Information **S**ciences and **C**omputing
Volume 2014, Number 2, Article ID ISC260314, 06 pages
Available online at http://www.infoscicomp.com/

Research Article

The Weak Convergence in Hilbert Spaces Concept Construction

Manuel Alberto M. Ferreira

Instituto Universitário de Lisboa (ISCTE – IUL), BRU - IUL, Lisboa, Portugal

Corresponding author: Manuel Alberto M. Ferreira, E-mail: manuel.ferreira@iscte.pt

Received 02 March 2014; Accepted 12 April 2014

Abstract: It is intended in this article to critically review the weak convergence concept in the real Hilbert (Note 1) spaces domain. It is studied mainly its construction process, since the main goal was to generalize the Bolzano (Note 2)-Weierstrass (Note 3) theorem. Then it is discussed in which conditions weak convergence implies convergence.

Keywords: Hilbert spaces; Bolzano-Weierstrass theorem; weak convergence, complexity spaces.

1. Introduction

In the Bolzano-Weierstrass theorem it is established that a bounded sequence of real numbers has at least one sublimit. This result remains true for any finite dimension space with inner product, that is: $\mathbb{R}^n$.

A result of this kind does not stand when infinite dimension spaces are considered. Actually, under those conditions it is possible to find a sequence in a Hilbert space H, orthonormal, designated $\{h_n\}$.So (Note 4) $\|h_n\| = 1$ and

$$\|h_n - h_m\|^2 = [h_n - h_m, h_n - h_m] = \|h_n\|^2 + \|h_m\|^2 = 1 + 1 = 2, \text{if } m \neq n.$$

Consequently this sequence is bounded and has not sublimits.

Then it is legitimate to ask which the generalization of the Bolzano-Weierstrass theorem is?

The problem of weak convergence in Hilbert Spaces is an important concept in theoretical mathematics, see [6], with multiple and interesting applications in real world problems.

A conceivable application of this notion is in the field of nonlinear time series. There stationarity and convergence are related concepts. In linear problems, weak stationarity plays a very important role. It is, however, uncommon to assume a truly nonlinear framework and the current article presents a thoughtful contribution to ease such applications. On this subject see, for instance, [2] and [7, 8].

It is therefore an interesting and important contribution to the knowledge of the mathematical properties of functionals defined in non-trivial complexity spaces; see [1] and [3]

2. Weak Convergence

For any $g \in H$ and for the orthonormal sequence seen above: $\|g\|^2 \geq \sum_{k=1}^{\infty} |[g, h_k]|^2$, according to Bessel's inequality. So

$$\lim_k [g, h_k] = 0 = [g, 0], \forall g \in H.$$

Founded on this example, a weaker notion of convergence will be introduced.

Definition 2.1: A sequence x_k in H converges weakly for x belonging to H if and only if $\lim_k [x_k, g] = [x, g]$ for any g in H. ■

Definition 2.2: A y is a weak limit of a set M if and only if $[x, y]$ is a limit point of $[x, M]$ for any x in H. ■

Definition 2.3: A set M is weakly closed if and only if contains all its weak limits. ■

Observation: Every set weakly closed is closed. The reciprocal proposition is not true.

Now two theorems at which important properties for the Hilbert spaces are established will be enounced without demonstration. The second is true in any Banach (Note 5) space. To demonstrate the first it would be necessary, in particular, the Riez (Note 6) representation theorem, see [4, 5]. For the second it would be necessary the Baire (Note 7) category theorem, see [1], true for any complete metric space.

Theorem 2.1 (Weak Compactness Property): Every bounded sequence of in a Hilbert space contains at least a subsequence weakly convergent. ■

Theorem 2.2 (Uniform Boundary Principle): Be $f_n(.)$ a sequence of continuous linear functionals in H such that $\sup_n |f_n(x)| < \infty$ for each x in H. Then $\|f_n(.)\| \leq M$ for any $M < \infty$.■

Two corollaries, very useful, from this theorem are:

Corollary 2.1: Be $f_n(.)$ a sequence of continuous linear functionals such that, for each $x \in H, f_n(x)$ converges. Then there is a continuous linear functional such that $f(x) = lim\, f_n(x)$ and $\|f(.)\| \leq \underline{lim}\, \|f_n(.)\|$.

Dem: By the Uniform Boundary Principle, it follows that

$\|f_n(.)\| \le M$ for any $M < \infty$. Define $g(x) = lim f_n(x)$. So $g(.)$ is evidently linear.

