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Stochastic Lotka-Volterra Model for Financial Dynamics

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Supervisor:
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Abstract

This work seeks to model financial time series dynamics using a stochastic Lotka-Volterra system as a stochastic diffusion process with an embedded dependence structure and the pricing of derivatives using this underlying dynamic with numerical methods. As an application we consider currency exchange rates.

Keywords: Mathematical Finance, Stochastic Processes, Dynamical Systems, Stochastic Lotka-Volterra System, Currency Exchange Rates

Resumo

Este trabalho procura modelar a dinâmica de séries temporais financeiras usando um sistema de Lotka-Volterra como processo de difusão estocástico com uma estrutura de dependência embutida e a valoração de derivados usando esta dinâmica subjacente com métodos numéricos. Como aplicação vamos considerar taxas de cambio monetário.

Keywords: Matemática Financeira, Processos Estocásticos, Sistemas Dinâmicos, Sistema de Lotka-Volterra Estocástico, Taxas Cambiais

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1 Introduction

Financial time series are often modeled with one dimensional stochastic differential equations, such as $dX(t) = \mu X(t)dt + \sigma X(t)dW(t)$, with $\mu, \sigma \in \mathbb{R}$ and $\{W(t)\}_{t\geq 0}$ a Brownian Motion, providing a means to describe the time evolution, or dynamics, of financial assets. By treating the drift μ and diffusion σ as stochastic differential equations in their own right, the former providing a dynamic for the tendency, $\{\mu(t)\}_{t\geq 0}$, and the latter providing a richer dynamic for volatility, $\{\sigma(t)\}_{t\geq 0}$, we get several several possible dynamics, but the simulated dynamics often do not resemble the dynamic of the financial asset being modeled, only the statistical properties taken over several simulations do so, which is enough to develop reliable models for risk management and derivatives pricing, but more can be done.

Financial assets are very sensible to each others movements, which hints at a way to further improve the descriptive quality of the previous models, by incorporating a dependence structure. The use of correlated Brownian Motions allows for pairwise dependence modeling, which is insufficient to address the whole set of financial assets, and uses statistical correlation, $\{\rho(t)\}_{t\geq 0}$, to determine the strength of this dependence, a linear exogenous variable which is not affected by the evolution of either the system's dynamics or structure, resulting in an incomplete and stiff model.

A more interesting approach is to consider the dependence structure as endogenous to the system by incorporating it in the drift terms of the equations modeling the dynamics of the financial assets, a structure which itself is subject to change, and assign volatility to the diffusion terms. This way we obtain a system of stochastic differential equations which incorporates the dependence structure and makes it the main driver of its dynamics, being itself subject to alteration in the case of significant shocks, which can be modeled with jump processes.

We will use a stochastic Lotka-Volterra system to capture this structure and dynamics. In Section 1 we introduce the basics of stochastic processes, stochastic calculus, and a conceptual overview of Ergodic Theory, going through basic definitions, some important theorems, used to prove some relevant propositions, and Markov Semigroups, used to study systems of stochastic differential equations. In Section 2 we draw a conceptual roadmap of the theory of Stochastic Persistence, and apply its propositions to an n dimensional stochastic Lotka-Volterra system, proving the existence and uniqueness of a solution and its persistence, showing its suitability to model currency exchange rates.

In Section 3 we use a correlation induced minimal spanning tree to find the dependence structure of the system, and embed *no triangular arbitrage* with cyclic dependence structures, making it a major driver for the assets dynamics. Then we fit the model with Ordinary Least Squares and find a strong correspondence between the historical and simulated data, evaluated per simulation and statistically, showing the descriptive capability of the model, in particular, of treating arbitrage as a systematic property. Finally, as a simple application, we use a Monte Carlo method with this underlying asset dynamic to price european options on currency exchange rates.

2 Theoretical Background

This chapter serves to introduce concepts of stochastic calculus that will be used throughout the work. It also contains a brief introduction to ergodic theory with some selected topics, which are necessary to understand recent developments in the qualitative study of systems of stochastic differential equations. Unless stated otherwise, we will be working in the typical filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$.

2.1 Stochastic Processes

A stochastic process is a sequence of random variables $\{X(t,\cdot)\}_{t\geq 0}$, with $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}, \eta)$, each realization of the process is a function $t \to X(t,\omega)$, describing how the variable evolves for a specific outcome $\omega \in \Omega$. A simple process that will be present throughout this work is the Markov process.

Definition 1 (***Markov Process***). *A Markov process is a stochastic process $\{X(t)\}_{t\geq 0}$ that satisfies the* ***Markov property***

$$\mathbb{P}(X(t) \in A|\mathcal{F}(s)) = \mathbb{P}(X(t) \in A|X(s)) \tag{2.1}$$

$\forall\, s < t$ and $\forall\, A \in \mathcal{B}$.

The **transition probability function** of this process is $\mathcal{P}(x, A)$ such that $\mathcal{P}(x,\cdot)$ is a probability measure on $(\mathbb{R}, \mathcal{B})$ and $\mathcal{P}(\cdot, A)$ is measurable $\forall x \in \mathbb{R}$, $\forall\, A \in \mathcal{B}$, with transition density p

$$\mathcal{P}(x, A) = \mathbb{P}(X(t) \in A|X(0) = x) = \int_A p(x, y)\eta(dy). \tag{2.2}$$

The building block of most diffusion processes is the Brownian Motion.

Definition 2 (***Brownian Motion***). *A standard Brownian motion (or Wiener process) is a stochastic process $\{W(t)\}_{t\geq 0}$ with the following properties:*

1. *$W(0) = 0$.*
2. *(independent increments) for $r < s < t < u$, the random variables $[W(s) - W(t)]$ and $[W(u) - W(t)]$ are independent.*
3. *(stationary increments) for $s \leq t$, the random variables $[W(t)-W(s)]$ and $[W(t-s)-W(0)]$ possess the same probability distribution.*
4. *For $s < t$, $[W(t) - W(s)] \sim N(0, t - s)$.*
5. *$\{W(t)\}_{t\geq 0}$ has continuous sample paths (i.e. with no jumps).*

Which is a Markov process, $\mathbb{P}(W(t) \in A|\mathcal{F}(s)) = \mathbb{P}(W(s) + (W(t) - W(s)) \in A|\mathcal{F}(s)) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(y-W(s))^2}{2(t-s)}\right) dy = \mathbb{P}(W(t) \in A|W(s))$.

The distribution of the process follows immediately from Definition 1 by setting $s = 0$, yielding $W(t) = W(t) - 0 = W(t) - W(0) \sim N(0, t)$.

Proposition 1. $W(t) \sim N(0, t)$.

Now we define the Itô process, a diffusion process with volatility driven by a Brownian motion, which serves as the basis for stochastic calculus.

Definition 3 (***Itô Process***)**.** *A stochastic process $\{X(t)\}_{t\geq 0}$ is an Itô process if it can be represented as*

$$dX(t) = \mu(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) \tag{2.3}$$

i.e.

$$X(t) = X(0) + \int_0^t \mu(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s) \tag{2.4}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion generating the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$, $X(0)$ is $\mathcal{F}(0)$-measurable and the stochastic processes $\mu(t, X(t))$ and $\sigma(t, X(t))$ are $\mathcal{F}(t)$-adapted.

The term $\mu(t, X(t))$ is called the **drift** and captures the trend of the process, and the term $\sigma(t, X(t))$ is called the **diffusion** and introduces volatility in the process as Gaussian noise.

To consider the analytical tractability of an Itô process the following result is of importance,

Theorem 1 (***Martingale Representation Theorem***)**.** *An Itô process $\{X(t)\}_{t\geq 0}$ is a martingale if and only if it can be represented by a SDE with no drift, i.e.*

$$dX(t) = \sigma(t, X(t))\, dW(t) \tag{2.5}$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion with filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ and $\sigma(t, X(t)) \in \mathcal{L}^2$.

however, in this work great emphasis is placed on the role of the drift term for modeling financial dynamics, so this line of study will not be pursued. From here we can define other stochastic processes widely used in financial time series modeling, such as

Definition 4 (***Geometric Brownian Motion***)**.** *A process $\{X(t)\}_{t\geq 0}$ is a Geometric Brownian motion if it can be represented as*

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, dW(t) \tag{2.6}$$

where $\mu \in \mathbb{R}$ is the drift term, $\sigma \in \mathbb{R}$ is the diffusion term, and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion.

We can relate two Geometric Brownian motions using correlation, allowing us to model stochastic pairwise dependence.

Definition 5 (***Correlated Geometric Brownian Motions***)**.** *Two Geometric Brownian motions* $\{X_1(t)\}_{t\geq 0}$ *and* $\{X_2(t)\}_{t\geq 0}$ *are said to be correlated if*

$$\mathbb{E}\left[dW_1(t)\cdot dW_2(t)\right] = \rho\, dt. \tag{2.7}$$

A common way to construct two correlated Brownian motions is by defining

$$dW_2(t) = \rho dW_1(t) + \sqrt{1-\rho^2}Z(t) \tag{2.8}$$

where $W_1(t)$ and $Z(t)$ are independent standard Brownian motions, and $\rho \in [-1, 1]$ is the correlation coefficient.

It is known that the constant volatility $\sigma \in \mathbb{R}$ of the GBM and the constant correlation ρ of the correlated GBM fail to accurately model observed financial time series, so stochastic terms $\{\sigma(t)\}_{t\geq 0}$ and $\{\rho(t)\}_{t\geq 0}$ have been introduced, so as to reflect the everchanging dynamics of financial time series. A thorough treatment of this subject, and further developments within the Black and Scholes (1973) framework, can be found in **[M09]**.

