Instituto Universitário de Lisboa

Departamento de Matemática

Optimization Problems

1 Unconstrained Optima

1. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{a} \in \mathbb{R}^n$,

   (a) What is the condition that defines $f(\boldsymbol{a})$ as an absolute [relative] maximum of f?

   (b) What is the condition that defines $f(\boldsymbol{a})$ as an absolute [relative] minimum of f?

2. Given a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, for a point $\boldsymbol{a} \in \mathbb{R}^2$ find the direction $\boldsymbol{u} \in \mathbb{R}^2$ which gives the maximum value for the directional derivative $f'_{\boldsymbol{u}}(\boldsymbol{a})$ at $\boldsymbol{a}$.

3. This exercise pretends to characterize a necessary property on relative extrema of a function $f : \mathbb{R}^n \to \mathbb{R}$.

   (a) From Linear Algebra, recall that the **inner product** (or **scalar product**, or even **dot product**) of two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ is given by

   $$\boldsymbol{u}|\boldsymbol{v} = (u_1, \dots, u_n)|(v_1, \dots, v_n) = u_1 v_1 + \dots + u_n v_n$$

   We say that two vectors $\boldsymbol{w}, \boldsymbol{h} \in \mathbb{R}^n$ are **orthogonal** (or **perpendicular**) when $\boldsymbol{w}|\boldsymbol{h} = 0$. Find the vectors of $\mathbb{R}^2$ which are orthogonal to the vector $(1, 1)$.

   (b) Verify that if a vector $\boldsymbol{u} \in \mathbb{R}^2$ is such that $\boldsymbol{u}|\boldsymbol{v} = 0$, for all $\boldsymbol{v} \in \mathbb{R}^2$, then $\boldsymbol{u} = \mathbf{0}$.

   (c) Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable function with a a relative maximum of h. Use the inequalities obtained in question 1 to deduce

   - $\frac{h(a+\epsilon)-h(a)}{\epsilon} \leq 0$ for $\epsilon > 0$
   - $\frac{h(a+\epsilon)-h(a)}{\epsilon} \geq 0$ for $\epsilon < 0$

   Then conclude that $h'(a) = 0$. In a similar way, deduce the condition for a to be a relative minimum of h.

   (d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2$. Prove that $\mathbf{0}$ is a minimum of f.

   (e) Let $g : \mathbb{R} \to \mathbb{R}^2$ be the function defined by $g(t) = \big(t, \sin(t)\big)$. Justify that $h = f \circ g$, a function from $\mathbb{R}$ to $\mathbb{R}$, has a minimum at $t = 0$ and, using (c) conclude that $h'(0) = 0$.

(f) If g from the previous question was defined by $g(t) = t\boldsymbol{v} = (tv_1, tv_2)$, for a fixed $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$, verify that we could obtain an analogous conclusion.

(g) Using the result obtained in (f) and the Chain Rule, prove that $\nabla f(\mathbf{0})|J_g(0) = 0$. What does this mean?

(h) Using (b) conclude that $\nabla f(\mathbf{0}) = 0$ at the minimum point of f.

(i) * [1] Generalize the result of the previous point for any differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ with a relative extremum at the point $\boldsymbol{a} \in \mathbb{R}^n$; consequently, conclude that $\nabla f(\boldsymbol{a}) = 0$.

4. The study of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = (y - 3x^2)(y - x^2)$ shows the difficulty of finding the sufficient conditions to conclude that a critical point is an extremum.

   (a) Verify that $(0, 0)$ is a critical point of f.

   (b) Determine and classify (relatively to the sign) the Hessean matrix of f at $(0, 0)$ and on a neighborhood of $(0, 0)$. What we can conclude from the necessary and sufficient conditions discussed in the class?

   (c) Show that $(0, 0)$ is a minimum of f in the direction on any line passing on the origin, that is, if $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$ and $g : \mathbb{R} \to \mathbb{R}^2$ is defined by $g(t) = (at, bt)$, then $t = 0$ is a minimum of $f \circ g$.

   (d) However, verify that $(0, 0)$ is not a minimum of f.

