current market volatility as studied by Pritsker (2006). Thus, Hull and White (1998) proposed a new approach that refines the traditional Historical VaR by incorporating volatility models, allowing the VaR estimates to be more sensible to recent market volatility changes. Unlike the traditional Historical VaR, which assumes that the distribution of past returns remains constant, Hull and White's methodology adjusts past returns based on volatility estimates, ensuring that when market conditions shift, the VaR estimates remains accurate. Empirical evidence from Hull and White's study suggests that this volatility-adjusted model outperforms the traditional Historical Simulation VaR, especially in periods of market uncertainty.

Building on the need for models that can adapt to changing market conditions, the Conditional Quantile Regression (QR) model emerged as an alternative model for estimating

the VaR. The QR model addresses many of the limitations in the RM and Historical VaR models. Although this is a non-parametric method such as the Historical VaR, it incorporates additional information that could improve the accuracy of the risk estimate. Instead of relying only on historical returns, QR VaR also incorporates explanatory variables, such as macroeconomic indicators. By estimating the quantiles of a distribution based on these variables, this model reflects how such indicators influence returns. This approach allows for a more flexible VaR estimation and is more accurate when market conditions change.

The study made by Sjur Westgaard et al. (2019) evaluates the performance of various VaR models, including the RM VaR, Historical VaR and QR VaR in the European energy market. They apply these models into nine different energy futures contracts with a sample size of 10 years. Their findings indicate that QR VaR provides superior risk estimation compared to Historical VaR and RM VaR, especially when lower significance levels are applied.

Unlike RM methodology, which due to its normal distribution assumptions fail in the accuracy of VaR estimates for lower significance levels, QR VaR offers more robust estimates across different significance levels. Even though it requires careful configuration of explanatory variables, QR VaR consistently outperforms the other models, especially in markets with non-normal distributions and high volatility.

CHAPTER 3

DATA AND PORTFOLIO COMPOSITION

In this work it will be analyzed a portfolio composed with both equities and bonds from the US, European, Japanese and Honk Kong markets. The Euro (EUR) will be used as the base currency for measuring VaR, while exposures to the U.S. Dollar (USD), Hong Kong Dollar (HKD), and Japanese Yen (JPY) introduce foreign exchange risk.

The equity segment of the portfolio includes twenty-eight positions, both long and short. All the three exchange rates – USD/EUR, HKD/EUR, JPY/USD – as well as the daily adjusted prices for splits and dividends for each stock, were downloaded from Yahoo Finance (https://finance.yahoo.com).

In terms of fixed income, the portfolio comprises four government bonds, each with a fixed coupon rate and different maturity dates. These bonds were issued in the U.S. and Netherlands and the details were downloaded from the Frankfurt Stock Exchange website (https://www.boerse-frankfurt.de/bonds). The dataset consists of daily observations for 17 years, from January 3rd, 2007, to February 2nd, 2024.

The strategy behind the choice of this portfolio was to balance growth, income, and stability for long-term wealth accumulation. By including a mix of leading global equities across technology, healthcare, financials, and industrials sectors, along with AAA and AA+ Dutch and US government bonds maturing between 2027 and 2042, the portfolio is well-positioned to benefit from growth opportunities across various sectors and regions. Companies like Apple and ASML drive capital appreciation in high-growth sectors, while stable firms like Walmart and Toyota provide reliable dividends to offset volatility. We also took short positions in underperforming assets, such as Wells Fargo and ORIX, expecting their value to decline due to sector challenges and weak economic signals (ORIX case – Japan), aiming to profit from their downward trajectory. The bonds serve as a hedge against the equity portfolio's volatility due to their low or negative correlation with stocks, ensuring steady income and capital preservation. The laddered bond approach minimizes interest rate fluctuations and supports long-term financial objectives, while the globally diversified equity mix safeguards against regional economic and currency risks, creating a robust portfolio for a multi decade horizon. As we can see in Table 1 and 2 below, the equity portfolio value is 3 330 477.51€ and the bond

Equity	Ticker	Currency	No. Shares	Value (EUR)	Allocation (%) (stocks+bonds)
Apple Inc.	AAPL	USD	2 500	332 136.56	4.94
Eli Lilly and Company	LLY	USD	1 000	309 874.28	4.61
JP Morgan Chase & Co	JPM	USD	1 000	124 497.04	1.85
Walmart Inc	WMT	USD	2 400	102 939.02	1.53
Advanced Micro Devices	AMD	USD	4 300	297 601.55	4.42
General Electric Company	GE	USD	2 000	121 267.02	1.80
Abbot Laboratories	ABT	USD	1 100	108 268.31	1.61
Oracle Corp	ORCL	USD	800	64 154.21	0.95
Wells Fargo & Co.	WFC	USD	-2 000	-81 059.05	-1.20
Comcast Corp	CMCSA	USD	2 300	80 710.96	1.20
Blackrock, Inc.	BLK	USD	310	208 675.10	3.10
Techtronic Industries Company Limited	0669.HK	HKD	20 000	225 305.87	3.35
ENN Energy Holdings Limited	2688.HK	HKD	9 000	122 873.83	1.83
CK Infrastructure Holdings Limited	1038.HK	HKD	-20 000	-93 552.29	-1.39
Nomura Holdings, Inc.	8604.T	JPY	-20 000	-70 001.36	-1.04
Toyota Motor Corporation	7203.T	JPY	15 000	199 682.81	2.97
ORIX Corporation	8591.T	JPY	-8 000	-122 669.50	-1.82
Takeda Pharmaceutical Company Limited	4502.T	JPY	3 700	101 590.51	1.51
ASML Holding N.V.	ASML.AS	EUR	500	305 796.94	4.55
BNP PARIBAS	BNP.PA	EUR	2 000	108 657.79	1.62
Airbus SE	AIR.PA	EUR	1 500	171 420.41	2.55
ASM International NV	ASM.AS	EUR	500	157 937.85	2.35
DNB Bank	DNB.OL	EUR	-600	-96 314.49	-1.43
EDP - Energias de Portugal, S.A.	EDP.LS	EUR	27 000	115 093.52	1.71
Capgemini SE	CAP.PA	EUR	-700	-116 681.82	-1.73
Alstom SA	ALO.PA	EUR	6 500	179 706.01	2.67
LVMH Moet Hennesy - Louis Vuitton	MC.PA	EUR	300	234 332.92	3.48
Ryanair Holdings Plc	RYA.IR	EUR	15 500	238 233.53	3.54
Total Equity	-	-	-	**3 330 477.51**	**49.50**

portfolio is 3 397 346.39€, totaling an initial portfolio value of 6 727 823.9€, as of 27 of January 2023.

Table 1. Equities composition in EUR as of 27 January 2023. Where necessary, investments originally denominated in USD/HKD/JPY were converted to EUR using the respective exchange rate as of 27 of January 2023. This segment represents 49.50% of our portfolio.

Bond	ISIN	Currency	Maturity	Coupon (%)	Coupon per Year	Face Value (EUR)	Fair Value (EUR)	Allocation (%)
Dutch Bond 2042	NL0009446418	EUR	15/01/2042	3.75	1	800 000	974 479.51	14.48
Dutch Bond 2037	NL0000102234	EUR	15/01/2037	4.00	1	750 000	898 583.04	13.36
US Treasury 2029	US91282CKP58	USD	30/04/2029	4.63	2	642 530	686 616.04	10.21
US Treasury 2027	US91282CGC91	USD	31/12/2027	3.88	2	826 110	837 667.80	12.45
Total Bonds	**-**	**-**	**-**	**-**	**-**	**-**	**3 397 346.39**	**50.50**

Table 2. Bonds characteristics in EUR as of 27 January 2023. The bond's fair value corresponds to the total present value of its future cash flows, discounted as of 27 January 2023, and converted to euros when applicable. This segment represents 50.50% of the portfolio exposure.

