Instituto Universitário de Lisboa

Departamento de Matemática

Exercises on Sequences and Series

1 Exercises: Sequences

1. Study the monotonicity of the following sequences.
   (a) $\frac{1}{n}$
   (b) $\sqrt{n+1}+\sqrt{n}$
   (c) $\frac{n+1}{n+2}$
   (d) $(-1)^n$
   (e) $\sqrt{n+1}-\sqrt{n}$
   (f) $\frac{(-1)^n}{n}$
   (g) $\frac{n^2+n}{n+4}$
   (h) $\frac{1}{2^n}$
   (i) $(-1)^n(1-\frac{n}{\sqrt{n}})$
   (j) $(-1)^n-(-1)^{n+1}$
   (k) $1+\frac{1}{n}+\frac{1}{n^2}$

2. Which sequences from exercise 1. are bounded? Justify.

3. Check the existence of limit for the following sequences. In the case the limit exists, compute its value.
   (a) $\frac{1}{n}$
   (b) $\frac{n+1}{n+2}$
   (c) $(-1)^n$
   (d) $\sqrt{n+1}-\sqrt{n}$
   (e) $\frac{(-1)^n}{n+1}$
   (f) $\frac{1}{2^n}$
   (g) $(2+\frac{1}{n})^n$
   (h) $\frac{2^{n+1}+3^n}{2^n+3^{n+1}}$
   (i) $\sin(n\frac{\pi}{2})$
   (j) $\cos(n\pi)+(-1)^{n+1}$
   (k) $\frac{n!}{(n-2)!(n^2+1)}$
   (l) $(\frac{n}{1+n})^{\frac{1}{n}}$
   (m) $\sqrt[n]{n}$
   (n) $\sqrt{n+\sqrt{n}}-\sqrt{n}$
   (o) $\frac{3n^{7/2}+2n^2}{n+4\sqrt{n+n^7}}$
   (p) $\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{n(n+1)}$
   (*Suggestion: note that* $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$)
   (q) $a_1=1, a_{n+1}=\frac{a_n}{2}+1, \forall n$
   (*Suggestion: is* $(a_n)_n$ *monotonic and bounded?*)
   (r) $a_1=1, a_{n+1}=1+\sqrt{a_n}, \forall n$

4. Let $a_n=1+2+\ldots+n=\sum_{k=1}^n k$. Following the next steps, prove by induction that $a_n=\frac{(n+1)n}{2}$:
   (a) First, check that $a_1=\frac{(1+1)1}{2}$.
   (b) Now, supposing that $a_n=\frac{(n+1)n}{2}$, deduce that $a_{n+1}=\frac{(n+2)(n+1)}{2}$, using the equality $a_{n+1}=a_n+(n+1)$.

5. Let $a_n = 1 + r + r^2 + r^n = \sum_{k=0}^{n} r^k$. Following the next steps, prove by induction that $a_n = \dfrac{1 - r^n}{1 - r}$:

   (a) First, check that $a_1 = \frac{1-r^1}{1-r}$.

   (b) Now, supposing that $a_n = \dfrac{1 - r^n}{1 - r}$, deduce that $a_{n+1} = \dfrac{1 - r^{n+1}}{1 - r}$, using the equality $a_{n+1} = a_n + r^{n+1}$.

6. If $a_n > 0$ for all n and $\lim \frac{a_{n+1}}{a_n} = L$, show that:

   (a) If $L > 1$ then $\lim a_n = \infty$.

   (b) If $L < 1$ then $\lim a_n = 0$.

