THE MODIFIED *PEAKEDNESS* AS A $M|G|\infty$ BUSY CYCLE DISTRIBUTION CHARACTERIZING PARAMETER [1]

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ABSTRACT

It is exposed that a parameter, then called θ,similar to the parameter η proposed in (1) to characterize the $M|G|\infty$ queue busy period distribution, that is a modification of the *peakedness* proposed in (2) is also useful to characterize the *M/G/∞* queue busy cycle distribution.

 Key words: $M|G|\infty$, busy cycle, *peakedness*, θ.

1. INTRODUCTION

In a $M|G|\infty$ queue system, λ is the Poisson process arrivals rate, α is the mean service time, $G(.)$ is the service time distribution function and so $\alpha = \int_0^\infty [1 - G(t)]\, dt$. The traffic intensity is defined as

$$\rho = \lambda\alpha \tag{1.1}$$

Upon its arrival a customer finds immediately an available server. And each customer service is independent from the other customers' services.

In the operation of this queue, as it happens with any other, there is a sequence of idle and busy periods. An idle period followed by a busy period is a busy cycle. Call *I*, *B* and *Z* the idle period, the busy period and the busy cycle, respectively, lengths random variables. Obviously,

$$Z = I + B \tag{1.2}$$

As *I* and *B* are independent, see (3), the following relation

$$\bar{Z}(s) = \bar{I}(s)\bar{B}(s) \tag{1.3}$$

among the respective Laplace transforms holds, where

$$\bar{I}(s) = \frac{\lambda}{\lambda + s} \tag{1.4}$$

since the idle period is exponentially distributed with parameter λ, as it happens with any queue with Poisson arrivals, and

$$\bar{B}(s) = 1 + \frac{1}{\lambda}\left(s - \frac{1}{\int_0^\infty e^{-st - \lambda - \int_0^t [1 - G(v)]dv}dt}\right) \tag{1.5}$$

The goal of this paper is to introduce a parameter,θ, which is a modification of the *peakedness*, for the *Z* random variable and show that it is useful to characterize the Z distribution. In section 2 it is introduced the *peakedness* for the $M|G|\infty$ queue system busy cycle distribution and studied its properties. Then, in section 3, taking in account the study of section 2, it is introduced the modified peakedness for the $M|G|\infty$ queue system busy cycle distribution, which is thought as an improvement of the *peakedness*. The paper ends with the conclusions and a list of references.

2. THE *Z* RANDOM VARIABLE *PEAKEDNESS*

For the random variable Z, the *peakedness,* here called *q,* is given by, see *(*1),

$$q = \bar{Z}\left(\frac{1}{\alpha}\right) \tag{2.1}$$

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and, being p the *peakedness* for B, it holds

$$q = \frac{\rho}{\rho+1}p \tag{2.2}$$

and, of course, $q \le p$.

It is easy to show that

$$q = \sum_{n=0}^{\infty}(-1)^n \frac{E[Z^n]}{n!\alpha^n} \tag{2.3}$$

and so q includes information of the Z every order centered at the origin moments such as p includes information of the B every order centered at the origin moments.

For the collection of service times' distribution[1], see (4),

$$G(t) = 1 - \frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}(e^{(\lambda+\beta)t}-1)+\lambda}, t \ge 0, -\lambda \le \beta \le \frac{\lambda}{e^{\rho}-1} \tag{2.4}$$

$$q = \alpha \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\beta}{(\rho+1)(e^{-\rho}(\rho+\alpha\beta)+1)}, -\lambda \le \beta \le \frac{\lambda}{e^{\rho}-1} \tag{2.5}$$

and

- $\beta = -\lambda$

$$q = 0 \tag{2.6}$$

- $\beta = 0$ ($M|G_1|\infty$ system)

$$q^{G_1} = \frac{\rho}{e^{\rho}+\rho} \tag{2.7}$$

- $\beta = \frac{\lambda}{e^{\rho}-1}$ ($M|G_2|\infty$ system)

$$q^{G_2} = \frac{\rho^2}{e^{\rho}(\rho+1)+\rho^2-1} \tag{2.8}$$

For the systems:

- $M|D|\infty$ (constant service times)

$$q^D = \frac{\rho}{e^{\rho+1}+\rho} \tag{2.9}$$

- $M|M|\infty$ (exponential service times)

$$q^M = \frac{e^{\rho}-\rho-1}{(\rho+1)(e^{\rho}-1)} \tag{2.10}$$

For the service times G_1, G_2, D and M, q depends only on ρ and is given by very simple expressions. Except

$$E[Z] = \frac{e^{\rho}}{\lambda} \tag{2.11}$$

that depends on service time only by its mean[2], it is the only parameter with simple expression, for the distribution of Z, to the $M|M|\infty$ systems, confer with (6).