Chapter 4

METHODOLOGY

The goal of this project is to measure the Value at Risk of a portfolio. It is centered on the selection and evaluation of the 4 most accurate VaR models from each VaR class – Normal VaR, SGSt VaR, Historical VaR and QR VaR – through a backtesting process. For this backtest, two statistical tests were applied: the Uncoverage Conditional (UC) test and the BCP test. In section 4.3 both tests will be explained with more detail.

At the end of the backtesting, we can conclude whether the models are indicated for our portfolio or not. In this work, the backtesting covers nearly 10 years, from 11 February 2013 to 27 January 2023 and the significance level chosen was 1%.

For each of the 4 chosen VaR models, a serie of daily VaR estimates will be computed between 30 January 2023 to 2 February 2024.

The classes of VaR models under consideration are the Normal, SGSt, Historical and QR. Within each of these four, different variations of these models will be employed to ensure we have a wide range of models to analyze and choose from.

Finally, based on the backtesting results, the four best-performing VaR models will be selected for further analysis and comparison.

4.1 Risk Factor Exposure Mapping

A risk factor is any variable or driver that can influence the return or value of an asset, portfolio, or financial system. In the case of portfolio management, risk factors are used to explain the sources of risk and return in investments. As so, identifying and quantifying these sources of risk, while linking them to every portfolio position and determine their exposure was extremely important as initial goal. The mapping of risk factors to each portfolio position is dependent on the type of asset. Typically, the risk factors driving stocks differ from those driving bonds. For stocks, the risk factor is the performance of the stock itself, whereas for bonds, it is predominantly the interest rate.

For each asset class in our portfolio, there will be present below the exposure mapping methodology.

4.1.1 Equity

In portfolios with stocks, the market price is the primary risk factor, as changes in price directly influence the portfolio's value. The risk factor mapping helps to quantify the exposure in euros to each stock, which is calculated as the capital invested (number of shares × market price). The table below shows the risk factors and corresponding mapped exposures for this segment as of 27 January 2023.

Risk Factor	Exposure (EUR)
AAPL	332 136.56
LLY	309 874.28
JPM	124 497.04
WMT	102 939.02
AMD	297 601.55
GE	121 267.02
ABT	98 425.74
ORCL	64 154.21
WFC	-81 059.05
CMCSA	80 710.96
BLK	208 675.10
0669.HK	225 305.87
2688.HK	122 873.83
1038.HK	-93 552.29
8604.T	-70 001.36
7203.T	199 682.81
8591.T	-122 669.50
4502.T	101 590.51
ASML.AS	305 796.94
BNP.PA	108 657.79
AIR.PA	171 420.41
ASM.AS	157 937.85
DNB.OL	-96 314.49
EDP.LS	115 093.52
CAP.PA	-116 681.82
ALO.PA	179 706.01
MC.PA	234 332.92
RYA.IR	238 233.53

Table 3. Equities - Risk factor exposures map in EUR as of 27 January 2023.

4.1.2 Bonds

Bonds are financial instruments that represent a loan from an investor, who expect to receive interest payments and return of the principal, to a borrower, and in our case, there are four different governments bonds.

The fair value of bonds is determined by discounting future cash flows at the current market interest rate – yield. When market interest rate rise, the present value of cash flows decreases, which causes bonds prices to fall. Given that interest rates fluctuate with market conditions, assessing the sensitivity of bond positions to these changes is crucial for effective risk measurement.

For a fixed-coupon bond, the cash flows consist of periodic coupon payments throughout the bond's term and a final repayment at maturity. The coupon payment is calculated as:

$$\mathrm{N} \times \frac{c_n}{n}. \tag{1}$$

N represents the bond's face value (amount invested), c_n is the annual coupon rate (the yearly interest paid until maturity), and n is the coupon frequency (the number of coupon payments per year).

At maturity, the final payment includes the repayment of the face value plus the coupons and is expressed as:

$$C_m = N \times \left(1 + \frac{c_n}{n}\right) \tag{2}$$

The present value (PV) of a future cash flow, denoted C_T, received after T years, is determined by discounting it at the continuously compounded interest rate r_T, typically expressed as:

$$PV_{C_T, r_T} = C_T \times e^{-r_T \times T} \tag{3}$$

.

To measure how sensitive this present value is to small changes in the interest rate, we use the present value of a basis point (PV01), which reflects the effect of a one basis point (0.01%) decrease in r_T. Using a first-order Taylor approximation, PV01 is expressed as:

$$PV01_{C_T,r_T} \approx \frac{\partial PV_{C_T,r_T}}{\partial r_T} \times (-0.01\%) = T \times PV_{C_T,r_T}. \quad (4)$$

where T represents the maturity of the cash flow.

The first-order approximation of the change in the PV of the cash flow, which represents its profit and loss (P&L), can be expressed in terms of PV01 as:

$$\Delta PV_{C_T,r_T} = - PV01_{C_T} \times \frac{\Delta r_T}{0.01\%} \quad (5)$$

.

where $\frac{\Delta r_T}{0.01\%}$ represents the absolute change in the interest rate, converted into basis points.

The P&L of this cash flow reflects its sensitivity to a small change in the interest rate, quantified by the negative of the present value of a one basis point (0.01%) increase, denoted $-PV01_{C_T}$, and the actual interest rate change, Δr_T. An increase in the interest rate results in a decrease in the present value, yielding a negative P&L. The same goes the other way: a decrease in the interest rate results in an increase in the present value, yielding a positive P&L. For a bond comprising *n* cash flows, the total P&L is computed by summing the P&L of each cash flow, where the change in present value for each is given by equation 5. This approach aggregates the individual impacts to assess the bond's overall sensitivity to interest rate fluctuations.

Managing a bond with *n* cash flows becomes complex, due to each cash flow being tied to a distinct interest rate, leading to n different risk factors. Additionally, there is further complexity when a cash flow's maturity lacks corresponding interest rate data, making it impossible to accurately determine its PV or compute its PV01. In our portfolio the maturities for the US bonds are 2037 and 2042 and for the Dutch bonds are 2027 and 2029.

To effectively manage a bond portfolio with these maturities, a structured approach is essential. To address these issues, following Alexander (2008), we adopt a vertices mapping approach. This method is the PV+PV01 invariant mapping. It maps cash flows with non-standard maturities to a set of standard maturity vertices where interest rate data is readily available. Each vertex represents a standard maturity with accessible interest rate data. With this approach, it is ensured that both the PV and the PV01 of the original cash flow are preserved.

To map a cash flow with a non-standard maturity T to standard maturity vertices, we denote the present value of the original cash flow as PV_{C_T}. Let T_1 and T_2 represent the standard maturities below and above T, respectively. The proportions of PV_{C_T} allocated to vertices T_1 and T_2 are denoted by $x_{T_1}$and x_{T_2}, respectively. To preserve the present value of the original cash flow, we ensure:

$$x_{T_1} + x_{T_2} = PV_{C_T} \tag{6}$$

.

The PV01 invariant mapping method guarantees that the sum of the PV01s of the mapped cash flows equals the PV01 of the original cash flow. As a result, under a parallel shift of one basis point in the interest rate curve, the total P&L of the mapped cash flows reflects the P&L of the original position. The preservation of the original PV01 is ensured by the following condition:

$$T_1 x_{T_1} + T_2 x_{T_2} = T\ PV_{C_T} \tag{7}$$

.

By solving these two equations simultaneously, we derive the proportions x_{T_1} and x_{T_2} that preserve both the PV and PV01:

$$x_{T_1} = \frac{T_2 - T_1}{T_2 - T_1} \times PV_{C_T}\ ; \tag{8}$$

$$x_{T_2} = 1 - \frac{T_2 - T_1}{T_2 - T_1} \times PV_{C_T} \tag{9}$$

For each cash flow of every bond in the portfolio, we apply the PV+PV01 invariant mapping process to align non-standard maturity cash flows with standard maturity vertices. The risk exposure of each mapped cash flow is determined by its sensitivity to changes in the corresponding standard maturity interest rate, quantified by its PV01.

