THE HAHN-BANACH THEOREM FOR THE NORMED SPACES

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DOI: 10.7813/einas.2013/3-1/1

ABSTRACT

It is intended, in this chapter, to present the Hahn-Banach theorem in its version for the normed spaces. This result is particularly important in optimization problems because of the separation theorems consequences of it.

Key words: Hahn-Banach theorem, normed spaces, separation theorems

1. THE HAHN-BANACH THEOREM

1.1. Convex Sets and Bodies

Be a real vector space \( L \).

Definition 1.1.1

A set \( K \subset L \) is convex if and only if

\[
\forall x, y \in K \quad \forall \theta \in [0,1] \quad \theta x + (1 - \theta) y \in K
\]  

(1.1.1)

\(^1\)This work was financially supported by FCT through the Strategic Project PEst-OE/EGE/UI0315/2011.
Definition 1.1.2
It is called **nucleus** of a set $E \subset L$, and designated $J(E)$, the set of the points $x \in E$ such that, given any $y \in L$, it is determined $\varepsilon = \varepsilon(y) > 0$ such that $x + ty \in E$ since $|t| < \varepsilon$.

Definition 1.1.3
A convex set with non-empty nucleus is called **convex body**.

**Theorem 1.1.1**
The nucleus of any convex set $K$ is also convex.

**Dem.:**
Suppose that $x, y \in J(K)$. Be $z = \theta x + (1 - \theta)y$, $0 \leq \theta \leq 1$. Then, given any $a \in L$, it is possible to determine $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that for $|t_1| < \varepsilon_1, |t_2| < \varepsilon_2$, $x + t_1 a$ and $x + t_2 a$ belong to $K$. So, the point

$$\theta(x + ta) + (1 - \theta)(y + ta) = z + ta$$

belongs to $K$ for $|t| < \varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $z \in J(K)$.

**Theorem 1.1.2**
The intersection of any convex sets family is a convex set.

**Dem:** Be $K = \bigcap_\alpha K_\alpha$, where each $K_\alpha$ is a convex set. Consider any two points $x$ and $y$ belonging to $K$. So $\theta x + (1 - \theta)y, 0 \leq \theta \leq 1$ belongs to any $K_\alpha$ and, in consequence, to $K$. Then $K$ is convex.

**Observation:**
The intersection of convex sets, being a convex set, is not necessarily a convex body.

**Definition 1.1.4**
Be $A$ any part of a vector space $L$. Among the convex sets that contain $A$ there is a minimal set: the intersection of the whole convex sets that contain $A^2$. This minimal convex set is called the **convex hull** of $A$.

**1.2. Homogeneous Convex Functionals**

**Definition 1.2.1**
A functional $p$ defined in $L$ is **convex** if and only if

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2 There is at least one convex set that contains $A$: the space $L$. 

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∀ x, y ∈ L θ ∈ [0, 1] p(θx + (1 − θ)y) ≤ θp(x) + (1 − θ)p(y) \hspace{1cm} (1.2.1). \blacksquare

**Definition 1.2.2**
A functional \( p \) is **positively homogeneous** if and only if

\[ ∀ x ∈ L \ α > 0 \ p(αx) = αp(x) \] \hspace{1cm} (1.2.2). \blacksquare

**Proposition 1.2.1**
For any convex positively homogeneous functional it always holds:

i) \( p(x + y) ≤ p(x) + p(y) \) \hspace{1cm} (1.2.3),

ii) \( p(0) = 0 \) \hspace{1cm} (1.2.4),

iii) \( p(x) + p(−x) ≥ 0, \ ∀ x ∈ L \) \hspace{1cm} (1.2.5),

iv) \( p(αx) ≥ αp(x), \ ∀ x ∈ L, α ∈ \mathbb{R} \) \hspace{1cm} (1.2.6).

**Dem:**

i) In fact, \( p(x + y) = 2p \left( \frac{x + y}{2} \right) ≤ 2 \left( p \left( \frac{x}{2} \right) + p \left( \frac{y}{2} \right) \right) = p(x) + p(y) \),

ii) In fact, \( p(0) = p(α0) = αp(0), \ ∀ α > 0 \). So, \( p(0) = 0 \),

iii) In this case, \( 0 = p(0) = p(x + (−x)) ≤ p(x)p(−x), \ ∀ x ∈ L \),

iv) The result is evident for \( α ≥ 0 \). For \( α < 0 \), \( 0 ≤ p(αx) + p(−αx) = p(αx) + p(|α|x) = p(αx) + |α|p(x) \). That is: \( p(αx) ≥ αp(x) \). \blacksquare

**1.3. Minkowsky Functionals**

**Definition 1.3.1**
Be \( L \) any vector space and \( A \) a convex body in \( L \) which nucleus contains 0. The \( A \) convex body **Minkowsky functional**, designated \( p_A(x) \), is the functional

\[ p_A(x) = \inf \{ r : \frac{x}{r} ∈ A, r > 0 \} \] \hspace{1cm} (1.3.1). \blacksquare

**Theorem 1.3.1**
A Minkowsky functional is convex, positively homogeneous and assumes only positive values. Reciprocally, if \( p(x) \) is a convex
positively homogeneous functional, assuming only positive values, and $k$ a positive number, then the set

$$A = \{x: p(x) \leq k\} \quad (1.3.2)$$

is a convex body with nucleus $\{x: p(x) < k\}$, which contains the point 0. If in (1.3.2) $k = 1$, the initial functional $p(x)$ is the A Minkowsky functional.

**Dem:** Given any element $x \in L$, $\frac{x}{r}$ belongs to $A$ if $r$ is great enough. Then, the number $p_A(x)$ defined by (1.3.1) is positive and finite.