For service times distributions related with the exponential distribution, important in the reliability theory:

-If the service time distribution is NBUE-**N**ew **B**etter than **U**sed in **E**xpectation with mean α, $\int_t^{\infty}[1-G(v)]dv \ge \int_t^{\infty}e^{-\frac{v}{\alpha}}dv$ (7, p. 273), and

$$q^{NBUE} \le \frac{e^{\rho}-\rho-1}{(\rho+1)(e^{\rho}-1)} \tag{2.12}$$

-If the service time distribution is NWUE-**N**ew **W**orse than **U**sed in **E**xpectation with mean α, $\int_t^{\infty}[1-G(v)]dv \le \int_t^{\infty}e^{-\frac{v}{\alpha}}dv$ (7, p. 273), and

$$q^{NWUE} \ge \frac{e^{\rho}-\rho-1}{(\rho+1)(e^{\rho}-1)} \tag{2.13}$$

In general, if G (.) and H (.) are service times distributions functions such that $1-G(t) \le 1-H(t)$

$$q^G \ge q^H \tag{2.14}$$

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[1] In this case the distribution function of Z is a mixture of exponentials, see (5):
$P(Z \le t) = 1 - \frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-e^{-\rho}(\lambda+\beta)t} + \frac{\beta}{\lambda-e^{-\rho}(\lambda+\beta)}e^{-\lambda t}, t \ge 0, -\lambda \le \beta \le \frac{\lambda}{e^{\rho}-1}$
[2] It is usual to say that it is "insensible" to the service time distribution.

3. THE PARAMETER θ

The modified *peakedness* for Z, called θ is given by, see (1), $\theta = \frac{q - \left(1 - \frac{e^{\rho}}{\rho}\right)}{\frac{e^{\rho}}{\rho} - 1}$, that is

$$\theta = q\frac{\rho}{e^{\rho} - \rho} + 1 \tag{3.1}$$

It was taken out of *q* the parts that do not depend on the service time's distribution form, check with the expression (2.3), and then it was standardized dividing by the common part.

After what was seen in the former section, it is possible to obtain simple expressions for θ^{G_1}, θ^{G_2}, θ^{D} and θ^{M} depending only on ρ. And evidently the properties of *q* extend to θ. It is easy to show that

$$1 \leq \theta \leq \frac{\rho^2}{(e^{\rho} + 1)(e^{\rho} - \rho)} + 1 \tag{3.2}$$

for any service time distribution, concluding so that

$$\lim_{\rho \to \infty} \theta = 1 \tag{3.3}.$$

Computing θ for some service times distributions, taking various values of ρ, were obtained the values presented in Table 1. In it *P* means the power distribution with parameter[1]c=.5.

Table 1. Values of θ for some service times distribution functions

G	ρ=.5	ρ=1	ρ=5	ρ=10	ρ=15	ρ=20
G_1	1.0573588	1.0841882	1.0000456	1.0000000	1.0000000	1.0000000
G_2	1.0357639	1.0575796	1.0000383	1.0000000	1.0000000	1.0000000
D	1.0247402	1.0373147	1.0000171	1.0000000	1.0000000	1.0000000
M	1.0376733	1.0654280	1.0002254	1.0000000	1.0000000	1.0000000
P	1.0359789	1.0646113	1.0004005	1.0000001	1.0000000	1.0000000

For this service time distribution θ^{P} was computed directly from (2.1) and (3.1).

When ρ assumes great values: above $\rho = 10$ for some service times distributions and $\rho = 20$ for others, under some conditions, the busy period is practically exponentially distributed (8, 9). In the same conditions θ is practically 1.

It is interesting to compare the values of θ, in Table 1, with the values of η, in Table 2, got from (1), computed in the same conditions, being the most relevant

Table 2. Values of η for some service times distribution functions

G	ρ=.5	ρ=1	ρ=5	ρ=10	ρ=15	ρ=20
G_1	3.3469730	1.7488465	1.0013731	1.0000002	1.0000000	1.0000000
G_2	2.4633636	1.5121659	1.0011518	1.0000002	1.0000000	1.0000000
D	2.0123054	1.3319113	1.0005158	1.0000001	1.0000000	1.0000000
M	2.5414941	1.5819767	1.0067837	1.0000454	1.0000000	1.0000000
P	2.4721612	1.5747122	1.0120525	1.0001526	1.0000000	1.0000000

to note that $\eta \geq \theta$, being the major differences for the lowest traffic intensities.

4. CONCLUSIONS

The parameter θ, defined in Expression (3.1), incorporates information on the whole moments of Z centered at the origin.

It assumes different values when the distributions of Z differ as functions of the service distribution.

And there are simple bounds for θ depending only on the traffic intensity ρ.

Even for the $M|M|\infty$ system, with analytically intractable parameters in the case of the distribution of Z, θ is written in a quite simple expression. This queue system, very important from the applications point of view, see (10-16), is a quite strange case because not even its markovian properties, owing to the exponential inter-arrival times and services, brings the expectable analytical simplicity.

It is even admissible that close values of θ, for different service time's distributions, indicate resemblance of the respective Z distributions behaviors.

So θ is a parameter that can help to characterize the distribution of Z, discriminating among different service time distributions.

This parameter, θ, may be included in the class of other composite parameters, as for instance the kurtosis or the skewness, see (17), that are used to characterize probability distributions putting an emphasis in their

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[1] $\alpha = \frac{c}{c+1}$.

probability functions or probability density functions graphic characteristics. But it is more complete and complex than any of the two examples here called, since incorporates the whole distribution centered at the origin moments. Because of that, its interpretation in probability functions or probability density functions graphic characteristics terms is not so obvious and it is an interesting open field of investigation.

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