5. Determine the relative extrema of the following functions:

   (a) $f(x, y) = xy$

   (b) $f(x, y) = x^2 + y^2$

   (c) $f(x, y) = x^2 - y^2$

   (d) $f(x, y) = x + 2e^y - e^x - e^{2y}$

   (e) $f(x, y) = 2^y \log x^2$

   (f) $f(x, y) = x^3 + y^2 + 2xy - 8x$

   (g) $f(x, y) = -x^3 + 4xy - y^2$

   (h) $f(x, y) = -3y^2 + x^3 + 2xy$

   (i) $f(x, y) = \sin(xy)$

6. Compute, if they exist, the unconstrained optima of the following functions:

   (a) $f(x, y) = 4x^2 - 12xy + 9y^2 + 36x - 54y + 90$

   (b) $f(x, y) = 2x^4 - 4x^2y^2 + 2y^4 + 34$

[1] Exercises with * are considered more difficult and demand more time and dedication for their resolution.

(c) $f(x,y) = x^4 + y^4 + 2x^2y^2 - 10x^2 - 10y^2 + 25$

(d) $f(x,y) = 4x^2y - 4x^4 - y^2 + 1$

(e) $f(x,y,z) = x^2 + yz + 5xz - xy - x^3 + 2y.$

7. Verify if the function $g(x,y) = e^{(x-y)}\left(x^2 - 2y^2\right)$ has relative extrema.

8. Discuss, in order to the parameter $\alpha \in \mathbb{R}$, the existence of extrema of the following function: $f(x,y) = xy(\alpha - x - y)$.

9. Let us consider the following classical problem on Statistics. Given a set of data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ over two variables x and y, find the line $y(x) = a+bx$ which better fits the observed relation between the variables. To define this relation, it is supposed that one of the variables determines the value of the other, to the first one we call **independent variable** and to the second **dependent variable**. In the case $y(x) = a+bx$, y is the dependent variable and x the independent one. One of the most common methods is the **Method of the Minimum Squares** which minimizes the distance between the observed values for the dependent variable y_i and the values estimated by the line $\hat{y}_i = a + bx_i$. In this way, the parameters defining the line can be found solving the following problem:

$$min_{a,b} \sum_{i=1}^{n} \left(y_i - (a + bx_i)\right)^2 \tag{1}$$

(a) For the set of data $\{(1,1),(3,3),(2,1)\}$, find the parameters a and b for the line of the minimum squares.

(b) * Find the expression of a and b of the line of the minimum squares for an abstract sample of data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$.

2 Constrained Optima

1. Determine which the following functions have extrema in the given domain. Use the Weierstrass Theorem for extrema, as possible. For the cases that you conclude that there are no extrema, identify the Theorem's condition(s) not satisfied.

(a) $w : \{(x,y) : 0 \leq x + y \leq 1, x, y \geq 0\} \to \mathbb{R}$, $w(x,y) = xy$

(b) $g : (1,2] \times [0,1] \to \mathbb{R}$, $g(x,y) = x$

(c) $f : (0,1] \times [0,1] \to \mathbb{R}$, $f(x,y) = \frac{y}{x}$

(d) $h : [0,1] \times [0,1] \to \mathbb{R}$, $h(x,y) = \begin{cases} x+y & \text{if } xy > 0 \\ x/2 & \text{if } xy = 0. \end{cases}$

(e) $w : \{(x,y) : 0 \leq x+y \leq 1\} \to \mathbb{R}$, $w(x,y) = xy$

(f) $z : [0,1] \to \mathbb{R}$, $z(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1/x & \text{if } 1 < x \leq 2. \end{cases}$

2. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and a set of points $S = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) = 0\}$ with $g : \mathbb{R}^n \to \mathbb{R}^m$. What do you understand by: $f(\boldsymbol{a})$, with $\boldsymbol{a} \in \mathbb{R}^n$, is a relative [absolute] maximum of f subject to S?

3. To find the constrained optima of $f(x,y) = x+y$ subject to $g(x,y) = x^2+y^2-4 = 0$, do as follows:

   (a) First, trace the set of points given by $g(x,y) = 0$ and the level curve $f(x,y) = 1$. Deduce in which way the level curve should move to increase/decrease the value of $f(x,y)$. Find the constrained optima.

   (b) Use the Lagrangean function to find the critical points and the bordered Hessian matrix to decide if they are optima.

4. Determine, tracing as in 3 and using the Lagrangean function, the extrema of the following functions:

   (a) $f(x,y) = x^2 + y^2 - 4$, subject to $x + y = 3$.

   (b) $f(x,y) = 4x^2 + y^2 - 4xy$, subject to $2y - x = 3$.

   (c) $f(x,y) = x^2y^2$, subject to $x + y = 20$.