Another widely used process to model financial time series, particularly interest rates, is the Ornstein-Uhlenbeck process.

Definition 6 (***Ornstein-Uhlenbeck Process***)**.** *The Ornstein–Uhlenbeck process* $\{X(t)\}_{t\geq 0}$ *is defined by the following stochastic differential equation*

$$dX(t) = -\theta X(t)\, dt + \sigma\, dW_t \tag{2.9}$$

where $\theta \in \mathbb{R}^+$ *and* $\sigma \in \mathbb{R}^+$ *are parameters, and* $\{W(t)\}_{t\geq 0}$ *is a standard Brownian motion. An additional term* $\mu \in \mathbb{R}$*, called the long term mean, is sometimes added*

$$dX(t) = \theta(\mu - X(t))\, dt + \sigma\, dW(t). \tag{2.10}$$

Equation (2.9) captures mean-reversion in the drift term, making it very useful for modeling financial time series that converge to a long term average.

2.2 Stochastic Calculus

Here we introduce the main tools of stochastic calculus. The function space that we'll be working in is $\mathcal{L}^2$.

Definition 7 ($\mathcal{L}^p$ ***Space***). *An $\mathcal{L}^p$ space on X is the space of functions f on X such that*

$$||f||_p = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} < \infty. \tag{2.11}$$

Definition 8 ($\mathcal{L}^2$ ***Space***). *The stochastic process $\{X(t)\}_{t \geq 0}$ belongs to the class $\mathcal{L}^2[a,b]$ if the following conditions are satisfied:*

1. *Square integrability condition:*

$$\int_a^b E[X^2(u)]du < \infty. \tag{2.12}$$

2. *The process $X(t)$ is adapted to the filtration $\mathcal{F}(t) = \sigma(W(u),\, a \leq u \leq b)$.*

The analytical tool that enables us to perform analytical computations with stochastic process is Itô's Integral.

Definition 9 (***Itô's Integral***). *The stochastic integral*

$$I(t) = \int_a^b g(u)\, dW(u) \tag{2.13}$$

is called an Itô integral if W(t) is a standard Brownian motion and $g(t) \in \mathcal{L}^2[a,b]$.

which has the following properties

Proposition 2 (***Properties of Itô's integral***).

1. ***($\mathcal{F}(t)$-measurable)*** *$I(t)$ is $\mathcal{F}(t)$-measurable.*
2. ***(Linearity)*** *For any constants $a, b \in \mathbb{R}$ and processes $f, g \in \mathcal{L}^2$,*

$$a \int_0^t f(u)dW(u) \pm b \int_0^t g(u)dW(u) = \int_0^t [a \cdot f(u) \pm b \cdot g(u)]dW(u). \tag{2.14}$$

3. ***(Expectation)***

$$\mathbb{E}\left[\int_0^t g(u)dW(u) \right] = 0. \tag{2.15}$$

4. ***(Itô Isometry)***

$$\mathbb{E}\left[\left(\int_0^t g(u)dW(u) \right)^2 \right] = \int_0^t E\left[g^2(u) \right] du. \tag{2.16}$$

5. ***(Martingale)*** *$I(t)$ is a martingale, i.e.*

$$\mathbb{E}\left[\int_0^t g(u)dW(u)\middle|\mathcal{F}(s)\right] = \int_0^s g(u)dW(u), \quad \forall s \leq t \tag{2.17}$$

6. ***(Variance)***

$$Var\left[\int_0^t g(u)dW(u)\right] = \int_0^t \mathbb{E}[g^2(u)]du. \tag{2.18}$$

7. ***(Quadratic Variation)***

$$\langle I, I\rangle(t) = \int_0^t g^2(u)du. \tag{2.19}$$

For our stochastic calculus toolset to be complete we need the inverse operator of Itô's integral, a stochastic derivative, this tool follows from Itô's lemma.

Lemma 1 (***Itô's Lemma***)**.** *Let X be an Itô process, i.e.*

$$dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t) \tag{2.20}$$

and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^{1,2}$ function, then $\forall\, t \geq 0$

$$df(t, X(t)) = \frac{\partial f(t, X(t))}{\partial t}dt + \frac{\partial f(t, X(t))}{\partial x}dx + \frac{1}{2}\frac{\partial^2 f(t, X(t))}{\partial x^2}d\langle X, X\rangle(t). \tag{2.21}$$

Alternatively, we can write the derivative as

$$\begin{aligned} df(t, X(t)) = &\left[\frac{\partial f(t, X(t))}{\partial t} + \mu(t, X(t))\frac{\partial f(t, X(t))}{\partial x} + \frac{1}{2}\sigma^2(t, X(t))\frac{\partial^2 f(t, X(t))}{\partial x^2}\right]dt \\ &+ \sigma(t, X(t))\frac{\partial f(t, X(t))}{\partial x}dW(t). \end{aligned} \tag{2.22}$$

Complete proofs for these results can be found in the reference work **[Ø10]**.

2.3 Ergodic Theory

According to **[W00]** *"ergodic theory is the study of the qualitative properties of actions of groups on spaces, e.g. a measure space (...) and each element of the group acts as a transformation on the space and preserves the given structure, e.g. each element acts as a measure preserving transformation (...)"*. We will be working on the dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, T)$, where T is a transformation that acts on Ω.

2.3.1 Definitions

Here we give a brief overview of important definitions and theorems of Ergodic Theory, following the treatment found in **[DD08]**.

Definition 10 (***Measure***)**.** *A measure μ is a map $\mu : \mathcal{F} \to \mathbb{R}$ over a space Ω, such that for $\forall A \in \mathcal{F}$*

1. $\mu(\emptyset) = 0$,
2. $\mu(A) \geq 0$,
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$, *with $\{A_i\}_{i\in\mathbb{N}}$ pairwise disjoint.*

If $\mu(\Omega) = 1$ then μ is called a probability measure, in which case it is denoted by $\mathbb{P}$.

Considering a transformation T that acts on Ω, that is $T : \Omega \to \Omega$, we say that it is measurable if $\forall A \in \mathcal{F} : T^{-1}(A) \in \mathcal{F}$, and we can classify it according to it's interaction with μ, such as

Definition 11 (***Measure Preserving Transformation***)**.** *Let $T : \Omega \to \Omega$ be a measurable transformation. T is said to be measure preserving with respect to μ if*

$$\forall\, A \in \mathcal{F} : \mu(T^{-1}(A)) = \mu(A). \tag{2.23}$$

*In which case we say that the measure μ is **invariant** under T.*

A stronger condition is ergodicity, a transformation with this property delineates the indivisible elements of a mesurable dynamical system, an ergodic system cannot be broken into further ergodic systems, but normal measure preserving ones can be broken into their ergodic components.

Definition 12 (***Ergodic Transformation***)**.** *Let T be a measure preserving transformation with respect to μ. The map T is said to be ergodic if*

$$A \in \mathcal{F} : T^{-1}(A) = A \implies \mu(A) \in \{0, 1\}. \tag{2.24}$$

*In which case we say that the measure μ is **ergodic**.*

Theorem 2. ***(Ergodic Decomposition Theorem)***

To study the behavior of a set $A \in \mathcal{F}$ under a measure preserving transformation T we need to study the behavior of the elements of A under T.

Definition 13. ***(Recurrent Point)*** *Let $T : \Omega \to \Omega$ be a measurable map. A point $x \in A$ is said to be A-recurrent if*

$$\exists\, k \geq 1 : T^k(x) \in A \tag{2.25}$$

One of the main results characterizing recurrence in a probability sapce $(\Omega, \mathcal{F}, \mu)$ is Poincaré's Recurrence Theorem. This result will not be used for the remainder of this work, however it provides a simple view into the object and methods of Ergodic theory, the goal of this section, hence it is stated with proof.

Theorem 3 (***Poincaré's Recurrence Theorem***)**.** *Let T be a measure preserving transformation, and let $A \in \mathcal{F}$. If $\mathbb{P}(A) > 0$ then a.e. $x \in A$ is A-recurrent.*

Proof. Consider the set F of points of A that are not A-recurrent,

$$F = \{x \in A : T^k(x) \notin A \, \forall k \geq 1\}$$

we want to show that $\mathbb{P}(F) = 0$. Notice that the pre-image $T^{-k}(F)$ consists of points $x \in F \subset A$ that are not mapped into A, hence $\forall\, k \geq 1 : F \cap T^{-k}(F) = \emptyset$. This relation can be generalized $\forall\, r \neq s : T^{-r}(F) \cap T^{-s}(F) = \emptyset$. Thus the sets F, $T^{-1}(F)$, $T^{-2}(F)$, ... are pairwise disjoint, and since T is measure preserving $\forall k \geq 1 : \mathbb{P}(T^{-k}(F)) = \mathbb{P}(F)$. IF $\mathbb{P}(F) > 0$, then

$$1 = \mathbb{P}(\Omega) \geq \mathbb{P}\left(\bigcup_{k \geq 0} T^{-k}(F)\right) = \sum_{k \geq 0} \mathbb{P}(F) = \infty$$

a contradiction, hence $\mathbb{P}(F) = 0$. □

2.3.2 The Ergodic Theorem

The theorem introduced in this section will be used to characterize the behavior of systems of stochastic differential equations, allowing us to determine if they are suitable for modeling financial time series. It can be interpreted as a generalization of the Strong Law of Large Numbers.

Theorem 4 (***Strong Law of Large Numbers***)**.** *For a sequence $\{X_i\}_{i=1}$ of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mu)$ with $E[X_i] < \infty$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E[X_1] \quad a.e.. \tag{2.26}$$

If we remove the i.i.d. requirement and consider a measure preserving transformation $T : \Omega \to \Omega$, e.g. a left shift $X_n = X_1 \circ T^{n-1}$, under what conditions does the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x)$ exists a.e., and assuming that it exists, what is it's value?