The risk factors of bonds are the standard maturity interest rates. The exposure to each of these rates is calculated as the sum of all −PV01s mapped to that specific maturity. The table 4 shows the risk factors and corresponding mapped exposures as of 27 January 2023:

Risk Factor	Exposure (EUR)
USD3M	-0.48
USD6M	-1.03
USD1Y	-4.40
USD2Y	-11.37
USD3Y	-32.18
USD5Y	-469.76
USD7Y	235.89
USD10Y	-
USD20Y	-
EUR3M	-
EUR6M	-0.19
EUR1Y	-5.66
EUR2Y	-11.42
EUR3Y	-24.99
EUR5Y	-53.56
EUR7Y	-88.91
EUR10Y	-300.05
EUR15Y	-1 032.21
EUR20Y	-900.05

Table 4. Interest Rate - Risk factor exposures map in EUR as of 27 January 2023

4.1.3 Currency

Currency risk comes from the eventual changes in exchanges rates between foreign currencies and our base currency – EUR. In our portfolio we are exposed to three (currency) risk factors – USD/EUR, JPY/EUR, HKD/EUR – which means that these three exchange rates directly impact the value of our foreign assets in EUR terms. The mapping process is based on determining how much exposure we have to each of these three currencies and multiplying for the respective exchange rate at the time we want to sell our positions. In the case of bonds, the exposure corresponds to the fair value of the bond positions. The table below shows the risk factors of currencies in our portfolio and corresponding mapped exposures as of 27 January 2023.

Risk Factor	Exposure (EUR)
USDEUR	3 259 222.42
HKDEUR	254 627.41
JPYEUR	108 602.46

Table 5. Currencies - Risk factor exposures map in EUR as of 27 January 2023.

4.2 Value-at-Risk Models

Value-at-Risk at an α significance level is the maximum potential loss over a specified time period with a given probability (1-α). With α = 1% as in our work, we are looking at the 99th percentile of the potential loss distribution. This means there is a 1% chance that the loss will exceed the VaR value.

As described by Alexander (2009), for any continuous random variable X representing returns or losses over a period h, the α-quantile x_α is defined such that the probability of X being less than x_α is α, such that:

$$P(X < x_\alpha) = \alpha \tag{10}$$

.

Here, x_α is the point on the distribution where the cumulative probability equals α.

If the distribution function F of X is known, the α-quantile can be found by taking the inverse of the cumulative distribution function:

$$x_\alpha = F^{-1}(\alpha) \tag{11}$$

.

This means that for any probability level α, we can find the corresponding value x_α on the distribution of X.

If F is the cumulative distribution function of our return distribution, then for a confidence level (1 - α), the VaR for an h-day horizon can be mathematically described as:

$$VaR_{h,\alpha} = -F^{-1}(\alpha) \tag{12}$$

.

Here, $-F^{-1}(\alpha)$ could also be $-x_\alpha$ and gives you the return level where the probability of worse outcomes (losses) is exactly α.

In the next subsections we explore the methodologies required to calculate the four classes of VaR models mentioned above.

4.2.1. Normal VaR

This model assumes that returns are normally distributed, which simplifies the calculations.

The model assumes that the portfolio returns, denoted by X, follow a normal distribution:

$$X_h = N\sim(\mu_h, \sigma_h), \quad (13)$$

.

where μ_h is the mean (expected return) and σ_h is the standard deviation for the period h.

The VaR for a given time horizon h and confidence level α is given by:

$$VaR_{h,a} = -\phi^{-1}(\alpha) \times \sigma_h - \mu_h, \quad (14)$$

.

Here: $-\phi^{-1}(\alpha)$ is the inverse of the cumulative distribution function for the standard normal distribution at the confidence level α. σ_h is the volatility or standard deviation of returns over the period h. μ_h is the mean return over the period h.

For daily VaR estimates, or small-time horizons, (where h=1), it's suggested by Alexander (2009) to set μ_h= 0 because for short horizons like in our work (daily data), the mean return is often negligible to estimate reliably. Thus, the equation simplifies to:

$$VaR_{h,\alpha} = -\phi^{-1}(\alpha) \times \sigma_h, \quad (15)$$

.

where σ_h is estimated using an EWMA approach, which prioritizes recent observations to better capture current market dynamics. The EWMA model estimates the variance of returns for day t, denoted ${\sigma^2}_t$ using a weighted average of the previous day's variance, ${\sigma^2}_{t-1}$, and the squared return from day t-1, ${r^2}_{t-1}$, where the smoothing factor λ (between 0 and 1) determines the influence of past variance versus recent market data:

$${\sigma^2}_t = \lambda {\sigma^2}_{t-1} + (1-\lambda){r^2}_{t-1} \quad (16)$$

4.2.2 Skewed Generalized Student-t (SGSt) VaR

This model is an advanced approach to risk modeling that addresses the limitations of the normal distribution, particularly in capturing fat tails and skewness.

The Skewed Generalized Student-t (SGSt) distribution, introduced by Theodossiou (1998), is designed to address normality in both skewness and tail behavior. The standardized SGSt distribution $T_{0,1,\lambda,p,q}$ incorporates three key parameters to model these deviations:

- $\lambda \in (-1,1)$ determines the skewness of the distribution. When $\lambda = 0$, the distribution is symmetric; positive values of λ indicate positive skewness, while negative values indicate negative skewness.
- $p > 0$ controls and shape of the central region of the distribution.
- $q > 0$ controls the shape of the tails, allowing for heavier or lighter tails depending on market conditions.

To apply the SGSt distribution to a portfolio, the parameters (λ, p, q) are estimated using the maximum likelihood method. By optimizing these parameters, the SGSt distribution is tailored to closely match the actual return distribution of the portfolio capturing its unique skewness and tail characteristics.

Formally, we calculate the h-day 100α% SGSt VaR as:

$$VaR_{h,a} = -T^{-1}{}_{0,1,\lambda,p,q}(\alpha) \times \sigma_h - \mu_h \quad (17)$$

Here, $T^{-1}{}_{0,1,\lambda,p,q}(\alpha)$ represents the quantile function (inverse cumulative density function) of the standard SGSt distribution. As described in the previous subsection, with Normal VaR, we assume a mean return of zero ($\mu_h = 0$) and estimate volatility (σ_h) using an Exponentially Weighted Moving Average (EWMA) model. Simplifying under these assumptions:

$$VaR_{h,a} = -T^{-1}{}_{0,1,\lambda,p,q}(\alpha) \times \sigma_h \quad (18)$$

This approach ensures that the SGSt VaR captures both asymmetry and tail risk in financial return distributions more effectively than models assuming normality.

4.2.3. Volatility-Adjusted Historical VaR

In the models explored above, we assume that portfolio returns follow specific parametric distributions. However, these assumptions can either oversimplify reality (as in the Normal VaR model) or become computationally complex (as in the SGSt VaR model). The Historical VaR offers a simpler, non-parametric alternative by directly using the empirical distribution of past returns to estimate risk. Instead of relying on a predefined distribution, the Historical VaR uses the α-quantile of historical returns to compute the VaR. To estimate the Historical VaR, we select a sample size *n*, calculate the portoflio's *h*-day returns using historical data while keeping the current portfolio exposure to risk factors constant during the sample period (this simulates how the current portfolio would have performed under historical market conditions), sort the returns from worst to best, assign each an equal probability of 1/n, and accumulate probabilities starting from the largest loss. Finally, we calculate the VaR as the negative of the return where the cumulative probability reaches the confidence level α (in our case 1%), expressed as a positive loss value.

