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Departamento de Matemática

Exercises on Ordinary Differential Equations

1 Exercises

1.1 ODE with Separable Variables

We say that an ordinary differential equation (ODE) has separable variables when it is of the form:

$$g(y)y' = f(t) \tag{1}$$

In fact, if f and g are continuous, they admit primitives F and G and, by the composition function Theorem, the problem can be rewritten as $\frac{d}{dt}\left(G(y(t)\right) = \frac{dF}{dt}$. Integrating both sides, we obtain the following solution

$$G(y(t)) = F(t) + K, \text{ for all } K \in \mathbb{R}. \tag{2}$$

1. Consider the equation $y' = y^2$.

   (a) Suppose that $y^2 \neq 0$. Show that this equation is an ODE with separable variables, identifying $g(y)$ and $f(t)$.

   (b) Solve the equation $y^{-2}dy = dt$ that is obtained from the given one by using the differential form $\frac{dy}{dt}$.

2. Solve the following problems.

   (a) $yy' = t$, $y(3) = -2$.

   (b) $y^2y' = t^2$.

   (c) $y' = 2e^{-y}t$.

   (d) $2yy' = -\sin t$, $y(0) = \sqrt{2}$.

   (e) $y' = 1 + y^2$, $y(0) = 0$.

   (f) $xdx = (3 + \dfrac{1}{y})dy$.

   (g) $(1 + e^x)yy' = e^x$, $y(0) = 1$

   (h) $(x + 1)ydx - 2x(y + 5)dy = 0$.

   (i) $(x^2 - yx^2)dy + (y^2 + xy^2)dx = 0$, $y(1) = 1$

3. The evolution of a population p can be determined by a differential equation. In the Verhulst (1836) model, the growth rate of a population $\kappa = \kappa(p)$ linearly depends on the population $\kappa = \kappa(p) = a - bp$ where a and b are positive constants. This leads to the equation with separable variables $p' = (a - bp)p$. Assume that $a = 2$ and $b = 1$, that is, the evolution of the population is given by

$$p' = (2-p)p. \tag{3}$$

(a) First, check that $p \equiv 0$ and $p \equiv 2$ are solutions. Give an interpretation.

(b) Now show that $\int \frac{1}{(2-p)p} dp = \frac{1}{2} \ln \left| \frac{p}{2-p} \right| + C$

(c) Using the previous point, conclude that the general solution of (3) satisfies $|p| = |2-p| e^{2(t+C)}$.

(d) Generalize the previous result to a and b arbitrary positive constants.

1.2 Linear ODE of 1st Order

Linear ODE are equations of the form

$$y' + a(t)y = b(t) \ . \tag{4}$$

with$a(t)$ and $b(t)$ continuous functions in some interval. The linear equation can be explicitly given. For that, notice that $(ye^{Pa})' = (y' + ay)e^{Pa}$, where a, b, and P denote the functions $a(t)$ and $b(t)$ and the primitive. Then, multiplying both sides of the equation (4) by e^{Pa}, we obtain

$$(ye^{Pa})' = be^{Pa} \Rightarrow ye^{Pa} = P\left(be^{Pa}\right),$$

and conclude that the general solution of (4) is given by

$$y = Ce^{-Pa} + e^{-Pa} P\left(be^{Pa}\right). \tag{5}$$

4. Let us solve the following initial value problem (IVP)

$$\begin{cases} y' = t(-2y+1) \\ y(0) = 3 \ . \end{cases} \tag{6}$$

(a) Write the problem in the form of (4). Identify a and b and compute Pa. Conclude that $P\left(be^{Pa}\right) = \frac{e^{t^2}}{2} + k$

(b) Deduce that the IVP solution is given by $y(t) = \frac{1}{2}\left(5e^{-t^2} + 1\right)$

(c) Check that in fact the previous solution solve the IVP.

5. Consider a tank with 500 liters capacity. Iniatilly containing 100 liters of water. At the instant $t = 0$, the tank starts to be filled up with water by opening a valve with flow rate of 4 liters per second and 50% of pollutants. Simultaneously, with the help of a bomb, 3 liters of the obtained mix of water and pollutants are taken in each second. We assume the mix is always homogeneous. The goal of this problem consists on finding the concentration of pollutants in the tank when it is completely filled up.

   Let $q(t) =$ quantity of pollutants at the instant t. The input pollutants flow rate is $v_e = 2$ liters per second. The output flow rate is $v_s = 3\frac{q(t)}{V(t)}$, where $V(t) = 100+t$ is the mix volume in the tank at the instant t. Then we have $q' = v_e - v_s = 2 - \frac{3q}{100+t}$. Therefore, since the tank reaches its maximum capacity at $t = 400$ (seconds), this situation has model given by the following IVP

$$\begin{cases} q' + \frac{3}{100+t}\, q = 2 \ , \ t \in]0, 400[\\ q(0) = 0 \ , \end{cases}$$

   (a) Find the solution of the IVP.

   (b) What is the pollutant concentration when that tank is completely filled up?

6. Solve the following IVP's

   (a) $y' + \sin(t)y = 0$, $y(0) = 3/2$.

   (b) $y' = te^{-t} - y$, $y(1) = 2$.

   (c) $y + 4\int_0^t y = 5t + 3$.

   (d) $\sqrt{1+x^2}\ y' + y = 2x$, $y(0) = 2$.