Suppose that $\|x_m - x\| \to 0$.

So $|g(x_m - x)| = \lim_n |f_n(x_m - x)| \le M\|x_m - x\| \to 0$.

Consequently $g(.)$ is continuous.

Also for any *x*, $\|x\| = 1, |g(x)| = lim|f_n(x)| \le \underline{lim}\ \|f_n(.)\|$.■

Corollary 2.2: Be $f_n(.)$ a sequence of continuous linear functional such that $\|f_n(.)\| \le M$ and $f_n(.)$ converges for each *x* in a dense subset of *H*. Then,

- There is a linear continuous functional $f(.)$ such that $\lim_n f_n(x) = f(x)$ since this limit exists,

- The limit linear functional is unique.

Dem: It will be stated that $f_n(x)$, in fact, converges for every *x* in *H*. For it, be x_n in the dense set (Note 8):

$$\|x - x_n\| \to 0;\ f_m(x_n) \text{ converges in } m.$$

Consider *p*, great enough such that, given $\varepsilon > 0, \|x - x_p\| \le \frac{\varepsilon}{4M}$.

And also *n* and *m* so that $|f_n(x_p) - f_m(x_p)| \le \frac{\varepsilon}{2}$. Then

$$|f_m(x) - f_n(x)| \le |f_m(x - x_p) - f_n(x - x_p)| + |f_m(x_p) - f_n(x_p)| \le 2M\|x - x_p\| + \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then, $f_n(x)$ converges and the conditions of the former corollary are fulfilled. ■

3. Weak Convergence and Convergence

It is obvious to pose the following question:

- Under which conditions weak convergence implies convergence?

The first result important to answer this question is:

Theorem 3.1: Suppose that x_n converges weakly for *x* and $\|x_n\|$ for $\|x\|$. Then x_n converges for *x*.

Dem: It is immediate that

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - [x_n, x] - [x, x_n] \to \|x\|^2 + \|x\|^2 - 2[x, x] = 2\|x\|^2 - 2\|x\|^2 = 0.$$

Consequently $\|x_n - x\|^2 \to 0$.■

Much more useful than the former one in the applications, on weak convergence, is the following result due to Banach-Saks (Note 9):

Theorem 3.2 (Banach-Saks): Suppose that x_n converges weakly for *x*. Then it is possible to determine a subsequence $\{x_{n_k}\}$ such that the arithmetical means $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converge for *x*.

Dem: Generality lossless, it may be supposed that *x* = 0. Consider x_{n_k} as follows:

- $x_{n_1} = x_1$,
- Due to the weak convergence, it is possible to choose x_{n_2}, such that $|[x_{n_1}, x_{n_2}]| < 1$,
- Having considered x_{n_1}, …, x_{n_k} it is evident that it is admissible to choose $x_{n_{k+1}}$ such that $|[x_{n_i}, x_{n_{k+1}}]| < \frac{1}{k}, i = 1, 2, \ldots, k$.

As, by the uniform boundary, it is possible to take $\|x_{n_k}\| \leq M$ for any $M < \infty$, with the inner products usual calculations rules it is obtained:

$$\left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}\right\|^2 \leq \left(\frac{1}{k}\right)^2 \left(kM + 2\sum_{i=2}^{k}\sum_{j=1}^{i-1}\left|\left[x_{n_j}, x_{n_i}\right]\right|\right) \leq \frac{1}{k^2}\left(kM + 2(k-1)\right) \to 0.$$

So $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converges to 0. ■

Observation: An alternative formulation of Theorem 3.2 is:

- Every closed convex subset is weakly closed.

Finally it is presented a Corollary of Theorem 3.2.

Corollary 3.1 (Convex Functionals Weak Inferior Semicontinuity): Be $f(.)$ a continuous convex functional in the Hilbert space *H*. So if x_n converges weakly to *x*, $\underline{lim}\, f(x_n) \geq f(x)$.

Dem: Consider a subsequence x_{n_m}, and put $x_m = x_{n_m}$, in order that $\underline{lim}\, f(x_n) = \lim f(x_m)$ and, still, that $\frac{1}{n}\sum_{m=1}^{n} x_m$ converges for *x*, in accordance with

Theorem 3.2. But, as $f(.)$ is convex,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right).$$

So, $\underline{\lim}\ f(x_n) = \lim \frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq \lim f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = f\ (x)$ ∎

4. Conclusion

The notion of weak convergence established in Definition 2.1 allows a possible Bolzano-Weierstrass theorem generalization. The Theorem 2.1 (Weak Compactness Property) and the Theorem 2.2 (Uniform Boundary Principle) help to understand that notion. Also in the Corollary 2.1 and in the Corollary 2.2 some operational properties are established. Finally with the help of Banach-Saks Theorem are presented conditions under which weak convergence implies convergence.

