THE VECTOR SPACE CONVEX PARTS SEPARATION AS A CONSEQUENCE OF THE HAHN-BANACH THEOREM

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ABSTRACT
The separation theorems are the key results for convex programming. They are important consequences of the Hahn-Theorem theorem. In this work begin to consider vector spaces, in general, then normed spaces and lastly Hilbert spaces. In the end, applications of these results in convex programming and in minimax theorem, two important tools in operations research, management and economics are presented.

Keywords: Hahn-Banach theorem, separation theorems, convex programming, minimax theorem.

1.INTRODUCTION

The Hahn-Banach theorem is presented, with great generality, together with an important separation theorem. Then these results are particularized: first for normed spaces and then for a subclass of these spaces, the Hilbert spaces.

The fruitfulness of these results is emphasized in the last sections where it is shown that they permit to obtain very important results in the applications. First, the Kuhn-Tucker theorem, the convex programming main result, so important in operations research. Then the minimax theorem, an important result in game theory, which consideration in management and economic models is becoming trivial.

Other work on this subject is (10).

2. THE HAHN-BANACH-THEOREM

Definition 2.1

Consider a vector space L and its subspace L_0. Suppose that in L_0 it is defined a linear functional f_0. A linear functional f defined in the whole space L is an extension of the functional f_0 if and only if $f(x) = f_0(x), \underset{x \in L_0}{\forall}$. ■

Theorem 2.1 (Hahn-Banach**)**

Be p a positively homogeneous convex functional, defined in a real vector space L, and L_0 an L subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition

$$f_0(x) \leq p(x), \underset{x \in L_0}{\forall} \quad (2.1),$$

there is an extension f of f_0 defined in L, linear, and such that f(x)$\leq p(x), \underset{x \in L}{\forall}$.

Demonstration:

Begin showing that if $L_0 \neq L$, there is an extension of f_0, f' defined in a subspace L' , such that $L \subset L'$, in order to fulfill the condition (2.1).

Be $z \in L - L_0$; if L' is the subspace generated by L_0 and z, each point of L' is expressed in the form $tz + x$, being $x \in L_0$. If f' is an extension (linear) of the functional f_0 to L', it will happen that $f'(tz + x) = tf'(z) + f_0(x)$ or, making $f'(z) = c$, $f'(tz + x) = tc + f_0(x)$. Now choose c, fulfilling the condition (2.1) in L' , that is: in order that the inequality $f_0(x) + tc \leq p(x + tz)$, for any $x \in L_0$ and any real number t, is accomplished. For $t > 0$ this inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right) \qquad (2.2).$$

For $t < 0$ it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \qquad (2.3).$$

Now it will be proved that there is always c satisfying simultaneously the conditions (2.2) and (2.3).

Given any $y'$and y''belonging to L_0,

$$-f_0(y'') + p(y'' + z) \geq -f_0(y') - p(-y' - z) \qquad (2.4),$$

since $f_0(y'') - f_0(y') \leq p(y'' - y') = p\big((y'' + z) - (y' + z)\big) \leq p(y'' + z) + p(-y' - z)$. Be $c'' = \inf_{y''}\big(-f_0(y'') + p(y'' + z)\big)$ and $c' = \sup_{y'}\big(-f_0(y') - p(-y' - z)\big)$. As $y'$and y'' are arbitrary, it results from (2.4) that $c'' \geq c'$. Choosing c in order that $c'' \geq c \geq c'$, it is defined the functional f' on L' through the formula $f'(tz + x) = tc + f_0(x)$. This functional satisfies the condition (2.1). So any functional f_0 defined in a subspace $L_0 \subset L$ and subject in L_0 to the condition (2.1), may be extended to a subspace L'. The extension f' satisfies the condition$f'(x) \leq p(x)$, $\underset{x \in L'}{\forall}$. If L has an algebraic numerable base $(x_1, x_2, \dots, x_n, \dots)$ the functional in L is built by finite induction, considering the increasing sequence of subspaces$L^{(1)} = (L_0, x_1), L^{(2)} = \big(L^{(1)}, x_2\big), \dots$ designating $\big(L^{(k)}, x_{k+1}\big)$ the L subspace generated by $L^{(k)}$ and x_{k+1}. In the general case, that is, when L has not an algebraic numerable base, it is mandatory to use a transfinite induction process, for instance the Haudsdorf maximal chain theorem.