As discussed in Chapter 2, the choice of sample size plays a crucial role. A larger sample size increases the diversity of observed returns but may fail to reflect current market conditions because older data are included. Conversely, a smaller sample size better captures recent market trends but may lack sufficient data for robust estimation. This trade-off is a key limitation of simple Historical VaR, as all observations are equally weighted, meaning that past volatility has the same impact as more recent volatility. To address this limitation, Hull and White (1998) proposed a volatility-adjusted Historical VaR method. This approach adjusts historical returns to reflect current market conditions while maintaining equal weights for all observations. First, we compute a series of volatility estimates ($\hat{\sigma}_t$) using an EWMA model. Then, we adjust each historical return (r_t) using:

$$\hat{r}_t = \frac{r_t}{\hat{\sigma}_t} \cdot \hat{\sigma}_T \tag{19}$$

where $\hat{r}_t$ is the adjusted return, *T* is the VaR measurement date, and $t < T$. This adjustment ensures that all historical returns reflect current market volatility (σ_t).

Formally, the *h*-day 100α% volatility-adjusted Historical VaR is computed as the negative of the α-quantile from the sample of volatility-adjusted returns. By adjusting for

volatility while maintaining simplicity and avoiding parametric assumptions, these methods enhance Historical VaR's ability to capture current market dynamics effectively.

4.2.4. Quantile Regression VaR

Remaining in the non-parametric framework, the Quantile Regression (QR) model provides a flexible approach for estimating VaR by modeling the relationship between portfolio returns and selected explanatory variables without assuming a specific distribution for returns. This method directly models the quantile of interest, which is particularly suitable for VaR estimation as VaR is defined as the negative α-quantile of a series of portfolio returns. Formally, the α-quantile (q_α) regression VaR is expressed as:

$$VaR = -q_{\alpha,y} = -(\hat{a} + \hat{b}x_i), \tag{20}$$

where:

- y represents the portfolio return,
- $\hat{a}$ and $\hat{b}$ are the estimated parameters of the quantile regression of y onto explanatory variables x.

The equation above is derived from the quantile regression model: $y = a + bx_i + \in$ which we can use to generate the estimated quantile regression equation for the α-quantile: $q_{\alpha,y} = (\hat{a} + \hat{b}x_i)$.

The parameters $\hat{a}$ and $\hat{b}$ are determined by solving a minimization problem (Koenker and Bassett Jr., 1978):

$$(\hat{a}, \hat{b}) = \arg \min_{a,b} \sum_{i=1}^{n} [y_i - (a + bx_i)](\alpha - I_{y_i-(a+bx_i)<0}), \tag{21}$$

where:

- $I_{y_i-(a+bx_i)<0}$ is an indicator function defined as:

$$I_{y_i-(a+bx_i)<0} = \begin{cases} 1, & if\ y_i - (a + bx_i) < 0 \\ 0, & otherwise \end{cases} \tag{22}$$

4.3 Backtesting tests

Two statistical tests were applied in the backtesting: the Uncoverage Conditional Test and the BCP test.

The Unconditional Coverage (UC) is a statistical method used to evaluate whether a Value-at-Risk (VaR) model accurately predicts the frequency of losses exceeding the VaR estimates. As explained before, VaR represents the maximum expected loss over a given period at a confidence level of (1 − α), meaning there is an α probability that actual losses will exceed the VaR estimate. For instance, with a 99% confidence level (α = 1%) and 100 daily VaR estimates, we expect losses to exceed the VaR on approximately 100 × α = 1 days, corresponding to an exceedance rate of 1%.

To perform the UC test, we analyze a sample of n observations by defining an indicator function for each day t:

$$I_{\alpha,t} = \begin{cases} 1, & if\ r_t < -VaR_{1,\alpha,t} \\ 0, & otherwise \end{cases} \quad (23)$$

.

Here, r_t is the actual return on day t, and $VaR_{1,\alpha,t}$ the VaR estimate for that day at the significance level α. This results in a series of n binary observations: a value of 1 indicates that there is exceedance, a value of 0 indicates otherwise. The UC Test evaluates whether the VaR model's exceedance probability aligns with its specified significance level α (Alexander, 2009). It tests the null hypothesis that the indicator function, assumed to follow an independent and identically distributed (i.i.d.) Bernoulli process, has a probability of exceedance equal to α. Thus, the null and alternative hypothesis are:

- $H_0 : = \pi_{exp} = \alpha$
- $H_1 : \pi_{obs} \neq \pi_{exp}$,

where π_{obs} is the observed exceedance rate and π_{exp} the expected exceedance rate.

To test this, we calculate the likelihood ratio test statistic:

$$LR_{UC} = \left(\frac{\pi_{exp}}{\pi_{obs}}\right)^{n1} \left(\frac{1-\pi_{exp}}{1-\pi_{obs}}\right)^{n0}, \quad (24)$$

.

where n_1 is the number of exceedances (days where $I_{\alpha,t} = 1$), and $n_0 = n - \mathrm{n}_1$ is the number of non-exceedances. The observed exceedance rate is $\pi_{obs} = \frac{\mathrm{n}_1}{n}$, and the

expected exceedance rate is $\pi_{exp} = \alpha$. Under the null hypothesis, the test statistic $-2\ln(LR_{UC})$ follows a chi-squared distribution with one degree of freedom (x_1^2).

The other test used in the backtesting is the BCP test, which evaluates whether a VaR model's exceedances are independent of each other. This implies that the occurrence of an exceedance on one day does not influence the likelihood of future exceedances. We can use as an example when we are in periods of high market volatility, the risk of having exceedances increases, and if these occur close together (clustering), it indicates the VaR model may not be adjusting quickly enough to adjust to this volatility. The BCP test is designed to detect such clustering by examining the autocorrelation structure of exceedances. Formally, the BCP test uses the same indicator function as the UC Test, thus the null and alternative hypotheses are:

- $H_0: \hat{\rho}_k = 0, for\ all\ k \in \{1, \ldots, K\}$
- $H_1: \exists k \in \{1, \ldots, K\}\ such\ that\ \hat{\rho}_k \neq 0,$

where $\hat{\rho}_k$ represents the autocorrelation at lag k for the series of VaR exceedance indicator function, and K is the maximum lag considered.

The test statistic is:

$$BCP\ (K) = n(n+2)\sum_{k=1}^{K}\frac{\hat{\rho}_k^2}{n-k} \tag{25}$$

.

where n is the sample size. Under the null hypothesis of no autocorrelation ($\hat{\rho}_k = 0$), the test statistic follows a chi-squared distribution with K degrees of freedom (x_k^2). The choice of K influences the test's properties. A larger K examines higher order autocorrelations, potentially capturing longer-term dependencies, but increases the degrees of freedom, raising the critical value and making it harder to reject the null hypothesis. Conversely, a smaller K enhances the test's sensitivity to short-term dependencies but may overlook autocorrelations at higher lags. To balance sensitivity and coverage, and to avoid these limitations, the BCP test was computed not only for K = 1, but from K = 1 up to K = 10.

Chapter 5

BACKTEST AND MODEL SELECTION

As referred in chapter 4, we select the 4 best models from different variants of each of VaR class. For each of the class, we tested innumerous configurations by varying the parameters with the final goal of reaching the best possible model. For Normal VaR, we adjusted the EWMA smoothing factor to estimate volatility. In SGSt VaR, we varied the EWMA smoothing factor, the sample size for estimation and the frequency of parameter reestimation (daily, weekly, biweekly, monthly). For Historical VaR, we tested configurations with and without volatility adjustment, alongside different EWMA smoothing factors and different sample sizes. For QR VaR, we tested many hypotheses by varying the inclusion or exclusion of a constant, one or two EWMA volatility variables with different smoothing factors, varying sample sizes, and adjusting the reestimation frequency. However, before starting to test all these configurations, we need to know what we define as an accurate/best model - what is the criteria behind this decision. Thus, we prioritized the UC test for the global period, followed by the BCP test for the global period, then UC for annual subperiods and BCP for annual subperiods, with ties resolved by selecting the model with an exceedance rate closest to 1%.

After this, we were able to find the best model of each class. These models have the following parameters: Normal VaR with $\lambda = 0.98$; SGSt VaR with $\lambda = 0.98$, sample size of 800 days, and daily reestimation; Historical VaR with volatility adjustment, $\lambda = 0.985$, and sample size of 700 days; and QR VaR without a constant, two EWMA volatility variables with $\lambda_1 = 0.95$ and $\lambda_2 = 0.9$, sample size of 700 days, and daily reestimation.