7. Using, if necessary, the Squeeze Theorem for sequences, prove that:

   (a) $\dfrac{a^n}{n!} \to 0$

   (b) $\dfrac{n^n}{n!} \to +\infty$

   (c) $\dfrac{n^\alpha}{a^n} \to 0$, for all α and $a > 0$;

   (d) $\dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{n+1}} + \cdots + \dfrac{1}{\sqrt{n+n}} \to +\infty$

   (e) $\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2} + \cdots + \dfrac{1}{(2n)^2} \to 0$

2 Exercises: Geometric and Mengoli Series

8. **Zeno's Paradox**

   (a) A person to go from the point A to the point B would need to pass by the middle point A_1 between the two points. At the point A_1, the person would need to pass by the middle point A_2 between A_1 and B. And so on. Zeno argued that in this way the person would never arrive to B.

   Using the geometric series, prove that Zeno was wrong.

   (b) If in the Zeno's argument the distance is consecutively divided in 3 parts instead of 2, do you think the conclusion of the previous point remains? First, define u_n as the walked distance after n steps and show that $u_n = u_{n-1} + \dfrac{1}{3}(1 - u_{n-1}) = \dfrac{1}{3} + \dfrac{2}{3}u_{n-1}$. Then deduce that $u_n = \sum_{k=1}^{n} \dfrac{2^{k-1}}{3^k}$ by sequential substitution (or induction). Conclude.

(c) And if $0 < r < 1$?

9. **St Petersbourg Paradox**

   Consider the following game in which two players participate, the "House" and the "Client". The Client invests K $m.u.$ and the House pays a prize g_k according to the following rule: Sequentially, a coin is flipped in the air until the output is "heads"; when the first "head"comes out in the kth flip, the prize received by the Client is $g_k = 2^k$.

   (a) What is the probability p_k to receive the prize in the kth flip, that is what is the probability the first "head"to come out in the kth flip?

   (b) Compute the value of the game (that is, the *expected value*) $E = \sum_{k=1}^{\infty} p_k g_k$.

   (c) What can you conclude from (b)?

10. The interest rate is $5\%/year$, which of the following options do you prefer?

    (a) Receive 100000 € immediately.

    (b) Receive 5000 € at the beginning of every year forever ever.

    (c) Receive $0,001 * (1,06)^t$ € at the beginning of every year forever.

11. One lets a ball to fall down from a height h; every time the ball reaches the floor it jumps until $2/3$ of the previous height. What is the total distance (up and down) covered by the ball?

12. In case of convergence, compute the sum of the following series:

    (a) $\sum_{n\geq 3}(\frac{1}{5})^n$

    (b) $\sum_{n\geq 1} 2^n$

    (c) $\sum_{n\geq 0} \frac{7}{2^{n+2}}$

    (d) $\sum_{n\geq 1}(\frac{2^{n+1}}{3^n} - \frac{5}{2^n})$

    (e) $\sum_{n\geq 1}(\frac{-3}{2^n} + \frac{2}{(-3)^{n+1}} - \frac{1}{4^{n+2}})$

13. Consider the following geometric series as functions of the parameter x. For each one, determine the ratio $r = r(x)$, the convergence range and the sum.

    (a) $\sum_{n\geq 1} \frac{x^n}{3^{n+1}}$

    (b) $\sum_{n\geq 0} \frac{(2x)^n}{4^{n-2}}$

    (c) $\sum_{n\geq 0} \frac{(x-1)^{n+1}}{2^{n+1}}$

    (d) $\sum_{n\geq 1} \frac{x^{n+1}}{2^n} - \frac{2^n}{3^{n+1}}$

    (e) $\sum_{n\geq 0} \frac{(2x)^n}{3^{n+1}} - \frac{7x^{n+1}}{4^n}$

    (f) $\sum_{n\geq 2} \frac{2^n}{x^{n+1}}$

    (g) $\sum_{n\geq 1}(\frac{x}{1-x})^n$

14. (a) Assuming the following equalities hold, for $0 < x < 1$:

i. $\sum_{k=0}^{\infty} x^{k+1} = \dfrac{1}{1-x}$

ii. $\dfrac{\partial\left(\sum_{k=0}^{\infty} x^{k+1}\right)}{\partial x} = \sum_{k=0}^{\infty} \dfrac{\partial(x^{k+1})}{\partial x}$,

check that $\sum_{k=0}^{\infty}(k+1)x^k = \dfrac{1}{(1-x)^2}$.

