Exponential Discounting Bias[*]

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- July 2012 -

Abstract

We address intertemporal utility maximization under a general discount function that nests the exponential discounting and the quasi-hyperbolic discounting cases as particular specifications. The suggested framework intends to capture one important anomaly typically found when addressing the way agents discount the future, namely the evidence pointing to the prevalence of decreasing impatience. The referred anomaly can be perceived as a bias relatively to what would be a benchmark exponential discounting setting, and is modeled as such. The general discounting framework is used to address a standard optimal growth model in discrete time. Transitional dynamics and stability properties of the corresponding dynamic setup are studied. An extension of the standard growth model to the case of habit persistence is also considered.

Keywords: Intertemporal Preferences, Exponential Discounting, Quasi-hyperbolic Discounting, Optimal Growth, Habit Persistence, Transitional Dynamics.

JEL Classification: C61, D91, O41.

1 Introduction

Typically, the benchmark utility maximization dynamic model takes a constant rate of time discounting and, thus, intertemporal discounting is modeled as being exponential. This is an analytically convenient assumption and it is logically consistent with the idea that a constant interest rate is often used to compare the value of money over time, for instance at the level of the evaluation of investment projects. However, there are psychological effects that must be taken into account when addressing intertemporal preferences. Such effects may have a huge impact on how we perceive the behavior of the representative agent in the context of conventional economic models since they tend to generate a departure relatively to exponential discounting.

In Xia (2011) three types of time preference anomalies that imply a deviation relatively to the standard exponential discounting framework are identified. These relate to the timing of the evaluation, the magnitude of the reward, and the sign of the reward. The sign effect was first highlighted by Kahneman and Tversky (1979) and basically states that gains are discounted more than losses. The magnitude effect is a matter that has received increasing attention on recent literature (see Noor, 2011 and Bialaszek and Ostaszewski, 2012) and relates to the evidence that there is an inverse relation between the amount of the reward and the steepness of discounting over time, i.e., agents tend to be more patient when larger rewards are under evaluation.

The most debated issue, though, is the one concerning changes on the degree of impatience as time elapses. This point relates essentially to the basic evidence that there is decreasing impatience over time – human beings tend to place much more weight on the difference between a reward to be received (or a cost to be incurred) today or tomorrow than on the difference between two consecutive dates in the far future. Thus, the rate of discount that we apply when measuring the present value of some near in time outcome is typically much larger than the discount rate applied to a distant in the future event. This is also the same as saying that the discount rate decreases in time. Such type of phenomenon is known as hyperbolic discounting and it has been widely discussed at various levels in recent years.

The discussion on the subject, from an economic point of view, has started with Strotz (1956) and Pollak (1968) and received influential contributions in the 1990s, with the work, among others, of Akerlof (1991), Laibson (1997, 1998) and O'Donoghue and Rabin (1999). These authors have raised some fundamental questions: Does the popularity of exponential discounting come from its time consistency or from analytical tractability? How can one incorporate into economic models an operational notion of decreasing impatience? If preferences are truly present-biased, how does this relate to important behavioral issues as self-control or procrastination? Are agents aware of their own intertemporal preferences, so that they adopt sophisticated plans of action or does unawareness lead to a naive interpretation about the future?

These interrogations continue today to be a rich source of debate on behavioral eco-

nomics and related fields. Part of the debate is still centered on justifying why hyperbolic discounting should be considered a rational way to form intertemporal preferences, more than exponential discounting. Prelec (2004), Dimitri (2005), Drouhin (2009), Farmer and Geanakoplos (2009), and Gollier (2010) argue that hyperbolic discounting is time consistent and rational. Decreasing impatience in a stochastic environment allows for a formal proof of such claim. Other authors are more skeptical about how hyperbolic discounting is being approached in the literature. While there is a tendency to search for analytical discount functions that may allow for an elegant treatment of economic models, one should take into account arguments as the ones by Rubinstein (2003) and Rasmussen (2008) who believe that modifying functional forms does not answer the main questions posed by the apparent lack of rationality in economic behavior. As stated by Ariel Rubinstein, a deeper understanding of intertemporal human decisions requires opening the black-box of decision making more than changing slightly the structure of the model used to address human behavior.

Other relevant contributions on the field of hyperbolic discounting relate the generalization of the concept and the exploitation of the corresponding implications. In Bleichrodt, Rohde, and Wakker (2009) the commonly used discount functions are modified in order to account for other kinds of time inconsistency on the formation of preferences besides decreasing impatience. Specifically, the proposed framework accommodates the possibilities of increasing impatience and strongly decreasing impatience. Also Benhabib, Bisin, and Schotter (2010) present a general version of the discount function, that contemplates the most common specifications of exponential and hyperbolic discounting found in the literature.

The powerful notion of hyperbolic discounting, and its most common specification in economics - Laibson's quasi-hyperbolic discounting concept - have been applied to study a wide range of relevant economic issues. Just to cite a few, we highlight the contributions of Gong, Smith, and Zou (2007), concerning consumption under uncertainty, Groom *et al.* (2005), Dasgupta (2008), Gollier and Weitzman (2010), and Hepburn, Duncan, and Papachristodoulou (2010) in the field of environment policy, namely when making the distinction between social and private discount rates, a paramount normative question in this field, Graham and Snower (2007) on short-run macroeconomics and inflation dynamics, and, Barro (1999) and Coury and Dave (2010) on the implications of non-exponential discounting to economic growth.

In this paper we generalize the quasi-hyperbolic discounting setting and apply the new framework of intertemporal preferences to a standard discrete time optimal growth problem. The setup differs from other approaches on the subject because we relate the shape of the discount function to issues of financial literacy, following the analysis on the exponential growth bias as developed by Stango and Zinman (2009) and Almenberg and Gerdes (2011). Our argument is that in the same way people tend to underestimate future values of variables that grow at constant rates, individuals also tend to overestimate close in time values (relatively to the ones more distant in the future) when discounting them to

the present. This reasoning allows us to present a discount function that is flexible enough to characterize different degrees of hyperbolic discounting and to nest the exponential discounting case as a possible limit outcome.

The proposed specification of intertemporal preferences is analytically convenient to address a discrete time optimal growth model. It enables us to derive explicit stability conditions and it serves to compare different degrees of deviation from the constant discount rate benchmark. Additionally, we extend the model to include habit persistence in consumption in order to demonstrate the flexibility of the exponential discounting bias concept when used in different settings.

The remainder of the paper is organized as follows. Section 2 discusses in detail the notion of exponential discounting bias relating it with financial literacy issues. In section 3 this concept is applied to compare different possibilities in terms of hyperbolic discounting. Section 4 approaches utility maximization under the general specification for intertemporal preferences. Section 5 sets up the growth model and analyzes the underlying dynamics. In section 6, an extension is explored; namely, the model is adapted in order to account for habit persistence. Finally, section 7 concludes.

2   Anomalies in Financial Evaluation

Recently, Stango and Zinman (2009) and Almenberg and Gerdes (2011) have carefully analyzed the evidence that points to a tendency to underestimate the future value of a given variable that grows at a constant rate. This exponential growth bias clearly exists in practice, for instance in what concerns household financial decision making.

The mentioned literature emphasizes the link between the extent of the bias and the degree of financial literacy. A poor ability to perform basic calculations and the lack of familiarity with elementary financial concepts and products will, in principle, imply a wider gap between individuals' calculations and the true future values, i.e., there is a negative correlation between financial literacy and the exponential growth bias.