Traditionally, we reject the null hypothesis when the p-value of the test statistic falls below 5%. Consequently, a model is considered to pass the UC or BCP tests if the null hypothesis is not rejected, which occurs when the p-value exceeds 5%. The tables below present the exceedance rates, the p-value for UC test as well as BCP statistics for both the global period and the annual sub-periods for the 4 chosen models.

Normal VaR				
Backtest Period	**Exceedance rate**	**UC Test (p-value)**	**BCP test**	
			Worst p-value	**Lag**
2013-2023	**1.58%**	**0.6%**	**0.01%**	**10**
2022-2023	1.92%	18.44%	2.69%	3
2021-2022	1.54%	41.87%	0.42%	4
2020-2021	2.31%	7.01%	0.00%	10
2019-2020	1.15%	80.77%	84.93%	1
2018-2019	1.54%	41.87%	9.91%	10
2017-2018	1.92%	18.44%	77.57%	1
2016-2017	0.77%	69.67%	89.96%	1
2015-2016	1.92%	18.44%	0.28%	1
2014-2015	1.54%	41.87%	0.05%	2
2013-2014	1.15%	80.77%	84.93%	1

Table 6. Normal VaR backtesting results over the global period (2013–2023) and annual subperiods, indicating the exceedance rate, the UC test p-value and the BCP test. The BCP test employs a battery of tests with lags ranging from 1 to 10 and reports the lowest p-value among these lags. The last column indicates the lag at which this lowest p-value occurred - in this case, at lag 10, with a p-value of 0.01%.

Looking at table 6, for the Normal VaR we can observe several periods have exceedance rates above 1% including the global period (2013-2023) which is 1.58%, suggesting the model underestimates risk. The p-value for the UC test is 0.6%, which means that the model fails this test (p-value <5%). For the BCP test, some periods like 2020-2021 have very low p-values, including the global period (0.01%).

Although this model is clearly not the best performer, it is the best variant of the Normal VaR and for analysis purposes it will be interesting to compare it with the 3 other models. Thus, the next table shows the SGSt VaR model:

SGSt VaR				
Backtest Period	**Exceedance rate**	**UC Test (p-value)**	**BCP test**	
			Worst p-value	**Lag**
2013- 2023	**1.00%**	**100.00%**	**0.00%**	**10**
2022-2023	0.00%	2.22%	100.00%	1
2021-2022	0.38%	25.44%	94.99%	1
2020-2021	1.92%	18.44%	0.00%	10
2019-2020	0.77%	69.67%	89.96%	1
2018-2019	0.77%	69.67%	0.00%	10
2017-2018	1.15%	80.77%	87.69%	1
2016-2017	0.77%	69.67%	89.96%	1
2015-2016	1.54%	41.87%	0.01%	1
2014-2015	1.54%	41.87%	0.05%	2
2013-2014	1.15%	80.77%	84.93%	1

Table 7. SGSt VaR backtesting results for the global and subperiods. A model passes the UC and BCP tests when the corresponding p-value is above 5%. Only the first passes the test with a 1% exceedance rate, while BCP fails due to a p-value below 5%.

Over the global period, regarding the UC test the SGSt VaR passes flawlessly with 1.00% exceedance rate, which suggests this model is well calibrated over the global period. However, despite the perfect exceedance rate, a BCP test p-value of 0.00% suggest that this model also fails. We can conclude that the exceedance rate shows that the model works well on average, but the fact that does not pass the BCP test, shows it might fail when markets become volatile.

The next model is the Historical VaR and the respective table is presented below:

Historical VaR				
Backtest Period	**Exceedance rate**	**UC Test (p-value)**	**BCP test**	
			Worst p-value	**Lag**
2013- 2023	**1.12%**	**56.15%**	**0.00%**	**10**
2022-2023	1.15%	80.77%	84.93%	1
2021-2022	0.00%	2.22%	100.00%	1
2020-2021	1.92%	18.44%	0.00%	10
2019-2020	0.77%	69.67%	89.96%	1
2018-2019	1.15%	80.77%	0.12%	10
2017-2018	1.54%	41.87%	82.60%	1
2016-2017	0.77%	69.67%	89.96%	1
2015-2016	1.54%	41.87%	0.01%	1
2014-2015	1.54%	41.87%	0.05%	2
2013-2014	0.77%	69.67%	89.96%	1

Table 8. Historical VaR backtesting results for the global and subperiods. The BCP test does not pass the model once again.

While the exceedance rate of 1.12% is close to ideal, it suggests that Historical VaR slightly underestimates risk. The unconditional coverage of 56.15% proves that the exceedances do not align with expectations. Similar to SGSt VaR, this model also fails the BCP test. Consequently, this model demonstrates limited consistency and reliability in estimating extreme losses over the long run.

Finally, the last model is the Quantile Regression which is shown below:

Quantile Regression VaR				
Backtest Period	**Exceedance rate**	**UC Test (p-value)**	**BCP test**	
			Worst p-value	**Lag**
2013- 2023	**1.00%**	**100.00%**	**14.25%**	**1**
2022-2023	0.77%	69.67%	89.96%	1
2021-2022	0.38%	25.44%	94.99%	1
2020-2021	1.15%	80.77%	0.00%	10
2019-2020	1.15%	80.77%	84.93%	1
2018-2019	0.77%	69.67%	89.96%	1
2017-2018	1.15%	80.77%	87.69%	1
2016-2017	0.77%	69.67%	89.96%	1
2015-2016	1.92%	18.44%	0.28%	1
2014-2015	0.77%	69.67%	89.96%	1
2013-2014	1.15%	80.77%	84.93%	1

Table 9. QR VaR backtesting results for the global and subperiods. Only model passing both tests. The exceedance rate for the global period successfully met the targeted rate.

The QR VaR with a 1% exceedance rate accurately predicts the expected frequency of exceedances. Regarding both tests, looking at the UC test we can conclude that this model indeed perfectly aligns with expected exceedances over the long run. The BCP test, since the p-value is above 5%, - 14.25% - indicates the model passes this test too. This suggests QR VaR produces exceedances that are more randomly distributed over time, making it more reliable for risk management.

These were the 4 best performing models of each class of VaR to analyze our portfolio. These were chosen because they outperformed the other variations of each VaR class with better exceedance rates and overall backtesting performance. It was selected one optimal version of each category (Normal, SGSt, Historical, and QR VaR) to ensure a fair and

meaningful comparison without redundancy while covering different risk modeling approaches.

Thus, a further analysis will be made between these 4 models, as studying them together instead of only one allows for a comprehensive evaluation of different risk management techniques and deeper understanding of which models perform the best, depending on investor's risk aversion.

Chapter 6

VaR MODEL ANALYSIS

In this section, we analyze the performance of the four selected VaR models by assessing their accuracy and reliability in predicting portfolio risk. We begin by examining their exceedances and backtesting results, followed by an evaluation of how well each model aligns with the portfolio's P&L movements relative to VaR estimates.

6.1 Review of Backtesting Data

This section evaluates the model's performance - Normal VaR, SGSt VaR, Historical VaR, and QR VaR - by reviewing their backtesting results across two periods: from 11 February 2013 to 27 January 2023 and from 30 January 2023 to 2 February 2024. The backtesting results for both periods are presented in the following tables:

Model	Exceedance Rate	UC test (p-value)	BCP Test (worst p-value)	Lag
Normal VaR	1.56%	0.6%	0.01%	10
SGSt VaR	1.00%	100%	0%	10
Historical VaR	1.12%	56.15%	0%	10
QR VaR	1%	100%	14.25%	1

Table 10. Backtesting results for the 4 models for the 2013-2023 period.