(b) Compute $\sum_{p=1}^{+\infty}\sum_{k=p-1}^{\infty} x^k$.

(c) What can you conclude from the previous equalities?

15. (a) Show that $0,99999\cdots = 0,(9) = 1$.

(b) Compute the rational number corresponding to the decimal number $3,666\cdots$.

(c) Compute the rational number corresponding to the decimal number $1,181818\cdots$.

16. Consider the autoregressive model for which the values of the variable y at the instant t are determined by the value of y at the instant $t-1$ together with the value of other variable x at the instant t;that is

$$y_t = x_t + \alpha y_{t-1}$$

.

(a) Prove that $y_t = \sum_{k=0}^{\infty} \alpha^k x_{t-k}$.

(b) What id the effect of the variable x at the instant j, x_j, within the value of y at the instant t, y_t, *i.e.* $\dfrac{\partial y_t}{\partial x_j}$?

(c) Determine the long term cumulative effect $\sum_{k=0}^{\infty} \dfrac{\partial y_t}{\partial x_j}$? What does it measure?

(d) Find an economic example modeled in this way.

17. Determine for which values of $a \in \mathbb{R}$ the following series converge and compute their sum.

(a) $\sum_{n\geq 0}(\frac{a}{a+1})^n$

(b) $\sum_{n\geq 0}(1-|a|)^n$

(c) $\sum_{n\geq 0} a$

(d) $\sum_{n\geq 0}(\frac{1}{|a|-1/2})^n$

18. Consider the following series

$$\sum_{n\geq 1} a_n - a_{n+1}$$

(a) Show that $S_n = a_1 - a_{n+1}$

(b) Conclude that $\sum_{n\geq 1} a_n - a_{n+1} = a_1 - \lim a_{n+1}$.

19. Consider the following series
$$\sum_{n\geq 1} \frac{1}{n(n+1)}$$

(a) Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

(b) Show that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$

(c) Conclude that $\sum_{n\geq 1} \frac{1}{n(n+1)} = 1$.

20. Generalize the previous result and, for an integer $k \geq 1$, compute the sum of the series.
$$\sum_{n\geq 1} \frac{1}{n(n+k)}$$

3 Series with non negative terms

21. Dirichelet's Series

(a) Using the equality

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^\alpha} &= 1 + \Big(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\Big) + \Big(\frac{1}{4^\alpha} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha}\Big) + \Big(\frac{1}{8^\alpha} + \dots \\ &\leq 1 + \Big(\frac{1}{2^\alpha} + \frac{1}{2^\alpha}\Big) + \Big(\frac{1}{4^\alpha} + \frac{1}{4^\alpha} + \frac{1}{4^\alpha} + \frac{1}{4^\alpha}\Big) + \Big(\frac{1}{8^\alpha} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{2^{(\alpha-1)}}\right)^n\end{aligned}$$

prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges to $\alpha > 1$.

(b) With the same technique, but using the inequality

$$\begin{aligned}&1 + \frac{1}{2^\alpha} + \Big(\frac{1}{3^\alpha} + \frac{1}{4^\alpha}\Big) + \Big(\frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha} + \frac{1}{8^\alpha}\Big) + \dots \geq \\ &1 + \frac{1}{2^\alpha} + \Big(\frac{1}{4^\alpha} + \frac{1}{4^\alpha}\Big) + \Big(\frac{1}{8^\alpha} + \frac{1}{8^\alpha} + \frac{1}{8^\alpha} + \frac{1}{8^\alpha}\Big) + \dots\end{aligned}$$

prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ diverges when $\alpha \leq 1$.