Well informed agents will be able to understand the basic notion of capitalization and to perceive the exponential path followed by any value that accumulates over time. However, many studies have been discovering serious flaws on the understanding, by the average citizen, of simple financial concepts and mechanisms. This was highlighted by Lusardi (2008) and Japelli (2010), among others. Financial literacy or, more precisely, the lack of it, can explain the kind of deficiency that consists in linearizing an exponential series in time.

The important argument concerning the lack of ability on accurately addressing the value of money in time is that incorrect answers are biased. As emphasized by Almenberg and Gerdes (2011), individuals are almost twice as likely to underestimate the correct amount than to overestimate it. Thus, on the aggregate it makes sense to state that in a society where a given degree of financial illiteracy exists, the future values of a series that grows at a constant rate will be underestimated. Exponential growth bias will then be

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common when assessing the future value of an investment that offers a return at a given annual constant interest rate.

It is reasonable to conceive the existence of a link between the interest rate and the rate of time preference. In Farmer and Geanakoplos (2009, pages 1,2), this link is explained in simple terms,

'A natural justification for exponential discounting comes from financial economics and the opportunity cost of foregoing an investment. A dollar at time s can be placed in the bank to collect interest at rate r, and if the interest rate is constant, it will generate $\exp(r(t-s))$ *dollars at time t. A dollar at time t is therefore equivalent to* $\exp(-r(t-s))$ *dollars at time s. Letting* $\tau = t - s$, *this motivates the exponential discount function* $D_s(\tau) = D(\tau) = \exp(-r\tau)$, *independent of s.'*

The above sentence establishes a possible direct connection between the interest rate and the discount rate of intertemporal preferences. Nevertheless, there is a substantial difference between the two. While the interest rate is obtained as a market outcome, and might not vary if market conditions do not change, the rate of intertemporal choice is a matter of perception and preferences. Agents may want to adopt a rate of time preference that is close to the interest rate, but if they fail in understanding how future values accumulate, the lack of financial literacy eventually helps in explaining why the subjective rate of time impatience possibly departs from a constant value or, in other words, why discounting possibly deviates from the benchmark exponential case under a constant discount rate.

To understand how financial illiteracy might contribute to deviate agents' preferences from exponential discounting, we just need to make the inverse path to the one that is present in the evaluation of the exponential growth bias, i.e., if individuals tend to underestimate future values when assessing them in the present, they will certainly overestimate current values when thinking about them as if they were taking decisions at some future time moment. In analytical terms, the idea of exponential growth bias is commonly presented as $FV = PV(1+r)^{(1-\theta)t}$, where FV is the future value, PV the present value, r the interest rate, t is time and $\theta \in (0,1)$ measures the magnitude of the bias. If one wants to address the present value given the future value, we just need to rearrange the previous expression and write it as $PV = FV/(1+r)^{(1-\theta)t}$.

The above relation implies decreasing impatience. Far in the future outcomes are much less valued than the ones occurring in the near future. Now the bias works on the opposite direction - near in time results are overestimated. We can call this effect **exponential discounting bias**, and we may define it as the tendency to overestimate close in time values of a variable that grows at a constant rate.

The exponential discounting bias will be bigger the larger is the extent of financial illiteracy and it constitutes an alternative explanation about why preferences in time tend to imply hyperbolic discounting: agents *want* to select a constant rate of time preference, namely a rate of time preference that follows the interest rate path, but their ability to

undertake the proper computations is biased, in such a way that far in time values are less considered than the ones near the current period.

Taking into consideration the notion of exponential discounting bias can be an analytically convenient way of approaching departures from strict exponential discounting. According to the distinction introduced by O'Donoghue and Rabin (1999) between naive and sophisticated agents, the discussed bias puts us closer to the naive evaluation of intertemporal preferences in Akerlof (1991) than to the sophisticated behavior that is implicit in Laibson's (1997, 1998) analysis. In this context, a sophisticated person will know exactly what the respective future selves' preferences will be, while naive individuals are not able to realize that as time evolves, preferences will evolve as well.

As a result of the understanding that a bias on discounting cannot be perceived by the agent, since it is the outcome of an anomaly on an otherwise intended constant discounting behavior, the representative agent in the models of the following sections will display a clearly naive behavior. Therefore, she will not be concerned with the possibility of tomorrow selves choosing options that are different from the ones chosen today. Since people are not aware of their own time inconsistency, it is legitimate to consider a dynamic optimal control problem where the representative agent maximizes at a given date $t = 0$ her future utility, and thus to design an optimal plan where the present bias exists but the agent acts as if it did not exist.

In short, the analysis in this paper finds support on two logical arguments:

First - Individuals desire to turn intertemporal preferences compatible with the opportunity cost of money. This is the benchmark time consistent behavior that the rational agent would like to adopt;

Second - Lack of a solid financial literacy eventually introduces a biased evaluation of intertemporal preferences, that makes the representative agent to act as if she was an exponential discounter, when in fact she is not.

3 Departures from Exponential Discounting

In order to account for decreasing impatience, Loewenstein and Prelec (1992) have proposed the following hyperbolic discount function: $D_H(t) = (1 + \alpha t)^{-\gamma/\alpha}$, where α and γ are two positive parameters. This discount function implies a decreasing discount rate: short-term discount rates are higher than long-term discount rates. Empirical evidence suggests that this is a much more appropriate and realistic way to approach intertemporal preferences then just considering a constant discount rate over time.

While empirically more suitable, hyperbolic discounting, considered as modeled above, is much less tractable from an analytical point of view than exponential discounting. Because of this, Laibson (1997, 1998), based on a previous formalization by Phelps and Pollak (1968), has proposed an approximation to hyperbolic discounting, that he dubbed quasi-hyperbolic discounting; this is straightforward to apply to the standard dynamic optimization models of economists. The discount function takes the following form:

$$D_{QH}(t) = \begin{cases} 1 \text{ if } t = s \\ \widehat{\beta}\widehat{\delta}^{t-s} \text{ if } t = s+1, s+2, ... \end{cases} \text{, with } s \text{ the time period in which the future}$$

is being evaluated; $\widehat{\beta} \in (0,1)$, $\widehat{\delta} \in (0,1)$. Note that in the limit case $\widehat{\beta} = 1$ we are back at exponential discounting.

As in the hyperbolic case, the quasi-hyperbolic discount function captures the idea that discount rates decline with the passage of time. Laibson proposes, in his studies, a small exercise to compare discount rates on each of the settings. He considers exponential discounting ($\widehat{\beta} = 1; \widehat{\delta} = 0.97$), quasi-hyperbolic discounting ($\widehat{\beta} = 0.6; \widehat{\delta} = 0.99$), and hyperbolic discounting ($\alpha = 10^5; \gamma = 5 \times 10^3$) and draws a graph where it is evident that $D_{QH}(t)$ generates a time trajectory that is considerably closer to $D_H(t)$ than the one originating in plain exponential discounting.

In the previous section, it was stated that the absence of a stable impatience level over time may be interpreted as an anomaly, something similar to the tendency that individuals have to linearize a series of values that accumulate at a constant rate (and, hence, truly exhibit an exponential path). In the proposed setting, this anomaly should be considered in the reverse way, i.e., if individuals tend to linearize exponential trajectories for the future, when discounting values to the present they will exacerbate the exponential nature of the series under analysis.