Model	Exceedance Rate	UC test (p-value)	BCP Test (worst p-value)	Lag
Normal VaR	0.38%	24.33%	95.04%	1
SGSt VaR	0.38%	24.33%	95.04%	1
Historical VaR	0.38%	24.33%	95.04%	1
QR VaR	0.38%	24.33%	95.04%	1

Table 11. Backtesting results for the 4 models for the 30/01/2023 - 02/02/2024 period.

6.1.1 Backtesting (2013–2023)

The 1% quantile, defined as the value below which 1% of portfolio losses lie, serves as the benchmark for VaR accuracy. SGSt VaR and QR VaR achieved an exceedance rate of 1.00%, indicating precise alignment with this quantile, where losses exceeded the VaR estimate 1% of the time over the 2013-2023 period.

Normal VaR, with an exceedance rate of 1.58%, underestimates risk, as actual breaches occurred more frequently than the 1% target, while Historical VaR (1.12%) showed moderate accuracy with a slight deviation. The UC test was 100% for SGSt and QR VaR, confirming their reliability. Normal VaR's 0.6% coverage suggests significant underestimation, and Historical VaR's 56.15% indicates partial alignment. The BCP test had p-values below 5% for Normal (0.01%), SGSt (0%), and Historical (0%) VaR, rejecting their reliability while QR VaR's p-value of 14.25% indicates the model passes the test.

6.1.2 Backtesting (2023–2024)

As we can see in Table 11, for this one-year period backtest, all models exhibited an identical exceedance rate of 0.38%, which corresponds to one exceedance, suggesting an overestimation of risk with fewer breaches than expected. Unconditional coverage was 24.33% for all models, significantly below the ideal 100%, indicating a misalignment with the 1% quantile and a potential underestimation of extreme losses. The BCP test p-value of 95.04% across all models, while suggesting no rejection of conditional coverage, is unusually high and indicates this was a one exceedance event.

6.1.3 Comparative Insights

In the 2013-2023 backtesting, QR VaR and SGSt VaR were the best models, with exceedance rates and coverage closely matching the 1% quantile, while Normal VaR underperformed with an exceedance rate of 1.58% and poor coverage (0.6%), and Historical VaR showed moderate performance (1.12%, 56.15%). In the 2023-2024 backtesting period, the uniform 0.38% exceedance rate and 24.33% coverage contrast with the 2013-2023 backtesting results. This shift indicates that the 2023–2024 backtesting may not fully reflect what will be observed in the VaR calculation in the same period, where portfolio adjustments occurred throughout the year. However, even if the portfolio had not been adjusted, one year period is too small to reliably distinguish model performance, as we can see by the 2016–2017

backtesting, where all four models recorded two exceedances, similar to the single exceedance observed across the four models in 2023-2024. This indicates that a one year timeframe is insufficient for accurate backtesting.

6.2 VaR Statistics

This section analyzes the Value at Risk (VaR) statistics - mean VaR, minimum VaR, maximum VaR, median VaR, and Risk-Adjusted Return on Capital (RORAC) - for the four models over the period of 30 January 2023 to 2 February 2024. The VaR statistics for this period are summarized as follows:

Model	Mean VaR (€)	Minimum VaR (€)	Maximum VaR (€)	Median VaR (€)	RORAC (%)
Normal VaR	101 664.12	80 301.02	141 399.60	99 474.69	11.14%
SGSt VaR	113 341.32	88 575.99	164 778.97	109 939.87	9.99%
Historical VaR	110 313.29	85 287.98	151 326.10	106 594.26	10.27%
QR VaR	95 748.36	74 942.66	140 454.57	91 685.23	11.83%

Table 12. VaR statistics results for the 4 models over the period of 30 January 2023 to 2 February 2024.

As shown in Table 12, SGSt VaR exhibited the highest Mean VaR (113 341.32€) and Maximum VaR (164 778.97€), indicating a more conservative risk estimate, which was expected due to the heavier-tailed Student-t distribution capturing greater tail risk. QR VaR, conversely, had the lowest Mean VaR (95 748.36 €) and Minimum VaR (74 942.66€), suggesting a less conservative approach. However, what was expected at this point was the QR VaR to have around the same Mean VAR as SGSt VaR, since both exceedance rates were 1%, indicating that both models were conservative. Not only had a much lower Mean VaR comparing to SGSt model, but also comparing to the other two models. Exceedances rates typically reflect Mean VaR levels, thus,

we would expect the QR and SGSt models to have approximately the same Mean VaR, followed by Historical VaR and Normal VaR, in that order. Since that was not what happened, we can conclude that, with this information, QR VaR, a model presumed to be conservative, is now, riskier than anticipated. Normal VaR and Historical VaR fell between these extremes, with Mean VaR values of 101 664.12€ and 110 313.29€, respectively. The Median VaR followed a similar trend (SGSt: 109 939.87€, QR: 91 685.23€), reinforcing SGSt VaR's conservatism and QR VaR's lower risk estimates.

RORAC, measuring return per unit of risk, provides insight into the risk-return trade-off. QR VaR achieved the highest RORAC (11.83%), indicating superior risk-adjusted returns due to its lower VaR estimates. Normal VaR followed closely with 11.14%, despite its higher Mean VaR, suggesting acceptable returns but still with relative high-risk exposure. SGSt VaR's RORAC was the lowest (9.99%), reflecting its conservative VaR estimates. Historical VaR's RORAC (10.27%) was moderate, aligning with its balanced VaR estimates. Based on the Mean VaR values, these RORAC results are somewhat expected, however, the differences between the models are not highly significant, indicating a comparable risk-adjusted performance across the models.

6.3 VaR Estimates



Figure 1. Daily VaR estimates evolution across models.

As we can see in Figure 1, all models exhibit a significant decline in VaR estimates during the first half of 2023, with values peaking in early February 2023 and reaching their lowest levels around October and early November.

As expected, the SGSt VaR starts at the highest level among all models in early 2023, reflecting its conservative approach to risk estimation. It exhibits a relatively smooth downward trend throughout 2023, with less pronounced fluctuations compared to other models.
Despite its initial high values, SGSt VaR converges with other models by mid-2023 and remains relatively stable during periods of volatility.

Historical VaR starts slightly below SGSt VaR and remains higher than Normal and QR VaR for most of the observed period, just like SGSt VaR. The Normal VaR begins at a lower level compared to SGSt and Historical VaR but follows a similar declining trend during early 2023. It demonstrates relatively smooth transitions and less sensitivity to short-term market shocks, making it one of the more stable models. The QR VaR starts at the lowest level among all models in early 2023 but exhibits sharper fluctuations throughout the observed period. It appears to be highly reactive to market changes, with significant spikes during late 2023 and early 2024, consistent with it being the only model to pass the BCP test.

6.4 Exceedances

When analyzing exceedances, we must know the daily P&L and the VaR estimate. We have an exceedance when our P&L exceeds the VaR estimate. Thus, if there were any doubts of which model had higher chance of having more exceedances after VaR statistics were computed in section 6.2, we can now conclude that, looking at Figure 1, the QR VaR had a lower VaR estimate for almost the entire period which makes it the most probable model to have (more) exceedances. After the QR VaR model, the Normal VaR is the most probable one to have exceedance/s since has lower VaR estimates compared to the SGSt and Historical models. These latter models had similar evolutions, with tiny deviations from each other throughout this period and presented no exceedances. Only the QR and Normal models had three and one exceedance, respectively (Figures 2 and 3). Below there are two tables, showing the exceedances dates and its amounts for this period (01/2023-02/2024):

Exceedance Date	Portfolio P&L	Total VaR (€)	Exceedance Amount (€)	Exceedance (% of Var)
		QR VaR		
06/07/2023	−87 526.31	80 883.82	−6 642.49	8.21
24/08/2023	−98 396.36	97 073.79	−1 322.57	1.36
21/09/2023	−77 015.67	76 102.35	−913.32	1.20
		Normal VaR		
24/08/2023	−98 396.36	93 224.46	−5 171.90	5.55

Table 13. Three exceedances for QR VaR and one exceedance for Normal VaR for the 01/2023-02/2024 period. The biggest one is on 06/07/2023, exceeding 8.21% of VaR and the smallest one was on 21/09/2023, exceeding only 1.20% of VaR totaling -913€.