22. Study if the following series with non negative terms are convergent or divergent:

(a) $\sum \frac{1}{\sqrt{n}+2}$

(b) $\sum \frac{1}{n^2+n}$

(c) $\sum \frac{1}{2^n+n}$

(d) $\sum \frac{n}{2^n+1}$

(e) $\sum \frac{1}{\sqrt{n+1}}$

(f) $\sum n \sin \frac{1}{n}$

(*Obs:* $\frac{\sin(a_n)}{a_n} \to 1$ *if* $a_n \to 0$.)

(g) $\sum \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

(h) $\sum \frac{1}{(3n-2)(2n+1)}$

23. Using the Ratio Test, determine if the following numerical series are convergent or divergent:

(a) $\sum \frac{2}{n!}$

(b) $\sum \frac{10^n}{n!}$

(c) $\sum \frac{(n!)^2}{(2n)!}$

(d) $\sum \frac{n!}{n^n}$

(e) $\sum \frac{3^n n!}{n^n}$

24. Using the Root Test, determine if the following numerical series are convergent or divergent:

(a) $\sum \frac{1}{n^n}$

(b) $\sum \frac{n^2}{3^n}$

(c) $\sum \left(1-\frac{1}{n}\right)^{n^2}$

(d) $\sum \frac{1}{n^{\frac{n}{2}}}$

(e) $\sum \frac{1}{(\log n)^{\frac{n}{2}}}$

4 Absolute Convergence

25. Check if the following series converge and, in the positive cases, say if the convergence is simple or absolute.

(a) $\sum \frac{(-1)^n}{n + \log n}$

(b) $\sum \frac{(-1)^n}{n^2}$

(c) $\sum \frac{\sin n}{n^2 + 1}$

(d) $\sum \frac{(-1)^n}{\sqrt{n}}$

(e) $\sum (-1)^n \frac{\log n}{n}$

(f) $\sum (-1)^n \frac{n}{\sqrt{n} + n^2}$

(g) $\sum \frac{\sin n}{n^3 + 1}$

(h) $\sum \sin(\frac{\pi}{2} n)$

(i) $\sum \frac{(-1)^n}{n}$

(j) $\sum \frac{(-1)^n}{n + \log n}$

(k) $\sum \frac{(-1)^n n}{n + 1}$

(l) $\sum (-1)^n \frac{\log n}{n}$

(m) $\sum (-1)^n (\sqrt{n^2 + 1} - \sqrt{n})$

26. Prove that if $\sum a_n$ is a series with strictly positive terms and $(b_n)_n$ is a bounded sequence, then $\sum a_n b_n$ is absolutly convergent.

27. Show that $\sum \frac{(-1)^n}{n^\alpha}$ is convergent, for any $\alpha > 0$.

5 Power Series

28. Study if the following series are convergent or divergent:

    (a) $\sum 2^{-n} x^n$

    (b) $\sum n!\, x^n$

    (c) $\sum x^n$

    (d) $\sum_{n \geq 2} \frac{3^n x^{n-1}}{2^{n+1}}$

29. Using the Taylor's series, conclude that

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \ , \quad \forall x \in \mathbb{R} \ ,$$

and

$$\log(1 + x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n \ , \quad \forall x \in] - 1, 1] \ .$$

30. Using the Taylor's series of the logarithm, compute the sum of the alternating harmonic series.

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \ .$$

31. Determine the Taylor's series of sine and cosine and its respective convergence radius?

32. Knowing that

$$f(x) = \frac{1}{1-x} = \sum_{n\geq 0} x^n \quad , \quad x \in]-1,1[\ .$$

(a) Prove that this equality holds.

(b) Determine the Taylor's series of $f'(x) = (1-x)^{-2}$ and its respective convergence radius.