So call $\mathcal{F}$ the set of the whole pairs(L', f'), at which L' is a L subspace that contains L_0 and f' is an extension of f_0 to L' that fulfills (2.1). Order partially $\mathcal{F}$so that $(L', f') \leq (L'', f'')$ if and only if $L' \subset L''$and $f''_{|L'} = f'$. By the Haudsdorf maximal chain theorem, there is a chain, that is: a subset of $\mathcal{F}$ totally ordered, maximal, that is: not strictly contained in another chain. Call it Ω. Be Φ the family of the whole L' such that $(L', f') \in \Omega$. Φ is totally ordered by the sets inclusion; so, the union T of the whole elements of Φ is a L subspace. If $x \in \mathrm{T}$ then $x \in L'$ for some $L' \in \Phi$; define$\tilde{f}(x) = f'(x)$, where $f'$is the extension of f_0 that is in the pair (L', f')- the definition of $\tilde{f}$ is obviously coherent. It is easy to check that $\mathrm{T} = L$ and that $f = f'$ satisfies the condition (2.1).■

Now it follows the Hahn-Banach theorem complex case, the Hahn contribution to the theorem.

Theorem 2.2 (Hahn-Banach**)**

Be p an homogeneous convex functional defined in a vector space L and f_0 a linear functional, defined in a subspace $L_0 \subset L$, fulfilling the condition$|f_0(x)| \leq p(x), x \in L_0$. Then, there is a linear functional *f* defined in *L*, satisfying the conditions

$|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0$.

Demonstration:

Call L_R and L_{0R} the real vector spaces underlying, respectively, the spaces L and L_0 . As it is evident, p is an homogeneous convex functional in L_R and $f_{0R}(x) = Ref_0(x)$ a real linear functional in L_{0R} fulfilling the condition $|f_{0R}(x)| \leq p(x)$ and so, $f_{0R}(x) \leq p(x)$. Then, owing to Theorem 2.1, there is a real linear functional f_R, defined in the whole L_R space, that satisfies the conditions $f_R(x) \leq p(x), x \in L_R$; $f_R(x) = f_{0R}(x), x \in L_{0R}$. But, $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$, and

$$|f_R(x)| \leq p(x), x \in L_R \qquad (2.5).$$

Define in L the functional f making $f(x) = f_R(x) - if_R(ix)$. It is immediate to conclude that f is a complex linear functional in L such that $f(x) = f_0(x), x \in L_0$; $Ref(x) = f_R(x), x \in L$.

It is only missing to show that $|f(x)| \leq p(x), \forall x \in L$.

Proceed by absurd: suppose that there is $x_0 \in L$ such that $|f(x_0)| > p(x_0)$. So, $f(x_0) = \rho e^{i\varphi}, \rho > 0$, and making $y_0 = e^{-i\varphi}x_0$, it would happen that $f_R(y_0) = Re[e^{-i\varphi}f(x_0)] = \rho > p(x_0) = p(y_0)$ that is contrary to (2.5).■

3. VECTOR SPACE CONVEX PARTS SEPARATION

The next theorem, a very useful consequence of the Hahn-Banach theorem, is about vector space convex parts separation. Beginning with

Definition 3.1

Be *M* and *N* two subsets of a real vector space *L*. A linear functional *f* defined in L separates *M* and *N* if and only if there is a number c such that$f(x) \geq c,$ for $x \in M$ and $f(x) \leq c,$ for $x \in N$ that is, if $\inf_{x \in M} f(x) \geq \sup_{x \in N} f(x)$. A functional f separates strictly the sets M and N if and only if $\inf_{x \in M} f(x) > \sup_{x \in N} f(x)$. ■

Theorem 3.1 (Separation**)**

Suppose that *M* and *N* are two convex subsets of a vector space *L* such that the kernel of at least one of them, for instance *M*, is non-empty and does not intersect the other set; so, there is a linear functional non-null on *L* that separates *M* and *N*.