QR VaR had three exceedances: on 06/07/2023, 24/08/2023, and 21/09/2023, with exceedance amounts ranging from 913.32€ to 6 642.49€. Normal VaR had one exceedance on 24/08/2023 (5 171.90€), coinciding with QR VaR's exceedance on the same day. SGSt VaR and Historical VaR reported no exceedances, consistent with their higher VaR estimates (e.g., SGSt VaR on 24/08/2023: 101 603.30€, Historical VaR: 100 044.63€ absorbed the loss of 98 396.36€).

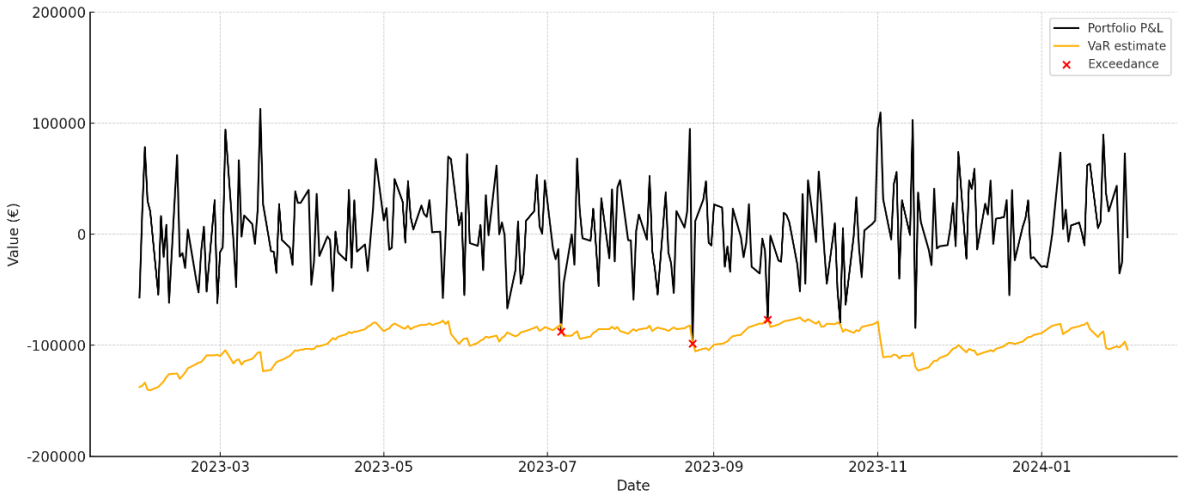


Figure 2. QR VaR model performance.

The red dots represent the P&L of the days in which the VaR was exceeded.

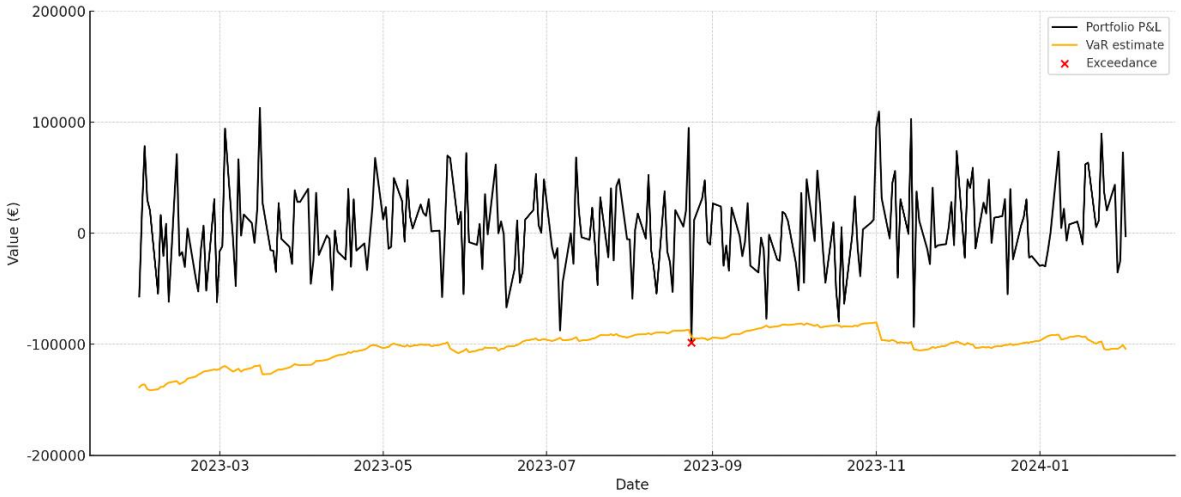


Figure 3. Normal VaR model performance.

There is one exceedance for this model on 24/08/2023.

As we can see in Figure 2 and Figure 3, these three days were some of the sharpest declines in Portfolio P&L which made the VaR estimate not be enough to cover the P&L losses. Although we had 3 exceedances in QR VaR and 1 in Normal VaR, the exceedance amounts were not that big, especially on 24/08/2023 and 21/09/2023 for the QR VaR (1.36% and 1.2% of the VaR estimate, respectively).

However, if we look at Table 14 and Figures 4 and 5, SGSt and Historical VaR models escaped to exceed the VaR by only 1.65% and 3.16%, respectively, on 24/08/2023 when the P&L was – 98 396.36 €.

VaR Model	VaR	Exceedance Amount (€)	Exceedance (% of Var)
Normal VaR	93 224.46	−5 171.90	5.55
SGSt VaR	101 603.30	3 206.94	**−3.16**
Historical VaR	100 044.63	1 648.27	**−1.65**
QR VaR	97 073.79	−1 322.57	1.36

Table 14: VaR models performance on 24/08/2023.

Both Normal VaR and QR VaR had exceedances, while SGSt VaR and Historical VaR avoided exceedances by 3 206.94€ and 1 648.27€, respectively.

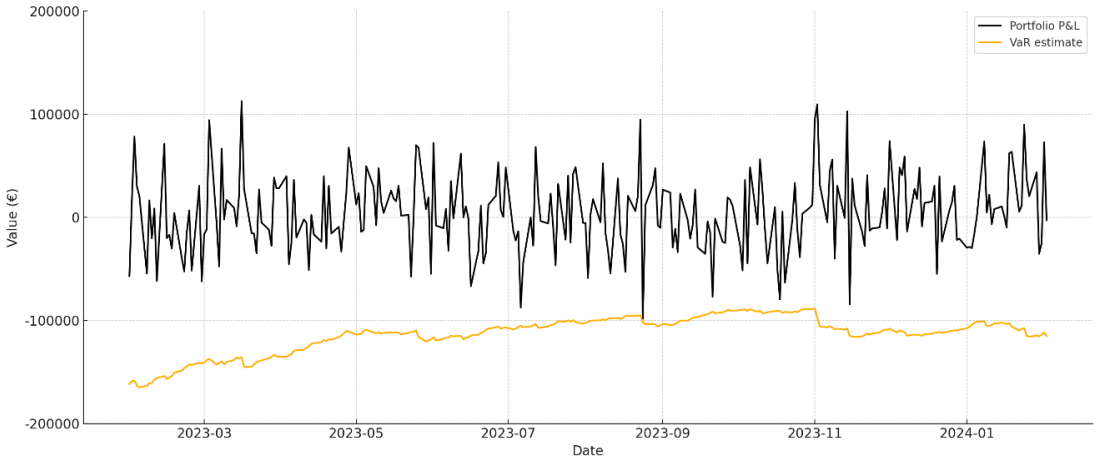


Figure 4: SGSt VaR model performance. On 24/08/2023 this model did not exceed the VaR, leaving a buffer of 3.16% of its estimate.