In this context, we will consider exponential discounting, $D_E(t) = \beta^{t-s}$, $\beta \in (0,1)$, but we add the possibility of an error of evaluation that increases short-run impatience, generating a kind of hyperbolic discounting. Let $\theta(t)$ be the anomaly term, which transforms $D_E(t)$ into a discount function with an exponential bias, i.e., $D_{EB}(t) = \beta^{(1+\theta(t))(t-s)}$. Function $\theta(t)$ will take the following form: $\theta(t) = \begin{cases} 0 \text{ if } t = s \\ \frac{\theta_1}{t-s} - \theta_0 \text{ if } t = s+1, s+2, ... \end{cases}$.

The assumption of $D_{EB}(t)$ as the discount function has two advantages. On one hand, it allows for an intuitive explanation on why we depart from exponential discounting. There is an error of evaluation by the agents; perhaps they want to adopt a constant discount rate but, relatively to the periods that are closer in time they do not have the capacity to make an objective evaluation of their priorities. As time goes by, such ability evolves and, in the long-run, the error in evaluation is much smaller. On the other hand, we introduce a more general and flexible approach to time discounting than the one underlying $D_{QH}(t)$; as we will see below, the values of β, θ_0, and θ_1 can be chosen in such a way that we obtain an approximation to $D_H(t)$ that is undoubtedly better than the one provided by quasi-hyperbolic discounting.

We consider $\theta_0 \in [0,1]$ and $\theta_1 \geq 0$. Naturally, exponential discounting holds for $\theta_0 = \theta_1 = 0$, while quasi-hyperbolic discounting is also a particular case of the more general setting provided by $D_{EB}(t)$ for $\widehat{\beta} = \beta^{\theta_1}$ and $\widehat{\delta} = \beta^{(1-\theta_0)}$. Recover Laibson's example and consider the following parameter values for the exponential bias discount function: $\beta = 0.97$, $\theta_0 = 0.95$, and $\theta_1 = 23$. Figure 1 displays a graph that is similar to the one in the original Laibson's analysis (50 periods are considered and hyperbolic and quasi-hyperbolic discount functions are displayed; pure exponential discounting is ignored

in the displayed figure). To this figure, we add the exponential bias case for the parameter values that were chosen.

It is evident that the new function generates results that offer a much better fit with the hyperbolic discount function than the ones generated by the quasi-hyperbolic case. After 15 periods there is almost a perfect match between $D_{EB}(t)$ and $D_H(t)$ (although, if we introduced additional periods - after 50 - we would start to see a departure of one of the series relatively to the other; nevertheless, this widening gap would never be as pronounced as the one regarding quasi-hyperbolic discounting).

Figure 2 allows for a closer look on this issue. The figure represents the distance (in percentage and in absolute value) between $D_{EB}(t)$ and $D_H(t)$ and between $D_{QH}(t)$ and $D_H(t)$. Only in three of the 50 time periods ($t = 1, t = 2$ and $t = 22$), the distance between $D_{EB}(t)$ and $D_H(t)$ exceeds the distance between $D_{QH}(t)$ and $D_H(t)$. It is notorious that the present proposal is well suited to address decreasing impatience and it is also well founded on the idea that agents lack the information, literacy, or ability to maintain a constant discount rate over time.

*** Fig. 1, 2 ***

4 Exponential Discounting Bias in Intertemporal Utility Functions

In many economic settings, discount functions are used to construct intertemporal utility functions. Their typical presentation is as follows,

$$U_s(c) = u(c_s) + \sum_{\tau=1}^{\infty} D(t)u(c_{s+\tau}) \tag{1}$$

Equation (1) represents the utility in the current period, $t = s$, from consuming today and in all future moments from $t = s + 1$ to an undefined future date. The term $u(c_s)$ is current consumption utility; the instantaneous utility function obeys conventional properties of continuity, smoothness, and concavity. Future utility is taken into account for all possible time moments but discounting implies that a larger weight is put on closer in time consumption opportunities. The discount function that we will consider is the one involving the exponential bias, $D(t) = D_{EB}(t)$.

We can take the same sequence of utility functions, but now initiating one period later. This becomes,

$$U_{s+1}(c) = u(c_{s+1}) + \sum_{\tau=1}^{\infty} D(t)u(c_{s+1+\tau}) \tag{2}$$

Taking into account $U_s(c)$ and $U_{s+1}(c)$ as presented above, we can address intertemporal utility under a recursive form. The following expression is straightforward to obtain

from the simultaneous consideration of (1) and (2), under exponential discounting bias. Now we denote time by t instead of s, in order to reflect that the important issue is that we are considering two consecutive time periods, independently of which the first in fact is:

$$U_t(c) = u(c_t) + \beta^{1-\theta_0}\left[U_{t+1}(c) - (1-\beta^{\theta_1})u(c_{t+1})\right] \tag{3}$$

The above expression is analytically useful, because one can apply to it, directly, dynamic programming techniques, in order to obtain optimal solutions.[1] Consider a simple budget constraint according to which a representative agent accumulates financial wealth (a_t) at a constant rate (r), besides receiving a constant labor income w. This constraint is

$$a_{t+1} = w + (1+r)a_t - c_t, \ a_0 \text{ given.} \tag{4}$$

The problem the representative agent will want to solve consists in maximizing utility subject to (4). It is crucial to remark, at this stage, that the intertemporal problem is solved under the implied assumption that the representative agent is naive. As discussed in section 2, we are not concerned with the tendency to procrastinate that an individual with decreasing impatience might display, because she will never realize that her intertemporal preferences are, in fact, not constant over time.

However, one must also highlight that the inability to understand how the future is effectively being discounted does not constitute an obstacle to the adoption of an optimal behavior; the agent solves an optimality problem and chooses the consumption path that best serves her purpose, which is the maximization of intertemporal utility. Putting it in other words, besides the budget constraint, the agent also faces a literacy constraint that affects the evaluation of time discounting; given these two constraints, the agent acts rationally by solving the dynamic optimization problem she faces. Financial illiteracy is not an impediment to the adoption of an optimizing behavior, although it can change the outcome of the problem at hand.

Solving the maximization problem requires defining a function $V(a_t)$ such that

$$V(a_t) = \max_c \left\{ u(c_t) + \beta^{1-\theta_0}\left[V(a_{t+1}) - (1-\beta^{\theta_1})u(c_{t+1})\right]\right\} \tag{5}$$

The corresponding first order conditions are

$$u'(c_t) + \beta^{1-\theta_0}[V(a_{t+1})]'\frac{\partial V(a_{t+1})}{\partial c_t} - \beta^{1-\theta_0}(1-\beta^{\theta_1})u'(c_{t+1}) = 0$$

$$\Rightarrow \beta^{1-\theta_0}[V(a_{t+1})]' = u'(c_t) - \beta^{1-\theta_0}(1-\beta^{\theta_1})u'(c_{t+1}) \tag{6}$$

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[1]The dynamic programming procedure used to solve the model was adapted from Walde (2011), and it is standard in terms of the analysis of deterministic discrete time optimization models.

and

$$[V(a_t)]' = \beta^{1-\theta_0} [V(a_{t+1})]' \frac{\partial V(a_{t+1})}{\partial a_t}$$
$$\Rightarrow [V(a_t)]' = (1+r)\beta^{1-\theta_0} [V(a_{t+1})]' \tag{7}$$

We must also take into account the transversality condition

$$\lim_{t\to\infty} a_t D_{EB}(t) V(a_t) = 0$$

Combining the two optimality conditions, (6) and (7), one obtains an equation of motion for consumption. In order to simplify the analysis, take a logarithmic utility function, $u(c) = \ln c$. For this functional form, the following difference equation is computed,

$$c_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})(1+r)c_t c_{t-1}}{(2-\beta^{\theta_1}+r)c_{t-1} - \frac{1}{\beta^{1-\theta_0}}c_t} \tag{8}$$

Expression (8) might be rewritten considering as endogenous variable the ratio $\psi_t := c_t/c_{t-1}$,

$$\psi_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})(1+r)}{(2-\beta^{\theta_1}+r) - \frac{1}{\beta^{1-\theta_0}}\psi_t} \tag{9}$$

From equation (9), we can determine the steady-state value of the ratio between two consecutive values of consumption.