   (d) $f(x,y) = x^2 + 4xy + 4y^2$ subject to the set $\{(x,y) : y = 2x+1\}$.

   (e) $f(x,y) = x + y$ subject to $\frac{x^2}{2} + \frac{y^2}{4} = 1$.

   (f) $f(x,y) = \frac{1}{x^2} + \frac{1}{y^2}$ subject to $xy + 1 = 0$.

5. (a) A company needs raw materials x and y to produce a specific good. The quantity produced of this good is determined by the production function $f(x,y) = xy$. The company has 6 m.u. to invest in raw materials and the cost of each unit of x is 1 m.u. and each unit of y is 2 m.u.. Formulate the problem for this company in order to find the optimal production. Solve it.

   (b) Now suppose that the company has κ m.u. to invest in raw materials. Solve the new problem of the company, finding x and y for the optimal production. Notice that $x = x(\kappa)$ and $y = y(\kappa)$ depend on κ.

(c) Define the maximum value to produce as a function on the available quantity to invest in raw materials, *i.e.* $f(\kappa) = f(x(\kappa), y(\kappa))$.

(d) Compute the rate of the optimal production caused by the rate of κ, $\frac{\partial f}{\partial \kappa}$. Compare with the Lagrange multiplier from the previous question.

(e) What is the minimum price (of the good produced) accepted by the company to invest an extra m.u.?

6. An economic agent has an utility function given by $U(x, y) = (x+2)(y+1)$, where x and y are two commodities consumed by it. The prices for these commodities are P_x, P_y and the available income is M.

   (a) Write the budget restriction of this agent in order to P_x, P_y and M.

   (b) Justify why the budget restriction saturates when the agent is maximizing his utility.

   (c) Write the Lagrangean function which allows to find the optimal solution for the consuming.

   (d) Find the optimal values for x, y and λ.

   (e) Verify that the second order condition is satisfied.

   (f) Observe what happen to the consumption of y when the price of x changes.

7. A factory produces two types of machines in quantities x and y. The total cost is given by the function $f(x, y) = x^2 + 2y^2 - xy$. To minimize the cost, how many machines of each type should be produced, if one wishes to produce a total of 8 machines?

8. Let $Q = 5x_1x_2$ (in tons) be the production function of a certain good, where x_1,x_2 are the inputs for the production. Let $p_1 = 2$, $p_2 = 4$ be the prices (in thousands of euros) of the inputs, respectively. Compute the minimum cost to reach the production level $Q = 40$, and the underlying inputs.

9. Find the constrained optima:

   (a) $f(x, y, z) = x^2 + 3y^2 + 5z^2$ subject to $2x + 3y + 5z = 24$

   (b) $f(x, y, z) = z$ subject to $x^2 + y^2 + z = 5$ and $x + y + z = 1$.

10. Which point in the plane of $\mathbb{R}^3$ defined by $x_1 + 2x_2 + 3x_3 = 1$ minimizes the distance to the point $(-1, 0, 1)$?

11. A company has a production function $f(x, y, z) = 1000x^{2/5}y^{1/5}z^{1/5}$, a budget of 1600 € and can buy a unit of x, y and z by the price of 80 €, 10 € and 20 €, respectively. Which input combination does maximize the production? If the output price is 2€ per unit, is it worthy the company to increase the investment on inputs?

12. Compute the constrained optima $f(x, y, z) = 2x^2 + y + z$ subject to the condition:
$$\begin{cases} x + y + z = 1/4 \\ x - 2y = 3/4 \end{cases}$$

13. Find the maximum of the function $2\ln x + \ln y + \ln z$ subject to the condition $x + y + z = 10$. Estimate the change of the optimal value when one changes the independent term to 11.

14. An economic agent wants to divide his savings in three different investments possibilities in a way that is risk is minimized but the average return is 12%. Investments 1, 2 e 3, give a return of 10 %, 10% and 15% and the share of each is x,y e z, respectively. With $x + y + z = 1$ the variance (the measure of risk used) is $400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz$. Therefore the investor's problem is

$$\begin{aligned} \min\ & 400x^2 + 800y^2 + 200xy + 1600z^2 + 400yz \\ \text{subject to: } & x + y + 1.5z = 1.2 \\ & x + y + z = 1 \end{aligned}$$

Using the Lagrangean find the optimal solution to the problem, and interpret the Lagrangean multipliers obtained.