Theorem 5 (***Birkhoff's Ergodic Theorem***)**.** *Let $(\Omega, \mathcal{F}, \mu, T)$ be a measure preserving dynamical system, for $\forall f \in \mathcal{L}^1$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i(x) = \bar{f}(x) \tag{2.27}$$

converges a.e. to a T-invariant function $\bar{f} \in \mathcal{L}^1_\mu$, where

$$\int f d\mu = \int \bar{f} d\mu \tag{2.28}$$

and if T is ergodic,

$$\int f d\mu = \bar{f}. \tag{2.29}$$

For an ergodic system determined by T, by observing the orbit of a single point, $\{T^k(x)\}_{k\geq 1}$, the time average of a function f along that orbit, $\bar{f}(x)$, equals the space average $\int f(x) d\mu$ for a.e. $x \in \Omega$, that is, by observing a single point over time, you see the same average behavior as if you looked at the entire space all at once.

2.3.3 Markov Semigroups

Now we introduce a class of transformations that act on function spaces, such as $\mathcal{L}^1$, called operators, with some special properties.

Definition 14 (***Markov Operator***)**.** *Denote by $\mathcal{D}$ the subset of the space $\mathcal{L}^1 = \mathcal{L}^1(\Omega, \mathcal{F}, \mu)$ which contains all densities*

$$\mathcal{D} = \{f \in \mathcal{L}^1 : f \geq 0, ||f|| = 1\}. \tag{2.30}$$

A linear mapping

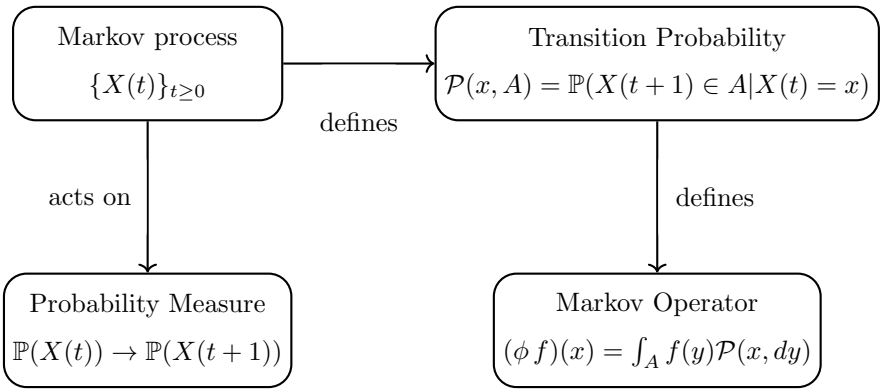
$$\phi : \mathcal{L}^1 \to \mathcal{L}^1 \tag{2.31}$$

is called a ***Markov operator*** *if $\phi(\mathcal{D}) = \mathcal{D}$.*

We can express the Markov operator in terms of the transition probability function of a Markov process, which gives the expected value of f after one transition from state x,

$$(\phi f)(x) = \int_A f(y)\mathcal{P}(x, dy) \tag{2.32}$$

showing the relation between the two concepts.



Markov semigoups are used to describe the time evolution of probability distributions of a Markov process.

Definition 15 (***Markov Semigroup***)**.** *Let $\Phi = \{\phi_i\}_{i\geq 0}$ be a set o Markov operators defined on the set of bounded measurable functions on $(\Omega, \mathcal{F})$. Φ is a Markov semigroup if*

1. *$\phi_0 = \mathbb{I}$.*
2. *$\phi_{t+s} = \phi_t \circ \phi_s, \quad \forall t, s \geq 0$.*
3. *$\forall f \in \mathcal{L}^1$ the function $t \mapsto \phi_t f$ is continuous.*

If $\forall f \in \mathcal{L}^1 : ||\phi_t f|| \leq ||f||$ then it is a contraction semigroup.

We denote the set of invariant probability measures under the action of this semigroup as

$$\mathcal{P}_{inv}(A) = \left\{\mathbb{P} : \mathbb{P}(\phi_t^{-1}(A)) = \mathbb{P}(A), \quad \forall \phi_t \in \Phi\right\}, \tag{2.33}$$

and if the semigroup is composed of ergodic operators as

$$\mathcal{P}_{erg}(A) = \{\mathbb{P} : \mathbb{P} \in \mathcal{P}_{inv}(A) \wedge \phi_t^{-1}(A) = A \Rightarrow \mathbb{P}(A) \in \{0, 1\}\} \tag{2.34}$$

Definition 16 (***Asymptotic Stability***)**.** *The Markov semigroup Φ is asymptotically stable if there exists an invariant density f^* such that*

$$\lim_{t\to\infty} ||\phi_t f - f^*|| = 0, \quad \forall\, f \in \mathcal{D}. \tag{2.35}$$

Now we present an instrument that we can use to study the behavior of a stochastic process $\{X(t)\}_{t\geq 0}$, interpreted as a process under the action of Φ.

Definition 17 (***Infinitesimal Generator of*** Φ)**.** *If the following limit exists*

$$\mathcal{L}f = \lim_{t\to 0^+} \frac{\phi_t f - f}{t} \tag{2.36}$$

then $\mathcal{L}$ is called the infinitesimal generator of the Markov semigroup $\Phi = \{\phi_t\}_{t\geq 0}$ with domain

$$\mathcal{D}(\mathcal{L}) = \{f \in \mathcal{D} : \exists\, \mathcal{L}f\}\ . \tag{2.37}$$

which has the following characterization

Proposition 3. *For all $f \in \mathcal{D}(\mathcal{L})$ and $t \geq 0$, $\phi_t f \in \mathcal{D}(\mathcal{L})$, and for all $x \in M : t \to \phi_t f(x)$ is C^1 and satisfies*

$$\frac{d}{dt}\phi_t f(x) = \mathcal{L}(\phi_t f)(x) = \phi_t(\mathcal{L}f)(x). \tag{2.38}$$

With these instruments we can characterize the behavior of a stochastic process $\{X_t\}_{t\geq 0}$,

Definition 18 (***Strong Law***)**.** *For a stochastic process $\{X(t)\}_{t\geq 0}$, governed by Φ, the function f satisfies a strong law if*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X(s))\, ds = \mu(f) \quad a.s. \tag{2.39}$$

where π is an invariant measure for the process.

that is, the Markov semigroup Φ captures the probability distribution of $\{X(t)\}_{t\geq 0}$.

3 Stochastic Differential Equations

Here we study the system of stochastic differential equations that we will use to model the dynamics of currency exchange rates.

3.1 Definitions and Fundamental Theorems

3.1.1 Definitions

Stochastic differential equations extend ordinary differential equations by incorporating random perturbations, typically modeled via Brownian motion. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and $W(t) = (W_1(t), ..., W_n(t)) \in \mathbb{R}^n$ be an n-dimensional Brownian Motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Definition 19 (***System of Stochastic Differential Equations***). *Let $X(t) = (X_1(t), ..., X_n(t)) \in \mathbb{R}^n$ be a stochastic process, a system of stochastic differential equations is given by*

$$dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) \tag{3.1}$$

where $b(t, X(t))$ is the drift coefficient and $\sigma(t, X(t))$ is the diffusion coefficient. A process $X(t)$ is a solution of the system if it satisfies the following integral form

$$X(t) = X(0) + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s) \tag{3.2}$$

$\forall t \in [0, T]$ *a.s..*

3.1.2 Existence and Uniqueness

Concerning the existence and uniqueness of the solution of the system we have the following theorem,

Theorem 6 (***Itô's Existence and Uniqueness Theorem***). *Suppose that $b(t, \cdot)$ and $\sigma(t, \cdot)$ are measurable functions satisfying the Lipschitz condition*

$$||b(t, x) - b(t, y)|| + ||\sigma(t, x) - \sigma(t, y)|| \leq L||x - y|| \tag{3.3}$$

and the linear growth condition

$$||b(t, x)||^2 + ||\sigma(t, x)||^2 \leq C(1 + ||x||^2) \tag{3.4}$$

for $L, C \in \mathbb{R}_0$ uniformly in $[0, T]$, then there exists a unique strong solution to the SDE starting from any initial value $X(0) \in \mathbb{R}^n$.

Now we have the necessary instruments to determine if system (3.1) has a unique solution. In the case where it is possible to determine it's analytical form a generalized version of Itô's formula is used,

Theorem 7 (***Itô's Formula***). *Let $f : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function of class $C^{1,2}$, then the process $f(t, X(t))$ satisfies*

$$df(t,X(t)) = \left[\frac{\partial}{\partial t}f + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\mu_i f + \sum_{i,j=1}^{n}\frac{\partial^2}{\partial x_i \partial x_j}a_{ij}f\right]dt + \left[\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\sigma_t f\right]dW(t) \tag{3.5}$$

where $a_{ij}(x) = \frac{1}{2}\sigma(t,X_t)\sigma(t,X_t)^T$.

Given the analytical solution we can obtain the distribution of the process $\{X(t)\}_{t\geq 0}$ using the Fokker-Planck equation.

Theorem 8 (***Fokker-Planck Equation***). *Consider the Itô process*

$$dX(t) = \mu(t,X(t))\,dt + \sigma(t,X(t))\,dW(t) \tag{3.6}$$

assume that $(X(t))_{t\geq 0}$ is a solution such that $X(0)$ has density $v(x)$, then $(X(t))_{t\geq 0}$ has density $u(t,x)$ which satisfies the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{n}\frac{\partial \mu_i u}{\partial x_i} \tag{3.7}$$

where $a_{ij}(x) = \frac{1}{2}\sigma(t,X(t))\sigma(t,X(t))^T$.