6 Solutions

1. (a) Decreasing
   (b) Increasing
   (c) Increasing
   (d) Non monotone
   (e) Decreasing
   (f) Non monotone
   (g) Increasing
   (h) Decreasing
   (i) Non monotone
   (j) Non monotone
   (k) Decreasing

2. (a) Bounded
   (b) Unbounded
   (c) Bounded
   (d) Bounded
   (e) Bounded
   (f) Bounded
   (g) Unbounded
   (h) Bounded
   (i) Unbounded
   (j) Bounded
   (k) Bounded

3. (a) 0
   (b) 1
   (c) Divergent
   (d) 0
   (e) 0
   (f) 0
   (g) Divergent
   (h) 1
   (i) 1
   (j) $\frac{1}{2}$
   (k) $\frac{1}{3}$
   (l) Divergent
   (m) 0
   (n) 1
   (o) $\frac{3}{4}$
   (p) 1
   (q) 2
   (r) $\frac{3+\sqrt{5}}{2}$

4.

5.

6.

7.

8. (a)
   (b)
   (c) For any r, the total distance is 1.

9. (a) $p_k = 2^{-}-k$.
   (b) $E = \infty$.
   (c) To play the game the "Client"would be available to invest all is wealth M, since $E > M$.

10. (c)

11. 5

12. (a) $\frac{1}{100}$

(b) $+\infty$

(c) $\frac{7}{2}$

(d) -1

(e) $\frac{-39}{12}$

13. (a) $r(x) = \frac{x}{3}$; $]-3,3[$; $\frac{x}{9-3x}$.

(b) $r(x) = \frac{x}{2}$; $]-2,2[$; $\frac{32}{2-x}$.

(c) $r(x) = \frac{x-1}{2}$; $]-1,3[$; $\frac{x-1}{3-x}$.

(d) $r(x) = \frac{x}{2}$; $]-2,2[$; $\frac{x^2}{2-x} - \frac{2}{3}$.

(e) $r_1(x) = \frac{2x}{3}, r_2(x) = \frac{x}{4}$; $]-\frac{3}{2},\frac{3}{2}[$; $\frac{1}{3-2x} - \frac{28x}{4-x}$.

(f) $r(x) = \frac{2}{x}$; $]-\infty,-2[\cup]2,+\infty[$; $\frac{4}{x^2(x-2)}$.

(g) $r(x) = \frac{x}{1-x}$; $]-\infty,\frac{1}{2}[$; $\frac{x}{1-2x}$.

14.

15. (a)

(b) $\frac{11}{3}$

(c) $\frac{13}{11}$

16.

17. (a) $a > \frac{-1}{2}$

(b) $-2 < a < 2 \wedge a \neq 0$

(c) $a = 0$

(d) $a < \frac{-3}{2} \vee a > \frac{3}{2}$

18.

19.

20.

21.

22. (a) Divergent

(b) Convergent

(c) Convergent

(d) Convergent

(e) Divergent

(f) Divergent

(g) Divergent

(h) Convergent

23. (a) Convergent

(b) Convergent

(c) Convergent

(d) Convergent

(e) Divergente

24. (a) Convergent

(b) Convergent

(c) Convergent

(d) Convergent

(e) Convergent

25. (a) Simply convergent

(b) Absolutely convergent

(c) Absolutely convergent

(d) Simply convergent

(e) Simply convergent

(f) Simply convergent

(g) Absolutely convergent

(h) Divergent

(i) Simply convergent

(j) Simply convergent

(k) Divergent

(l) Simply convergent

(m) Divergent

26.

27.

28. (a) Absolutely convergent at $x \in [-2, 2]$, divergent otherwise.

(b) Absolutely convergent at $x = 0$, divergent otherwise.

(c) Absolutely convergent at $x \in]-1, 1[$, divergent otherwise.

(d) Absolutely convergent at $x \in]-2/3, 2/3[$, divergent otherwise.

29.

30. $-\log 2$

31. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, $r = \infty$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $r = \infty$.

32. (a)

(b) $\sum_{n \geq 0} n\, x^{n-1}$, $r = 1$.