Note 1: David Hilbert, (German:['da:vit 'hilbet]; January 23, 1862 – February 14, 1943) was a German mathematician. He is recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. Hilbert discovered and developed a broad range of fundamental ideas in many areas, including invariant theory and the axiomatization of geometry. He also formulated the theory of Hilbert spaces one of the foundations of functional analysis. (From Wikipedia)

Note 2: Bernhard Placidus Johann Nepomuk Bolzano (*Bernard Bolzano* in English; October 5, 1781 – December 18, 1848) was a Bohemian mathematician, logician, philosopher, theologian, Catholic priest and antimilitarist of German mother tong. (From Wikipedia)

Note 3: Karl Theodor Wilhelm Weierstrass (German:*Weierstraß*; 31 October 1815 – 19 February 1897) was a German mathematician who is often cited as the "father of modern analysis".(From Wikipedia)

Note 4: $[.,.]$ is the symbol for inner product and $\|.\|$ for norm.

Note 5: Stefan Banach (['stɛfan 'banax]); March 30, 1892 – August 31, 1945) was a Polish mathematician. He is generally considered to have been one of the 20th century's most important and influential mathematicians. Banach was one of the founders of modern functional analysis and one of the original members of the Lwów School of Mathematics. His major work was the 1932 book, *Théorie des opérations linéaires* (Theory of Linear Operations), the first monograph on the general theory of functional analysis. (From Wikipedia)

Note 6: Frigyes Riesz (Hungarian: *Riesz Frigyes*, Hungarian pronunciation: [ˈriːs ˈfriɟɛʃ]; January 22, 1880 – February 28, 1956) was a Hungarian mathematician who made fundamental contributions to functional analysis. (From Wikipedia)

Note 7: René-Louis Baire (French: [bɛʁ]; 21 January 1874 – 5 July 1932) was a French mathematician most famous for his Baire category theorem, which helped to generalize and prove future theorems. His theory was published originally in his dissertation *Sur les fonctions de variable réelles* ("On the Functions of Real Variables") in 1899. (From Wikipedia)

Note 8: That is: be x_n, elements of the dense set, such that $x_n \to x$.

Note 9: The Polish mathematician and university tutor Stanisław Saks (December 30, 1897 – November 23, 1942) was known primarily for his membership in the Scottish Café circle, an extensive monograph on the theory of integrals, his works on measure theory and the Vitali-Hahn-Saks theorem. (From Wikipedia)

Acknowledgement

The author wishes to gratefully thank the comments produced in the first referee report which were very helpful in improving this article.

References

[1] H.L. Royden, Real Analysis (3rd edition), New York: MacMilan Publishing Co. Inc., 1968.
[2] J.Y. Campbell, A.W. Lo and A.C. MacKinlay, The Econometrics of Financial Markets, Princeton, New Jersey: Princeton University Press, 1977.
[3] L.V. Kantorovich and G.P. Akilov, Functional Analysis (2nd edition), Oxford: Pergamon Press, 1982.
[4] M.A.M. Ferreira, The Hahn-Banach theorem for the normed spaces in: J. Jafarov, A. Khankishiyev et al (Eds), Emerging Issues in the Natural and Applied Sciences, Azerbaijan: Progress IPS LLC and IJAR, 3(1) (2013), 5-17.
[5] M.A.M. Ferreira and M. Andrade, Riez representation theorem in Hilbert spaces separation theorems, International Journal of Academic Research, 3(6) (II Part) (2011), 302-304.
[6] M.A.M. Ferreira, M. Andrade and J.A. Filipe, Weak convergence in Hilbert spaces, International Journal of Academic Research, 4(4) (Part A) (2012), 34-36.
[7] R.R. Nigmatullin, J.T. Machado and R. Menezes, Self-similarity principle: The reduced description of randomness, Central European Journal of Physics, 11(6) (2013), 724-739.
[8] S.R. Bentes and R. Menezes, On the predictability of realized volatility using feasible GLS, Journal of Asian Economics, 28(2013), 58-66.