This equation can be studied with Markov semigroups. Following **[RPT02]**, consider the operator

$$A\,f = \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}\frac{\partial f}{\partial x_j}\right) - \sum_{i=1}^{n}\frac{\partial \mu_i f}{\partial x_i} \tag{3.8}$$

on the set $E = \{f \in \mathcal{L}^1(\mathbb{R}^n) \cap C_b^2(\mathbb{R}^n) : A\,f \in \mathcal{L}^1(\mathbb{R}^n)\}$, where $C_b^2(\mathbb{R})$ is the set of twice continuously differentiable functions with all derivatives up to order 2 being bounded. If the density of $X(0)$, $v(x) \in C_b^2(\mathbb{R}^n)$, then (3.7) has in any time interval $[0,T]$ a unique solution u such that $u(x,0) = v(x)$. If $v(x) \in C_c^2(\mathbb{R}^n)$, then $\lim_{||x||\to\infty} u(x,t) = 0$, the same holds for it's spatial derivatives. By the Gauss-Ostrogradski theorem it follows that the integral $\int u(x,t)dx$ is constant. Let $\phi_t\, v(x) = u(x,t)$, the operators $\{\phi_t\}_{t\geq 0}$ form a Markov semigroup. The closure of the operator A generates $\{\phi_t\}_{t\geq 0}$.

$$\phi_t\, f(x) = \int_{\mathbb{R}^n} f(y)\mathcal{P}_t(x,dy) \tag{3.9}$$

where $\mathcal{P}_t(x,dy)$ is the transition probability function for the process $\{X(t)\}_{t\geq 0}$, i.e. $\mathcal{P}_t(x,dy) = \mathbb{P}(X(t) \in A | X(0) = x)$. However, for our system, no closed form analytical solution is known.

3.1.3 Persistence

Without an analytical solution we must use qualitative methods to characterize the behavior of $\{X(t)\}_{t\geq 0}$, in particular, the system's persistence, i.e. the solution does not converge to $0 \in \mathbb{R}^n$. To study this property we use methods developed in **[B19]**. Here $\{X(t)\}_{t\geq 0}$ is a continuous time Markov process on a metric space M with an invariant extinction set $M_0 \subset M$, and $C_b(M)$ is the Banach space of real-valued bounded continuous functions on M. Let $\mathcal{L}$ denote the infinitesimal generator of Φ such that $\mathcal{D}(\mathcal{L}) \subset C_b(M)$ is it's domain.

Definition 20 (***Extinction Set***)**.** *A set $M_0 \subset M$ is called an extinction set if*

$$X_t \in M_0 \Rightarrow X_{t+h} \in M_0, \quad \forall h \geq 0 \tag{3.10}$$

and the non-extinction set M^+ is

$$M^+ = M \setminus M_0. \tag{3.11}$$

The methods developed to study persistence work under the following hypothesis.

Hypothesis 1 (***Extinction Set Invariance***)**.** *$\exists\, M_0$, called the extinction set of Φ, closed and invariant under the action of the semigroup $\Phi = \{\phi_t\}_{t\geq 0}$ of Markov operators*

$$\phi_t \delta_{M_0} = \delta_{M_0}, \quad \forall t \geq 0\,. \tag{3.12}$$

The Markov semigroup Φ associated with $\{X(t)\}_{t\geq 0}$ is $C_b(M)$-Feller if $\phi_t(C_b(M)) \subset C_b(M)$, that is, each operator maps the space $C_b(M)$ into itself. The following hypothesis claims that this mapping is continuous.

Hypothesis 2 ($C_b(M)$***-Feller Continuity***)**.** *For $\forall f \in C_b(M)$, $(t,x) \in \mathbb{R}_+ \times M \longrightarrow \phi_t f(x)$ is continuous.*

When Φ is a strongly continuous semigroup on a Banach subspace $E \subset C_b(M)$, guaranteed by Feller continuity, then the set $\mathcal{G} = \{(f,g) \in E \times E :\, f \in \mathcal{D}(\mathcal{L}) \wedge g = \mathcal{L}f\}$ equals the graph of the infinitesimal generator of Φ restricted to E, $\mathcal{L}_\Phi|_E$.

Definition 21 (***Extended Generator***)**.** *We define the extended generator of Φ on M as the set $\mathcal{G}$ with $(f,g) : M \to \mathbb{R}^2$ such that $\forall\, x \in M$ the process $\{M_f^x(t)\}_{t\geq 0}$ defined as*

$$M_f^x(t) = f(X^x(t)) - f(x) - \int_0^t g(X^x(s))ds \tag{3.13}$$

is a $\mathbb{P}$-martingale.

In particular, $\forall\, f \in \mathcal{D}(\mathcal{L}) : (f, \mathcal{L}f) \subset \mathcal{G}$. If $\forall x \in M :\, \lim_{t\to\infty} \frac{M_f^x(t)}{t} = 0$, $\mathbb{P} - a.s.$, then (f,g) satisfies the strong law of large numbers, giving us an analytically tractable instrument which allows us to study the behavior of the process near infinity, this is the subject of the next hypothesis.

A continuous function $W : M \to \mathbb{R}$ is proper if $\{x \in M : W(x) \leq R\}$ is compact $\forall\, R > 0$.

Hypothesis 3 (***Lyapunov Function for Tightness***). *There exist proper functions $W, \tilde{W} : M \longrightarrow \mathbb{R}$ and a continuous function $LW : M \longrightarrow \mathbb{R}$ such that*

1. *(W, LW) is the extended generator of Φ and satisfies the strong law of large numbers.*
2. *$LW \leq -\tilde{W} + C$ for some $C \in \mathbb{R}$.*

Proving or disproving stochastic persistence requires to control the behavior of $\{X(t)\}_{t\geq 0}$ near the extinction set, such as ∂M_0. This will be done by assuming the existence of another suitable type Lyapunov function.

Hypothesis 4 (***Lyapunov Function for Persistence***). *There exist continuous maps $V : M_+ \to \mathbb{R}_+$ and $H : M \to \mathbb{R}$ enjoying the following properties:*

1. *the pair $(V, H|_{M_+})$ lies in the extended generator of Φ on M_+ and satisfies the strong law.*
2. *the map $\frac{\tilde{W}}{1+|H|}$ is proper, where $\tilde{W}$ is like in Hypothesis 3.*

The following measure is used to determined the amount of time that a process spends on a set.

Definition 22 (***Occupation Measure***). *The occupation measure is defined as*

$$\Pi_t^x(A) = \frac{1}{t}\int_0^t \delta_{\{X_s^x \in A\}} ds, \quad \forall A \in \mathcal{B}(M). \tag{3.14}$$

If Hypothesis 2 and 3 are satisfied, then (Π_t^x) is relatively compact and its limit points are invariant, $\Pi \in \mathcal{P}_{inv}(M)$, by **[B19] Theorem 2.2**. With this measure we can rigorously define stochastic persistence.

Definition 23 (***Stochastic Persistence***). *The family of stochastic processes $\{(X_t^x)_{t\geq 0} : x \in M_+\}$ is stochastically persistent if for all $\epsilon > 0$ there exists a compact set $K_\epsilon \subset M_+$ such that*

$$\mathbb{P}\left(\lim_{t\to\infty} \inf \Pi_t^x(K_\epsilon) \geq 1 - \epsilon\right) = 1. \tag{3.15}$$

By the definition of compactness, K_ϵ contains all of it's limit points, hence the process never leaves M_+.

However, finding such a set might prove difficult, so we use H-Persistence to aid us in determining stochastic persistence.

Definition 24 (***H-Exponents***)**.** *If there exist V, H as in Hypothesis 4, the H-exponents of $\{X(t)\}_{t\geq 0}$ are defined as*

$$\Lambda^{-}(H) = -\sup_{\mu\in P_{erg}(M_0)} \int H d\mu \tag{3.16}$$

and

$$\Lambda^{+}(H) = -\inf_{\mu\in P_{erg}(M_0)} \int H d\mu \tag{3.17}$$

with which we define H-Persistence,

Definition 25 (***H-Persistence***)**.** *A stochastic process $\{X^x(t)\}_{t\geq 0}$ is H-Persistent if*

$$\Lambda^{-}(H) > 0, \tag{3.18}$$

where $x = X(0) \in M_+$.

from here we can relate H-Persistence and Stochastic Persistence.

Theorem 9 (***H-Persistence implies Stochastic Persistence***)**.** *If $\{X^x(t)\}_{t\geq 0}$ is H-Persistent then*

1. *every weak limit point of Π_t^x lies in $P_{inv}(M_+)$ a.s., and*
2. *the process is stochastically persistent.*

This result is of central importance for our work, if we can determine that our system is H-persistent then it follows that it is also stochastically persistent, that is, it's solution does not converge to $0 \in \mathbb{R}^n$, making it suitable to model financial time series, such as currency exchange rates, that do not assume null values. Its proof can be found in **[B19]**, requiring that Hypothesis $1-4$ are satisfied.

3.2 Deterministic Lotka-Volterra System

The deterministic Lotka-Volterra system is a system of ordinary differential equations widely used in mathematical biology to model the dynamics of n interacting species that prey, compete, or cooperate with each other. The first mathematical study of the 2-dimensional prey-predator system can be found in [**V28**]. In it's n-dimensional form it can be written as

$$\frac{dx_i(t)}{dt} = x_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}x_j(t)\right), \quad i = 1, 2, \ldots, n \tag{3.19}$$

where

- $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^{1\times n}$ is the vector of species populations,
- $r = [r_1, r_2, \ldots, r_n]^T \in \mathbb{R}^{1\times n}$ is the intrinsic growth vector, and
- $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the interaction matrix, with a_{ij} representing the effect of species j on species i.