Demonstration:

Less than on translation, it is supposable that the point 0 belongs to the kernel of *M*, which is designated $\dot{M}$. So, given $y_0 \in N$, $-y_0$ belongs to the kernel of $M - N$ and 0 to the kernel of $M - N + y_0$. As $\dot{M} \cap N = \emptyset$, by hypothesis, 0 does not belong to the kernel of $M - N$ and y_0 does not belong to the one of $M - N + y_0$. Put $K = M - N + y_0$ and be p the Minkovsky functional of $\dot{K}$. So $p(y_0) \geq 1,$ since $y_0 \notin \dot{K}$. Define, now, the linear functional $f_0(\alpha y_0) = \alpha p(y_0)$. Note that f_0 is defined in a space with dimension1, constituted by elements αy_0, and it is such that $f_0(\alpha y_0) \leq p(\alpha y_0)$. In fact, $p(\alpha y_0) = \alpha p(y_0)$, when $\alpha \geq 0$ and $f_0(\alpha y_0) = \alpha f_0(y_0) < 0 < p(\alpha y_0)$, when $\alpha > 0$. Under these conditions, after the Hahn-Banach theorem, it is possible to state the existence of linear functional f , defined in L, that extends f_0, and such that $f(y) \leq p(y), \underset{y \in L}{\forall}$. Then it results $f(y) \leq 1, \underset{y \in K}{\forall}$ and $f(y_0) \geq 1$. In consequence:

-f separates the sets K and $\{y_0\}$, that is

- f separates the sets *M*-*N* and $\{y_0\}$, that is

-f separates the sets *M* and *N*.■

4. THE HAHN-BANACH THEOREM FOR NORMED SPACES

Definition 4.1

Consider a continuous linear functional *f* in a normed space *E*. It is called *f* norm, and designated $\|f\|$: $\|f\| = \sup_{||x|| \leq 1} |f(x)|$ that is: the supreme of the values assumed by $|f(x)|$ in the *E* unitary ball.■

Observation:

-The class of the continuous linear functionals, with the norm above defined, is a normed vector space, called the E dual space, designated E'.

The Theorem 2.1 in normed spaces is:

Theorem 4.1 (Hahn-Banach**)**

Call L a subspace of a real normed space E. And f_0 a bounded linear functional in L. So, there is a linear functional defined in E, extension of f_0, such that $\|f_0\|_{L'} = \|f\|_{E'}$.

Demonstration:

It is enough to think in the functional K satisfying $K\|x\| = \|f_0\|_{L'}$. As it is convex and positively homogeneous, it is possible to put $p(x) = K\|x\|$ and to apply Theorem 2.1.■

Observation:

-To see an interesting geometric interpretation of this theorem, consider the equation $\|f_0(x)\| = 1$. It defines, in L, an hiperplane at distance $\frac{1}{\|f_0\|}$ of 0. Considering the $f_0$extension f, with norm conservation, it is obtained an hiperplane in E, that contains the hiperplane considered behind in L, at the same distance from the origin.

The Theorem 2.2 in normed spaces is:

Theorem 4.2 (Hahn-Banach**)**

Be E a complex normed space and f_0 a bounded linear functional defined in a subspace $L \subset E$. So, there is a bounded linear functional f, defined in E, such that $f(x) = f_0(x), x \in L; \|f\|_{E'} = \|f_0\|_{L'}$. ■

Two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, are then presented:

Theorem 4.3 (Separation**)**

Consider two convex sets A and B in a normed space E. If one of them, for instance A, has at least on interior point and $(intA) \cap B = \emptyset$, there is a continuous linear functional non-null that separates the sets A and B.■

Theorem 4.4 (Separation**)**

Consider a closed convex set A, in a normed space E, and a point $x_0 \in E$, not belonging to A. So, there is a continuous linear functional, non-null, that separates strictly $\{x_0\}$ and A.■

5. SEPARATION THEOREMS IN HILBERT SPACES

In a Hilbert space H,

Theorem 5.1 (Riesz representation**)**

Every continuous linear functional $f(\cdot)$ may be represented in the form $f(x) = [x, \tilde{q}]$ where $\tilde{q} = \frac{\overline{f(q)}}{[q,q]} q$. ■

From now on, only real Hilbert spaces will be considered.