3 Linear Programming

1. Using the graphical method, solve the following Linear Programming problems:

(a)
$$\begin{aligned} \text{Max Z} =& 4x + 3.5y \\ \text{s.t.}\quad & x + y \leq 5 \\ & x \leq 3 \\ & x, y \geq 0 \end{aligned}$$

(b)
$$\begin{aligned} \text{Max Z} =& 7x + 3y \\ \text{s.t.}\quad & 2x + 5y \leq 20 \\ & x + y \leq 8 \\ & x, y \geq 0 \end{aligned}$$

(c)
$$\begin{aligned} \text{Max Z} =& 2x + 3y \\ \text{s.t.}\quad & x + 2y \leq 2 \\ & 6x + 4y \geq 24 \\ & x, y \geq 0 \end{aligned}$$

(d) (e)

$$\begin{aligned} \text{Max Z} =& 6x + 5y \\ \text{s.t.} \quad & 5x + 6y \leq 40 \\ & 12x + 9y \leq 144 \\ & x = 4y \\ & x, y \geq 0 \end{aligned} \qquad \begin{aligned} \text{Max Z} =& 40x + 30y \\ \text{s.t.} \quad & x \leq 16 \\ & y \leq 8 \\ & x + 2y \leq 24 \\ & x, y \geq 0 \end{aligned}$$

2. Using the Simplex Algorithm, solve the Linear Programming problems from the previous point.

3. Consider the set of feasible solutions defined by the following constraints:

$$\begin{aligned} & 2x - y \geq -2 \\ & x + 2y \leq 8 \\ & x, y \geq 0 \end{aligned}$$

For this set, graphically determine:

Max Z= y Min Z=$2x - 2y$ Max Z= $2x - 2y$

4. (P1) (P2)

$$\begin{aligned} \text{Max Z} =& 20x_1 + 6x_2 + 8x_3 \\ \text{s.t.} \quad & 8x_1 + 2x_2 + 3x_3 \leq 200 \\ & 4x_1 + 3x_2 \leq 150 \\ & x_3 \leq 20 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \qquad \begin{aligned} \text{Max Z} =& 2x_1 - 3x_2 - 4x_3 \\ \text{s.t.} \quad & x_1 + 5x_2 - 3x_3 \leq 15 \\ & x_1 + x_2 + x_3 \leq 5 \\ & 5x_1 + -6x_2 + 4x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   (a) Solve the Linear Programming problems using the Simplex Algorithm.
   (b) Identify and interpret the value of the Lagrange multipliers/shadow prices.

5. Consider the following Linear Programming problems:

(P1)

$$\begin{aligned} \text{Max Z} =& 2x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(P2)

$$\begin{aligned} \text{Max Z} =& 3x_1 + 7x_2 \\ \text{s.t.} \quad & x_1 - x_2 \geq 0 \\ & -3x_1 + 2x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(P3)

$$\begin{aligned} \text{Max Z} =& x_1 + x_2 \\ \text{s.t.} \quad & 4x_1 + 3x_2 \geq 12 \\ & 2x_1 - 3x_2 \leq 5 \\ & x_1 \geq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Solve the problems P1, P2 and P3.

(b) Identify the type of solution obtained for each problem. Justify your answer.

(c) For problems with multiple optimal solutions, give another optimal solution.

6. A company produces and sells two types of carpets. For the production, it uses two machines, A and B, each one working 12 and 14 hours per day, respectively. To produce $100\,m^2$ of the first type of carpet, it is necessary 3 hours of work in the machine A and 7 hours of work in the machine B. For same quantity of carpet of the type 2, it is required 4 hours and 2 hours in the machines A and B, respectively. It is known that the profit obtained by the production of $100\,m^2$ of the first type of carpet is $4\,m.u.$ and the correspondent value for the second type of carpet is $3\,m.u.$. What daily production plan should the company follow to achieve the maximum profit? How should the company use the machines daily? Formulate and solve the problem, and interpret the obtained solution.

7. Consider the following Linear Programming problem:

$$\begin{aligned} \text{Max Z} =& 7x + 3y \\ \text{s.t.} \quad & 2x + y \leq 10 \\ & 8x + y \leq 20 \\ & 3x - y \leq 9/2 \\ & x, y \geq 0 \end{aligned}$$

(a) Solve it using the graphical method.

(b) Determine the variation interval for the coefficient of x within the objective function (c1) for which the optimal point is invariant.