Proposition 1 *The typical intertemporal optimization problem of the representative agent under exponential discounting bias has two equilibrium points:* $\psi^* = \beta^{1-\theta_0}(1+r) \vee \psi^* = \beta^{1-\theta_0}(1-\beta^{\theta_1})$.

Proof. Solve (9), for $\psi^* := \psi_{t+1} = \psi_t$ ∎

The found values have direct correspondence in the exponential case with the solutions $\psi^* = \beta(1+r) \vee \psi^* = 0$. Observe that the steady-state values are as much larger as the wider is the discounting bias, meaning that the deviation from exponential discounting promotes a faster steady-state growth of consumption. This is the obvious result of taking a discount function with a corresponding steady-state value, $\beta^{1-\theta_0}$, that is a value higher than the benchmark constant discount factor β.

Note that the two solutions have a different nature: the first one is unstable and the second one is stable,

$$\left.\frac{d\psi_{t+1}}{d\psi_t}\right|_{\psi^*=\beta^{1-\theta_0}(1+r)} = \frac{1+r}{1-\beta^{\theta_1}} > 1;$$
$$\left.\frac{d\psi_{t+1}}{d\psi_t}\right|_{\psi^*=\beta^{1-\theta_0}(1-\beta^{\theta_1})} = \frac{1-\beta^{\theta_1}}{1+r} \in (0,1).$$

This is precisely a same stability outcome as the one achieved under exponential discounting. Since consumption is a control variable, the representative agent has the possibility of selecting the unstable solution as the long-run path of consumption (it is the solution

that allows for positive growth, namely if the rate of discount is lower than the interest rate). Therefore, the long-run growth rate of consumption is $\psi^* - 1 = \beta^{1-\theta_0}(1+r) - 1$; the larger the value of θ_0, the more consumption will grow in the steady-state.

5 Exponential Discounting Bias in the Neoclassical Growth Model

5.1 The Setup

We now characterize the dynamics of a neoclassical growth model under exponential discounting bias. The maximization problem is the same as in the previous section (and the same remarks on naive intertemporal preferences and on the ability to optimize even under eventual financial literacy flaws continue to be valid). However, the constraint on the problem differs. We take capital accumulation and a production function involving decreasing marginal returns. Let k_t represent the capital stock and assume the following parameters: $A > 0$ (technology index), $\delta \in (0,1)$ (depreciation rate), $\eta \in (0,1)$ (output-capital elasticity). The resource constraint takes the form

$$k_{t+1} = Ak_t^\eta - c_t + (1-\delta)k_t, \; k_0 \text{ given} \tag{10}$$

Again, we compute first order conditions to encounter an optimal dynamic relation for consumption. If $\eta = 1$ (endogenous growth model with an AK production function), we end up with exactly the same dynamics as in the previous section (with $r = A - \delta$). Under decreasing marginal returns, we will be able to find constant steady-state values for both the state and the control variable, i.e., k_t and c_t. Repeating the same procedure of calculus to find optimality conditions, we arrive to the difference equation for consumption

$$c_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})\left[1 + \eta Ak_t^{-(1-\eta)} - \delta\right]c_t c_{t-1}}{\left[2 - \beta^{\theta_1} + \eta Ak_t^{-(1-\eta)} - \delta\right]c_{t-1} - \frac{1}{\beta^{1-\theta_0}}c_t} \tag{11}$$

Because one cannot address consumption dynamics independently of capital accumulation on the present setting, we end up with a system of three difference equations to be analyzed; the system is,

$$\begin{cases} k_{t+1} = Ak_t^\eta - c_t + (1-\delta)k_t \\ c_{t+1} = \frac{\beta^{1-\theta_0}(1-\beta^{\theta_1})\left[1+\eta Ak_t^{-(1-\eta)}-\delta\right]c_t z_t}{\left[2-\beta^{\theta_1}+\eta Ak_t^{-(1-\eta)}-\delta\right]z_t - \frac{1}{\beta^{1-\theta_0}}c_t} \\ z_{t+1} = c_t \end{cases} \tag{12}$$

Next, we proceed to the full characterization of the dynamics of system (12). This requires finding the steady-state and looking at local dynamics.

Proposition 2 *The steady-state of the neoclassical optimal growth problem under expo-*

nential discounting bias corresponds to a unique equilibrium point:

$$(k^*; c^*) = \left(\left[\frac{\eta A}{1/\beta^{1-\theta_0} - (1-\delta)} \right]^{1/(1-\eta)} ; A(k^*)^\eta - \delta k^* \right)$$

Proof. The steady-state is defined as the pair of values $(k^*; c^*)$ such that $k_{t+1} = k_t$ and $c_{t+1} = c_t = c_{t-1}$. Applying these conditions to system (12), it is straightforward to determine the values in the proposition. ∎

The steady-state value of the capital stock increases with the output-capital elasticity and with the value of the technology index. It falls with a larger depreciation rate. It is also straightforward to observe that a higher θ_0 (stronger deviation relatively to exponential discounting) implies a larger long-run value for the capital stock; the same is true for the value of β. As for parameter θ_1, this has no influence over the steady-state values of the endogenous variables.

We illustrate the results with a small numerical example. Let $\eta = 1/3$, $A = 1$ and $\delta = 0.05$. The steady-state for capital and consumption is explored for values of θ_0 in the range $0-1$ and values of β in the range $0.75-1$. Figure 3 shows how the amounts of capital and consumption vary with different values of the parameters that define intertemporal impatience. The higher is θ_0, the larger are the equilibrium values of the two variables; the same occurs for a higher β. The horizontal axis respects to β, the depth axis represents different values of θ_0 and the vertical axis gives the values of each of the variables for the selected parameter values.

*** Fig. 3 ***

Although discounting is important in terms of the dynamics of the growth model, we conclude that it has a limited impact on the steady-state: parameter θ_1 does not have any influence in long-run equilibrium, while a change on θ_0 disturbs the steady-state slightly by making the discount factor value to change in the same direction. A larger discount factor is synonymous of increased patience, which benefits the economy in terms of long-run accumulated capital and consumption levels.

5.2 Local Dynamics

In order to address stability properties, one needs to linearize the system in the vicinity of the steady-state point. Computation leads to

$$\begin{bmatrix} k_{t+1} - k^* \\ c_{t+1} - c^* \\ z_{t+1} - c^* \end{bmatrix} = \begin{bmatrix} 1/\beta^{1-\theta_0} & -1 & 0 \\ j & 1 + \frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} & -\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} k_t - k^* \\ c_t - c^* \\ z_t - c^* \end{bmatrix} \qquad (13)$$

with $j = \frac{1}{\beta^{1-\theta_0}} \left[\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} - 1 \right] \frac{1-\eta}{\eta} \left[1 - (1-\delta)\beta^{1-\theta_0} \right] \left[1 - (1-(1-\eta)\delta)\beta^{1-\theta_0} \right] > 0.$

Proposition 3 *The system is saddle-path stable. There exists one stable dimension, in the three dimensional space of the model.*

Proof. The existence of one stable dimension implies that one of the eigenvalues of the Jacobian matrix locates inside the unit circle, while the other two fall outside the unit circle. Let the eigenvalues be $\lambda_1, \lambda_2, \lambda_3$. We want to prove that $|\lambda_1| < 1$, $|\lambda_2| > 1$ and $|\lambda_3| > 1$.