   (e) $\dfrac{1}{2}y' = y\tan(2x) + 1 + \sec(2x)$, $y(\pi/2) = 1$.

7. Solve the following linear ODE:

   (a) $(x+1)\,y' - y = 3x^4 + 3x^3$

   (b) $y' - \dfrac{y}{x} = x$.

   (c) $y^2 dx - (2xy + 3)\, dy = 0$.

   (d) $y' + y\cos(x) = \dfrac{1}{2}\sin(2x)$.

   (e) $y' = y\tan(x) + \cos(x)$.

8. Let $p = p(t)$ (euros/year), $t \in]0, T]$ (years), the payment rate in the consider time. Show that the deposit value to be done in a continuous investment account with an annual tax rate r, to cover all the expenses, is given by

$$\int_0^T p(t)e^{-rt}dt.$$

1.3 Bernoulli ODE

A *Bernoulli* ODE of 1st order is an equation of the form

$$y' + A(x)y = B(x)y^n.$$

Dividing by y^n, supposing that $y \neq 0$, the ODE is equivalent to $\frac{1}{y^n}y' + A(x)\frac{1}{y^{n-1}} = B(x)$. Making the change of variable $z = y^{1-n}$, a Bernoulli ODE can be written as a linear ODE

$$\frac{1}{1-n}z' + A(x)z = B(x).$$

9. Consider the following ODE of 1st order

$$y' + xy = x^3y^3.$$

   (a) Divide the ODE by y^3.
   (b) Make the change of variable $y^{-2} = z$ and classify the obtained ODE .
   (c) Solve the obtained ODE.

10. Solve the following Bernoulli ODE:

   (a) $3y^2y' - ay^3 = x + 1$.
   (b) $yx' + x = x^2 \ln(y)$.
   (c) $xy' = y + 2xy^2$.
   (d) $y' \cos(x) + y \sin(x) + y^3 = 0$.

11. Solve the following IVP:

   (a) $y - \cos(x)y' = (1 - \sin(x))y^2 \cos(x)$, $y(0) = 1$.
   (b) $y'\left(x^2y^3 + xy\right) = 1$,$y(1) = 0$.

2 Solutions

2.1 ODE with Separable Variables

1. (a) $y^{-2}y' = 1$, $g(y) = y^{-2}$ e $f(t) = 1$.

   (b) $-y^{-1} = t + k$, for all $k \in \mathbb{R}$.

2. (a) $y^2 = t^2 - 5$

   (b) $y^3 = t^3 + k$.

   (c) $e^y = -t^2 + k$, for all $k \in \mathbb{R}$.

   (d) $y^2 = \cos(t) + 1$.

   (e) $\arctan(y) = t$.

   (f) $x^2 - 3y - \ln|y| = k$, for all $k \in \mathbb{R}$.

   (g) $y^2/2 = \ln(1 + e^x) + 1/2 - \ln 2$.

   (h) $x + \ln|x| - 2y - 10\ln|y| = k$, for all $k \in \mathbb{R}$.

   (i) $ln|x/y| - (1/x + 1/y) = -2$, for all $k \in \mathbb{R}$.

3. (a)

   (b)

   (c)

   (d)

2.2 Linear ODE of 1st Order

4.

5. (a) $q(t) = 1/2\Big(100 + t - \frac{10^8}{(100+t)^3}\Big)$

   (b) $q(400)/400 = 0,624$, $q(400) = 2\frac{5^4-1}{5} = 249.6(liters)$.

6. (a) $y = \frac{3}{2}e^{-\cos(t)}$.

   (b) $y = \frac{t^2}{2}e^{-t} + ke^{-t}$, for $k \in \mathbb{R}$.

   (c) $y = 1/4(5 + 7e^{-4t})$.

   (d) $(x + \sqrt{1+x^2})y = x\sqrt{1+x^2} - \ln(x + \sqrt{1+x^2}) + x^2 + 2$.

   (e) $y = \tan(2x) + 2x\sec(2x) + (-1-\pi)\sec(2x)$.

7. (a) $y = 3(x+1)(x^3/3 - x^2/2 + x - \ln(x+1)) + C(x+1),\ k \in \mathbb{R}$.
   (b) $y = x^2 + kx,\ k \in \mathbb{R}$.
   (c) $x = -1/y + ky^2,\ C \in \mathbb{R}$.
   (d) $y = \sin(x) - 1 + ke^{-\sin(x)},\ C \in \mathbb{R}$.
   (e) $y = \sec(x)(x/2 + \sin(2x)/4) + k,\ k \in \mathbb{R}$.

8.

2.3 Bernoulli ODE

9. (a) $(1/y^3)y' + x/y^2 = x^3$.
   (b) Linear ODE, $z' - 2xz = -2x^3$.
   (c) $1/y^2 = x^2 + 1 + ke^{x^2},\ k \in \mathbb{R}$.

10. (a) $y^3 = -(x+1)/a - 1/a^2 + ke^{ax},\ k \in \mathbb{R}$.
    (b) $1/x = \ln y + 1 + ky,\ k \in \mathbb{R}$.
    (c) $1/y = -x + k/x,\ k \in \mathbb{R}$.
    (d) $y = 1/\sqrt{2\tan(x)\sec(x) + k\sec^2(x)},\ k \in \mathbb{R}$.

11. (a) $y = (\sec(x) + \tan(x)) / (\sin(x) + 1)$
    (b) $1/x = -y^2 + 2 - e^{-y^2/2}$