(c) Determine the variation interval for the coefficient of y within the objective function (c2) for which the optimal point is invariant.

(d) Determine the possible variations in the independent terms of the constraints such that the optimal base is invariant.

8. Consider the following Linear Programming problem:

$$
\begin{aligned}
\text{Max } Z = & x_1 + x_2 \\
\text{s.t.} \quad & 4x_1 + 3x_2 \geq 12 \\
& 2x_1 + 3x_2 \geq 6 \\
& 5x_1 + 4x_2 \leq 20 \\
& 4x_1 + 5x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

(a) Solve it using the graphical method.

(b) Make the Sensitivity Analysis for this problem.

4 Solutions for the presented exercises

4.1 Unconstrained Optima

1.

2.

3.

4.

5. (a) $\bar{\mathbf{0}}$ saddle point
   (b) $\bar{\mathbf{0}}$ minimum.
   (c) $\bar{\mathbf{0}}$ saddle point.
   (d) $\bar{\mathbf{0}}$ maximum.
   (e) There are no relative extrema.
   (f) $(2,-2)$ minimum; $(-4/3,4/3)$ saddle point.
   (g) $^{8}/_{3}\,(1,2)$ maximum; $\mathbf{0}$ saddle point.
   (h) $-^{2}/_{9}\left(1,3^{-1}\right)$ maximum; $\mathbf{0}$ saddle point.
   (i) $\left(\dfrac{\frac{\pi}{2}+k\pi}{y},y\right)$, $y\in]-\infty,-1\,[\cup]\,1,+\infty[$, represents a family of points of maximum and $\left(\dfrac{-\frac{\pi}{2}+k\pi}{y},y\right)$, $y\in]-\infty,-1\,[\cup]\,1,+\infty[$, represents a family of points of minimum.

6. We have the following families of extrema:
   (a) $\left(\frac{3y-9}{2},y\right)$, $y\in\mathbb{R}$, family of points of minimum represented by the line with equation $2x-3y+9=0$ (positive function)
   (b) $(\pm y,y)$, $y\in\mathbb{R}$, family of points of minimum represented by the lines with equations $x=\pm y$
   (c) $\left(\pm\sqrt{5-y^2},y\right)$, $y\in\,]-\sqrt{5},\sqrt{5}[$, family of points of minimum represented by the circumference with equation $x^2+y^2=5$
   (d) $\left(x,2x^2\right)$, $x\in\mathbb{R}$, family of points of maximum represented by the parabola with equation $y=2x^2$.

(e) $\left(-2 \pm \frac{\sqrt{6}}{3}, 10 \mp \frac{5\sqrt{6}}{3}, -4 \pm \frac{\sqrt{6}}{3}\right)$ saddle points.

7. $(2, -4)$ maximum; $\mathbf{0}$ saddle point.

8. $(0,0), (\alpha, 0), (0, \alpha)$ neither maxima nor minima. $(\alpha/3, \alpha/3)$ minimum if $\alpha < 0$, and it is maximum if $\alpha > 0$.

9.

4.2 Constrained Optima

1. (a) The function w has maximum and minimum, by Weierstrass's Theorem.
   (b) The function g has maximum, but no minimum. The domain of g, D_g, is not closed.
   (c) The function has minimum, but no maximum. D_f is not closed, (but f é continuous on D_f).
   (d) the function h has maximum and minimum. h is not continuous.
   (e) The function has maximum, but no minimum. D_w is not bounded.
   (f) The function has maximum and minimum, by Weierstrass's Theorem.

2.

3. $(x, y) = 2^{\frac{1}{2}}(1, 1)$ maximum; $(x, y) = -2^{\frac{1}{2}}(1, 1)$ minimum.

4. (a) $(3/2, 3/2)$ constrained minimum.
   (b) $(1, 2)$ constrained minimum.
   (c) $(10, 10)$ constrained maximum; $(0, 20)$ and $(20, 0)$ constrained minima.
   (d) $(2/3, -1/3)$ constrained minimum.
   (e) $\sqrt{2/3}\,(-1, -2)$ constrained minimum; $\sqrt{2/3}\,(1, 2)$ constrained maximum.
   (f) Constrained minima at $(-1, 1)$ and $(1, -1)$

5. (a)

$$\begin{aligned} \text{Max Z} = & xy \\ \text{s.t.} \quad & x + 2y \leq 6 \\ & x, y \geq 0 \end{aligned}$$

   $(x, y) = 3(1, \frac{1}{2})$

(b) $\big(x(\kappa), y(\kappa)\big) = \left(\frac{\kappa}{2}, \frac{\kappa}{4}\right)$

(c) $f(\kappa) = f\big(x(\kappa), y(\kappa)\big) = x(\kappa)y(\kappa) = \frac{k^2}{8}$

(d) $\frac{\partial f(\kappa)}{\partial \kappa} = \frac{\kappa}{4} = \lambda$

(e) $p > \frac{1}{\lambda}$.