Note that the separation theorems, seen in the former section, are effective in Hilbert spaces. But, due to the Riesz representation theorem, they may be formulated in the following way:

Theorem 5.2 (Separation**)**

Consider two convex sets *A* and *B* in a Hilbert space *H*. If one of them, for instance *A*, has at least one interior point and $(intA) \cap B = \emptyset$, there is a non-null vector *v* such that $\sup_{x\in A}[v, x] \leq \inf_{y\in B}[v, y]$. ■

Theorem 5.3 (Separation**)**

Consider a closed convex set *A*, in a Hilbert space *H*, and a point $x_0 \in H$, not belonging to *A*. So, there is a non-null vector *v*, such that $[v, x_0] < \inf_{x\in A}[v, x]$. ■

Another separation theorem:

Theorem 5.4 (Separation**)**

Two closed convex subsets *A* and *B*, in a Hilbert space, at finite distance, that is: such that: $\inf_{x\in A, y\in B}\|x - y\| = d > 0$ may be strictly separated: $\inf_{x\in A}[v, x] > \sup_{y\in B}[v, y]$. ■

It is also possible to establish that:

Theorem 5.5 (Separation**)**

Being *H* a finite dimension Hilbert space, if *A* and *B* are disjoint and non-empty convex sets they always may be separated. ■

6. CONVEX PROGRAMMING

A class of convex programming problems, at which it is intended to minimize convex functionals subject to convex inequalities, is outlined now. Begin presenting a basic result that characterizes the minimum point of a convex functional subject to convex inequalities. Note that it is not necessary to impose any continuity conditions.

Theorem 6.1 (Kuhn-Tucker**)**

Be *f(x)*, $f_i(x), i = 1, \ldots, n$, convex functionals defined in a convex subset *C* of a Hilbert space. Consider the problem $\min_{x \in C} f(x)$, $sub.: f_i(x) \leq 0, i = 1, \ldots$ Be x_0 a point where the minimum, supposed finite, is reached. Suppose also that for each vector *u* in E_n, Euclidean space with dimension *n*, non-null and such that $u_k \geq 0$, there is a point *x* in *C* such that $\sum_1 u_k f_k(x) < 0$,designating u_k the components of *u*. So,

i) There is a vector *v*, with non-negative components $\{v_k\}$, such that

$$\min_{x \in C} \left\{ f(x) + \sum_1^n v_k f_k(x) \right\} = f(x_0) + \sum_1^n v_k f_k(x_0) = f(x_0) \qquad (6.1),$$

ii) For every vector u in E_n with non-negative components, that is: belonging to the positive cone of E_n ,

$$f(x) + \sum_1^n v_k f_k(x)$$

$$\geq f(x_0) + \sum_1^n v_k f_k(x_0) \geq f(x_0) + \sum_1^n u_k f_k(x_0) \quad (6.2). \blacksquare$$

Corollary 6.1 (Lagrange duality**)**

In the conditions of Theorem 6.1 $f(x_0) = \sup_{u \geq 0} \inf_{x \in C} f(x) + \sum_1^n u_k f_k(x)$. ■

Observation:

-This corollary is useful supplying a process to determine the problem optimal solution,

-If the whole v_k in expression (6.2) are positive, x_0 is a point that belongs to the border of the convex set defined by the inequalities,

-If the whole $v_k$are zero, the inequalities do not influence the problem, that is: the minimum is equal to the one of the restrictions free problem.