For the 2-dimensional system a closed form solution can be found

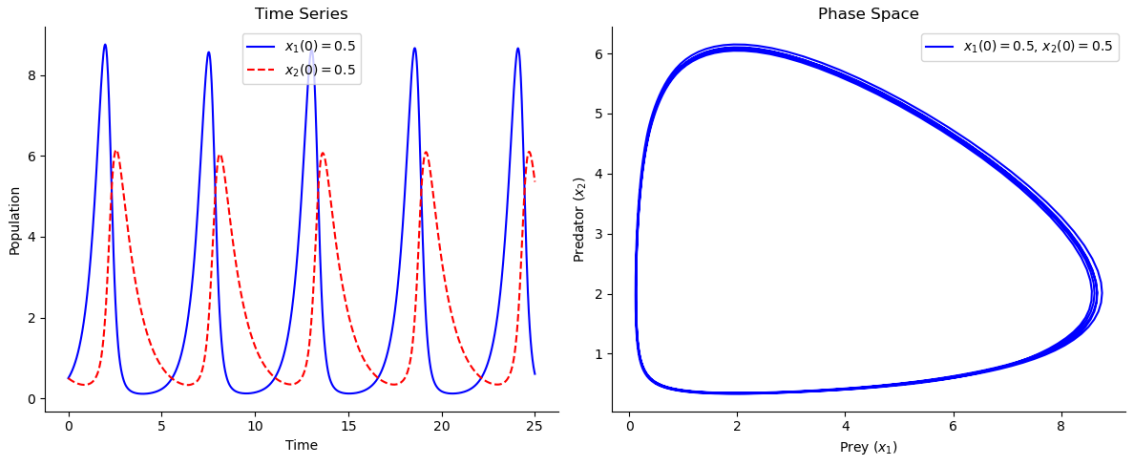


Figure 3: periodic solution for a 2-dimensional system

however, for the n-dimensional system a closed form solution is not known, but for most cases a numerical approximation can be obtained using, for example, Runge-Kutta methods. Despite of these difficulties, the qualitative analysis of the system provides us with a good description of it's dynamics. A contemporary survey of the system can be found in [**B10**].

3.3 Stochastic Lotka-Volterra System

Considering the n-dimensional Lotka-Volterra system, if we add environmental noise, which does not affect the inter-species interactions, we obtain the following stochastic analogue

$$dX_i(t) = X_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}X_j(t)\right)dt + X_i(t)\sigma_i dW_i(t), \quad i = 1, 2, \ldots, n \tag{3.20}$$

where

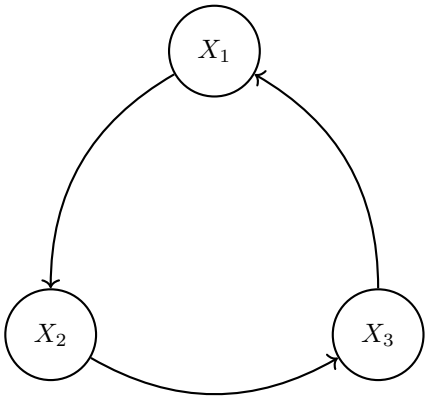
- $X = [X_1, X_2, ..., X_n]^T \in (\Omega, \mathcal{F}, \mathbb{P})$ is the random vector of species populations,
- $r = [r_1, r_2, ..., r_n]^T \in \mathbb{R}^{1\times n}$ is the intrinsic growth vector,
- $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the interaction matrix,
- $\sigma = [\sigma_1, \sigma_2, ..., \sigma_n]^T \in \mathbb{R}^{1\times n}$ is is the diffusion parameters vector, and
- $W_1, W_2, ..., W_n$ are Brownian motions.

The drift of the system captures the interactions between the species, and is the main driver of it's dynamics, whereas it's diffusion term is a simple constant, and serves to introduce environmental variability in the system.

For our modeling purposes we are interested in a three equation cyclic stochastic Lotka-Volterra system, such as

$$\begin{cases} dX_1(t) = X_1(t)(r_1 + a_{1,2}X_2(t))dt + X_1(t)\sigma_1(t)dW_1(t) \\ dX_2(t) = X_2(t)(r_2 + a_{2,3}X_3(t))dt + X_2(t)\sigma_2(t)dW_2(t) \\ dX_3(t) = X_3(t)(r_3 + a_{3,1}X_1(t))dt + X_3(t)\sigma_3(t)dW_3(t) \end{cases} \tag{3.21}$$

which has a cyclic dependence structure



This system is a particular case of a Kolmogorov system, which is studied in [**B19**], so it satisfies the assumptions stated in the paper and is stochastically persistent. Here we carry out the study of this system in the order of Section 3.1 with proofs, to ensure that the system is well behaved and for completeness of exposition.

To start we must determine if the system has a solution, which is done with an application of ***Itô's Existence and Uniqueness Theorem***.

Proposition 4 (***Existence and Uniqueness***)**.** *Let $X_n(0) \in \mathbb{R}^n_+$, then system (3.21) has a unique solution $X_n(t) \in \mathbb{R}^n_+$.*

Proof. Following ***Itô's Existence and Uniqueness Theorem***, we must show that the **Lipschitz condition** is satisfied,

$$\forall \epsilon > 0 \, \exists \, \delta > 0 \, \forall x, y \in \mathbb{R}^n : \, ||x||, ||y|| \leq \delta \implies ||b(x) - b(y)|| + ||\sigma(x) - \sigma(y)|| \leq \epsilon ||x - y||.$$

For the drift term

$$\begin{aligned}
& b_i(x) - b_i(y) = x_i \left(r_i + \sum_{j=1}^{n} a_{i,j} x_j \right) - y_i \left(r_i + \sum_{j=1}^{n} a_{i,j} y_j \right) \\
& \iff b_i(x) - b_i(y) = (x_i - y_i) \left(r_i + \sum_{j=1}^{n} a_{i,j} x_j \right) + y_i \left(r_i + \sum_{j=1}^{n} a_{i,j} (x_j - y_j) \right) \\
& \implies |b_i(x) - b_i(y)| \overset{\triangle}{\leq} |x_i - y_i| \left(|r_i| + \sum_{j=1}^{n} |a_{ij}| \delta \right) + \delta \sum_{j=1}^{n} |a_{i,j}| |x_i - y_i| \\
& \implies |b_i(x) - b_i(y)| \overset{(1)}{\leq} \left(|r_i| + \sum_{j=1}^{n} |a_{ij}| \delta + \delta \sqrt{n} \sum_{j=1}^{n} |a_{ij}| \right) ||x - y|| \\
& \iff |b_i(x) - b_i(y)| \leq \epsilon_1 \, ||x - y|| \implies \sum_{i=1}^{n} |b_i(x) - b_i(y)|^2 \leq \sum_{i=1}^{n} \epsilon_1^2 \, ||x - y||^2 \\
& \iff ||b(x) - b(y)||^2 \leq n \, \epsilon_1 \, ||x - y||^2 \\
& \iff ||b(x) - b(y)|| \leq \epsilon_1 \, ||x - y||
\end{aligned}$$

where (1) is given by $|x_i - y_i| \left(|r_i| + \sum_{j=1}^{n} |a_{ij}| \delta \right) \leq ||x - y|| \left(|r_i| + \sum_{j=1}^{n} |a_{ij}| \delta \right)$ and $\sum_{j=1}^{n} |x_i - y_i| \leq \sqrt{n} ||x - y||$ (Cauchy-Schwarz), and for the diffusion term

$$\begin{aligned}
& ||\sigma(x) - \sigma(y)||^2 = \sum_{i=1}^{n} \sigma_i^2 (x_i - y_i)^2 \\
& \iff ||\sigma(x) - \sigma(y)|| = \sqrt{\sum_{i=1}^{n} \sigma_i^2 (x_i - y_i)^2} \leq (\max_i |\sigma_i|) \, ||x_i - y_i|| = \epsilon_2 \, ||x_i - y_i||
\end{aligned}$$

so $||b(x) - b(y)|| + ||\sigma(x) - \sigma(y)|| \leq (\epsilon_1 + \epsilon_2) ||x - y|| = \epsilon \, ||x - y||$. Now me must show that the **linear growth condition** is satisfied,

$$||b(x)||^2 + ||\sigma(x)||^2 \leq C(1 + ||x||^2).$$

For the drift term,

$$|b_i(x)| \leq |x_i| \left(|r_i| + \sum_{j=1}^{n} |a_{ij}||x_j| \right) = \underbrace{|x_i||r_i|}_{(2)} + \underbrace{|x_i| \underbrace{\sum_{j=1}^{n} |a_{ij}||x_j|}_{3}}_{(4)}$$

$$\implies ||b(x)|| \leq C_1||x|| + C_2||x||^2$$

$$\implies ||b(x)||^2 \leq 2C_1^2||x||^2 + 4C_2^2(1 + ||x||^2) \leq C(1 + ||x||^2)$$

where we norm (2) and (4),

$$(2): \sqrt{\sum_{i=1}^{n} r_i^2 x_i^2} \leq \max_i (|r_i|) \sqrt{\sum_{i=1}^{n} x_i^2} = \max_i (|r_i|)\, ||x|| = C_1||x||$$

$$(3): \sum_{j=1}^{n} |a_{ij}||x_j| \leq \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \leq \max_i \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} ||x|| = C_2||x||$$

$$(4): \sqrt{\sum_{i=1}^{n} (|x_i| \cdot C_2||x||)^2} = \sqrt{\sum_{i=1}^{n} (|x_i|)^2} \cdot C_2||x|| = C_2||x||^2$$

and for the diffusion term,

$$||\sigma(x)||^2 = \sum_{i=1}^{n} \sigma_i^2 x_i^2 \leq \left(\max_i \sigma_i^2 \right) ||x||^2 = C||x_i||^2 \leq C(1 + ||x||^2)$$

so $||b(x)||^2 + ||\sigma(x)||^2 \leq C(1 + ||x||^2)$. □

Hence the system has a unique solution, but we can't obtain it's analytical expression. However, we can still achieve a good understanding of the system's solution using qualitative methods. Of the utmost importance is to guarantee that the solution does not converge to zero, that is, the system is stochastically persistent.