We start by presenting trace, Tr, determinant, Det, and sum of principal minors, ΣM, of the Jacobian matrix; these are:

$Tr = \lambda_1 + \lambda_2 + \lambda_3 = 1/\beta^{1-\theta_0} + 1 + \frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})}$;

$Det = \lambda_1 \lambda_2 \lambda_3 = \frac{1}{\left(\beta^{1-\theta_0}\right)^2 (1-\beta^{\theta_1})}$

$\Sigma M = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = Tr + Det + j - 1$

It is straightforward to observe that $Tr > 3$, $Det > 1$ and $\Sigma M > 3$. The constraint on the determinant implies that the eingenvalues are all positive or that $\lambda_1 > 0, \lambda_2, \lambda_3 < 0$. In this second scenario, the conditions involving the trace and the sum of principal minors imply $\lambda_1 > 3 - (\lambda_2 + \lambda_3)$ and $\lambda_1 < \frac{3-\lambda_2\lambda_3}{\lambda_2+\lambda_3}$; these inequalities cannot be simultaneously satisfied for the constraints on the values of the eigenvalues. Thus, the only feasible possibility is the one under which the eigenvalues are all positive: $\lambda_1, \lambda_2, \lambda_3 > 0$. If all the eigenvalues are larger than zero, then the constraints involving the trace and the determinant allow to perceive that full stability (all eigenvalues below one) is not a possible outcome. At least one eigenvalue must be larger than 1.

Next, we resort to Brooks (2004) to identify how many eigenvalues effectively fall inside the unit circle. According to the mentioned author, an evaluation of the characteristic polynomial allows to state that: if condition

$$-(1 + \Sigma M) < Tr + Det < 1 + \Sigma M$$

is met, there exists one real eigenvalue λ_1 of magnitude less than 1 and either:

  - a pair of complex conjugate eigenvalues $\lambda_2, \lambda_3 = a \pm ib$, with $|a \pm ib| < 1$;
  - two more real eigenvalues of magnitude less than 1; or
  - a pair of real eigenvalues of magnitude greater than 1 and having the same sign.

Since we have remarked that at least one of the eigenvalues is larger than 1, the only possibility that can hold from the three above is the last one. Thus, if the displayed double inequality is satisfied, we confirm that $0 < \lambda_1 < 1$ and $\lambda_2, \lambda_3 > 1$. It is straightforward to verify the validity of the condition since it is equivalent to

$$-(Tr + Det + j) < Tr + Det < Tr + Det + j$$

Therefore, we confirm the existence of a single stable dimension, in the three dimensional space of the assumed system ∎

The above result can be illustrated through a numerical example. Recover the benchmark values for the exponential discounting bias case, i.e., $\beta = 0.97, \theta_0 = 0.95, \theta_1 = 23$. For these, one computes the eigenvalues of the Jacobian matrix for various possibilities in terms of the values of $\eta \in (0,1)$ and $\delta \in (0,1)$, namely, we consider η between $1/6$ and $5/6$ and δ between 0.025 and 0.125. The values that the eigenvalues possess for each combination of parameters are displayed in Figure 4. The figure involves three panels, one for each eigenvalue. In the horizontal axis we have the values of δ, in the depth axis the value of η and in the vertical axis the values assumed by each eigenvalue. Panel a respects to λ_1 and the other two to the eigenvalues that fall outside the unit circle, independently of the values of the parameters.

*** Fig. 4 ***

With saddle path stability, we have a result that is qualitatively similar to the one of the original Ramsey model with a constant discount rate. There is convergence towards the unique steady-state point, along a one-dimensional stable path; this trajectory is followed because the representative agent has the possibility of adapting its initial consumption level in order to place the system over the stable path, since consumption is a control variable. The expression of the stable trajectory can be presented in general terms.

Proposition 4 *Consider a point (k_0, c_0) in the vicinity of the steady-state $(k^*; c^*)$. In the convergence from the first to the second point, contemporaneous values of consumption and capital evolve following the stable path*

$$c_t - c^* = \left(1/\beta^{1-\theta_0} - \lambda_1\right)(k_t - k^*)$$

Proof. The saddle-path stable trajectory can be obtained by computing the eigenvector associated to the eigenvalue inside the unit circle. Thus, we can solve the system

$$\begin{bmatrix} 1/\beta^{1-\theta_0} - \lambda_1 & -1 & 0 \\ j & 1 + \frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} - \lambda_1 & -\frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})} \\ 0 & 1 & -\lambda_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Letting $p_1 = 1$, the eigenvector might be written as

$$P = \begin{bmatrix} 1 \\ 1/\beta^{1-\theta_0} - \lambda_1 \\ 1/(\lambda_1\beta^{1-\theta_0}) - 1 \end{bmatrix}$$

The slope of the contemporaneous relation between consumption and capital is given by the ratio p_2/p_1, i.e., $c_t - c^* = (p_2/p_1)(k_t - k^*)$, which corresponds to the expression in the proposition ∎

As in the original Ramsey model, the convergence relation between capital and consumption is of positive sign; thus, values of both variables are likely to simultaneously increase towards their long-term values. Note that this is a generic stable path expression that can be applied to specific forms of the discount function, namely the quasi-hyperbolic case and also the pure exponential discounting case. Observe, as well, that another stable trajectory emerges from the analysis: one can also relate consumption at $t - 1$ to the capital stock at t. This convergence relation is also of positive sign.

Let us return to the numerical example. Take $\eta = 1/3$ and $\delta = 0.05$ and recover the remaining parameter values of the benchmark exponential discounting bias exercise. The eigenvalue inside the unit circle is, in this case, $\lambda_1 = 0.944\,14$. The stable trajectory is, then, $c_t - c^* = 5.738\,4 \times 10^{-2}(k_t - k^*)$. For the assumed parameter values, in the convergence towards the steady-state, when the capital stock increases by one unit, consumption will increase $5.738\,4 \times 10^{-2}$ units. This rate of convergence can be compared to the one of the quasi-hyperbolic discounting setting. Recall that hyperbolic discounting was discussed for $\widehat{\beta} = 0.6; \widehat{\delta} = 0.99$, with $\widehat{\beta} = \beta^{\theta_1}$ and $\widehat{\delta} = \beta^{(1-\theta_0)}$.

Continue to consider $\beta = 0.97$; to be in the conditions of the QHD case, we must now take $\theta_0 = 0.67004$ and $\theta_1 = 16.771$. For these parameter values, the lower than 1 eigenvalue comes $\lambda_1 = 0.936\,68$ and the stable trajectory is now $c_t - c^* = 7.342\,1 \times 10^{-2}(k_t - k^*)$. In the QHD case, the saddle path is steeper than in the EDB case. If we take EDB as a closer approximation to pure hyperbolic discounting, one possible error in using QHD consists in achieving a larger change in consumption as the capital stock evolves than the one that should, in fact, be obtained.

Next, we consider the constant discount rate case. This is the case for which θ_0 and θ_1 are zero. As θ_1 approaches zero, the eigenvalue lower than 1 approaches $0.917\,99$. Thus, the stable trajectory is $c_t - c^* = 0.11294(k_t - k^*)$. This case departs even more from the hyperbolic discounting case and thus the relation between k and c is even more steeper.