Considering non-finite inequalities:

Theorem 6.2 (Kuhn-Tucker in infinite dimension**)**

Be *C* a convex subset of a Hilbert space *H* and *f* (*x*) a real convex functional defined in *C*. Be *I* a Hilbert space with a closed convex cone $\wp$, with non-empty interior, and *F*(*x*) a convex transformation from *H* to *I* (convex in relation to the order introduced by cone $\wp$: if $x, y \in \wp, x \geq y \ if \ x - y \in \wp$). Be x_0 a *f* (*x*) minimizing in C subjected to the inequality $F(x) \leq 0$.Consider $\wp^* = \left\{x: [x,p] \geq 0, \begin{smallmatrix}\forall \\ x \in \wp\end{smallmatrix}\right\}$ (dual cone). Admit that given any $u \in \wp^*$ it is possible to determine x in C such that $[u, F(x)] < 0$. So, there is an element v in the dual cone $\wp^*$, such that for x in C $f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)]$, being u any element of $\wp^*$. ■

Corollary 6.2 (Lagrange duality in infinite dimension**)**

$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C}(f(x) + [v, F(x)])$ in the conditions of Theorem 6.2.■

7. MINIMAX THEOREM

In a two players game with null sum be $\Phi(x,y)$ a real function of two variables $x, y \in H$ and A and B convex sets in H. One of the players chooses strategies (points) in A in order to maximize $\Phi(x,y)$ (or minimize $-\Phi(x,y)$): it is the maximizing player. The other player chooses strategies (points) in B in order to minimize $\Phi(x,y)$ (or maximize $-\Phi(x,y)$): it is the minimizing player. The function $\Phi(x,y)$ is the payoff function. $\Phi(x_0, y_0)$ represents, simultaneously, the gain of the maximizing player and the loss of the minimizing player in a move at which they chose, respectively the strategies x_0 and y_0. So, the gain of one of the players is equal to the other's loss. That is why the game is a null sum game. A game in these conditions value is *c* if

$$\sup_{x \in A} \inf_{y \in B} \Phi(x,y) = c = \inf_{y \in B} \sup_{x \in A} \Phi(x,y) \qquad (7.1).$$

If, for any (x_0, y_0), $\Phi(x_0, y_0) = c$, (x_0, y_0) is a pair of optimal strategies. There will be a saddle point if also

$$\Phi(x, y_0) \leq \Phi(x_0, y_0) \leq \Phi(x_0, y), x \in A, y \in B \qquad (7.2).$$

Theorem 7.1

Consider A and B closed convex sets in *H*, being A bounded. Be $\Phi(x,y)$a real functional defined for *x* in *A* and *y* in *B* fulfilling:

-$\Phi(x, (1-\theta)y_1 + \theta y_2) \leq (1-\theta)\Phi(x, y_1) + \theta\Phi(x, y_2)$ for *x* in *A* and y_1, y_2 in *B*, $0 \leq \theta \leq 1$ (that is: $\Phi(x,y)$ is convex in *y* for each *x*),

- $\Phi\big((1-\theta)x_1 + \theta x_2, y\big) \geq (1-\theta)\Phi\big(x_{1,}, y\big) + \theta\Phi(x_2, y)$ for y in B and x_1, x_2 in *A*, $0 \leq \theta \leq 1$ (that is: $\Phi(x,y)$ is concave in *x* for each *y*),

- $\Phi(x,y)$ is continuous in x for each y,

so (7.1) holds, that is: the game has a value. ■

The next corollary follows from the Theorem 7.1 hypothesis strengthen:

Corollary 7.1(Minimax**)**

Suppose that the functional $\Phi(x, y)$ of Theorem 7.1 is continuous in both variables, separately, and that B is also bounded. Then, there is an optimal pair of strategies, with the property of being a saddle point.■

8. CONCLUSIONS

The Hahn-Banach theorem was presented with great generality, real and complex version, followed by an important separation theorem, consequence of it.

These results were specified for normed spaces and then for a subclass of these spaces: the Hilbert spaces. Better saying, they were reformulated for Hilbert spaces using the Riesz representation theorem.

Examples of the fruitfulness of the results presented are patent in the last two sections, where it is shown that they permit to obtain important results, for the applications, as the Kuhn-Tucker and the Minimax theorems. Now the structures considered were the real Hilbert spaces. The problems studied were convex optimization problems in which, it is well known, the separation theorems are a key tool.

The Kuhn-Tucker theorem is the convex programming main result so important in operations research. The Minimax theorem is an important result in game theory, which consideration in management and economic problems resolution is greater and greater.

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