For stochastic Lotka-Volterra systems **Theorem 5.8 [B19]** relates the behavior of the **invasion rates**

$$\lambda_i(\mu) = \int_{\partial M_+} \left(r_i + \sum_{i=1}^{n} a_{ij} x_j - \frac{\sigma_i^2(x)}{2} \right) \mu(dx), \quad i = 1, ..., n \tag{3.22}$$

with stochastic persistence, saying that if there exist weights $w_i > 0$ such that $\sum_{i=1}^{n} w_i \mu \lambda_i > 0$ for every measure $\mu \in \mathcal{P}_{erg}(M_0)$ then the process is H-persistent, hence stochastically persistent. This is the method that is employed in practice.

Proposition 5 (***Persistence***). *System (3.21)*

$$dX_i(t) = X_i(t)\left(r_i + \sum_{j=1}^{n} a_{ij}X_j(t)\right)dt + X_i(t)\sigma_i dW_i(t), \quad i = 1, 2, \ldots, n$$

is stochastically persistent.

Proof. (Outline)

1. (Extinction Set) Consider $M = \mathbb{R}^n_+$ and $M_0 = \{X \in M : \prod_{i=1}^n X_i(t^*) = 0\}$, if $X(t_*) \in M_0$, then $\forall t \geq t_* : X(t) = 0$.

2. ($C_b(M)$-Feller Continuity) $b(x)$ and $\sigma(x)$ are locally Lipschitz with linear growth, hence the semigroup Φ is Feller.

3. (Lyapunov Function for Tightness) Let

$$W(x) = ||x||^2 = \sum_{i=1}^{n} x_i^2$$

we have $\lim_{||x||\to\infty} W(x) = \infty$ so W is proper. Define $LW = \mathcal{L}W$ the infinitesimal generator of Φ, which amounts to the Ito derivative of W,

$$LW(x) = \sum_{i=1}^{n} 2x_i\left(r_i + \sum_{j=1}^{n} a_{ij}x_j\right) + \sum_{i=1}^{n} \sigma_i^2 x_i^2$$

which is bounded above by

$$LW(x) \leq C - ||x||^3 = C - \tilde{W}, \quad C \in \mathbb{R}.$$

4. (Lyapunov Function for Persistence) Let

$$V(x) = \sum_{i=1}^{n} w_i \log(x_i)$$

Define $LV = \mathcal{L}V$,

$$LV(x) = \sum_{i=1}^{n} w_i\left(r_i + \sum_{j=1}^{n} a_{ij}x_j - \frac{1}{2}\sigma_i^2\right).$$

For $x \in M_0$, the invasion rate of x_i with respect to an invariant measure $\mu \in \mathcal{P}_{erg}(M_0)$ is

$$\lambda_i(\mu) = \int_{\partial M_+}\left(r_i + \sum_{i=1}^{n} a_{ij}x_j - \frac{\sigma_i^2(x)}{2}\right)\mu(dx)$$

We have $\sum_{i=1}^n w_i\lambda_i(\mu) > 0$, hence the system is H-persistent, and by ***H-Persistence implies Stochastic Persistence*** it's stochastically persistent. □

4 Currency Exchange Rates Dynamics

This section is dedicated to the modeling of the financial dynamics of currency exchange rates with a stochastic Lotka-Volterra system. We start by providing a financial interpretation of the system, an analytically derived property, and a possible generalization of the system's structure, which allows for richer dynamics.

4.1 Stochastic Lotka-Volterra Model

We can rewrite system (3.21) in the context of currency exchange rates,

$$\begin{cases} dr_{A/B}(t) = r_{A/B}(t)(r_1 + a_{1,2}r_{B/C}(t))dt + r_{A/B}(t)\sigma_1(t)dW_1(t) \\ dr_{B/C}(t) = r_{B/C}(t)(r_2 + a_{2,3}r_{C/A}(t))dt + r_{B/C}(t)\sigma_2(t)dW_2(t) \\ dr_{C/A}(t) = r_{C/A}(t)(r_3 + a_{3,1}r_{A/B}(t))dt + r_{C/A}(t)\sigma_3(t)dW_3(t) \end{cases} \tag{4.1}$$

which can be interpreted as:

- $r_{A/B}$, $r_{B/C}$, $r_{C/A}$ are the currency exchange rates,
- $r = [r_1, r_2, r_3]^T \in \mathbb{R}^{1\times 3}$ is the intrinsic growth vector, which can represent currency (de)appreciation due to underlying economic factors,
- $A = [a_{i,j}] \in \mathbb{R}^{3\times 3}$ is the interaction matrix, whose cyclic dependence structure seeks to guarantee no-arbitrage,
- $\sigma = [\sigma_1, \sigma_2, \sigma_3]^T \in \mathbb{R}^{1,3}$ is the volatility vector, that modifies the intensity of economic shocks,
- W_1, W_2, W_3 are Brownian motions, that model the economic shocks.

4.1.1 No-Arbitrage

Here no-arbitrage refers to no-triangular arbitrage, for example, if $r_{A/B} = 0.85$ and $r_{B/C} = 0.87$ then we must have $r_{A/C} = \frac{1}{r_{A/B}\times r_{B/C}} = 0.85 \times 0.87 = 1.35$.

The model seeks to capture the no-arbitrage condition within the circular dependence structures, however it does not guarantee that the condition is completely satisfied at every step, rather, whenever an arbitrage opportunity arises from a volatility induced shock, the drift components start converging to an equilibrium state of no-arbitrage.

4.1.2 Mean Reversion

One aspect of currency exchange rates that previous studies have modeled is mean reversion, the Ornstein-Uhlenbeck process captures this dynamic very well [**M09**]

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t) \tag{4.2}$$

where

- $\theta, \sigma \in \mathbb{R}_+$ are model parameters, and
- $\mu \in \mathbb{R}$ is the long term mean of the process.

notice that we can rewrite the drift term of this process as

$$\begin{aligned} b_{OU}(t, X(t)) &= \theta(\mu - X(t)) \\ &= \theta\mu - \theta X(t) \\ &= X(t)\left(\frac{\theta\mu}{X(t)} - \theta\right) \\ &= X(t)(-\theta + \theta\mu Y(t)) \\ &= X(t)(-\theta + aY(t)) \end{aligned} \tag{4.3}$$

which very closely resembles the drift term of the equations of our stochastic Lotka-Volterra system

$$b_{LV}(t, X_i(t)) = X_i(t)(r_i + a_{i,j}X_j(t)) \tag{4.4}$$

so, in a narrow sense, the stochastic Lotka-Volterra system can also capture mean reversion, but this mean reversion is dependent on the other processes in the system.

4.1.3 Generalization

The system can be generalized to $n \in \mathbb{N}$ currencies if we consider the **network topology** of the foreign exchange market. Take for example the following **minimal spanning tree** of twenty currencies in 2024, obtained with the correlation induced metric as in [**KGDG09**]

$$d(x, y) = \sqrt{2(1 - \rho_{x,y})} \tag{4.5}$$

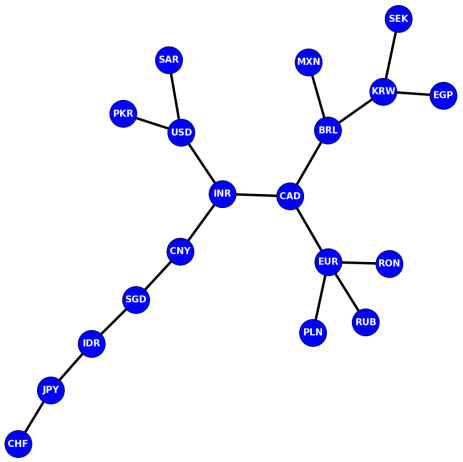


Figure 4: Minimal Spanning Tree

	USD	EUR	CAD	INR	⋯	SAR	PLN	RUB	RON
USD	0	0	0	1	⋯	1	0	0	0
EUR	∗	0	1	0	⋯	0	1	1	1
CAD	∗	∗	0	1	⋯	0	0	0	0
INR	∗	∗	∗	0	⋯	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
SAR	∗	∗	∗	∗	⋯	0	0	0	0
PLN	∗	∗	∗	∗	⋯	∗	0	0	0
RUB	∗	∗	∗	∗	⋯	∗	∗	0	0
RON	∗	∗	∗	∗	⋯	∗	∗	∗	0

Figure 5: MST Adjacency Matrix

we can also add circular dependencies for all nodes, yielding an arbitrage free interaction matrix

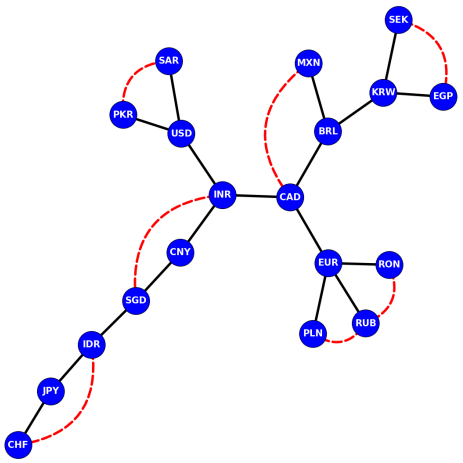


Figure 6: MST with Triangular Dependence

	USD	EUR	CAD	INR	⋯	SAR	PLN	RUB	RON
USD	0	0	0	1	⋯	1	0	0	0
EUR	∗	0	1	0	⋯	0	1	1	1
CAD	∗	∗	0	1	⋯	0	0	0	0
INR	∗	∗	∗	0	⋯	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮
SAR	∗	∗	∗	∗	⋯	0	0	0	0
PLN	∗	∗	∗	∗	⋯	∗	0	**1**	0
RUB	∗	∗	∗	∗	⋯	∗	∗	0	**1**
RON	∗	∗	∗	∗	⋯	∗	∗	∗	0

Figure 7: MST-TD Adjacency Matrix

This way we can incorporate the structure of the foreign exchange market in the interaction matrix of the system, A, and determine it's dynamics by fitting it's parameters to historical data.