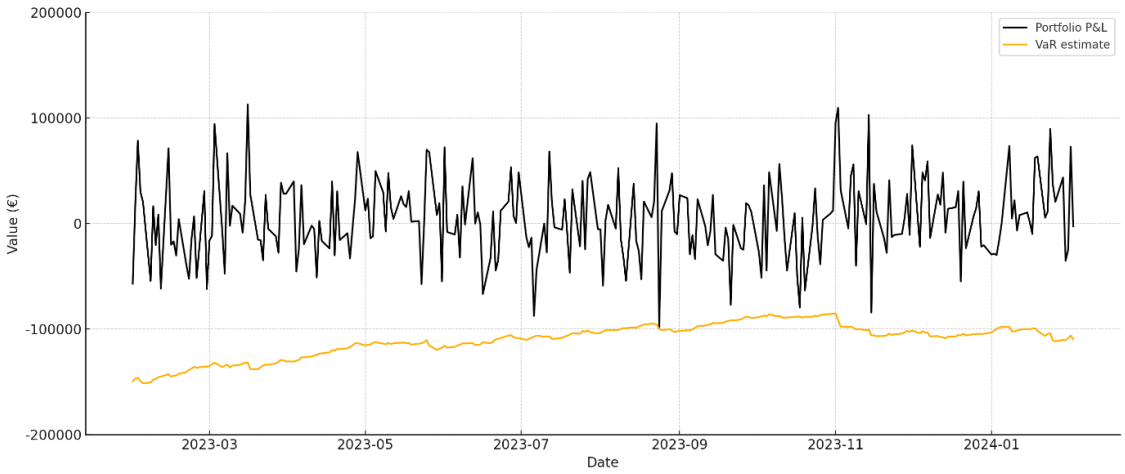


Figure 5: Historical VaR model performance. On 24/08/2023 this model did not exceed the VaR with a remaining buffer of 1.65% of its 100 044.63€ estimate.

However, as shown in chapter 5, the backtesting for the 2013-2023 period revealed exceedance rates of 1% and 1.12% for SGSt and Historical VaR, respectively. This means that for a 99% confidence level were expected 3 exceedances, since 1% of 265 trading days are ≈ 3. For the Normal VaR we had 1.58% of exceedance rate, hence it would be expected around 4 exceedances per year, which was also not the case. Lastly, the QR VaR had an exceedance rate of 1%, which means that the 3 exceedances we had, were expected. This suggests that all models except the QR, are overestimating risk in this period – being too conservative. After

all, the QR VaR is the only model that corresponds to the backtesting results and is not being conservative. Despite initial perceptions based on the Mean VaR, discussed in section 6.2, where QR VaR appeared misaligned, it is now, in fact, the other three models that were misaligned.

Going back to the 2023-2024 backtesting results, we saw that all models had an exceedance rate of 0.38%, which means that 1 exceedance was expected for each of models if we take this backtesting in consideration. This was clearly, not what happened. The backtesting at the end of 2024 serves as a snapshot of the portfolio as it stood then, reflecting an evolved risk exposure due to changing market conditions throughout 2023, even if its composition had remained static. It did not remain static because there were reinvestments of coupon bonds payments into stocks, which helped to slightly increased our equity weight from 49.50% at the start of 2023 to 57.6%, one year later (02/02/2024) - a shift of 8.10%%. However, without these reinvestments, the equity weight at the end of the period would be 57%. This indicates that these adjustments had a limited and almost unsignificant overall impact in our portfolio (+0.6%). We can see the details of these coupon payments below:

Date	Value Invested (€)	Stock	Quantity added	Share Price (€)	Price Change by 2/2/2024
02/05/2023	14 673.16	ABT	122.37	98.90	3.7%
05/07/2023	16 044.77	ORCL	127.37	105.29	0.8%
31/10/2023	15 322.60	ORCL	143.74	95.56	11.1%
02/01/2023	15 754.61	CMCSA	329.10	39.16	5.1%
17/01/2023	30 000.00	ALO.PA	2 622.38	11.44	4.55%
17/01/2023	30 000.00	MC.PA	45.45	660.03	16.7%

Table 15. Reinvestment of coupon payments into equities during 01/2023-02/2024 period.

The equity value increased from 3 330 477.51€ (27/01/2023) to 4 527 364.84€ (02/02/2024), a 36% rise. At the end of the period, the total value of the portfolio was 7 858 331.23€.

6.5 Final Conclusions: Risk-Return Dynamics of VaR Models

QR VaR achieves the highest RORAC (11.83%) due to its lower Mean VaR (95 748.36€), which made it more prone to exceedances when equity volatility increased. Notably, this was the only model that was aligned with the 10 year backtesting, which neither overestimate it nor underestimate the VaR. Additionally, this was also the highest RORAC among the models. Despite potential weaknesses in capturing tail risk, QR VaR proved to be suitable for both stable portfolios and dynamic contexts. This was, by far, the best performing model.

Normal VaR was a quite surprising model, as it was expected to underestimate risk but instead, ended up overestimating it with only one exceedance. With a Mean VaR of 101664.12€ it achieved a RORAC of 11.14%, reflecting a balanced risk-return profile. Figure 1 highlights its relative stability, which can make Normal VaR a versatile choice for dynamic portfolios. This was the second best model, as its VaR estimates were less conservative than those of the SGSt and Historical VaR models, resulting in a better RORAC.

SGSt VaR with a high Mean VaR (113 341.32€) estimate led to a low RORAC of 9.99% due to excessive capital allocation. With the same exceedance rate as QR VaR, the Mean VaR was excessively high, which led to a significant overestimation of VaR and to generate the worst RORAC across all models. In a dynamic portfolio context, SGSt VaR's conservatism ensured robust risk coverage, making it good for regulatory compliance or risk-averse strategies. However, its inefficiency in capital utilization limits its suitability for growth-oriented portfolios where maximizing returns is a priority.

Historical VaR (Mean: 110 313.29€) also avoided exceedances due to their conservative VaR estimates. Like SGSt VaR, this high mean VaR led to a low RORAC of 10.27%, slightly better than SGSt VaR. This stability makes it a reliable choice for consistent risk management in a dynamic portfolio. However, just like SGSt VaR, its higher estimates reduce efficiency, constraining risk-adjusted returns and making it less optimal for portfolios prioritizing growth. Between these last 2 models, while both are over-conservatives, for this portfolio and for this period, the Historical VaR had a tiny advantage only because benefits a slightly higher RORAC comparing to SGSt VaR.

Chapter 7

CONCLUSION

The goal of this project was to measure the Value at Risk of a portfolio and select and analyze the 4 best performers VaR models from a wide range of different models, using a backtesting process. The backtesting was driven by UC and BCP tests which its results were essential to indicate our 4 best variants of each class of VaR models: Normal, SGSt, Historical and QR.

From 2013 to 2023, backtesting results revealed that all models, except Normal VaR, exhibited low exceedance rates. However, their failure to pass the UC and BCP tests indicated they were unsuitable for our portfolio. The QR VaR was the only model thar fit the portfolio, with a low exceedance rate and a p-value above 5%, passing both the UC and BCP tests. Despite QR VaR's suitability, we also analyze the other three models to assess whether their performance was, in fact, worse than QR VaR.

We proceed to calculate the VaR and different metrics - mean, median, max, min, RORAC - for the 2023-2024 period. After that, we considered that the QR VaR was, indeed, the most viable choice for the portfolio given the risk-reward. Contrary to expectations, Normal VaR ended up to overestimate risk, with only one exceedance, while QR VaR, with three exceedances was aligned with expectations. Given the lower mean VaR, QR model demonstrated a higher RORAC, offering a balanced risk-reward profile. The Historical VaR and the SGSt VaR are not viable choices due to their conservatism. These had low RORAC numbers and great risk coverage.

Investors favoring Normal VaR over QR VaR tend to be more cautious, accepting a slightly lower RORAC for reduced exceedance risk. Conversely, those preferring QR VaR prioritize higher RORAC, accepting a greater risk of exceedances. The other two models, with zero exceedances, overestimated risk and are better suited for highly risk-averse investors.

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