The above results point to the conclusion that the further we are from the exponential discounting case the less consumption will vary, in the convergence towards the steady-state, as the capital stock evolves. To emphasize this outcome, let us present the slope of the stable trajectory for different values of θ_0 and θ_1. Consider, again, $\eta = 1/3$, $\delta = 0.05$, $\beta = 0.97$. Figure 5 takes θ_1 in the horizontal axis and θ_0 in the depth axis, in order to quantify the slope of the stable arm in the vertical axis. It is evident from the figure that a larger bias relatively to exponential discounting (measured by higher values of θ_0 and θ_1) implies a smaller change in c relatively to k in the process of adjustment towards the steady-state.

*** Fig. 5 ***

6 Habit Persistence

In this section, we extend the discounting bias analysis to a setting where consumption choices are subject to habit persistence. This extension serves the purpose of showing that the adaptation of the standard intertemporal optimization model in order to include the discounting bias is flexible enough to approach other meaningful issues concerning the analysis of utility dynamics.

Habit persistence is modeled through the consideration of the following utility function,[2]

$$u(c_t, c_{t-1}) = \ln(c_t - bc_{t-1}), \quad b \in [0,1) \tag{14}$$

According to (14), when $b = 0$, we have the conventional version of the model without habit persistence; a positive b indicates that utility is directly dependent on how much more the individual consumes today, relatively to consumption in the previous period. As it is obvious, the larger the value of b the stronger is the habit persistence effect. Constraint $c_t > bc_{t-1}$ must hold, in order to guarantee a feasible solution.

Consider the problem in section 4, relating utility maximization under the exponential discounting bias, subject to resource constraint (4). Now, the problem takes the form

$$V(a_t) = \max_c \left\{ u(c_t, c_{t-1}) + \beta^{1-\theta_0} \left[V(a_{t+1}) - (1 - \beta^{\theta_1}) u(c_{t+1}, c_t) \right] \right\} \tag{15}$$

Following the same procedure for the computation of first-order conditions as before, one arrives to the dynamic equation of consumption,

$$c_{t+1} = bc_t + \frac{\beta^{1-\theta_0}(1 - \beta^{\theta_1})(1 + r)}{\frac{2 - \beta^{\theta_1} + r}{c_t - bc_{t-1}} - \frac{1}{\beta^{1-\theta_0}(c_{t-1} - bc_{t-2})}} \tag{16}$$

which is equivalent to

$$\psi_{t+1} = b + \frac{\beta^{1-\theta_0}(1 - \beta^{\theta_1})(1 + r)}{\frac{2 - \beta^{\theta_1} + r}{\psi_t - b} - \frac{1}{\beta^{1-\theta_0}(1 - b/\psi_{t-1})}} \frac{1}{\psi_t} \tag{17}$$

Proposition 5 *The optimal control problem of the representative household under exponential discounting bias and habit persistence has three steady-state points: $\psi^* = \beta^{1-\theta_0}(1 + r) \vee \psi^* = \beta^{1-\theta_0}(1 - \beta^{\theta_1}) \vee \psi^* = b$.*

Proof. By taking $\psi^* := \psi_{t+1} = \psi_t$, one transforms difference equation (17) into

$$\ell_1 (\psi^*)^3 - (\ell_2 + b\ell_1)(\psi^*)^2 + (\ell_3 + b\ell_2)\psi^* - b\ell_3 = 0,$$

with $\ell_1 := 1/\beta^{1-\theta_0}$, $\ell_2 := 2 - \beta^{\theta_1} + r$ and $\ell_3 := \beta^{1-\theta_0}(1 - \beta^{\theta_1})(1 + r)$.

---

[2]See Heer and Maussner (2005) for a more general presentation of the utility function with habit persistence and corresponding analytical treatment.

The solutions of the equation are $\psi^* = \frac{\ell_2 \pm \sqrt{\ell_2^2 - 4\ell_1 \ell_3}}{2\ell_1} \vee \psi^* = b$, which correspond to the ones in the proposition ∎

Comparing with the problem without habit persistence, we have now an additional solution, $\psi^* = b$, but the other two remain exactly the same. Again, the representative agent chooses the path that opens the door for a possible positive steady-state growth rate of consumption, i.e., $\psi^* - 1 = \beta^{1-\theta_0}(1+r) - 1$.

If we include the habit persistence feature in the growth model with capital accumulation, the result is that, once more, this extension does not interfere with the steady-state outcome and with the stability properties of the system. The steady-state continues to correspond, as before, to a saddle-path stable equilibrium. Under habit persistence, system (12), which characterized growth dynamics, takes now the form of a four-dimensional set of relations,

$$
\begin{cases}
k_{t+1} = Ak_t^\eta - c_t + (1-\delta)k_t \\
c_{t+1} = bc_t + \dfrac{\beta^{1-\theta_0}(1-\beta^{\theta_1})\left[1+\eta Ak_t^{-(1-\eta)} - \delta\right]}{\dfrac{2-\beta^{\theta_1}+\eta Ak_t^{-(1-\eta)}-\delta}{c_t - bz_t} - \dfrac{1}{\beta^{1-\theta_0}(z_t - bv_t)}} \\
z_{t+1} = c_t \\
v_{t+1} = z_t
\end{cases}
\tag{18}
$$

In the steady-state, $c^* := c_{t+1} = c_t = z_t = v_t$ and $k^* := k_{t+1} = k_t$. The evaluation of (18) under the previous conditions leads to an exact same outcome as the one in proposition 2. Habit persistence does not change the unique equilibrium point towards which the economy converges in the long-run.

Stability could be addressed as in the case $b = 0$. We present the linearized system, in the steady-state vicinity, but we do not pursue a generic discussion on the signs of the eigenvalues. Instead, we just characterize stability resorting to a small example. The linearized system is:

$$
\begin{bmatrix}
k_{t+1} - k^* \\
c_{t+1} - c^* \\
z_{t+1} - c^* \\
v_{t+1} - c^*
\end{bmatrix}
=
\begin{bmatrix}
1/\beta^{1-\theta_0} & -1 & 0 & 0 \\
j(1-b) & 1 + \widehat{j} + b & -\widehat{j}(1+b) - b & \widehat{j}b \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
k_t - k^* \\
c_t - c^* \\
z_t - c^* \\
v_t - c^*
\end{bmatrix}
\tag{19}
$$

where j is the same combination of parameters as in section 5 and $\widehat{j} = \frac{1}{\beta^{1-\theta_0}(1-\beta^{\theta_1})}$.

Reconsider the numerical example of the previous sections ($\beta = 0.97, \theta_0 = 0.95, \theta_1 = 23, \eta = 1/3, A = 1, \delta = 0.05$). For these values, independently of the degree of habit persistence (i.e., of the value of b), one finds, for the Jacobian matrix in (19), a pair of eigenvalues inside the unit circle. Therefore, habit persistence does not change the stability result previously found: saddle-path stability holds, what implies that the representative agent will be able to control the consumption path in order to place the system on the stable arm, through which the economy converges towards the steady-state.

7   Conclusion

People do not evaluate future outcomes as if they were computers or calculators. Measuring the future value of some current event or the present value of some future event is many times an intuitive process in which individuals engage. In the same way there is evidence of an exponential growth bias, according to which individual agents tend to linearize the sequence of accumulated future outcomes, we can conceive a kind of exponential discounting bias, according to which we may explain the evidence that points to decreasing impatience and that is analytically translated in the concept of hyperbolic discounting.

The notion of exponential discounting bias is more general than the one commonly used by economists to characterize observed intertemporal preferences, i.e., the notion of quasi-hyperbolic discounting. This allows for a flexible analysis, where we can shape the trajectory of the discount factor in the way we find more reasonable in order to be as close as possible to what evidence reveals.