4.2 Fitting the Model

Here we set our focus on the 3-dimensional cyclic stochastic Lotka-Volterra system. We fit the model on the 1-day frequency exchange rates USD/EUR, EUR/GBP, and GBP/USD observed during 2024, yielding a system of the form

$$\begin{cases} dr_{USD/EUR}(t) = r_{USD/EUR}(t)(r_1 + a_{1,2} r_{EUR/GBP}(t))\, dt + r_{USD/EUR}(t)\sigma_1(t)\, dW_1(t) \\ dr_{EUR/GBP}(t) = r_{EUR/GBP}(t)(r_2 + a_{2,3} r_{GBP/USD}(t))\, dt + r_{EUR/GBP}(t)\sigma_2(t)\, dW_2(t) \\ dr_{GBP/USD}(t) = r_{GBP/USD}(t)(r_3 + a_{3,1} r_{USD/EUR}(t))\, dt + r_{USD/EUR}(t)\sigma_3(t)\, dW_3(t) \end{cases} \tag{4.6}$$

4.2.1 Euler-Maruyama Discretization

Since we don't have a closed form solution of system (3.21) we use the following discretization scheme to obtain an approximate solution

Definition 26 (***Euler-Maruyama Discretization Scheme***)**.** *Let $0 < t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0, T]$ with step size $\Delta t = t_{n+1} - t_n$. Then the Euler-Maruyama approximation of the solution $\{X_n\}_{n=0}^N$ of a SDE is defined recursively by:*

$$X_{n+1} = X_n + \mu(X_n, t_n)\Delta t + \sigma(X_n, t_n)\Delta W(t)\,. \tag{4.7}$$

Now consider the equation belonging to system (3.21)

$$dX_i(t) = X_{i+1}(t)(r_i + a_{i,i+1} X_{i+1}(t))\, dt + X_i(t)\sigma_i\, dW_i(t) \tag{4.8}$$

for analytical tractability, and to lend easily to parameter estimation, apply the log transform $Y(t) = \log(X(t))$, then use ***Itô's Lemma*** to find it's differential form

$$\begin{aligned} dY_i(t) &= \frac{1}{X_i(t)}\, dX_i(t) - \frac{1}{2}\frac{1}{X_i(t)^2}\, (dX_i(t))^2 \\ &= \frac{1}{X_i(t)}[X_i(t)(r_i + a_{i,i+1}X_{i+1}(t))\, dt + X_i(t)\sigma_i\, dW_i(t)] \\ &\quad - \frac{1}{2}\frac{1}{X_i(t)^2}[X_i(t)(r_i + a_{i,i+1}X_{i+1}(t))\, dt + X_i(t)\sigma_i\, dW_i(t)]^2 \\ &= (r_i + a_{i,i+1}X_{i+1}(t))\, dt + \sigma_i\, dW_i(t) - \frac{1}{2}\sigma^2\, dt \\ &= (r_i - a_{i,i+1}X_{i+1}(t) - \frac{1}{2}\sigma_i^2)\, dt + \sigma_i\, dW_i(t) \end{aligned}$$

with the Euler-Maruyama discretization scheme we get the following discrete approximation

$$\log(X_i(t + \Delta t)) \approx \log(X_i(t)) + (r_i - a_{i,i+1}X_{i+1}(t) - \frac{1}{2}\sigma_i^2)\, \Delta t + \sigma_i\, \Delta W_i(t)\,. \tag{4.9}$$

4.2.2 Ordinary Least Squares Fitting

Definition 27 (***OLS Estimator***). *Let $y \in \mathbb{R}^n$ be the observed vector, the OLS estimator seeks to find $\hat{\alpha}, \hat{\beta} \in \mathbb{R}^n$ that minimizes the residual sum of squares*

$$(\hat{\alpha}, \hat{\beta}) = \arg \min_{\alpha,\beta \in \mathbb{R}^n} ||y - \alpha - X\beta||_2^2 \,. \tag{4.10}$$

To estimate equation (3.21) parameters define the regression-like form

$$\Delta \log(X_i(t)) \approx (\alpha_i + X_{i+1,t}\beta_i) + \epsilon_{i,t}, \quad \epsilon_{i,t} \sim \mathcal{N}(0, \frac{\sigma_i^2}{\Delta t}) \tag{4.11}$$

where:

$$\alpha_i = r_i \Delta t - \frac{1}{2}\sigma^2 \Delta t \,, \tag{4.12}$$

$$\beta_i = -a_{i,i+1}\Delta t \,. \tag{4.13}$$

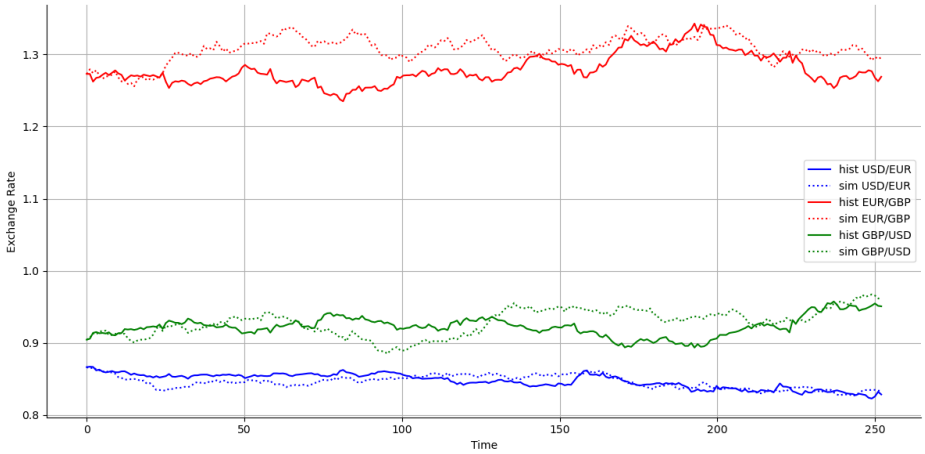
Now we perform a regression based estimation for each equation (4.1) on the 1-day frequency exchange rates USD/EUR, EUR/GBP, and GBP/USD for the year 2024, using ordinary least squares (OLS), and estimate its parameters by setting

$$\hat{\beta}_i \Rightarrow \hat{a}_{i,i+1} = -\frac{\hat{\beta}_i}{\Delta t} \,, \tag{4.14}$$

$$\hat{\alpha}_i \Rightarrow \hat{r}_i = \frac{(\hat{\alpha}_i + \frac{1}{2}\sigma_i^2 \Delta t)}{\Delta t} \,, \tag{4.15}$$

$$\text{and} \quad \hat{\sigma}_i^2 = \frac{1}{N}\sum_t^N \frac{\epsilon_{i,t}^2}{\Delta t} \,. \tag{4.16}$$

The following figure shows the plot of the historical exchange rates in solid line, and simulated data generated by the fitted system (4.1) in dotted line, along with the estimated parameters



term	value
$\hat{a}_{1,2}$	0.0041
$\hat{r}_1$	0.0051
$\hat{\sigma}_1^2$	0.0025
$\hat{a}_{2,3}$	0.0001
$\hat{r}_2$	0.0000
$\hat{\sigma}_2^2$	0.0039
$\hat{a}_{3,1}$	0.0286
$\hat{r}_3$	0.0245
$\hat{\sigma}_3^2$	0.0037

Figure 8: Fitted Cyclic Lotka-Volterra

4.2.3 Observations

This section is dedicated to study of the fitted system regarding it's ability to enforce no-arbitrage.

$$\begin{cases} dr_{USD/EUR}(t) = r_{USD/EUR}(t)(0.0051 + 0.0041\, r_{EUR/GBP}(t))\, dt + r_{USD/EUR}(t)\, 0.001\, dW_1(t) \\ dr_{EUR/GBP}(t) = r_{EUR/GBP}(t)(0.0000 + 0.0001\, r_{GBP/USD}(t))\, dt + r_{EUR/GBP}(t)\, 0.000\, dW_2(t) \\ dr_{GBP/USD}(t) = r_{GBP/USD}(t)(0.0245 + 0.0286\, r_{USD/EUR}(t))\, dt + r_{USD/EUR}(t)\, 0.007\, dW_3(t) \end{cases} \tag{4.17}$$

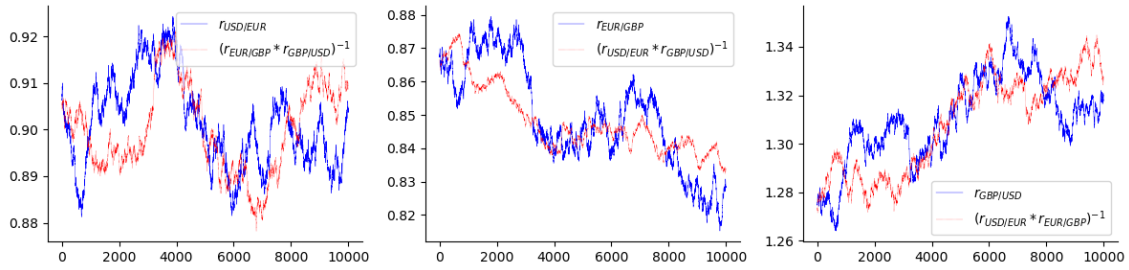


Figure 9: Cyclic No-Arbitrage

In this instance the system fails to enforce no-arbitrage, $r_{USD/EUR} \times r_{EUR/GBP} \times r_{GBP/USD} \neq 1$. The series seem to follow the same trend, but their movements are somewhat lagged, which follows from the reactive nature of Lotka-Volterra systems. Some work on the drift terms and a more taylored approach to parameter fitting might yield a system closer to satisfying no-arbitrage.