6. (a)

$$\begin{aligned}\text{Max Z} =& (x+2)(y+1) \\ \text{s.t.} \quad & P_x x + P_y y \leq M \\ & x, y \geq 0\end{aligned}$$

(b) $\nabla f(x,y) \geq 0$

(c)

(d)

$$\begin{cases} x^* = & \frac{M-2P_x+P_y}{2P_x} \\ y^* = & \frac{M+2P_x-P_y}{2P_y} \\ \lambda^* = & \frac{M+2P_x+P_y}{2P_x P_y} \end{cases}$$

(e)

(f) $\frac{\partial x^*}{\partial P_y} = \frac{1}{2P_x}$

7.

8. Minimum cost at $(x, y) = (5, 3)$.

9. Minimum at $(4, 2)$ and the minimum cost is 16000.

(a) $(2, 1, 1)$ minimum.

(b) $(-1, -1, 3)$ minimum; (2,2,3) maximum.

10. $(-{}^{15}/_{14}, -{}^{1}/_{7}, {}^{11}/_{4})$.

11. $(x, y, z; \lambda) = (10, 40, 20; 200)$ maximum.

12. $(x, y, z; \lambda_1, \lambda_2) = (1/4, -1/4, 1/4; 1, 0)$ minimum.

13. $(x, y, z; \lambda) = (4, 2, 2; 32)$ maximum.

14. $(x, y, z) = (1/2, 1/10, 2/5)$ minimum.

4.3 Linear Programming

1. (a) Optimal Solution: $x = 3$, $y = 2$; Optimal value: $z = 19$.
   (b) Optimal Solution: $x = 20/3$, $y = 4/3$; Optimal value: $z = 152/3$.
   (c) Impossible problem.
   (d) Optimal Solution: $x = 80/13$, $y = 20/13$; Optimal value: $z = 880/13$.
   (e) Optimal Solution: $x = 16$, $y = 4$; Optimal value: $z = 760$.

2.

3. (a) Optimal Solution: $x = 4/5$, $y = 18/5$; Optimal value: $z = 18/5$.
   (b) Optimal Solution: $x = 4/5$, $y = 18/5$; Optimal value: $z = -28/5$.
   (c) Optimal Solution: $x = 8$, $y = 0$; Optimal value: $z = 16$.

4. (a) (P1) Optimal Solution: $x_1 = 15/2$, $x_2 = 40$, $x_3 = 20$, $s_1 = s_2 = s_4 = 0, s_3 = 15$; Optimal value: $z = 550$;
   (P2) Optimal Solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 4$, $s_1 = 22$, $s_2 = s_3 = 0$; Optimal value: $z = -19$
   (b)

5. (a) Multiple optimal solutions; Optimal solution: $(x_1, x_2) = \alpha(0, 5/2) + (1 - \alpha)(3, 1)$ for $\alpha \in [0, 1]$
   (b) Impossible problem.
   (c) Not bounded solution.

6.

$$\begin{aligned} \text{Max Z } =& 4x + 3y \\ \text{s.t.} \qquad & 3x + 4y \geq 12 \\ & 7x + 2y \geq 14 \\ & x, y \geq 0 \end{aligned}$$

   Optimal solution: $x = 16/11$; $y = 21/11$; Optimal value: $z = 127/11$ (that is, optimal daily production of carpets of type 1 and 2: $16/11m^2$ and $21/11m^2$, respectively; Daily maximum profit: $127/11m.u.$; daily, the available work time of the machines is totaly consumed).

7. (a) Optimal value: $z = 95/3$; Optimal solution: $x = 5/3$ e $y = 20/3$;

(b) $c_1 \in [6, 24]$

(c) $c_2 \in [7/8, 7/2]$

8. (a) Optimal value: $z = 40/9$; Optimal solution: $x_1 = 20/9$ e $x_2 = 20/9$;

   (b) $c_1 \in [4/5, 4/5]$