Furthermore, the new specification has appealing features from an analytical tractability point of view: because the bias originates on a misperception about how to evaluate the future that does not introduce any kind of sophistication on individual behavior, i.e., any kind of ability to understand that the perception of the future will change as time evolves, the optimization model can be approached similarly to what is done in the exponential discounting case.

When assessing the dynamics of an intertemporal representative consumer growth model in discrete time, the exponential discounting bias assumption has allowed to construct a three dimensional dynamic system, from which it is straightforward to analyze steady-state properties and transitional dynamics. The analysis makes it possible to proceed with a thorough characterization of how different intertemporal preferences may shape the optimal relation between capital accumulation and consumption.

The exponential discounting bias concept is adaptable to other features of the benchmark utility analysis. Specifically, in this paper, one has explored the implications of introducing habit persistence into the utility function; the conclusion is that steady-state and stability results remain basically the same when the new assumption is taken into consideration.

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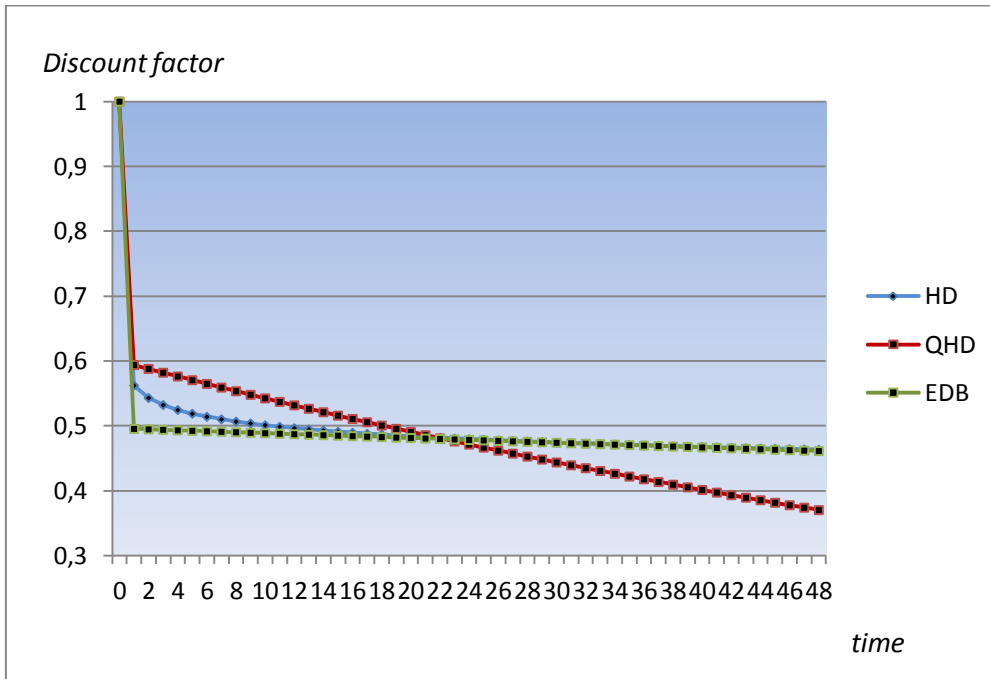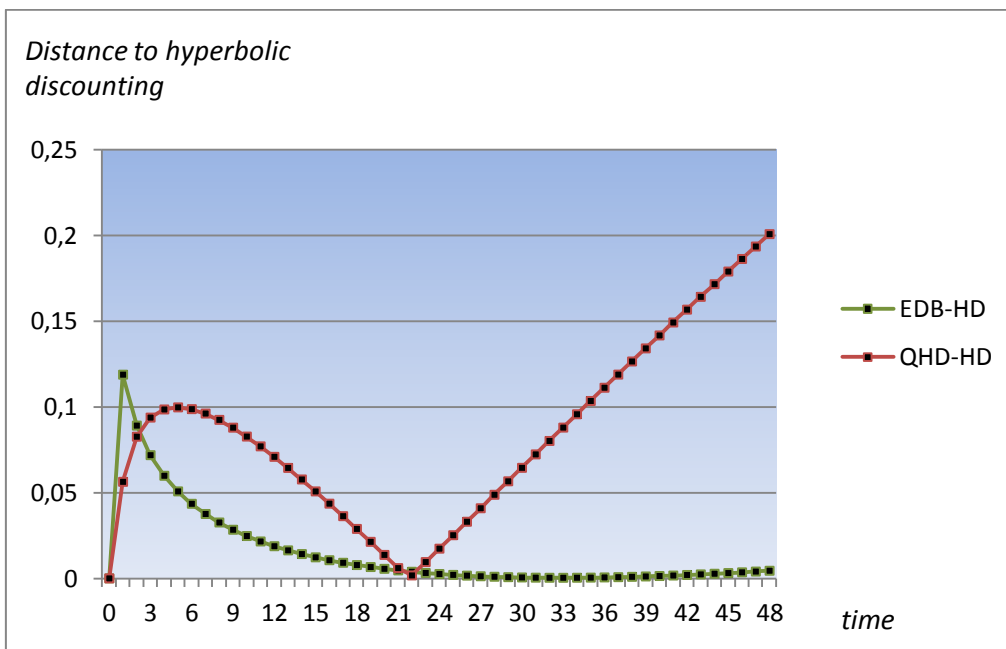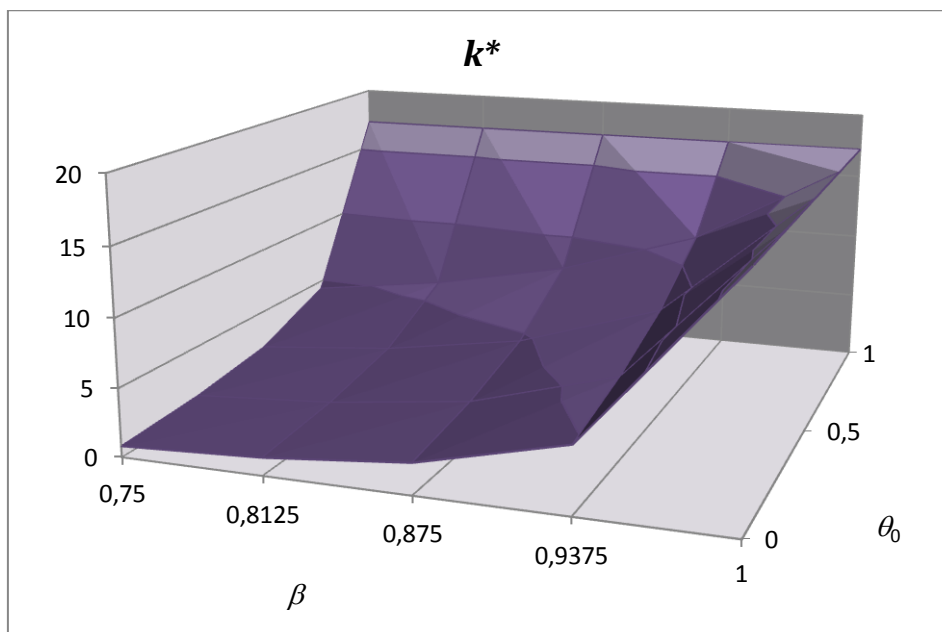**Fig.1 – Discount factors for hyperbolic discounting, quasi-hyperbolic discounting and exponential discounting bias.**