An immediate way to solve this issue is to make the $r_{GBP/USD}$ completely dependent on $r_{USD/EUR}$ and $r_{EUR/GBP}$, without any intrinsic growth and volatility parameters, serving only to enforce no-arbitrage

$$\begin{cases} dr_{USD/EUR}(t) = r_{USD/EUR}(t)(0.0051 + 0.0041\, r_{EUR/GBP}(t))\, dt + r_{USD/EUR}(t)\, 0.001\, dW_1(t) \\ dr_{EUR/GBP}(t) = r_{EUR/GBP}(t)(0.0000 + 0.0001\, r_{GBP/USD}(t))\, dt + r_{EUR/GBP}(t)\, 0.000\, dW_2(t) \\ dr_{GBP/USD}(t) = \dfrac{1}{dr_{USD/EUR}(t) * dr_{EUR/GBP}(t)} \end{cases}$$

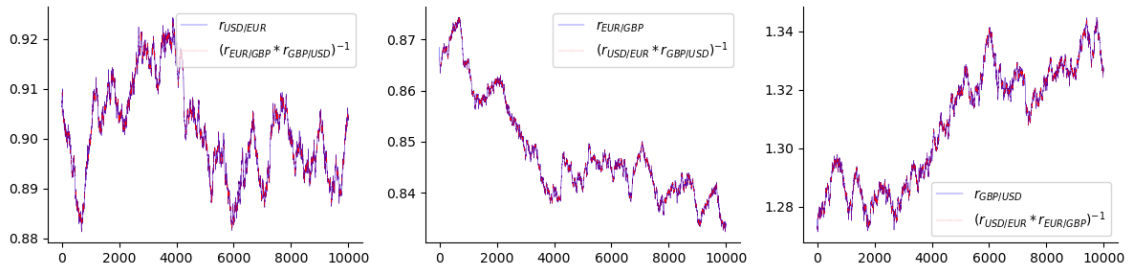
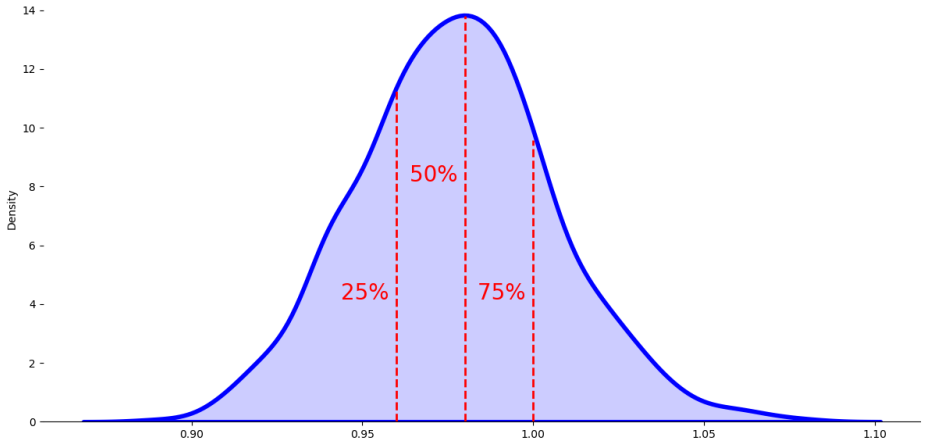


Figure 10: Tight No-Arbitrage

Statistically, the cyclic system is close to achieving no-arbitrage, as we can see after running 1000 simulations, however it has a negative bias $\mathbb{E}[\hat{\alpha}] - 1 = -0.033$,

$$\alpha = \frac{1}{N} \sum_{i=1}^{N} \overline{X_1(t)} * \overline{X_2(t)} * \overline{X_3(t)} \tag{4.18}$$



term	value
count	1000
mean	0.977
std	0.028
min	0.890
25%	0.960
50%	0.980
75%	1.000
max	1.080

Figure 11: Empirical Distribution of α

4.3 Derivatives Pricing

This section presents a numerical technique for pricing European options using system (3.21) as the underlying asset dynamic.

4.3.1 European Options

Options are financial derivatives that grant the holder the right, but not the obligation, to buy (call) or sell (put) an underlying asset at a strike price K at a maturity date T. While american options can be exercised for $\forall t \leq T$, european options can only be exercised at $t = T$. Although closed-form pricing solutions such as the Black–Scholes formula exist for european options under idealized assumptions, most real-world scenarios—such as non-standard payoffs, stochastic volatility, or path-dependence—require numerical methods for accurate valuation.

Under the standard Black–Scholes framework, the price $\{X(t)\}_{t\geq 0}$ of the underlying asset follows a geometric Brownian motion

$$dX(t) = \mu X(t)\, dt + \sigma(t)\, dW(t)\,. \tag{4.19}$$

Under this dynamic the value of an European call $c(S, K, T)$ and European put $p(X, K, T)$, where X is the underlying price, K the strike price, T the maturity, and r is the risk-free interest rate, at time t is given by

$$c_t(X, K, T) = \mathcal{N}(d_1)X(t) - \mathcal{N}(d_2)Ke^{-r(T-t)} \tag{4.20}$$

$$p_t(X, K, T) = \mathcal{N}(-d_2)Ke^{-r(T-t)} - \mathcal{N}(-d_1)X(t) \tag{4.21}$$

where Φ represents the normal cumulative distribution function, and

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[ln\left(\frac{X(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{4.22}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}\,. \tag{4.23}$$

4.3.2 Monte Carlo Simulation

However, system 3.21 does not yield a closed form analytical solution, so we must employ numerical methods to find the price of derivative products. Monte Carlo methods estimate option prices by simulating a large number of possible price paths of the underlying asset, in our case

$$\log(X_i(t+\Delta t)) \approx \log(X_i(t)) + \left(r_i - a_{i,i+1}X_{i+1}(t) - \frac{1}{2}\sigma_i^2\right)\Delta t + \sigma_i\, \Delta W_i(t)\,. \tag{4.24}$$

and then the option value is computed as the discounted payoff across N simulated paths:

$$\begin{aligned} c_0(X, K, T) &= e^{rT}\mathbb{E}[\Phi(X(T))] \\ &\approx e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\max(X_i(T) - K, 0) \end{aligned} \tag{4.25}$$

Monte Carlo is especially useful for high-dimensional problems (e.g. basket options), and path-dependent payoffs (e.g. Asian options).

4.3.3 Results

Running 1000 simulations of system (3.21) and computing the price of an European call on the exchange rate $r_{USD/EUR}$, we see that the the variable that affects the option price the most is the strike price K, while the maturity date T has a low influence on the price, since the fitted process deviates within a small interval over time, which is a consequence of the stable nature of the exchange rate that we considered here.

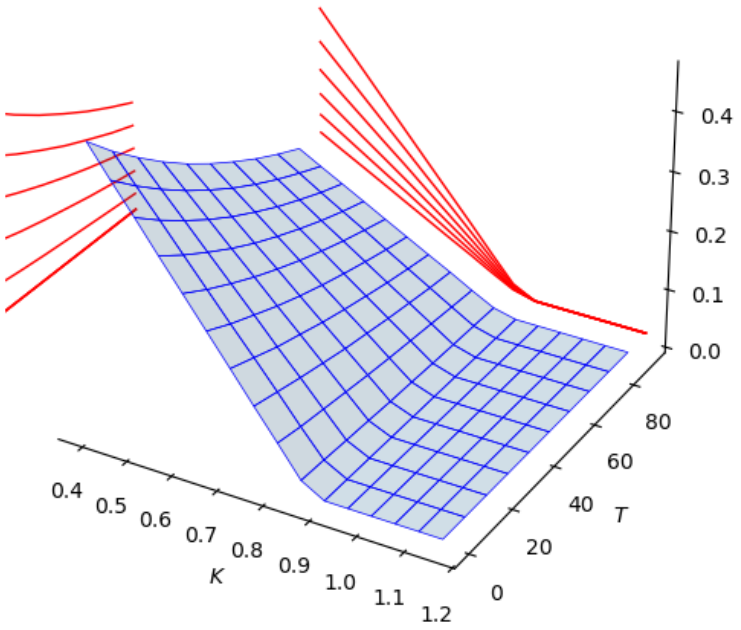


Figure 12: $c_0(X, K, T)$

5 Conclusion

The Stochastic Lotka-Volterra model generates a dynamic which closely resembles observed currency exchange rates, exhibiting mean-reversion and systematic no-arbitrage, due to it's dependence structure.

The model can be generalized by modeling volatility with a diffusion process $\{\sigma(t)\}_{t\geq 0}$, and a regime switching with a jump process $\{(\omega(t)\}_{t\geq 0}$, that can effect changes in the dependence structure. However, the introduction of these terms would require a new study on the existence, uniqueness, and persistence of solutions.

For further applications, the model can be applied to other asset categories, such as stocks, with appropriate modifications to fit the assets dynamics.

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