Fig.2 – Comparison between hyperbolic discounting and the two approximations (quasi-hyperbolic discounting and exponential discounting bias).
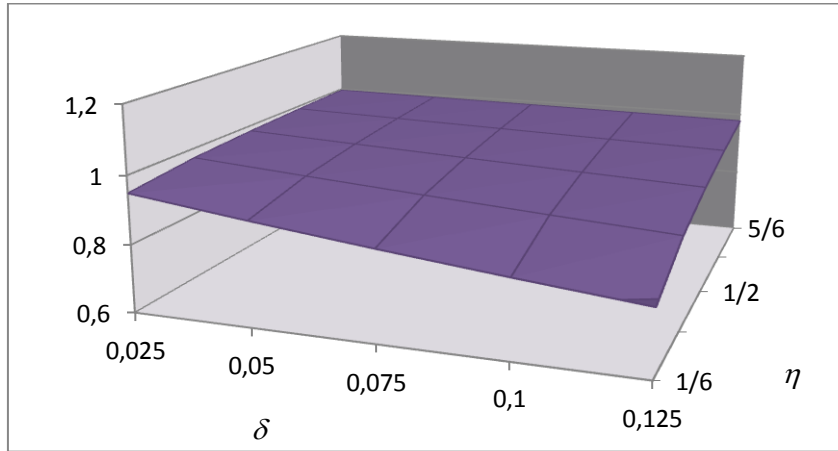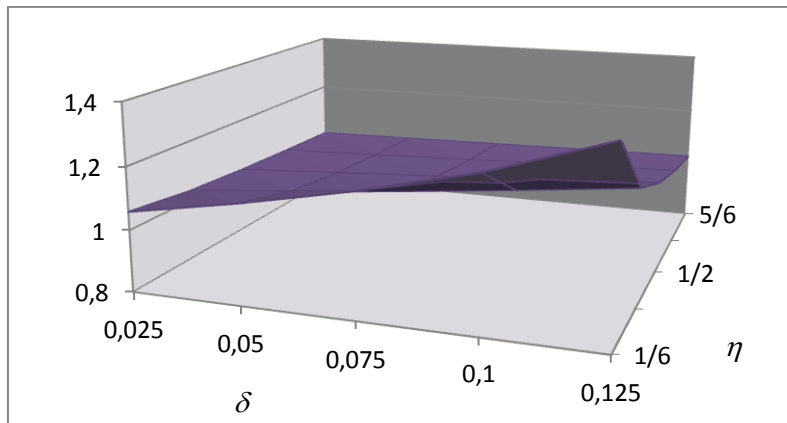
Panel a – Capital stock steady-state value.
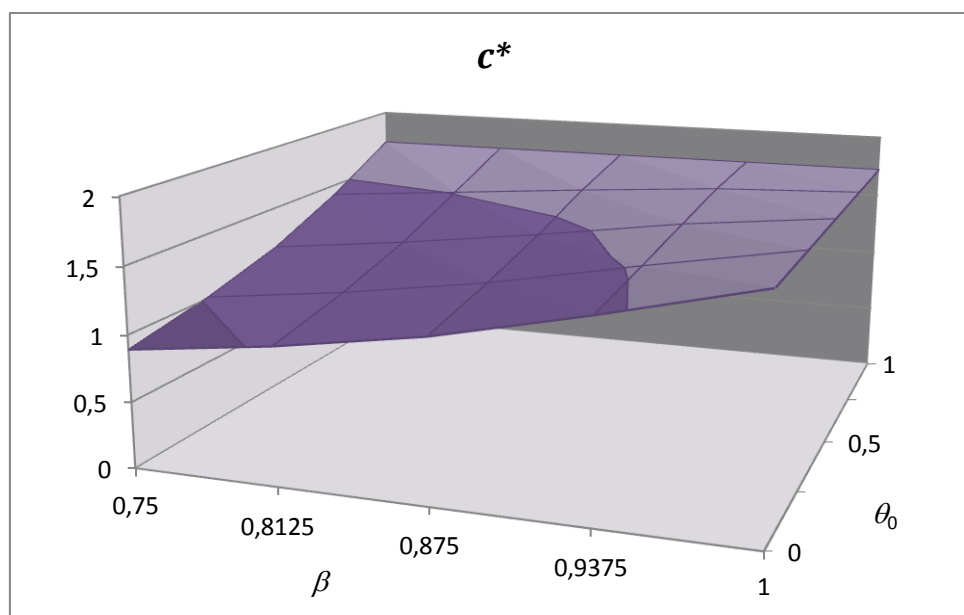
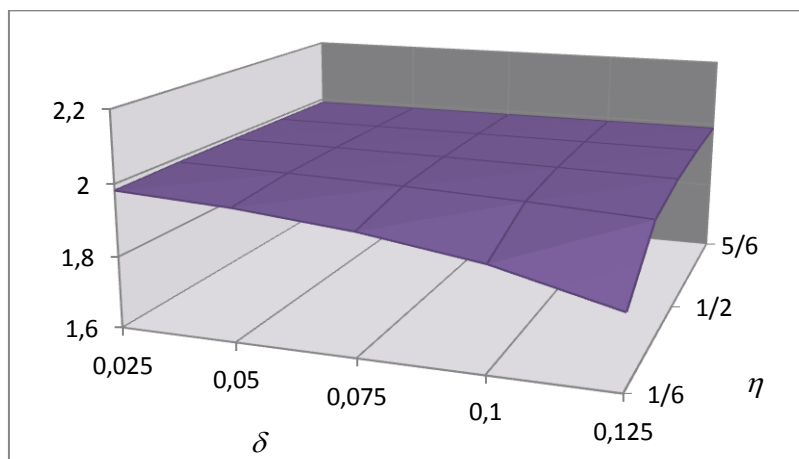Panel b – Consumption steady-state value.

Fig. 3- Steady-state values of capital and consumption for different values of parameters β and θ_0.

Panel $a - \lambda_1$

Panel $b - \lambda_2$

Panel $c - \lambda_3$

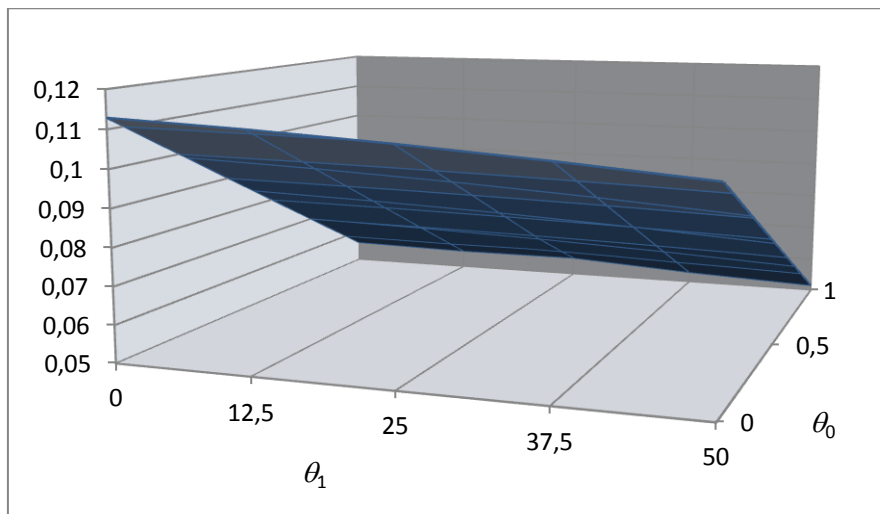**Fig.4 – Eigenvalues for different values of parameters η and δ (exponential discounting bias case).**

Fig. 5- Slope of the stable trajectory for different values of parameters θ_0 and θ_1.