But, given $t > 0$ and $y = tx$, $p_A(y) = \inf \{r > 0: \frac{y}{r} \in A\} = \inf \{r > 0: \frac{tx}{r} \in A\} = \inf \{rt > 0: \frac{x}{r} \in A\} = t \inf \{tr > 0: \frac{x}{r} \in A\} = t p_A(x)$.

So,

$$p_A(tx) = tp_A(x), \quad \forall t > 0 \quad (1.3.3)$$

and $p_A(x)$ is positively homogeneous.

Suppose now that $x_1, x_2 \in L$. Given any $\epsilon > 0$, choose the numbers $r_i$ $i = 1,2$) in the way that $p_A(x_i) < r_i < p_A(x_i) + \epsilon$. Then $\frac{x_i}{r_i} \in A$. Then, defining $r = r_1 + r_2$, the point $\frac{x_1 + x_2}{r} = \frac{r_1}{rr_1}x_1 + \frac{r_2}{rr_2}x_2$ will belong to the set of points $S = \left\{z: z = \theta \frac{x_1}{r_1} + (1 - \theta) \frac{x_2}{r_2}, \theta \in [0,1]\right\}$.

As $A$ is a convex set, $S \subset A$ and, in particular, $\frac{x_1 + x_2}{r} \in A$. So, $p_A(x_1 + x_2) \leq r = r_1 + r_2 < p_A(x_1) + p_A(x_2) + 2\epsilon$. As $\epsilon$ is arbitrary,

$$p_A(x_1 + x_2) \leq p_A(x_1) + p_A(x_2).$$

So, $p_A(\theta x + (1 - \theta)y) \leq p_A(\theta x) + p_A((1 - \theta)y) = \theta p_A(x) + (1 - \theta) p_A(y)$, $\forall x, y \in L', \theta \in [0,1]$; since it was already shown that $p_A(x)$ is positively homogeneous.

Look now to the set defined by (1.3.2). If $x, y \in A$ and $\theta \in [0,1]$, so $p(\theta x + (1 - \theta)y) \leq \theta p(x) + (1 - \theta)p(y) \leq K$. In consequence, $A$ is a convex set. Suppose now that $p(x) < K, t > 0$ and $y \in L$. Under these conditions, $p(x \pm ty) \leq p(x) + tp(\pm y)$. If $p(-y) = p(y) =$
so \( x \pm ty \in A \) for any \( t \). If at least one of the numbers (positive) \( p(y), p(-y) \) is not null, so \( x \pm ty \in A \) for

\[
t < \frac{K - p(x)}{\max\{p(y), p(-y)\}}.
\]

From the definitions it results that \( p \) is the Minkowsky functional of the set \( \{ x : p(x) \leq 1 \} \). ■

**Observation:**
Taking in account the Theorem 1.3.1, the Minkowsky functional allows to establish a correspondence between the positively homogeneous convex functionals, assuming only positive values, and the convex bodies to which nucleus the origin belongs.

### 1.4. The Hahn-Banach-Theorem

**Definition 1.4.1**

Consider a vector space \( L \) and its subspace \( L_0 \). Suppose that in \( L_0 \) is defined a linear functional \( f_0 \). A linear functional \( f \) defined in the whole space \( L \) is an **extension** of the functional \( f_0 \) if and only if

\[
f(x) = f_0(x), \quad \forall x \in L_0.
\]

The Hahn-Banach theorem is essential in the in the resolution of the problem of finding an extension of a linear functional.

**Theorem 1.4.1 (Hahn-Banach)**

Be \( p \) a positively homogeneous convex functional defined in a real vector space \( L \) and \( L_0 \) an \( L \) subspace. If \( f_0 \) is a linear functional defined in \( L_0 \), fulfilling the condition

\[
f_0(x) \leq p(x), \quad \forall x \in L_0
\]

so there is an extension \( f \) of \( f_0 \) defined in \( L \), linear, and such that

\[
f(x) \leq p(x), \quad \forall x \in L.
\]
**Dem:**

Begin showing that if $L_0 \neq L$, there is an extension of $f_0$, $f'$, defined in a subspace $L'$ such that $L \subset L'$, in order to fulfill the condition (1.4.1).

Be $z$ any element of $L$ not belonging to $L_0$; if $L'$ is the subspace generated by $L_0$ and $z$, each element of $L'$ is expressed in the form $tz + x$, being $x \in L_0$. If $f'$ is an extension (linear) of the functional $f_0$ to $L'$, it will happen that $f'(tz + x) = tf'(z) + f_0(x)$ or, making $f'(z) = c$,

$$f'(tz + x) = tc + f_0(x).$$

Now choose $c$, fulfilling the condition (5.1) in $L'$, that is: in order that the inequality $f_0(x) + tc \leq p(x + tz)$, for any $x \in L_0$ and any real number $t$, is accomplished.

For $t > 0$ this inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right). \tag{1.4.2}$$

For $t < 0$ it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right). \tag{1.4.3}$$

Now it will be proved that there is always a number $c$ satisfying simultaneously the conditions (1.4.2) and (1.4.3).

Given any two elements $y$ and $y''$ belonging to $L_0$,

$$-f_0(y'') + p(y'' + z) \geq -f_0(y') - p(-y' - z) \tag{1.4.4},$$

since

$$f_0(y'') - f_0(y') \leq p(y'' - y) = p((y'' + z) - (y' + z)) \leq p(y'' + z) + p(-y' - z).$$

Be $c'' = \inf_y \left(-f_0(y'') + p(y'' + z)\right)$ and $c' = \sup_y \left(-f_0(y') - p(-y' - z)\right)$. As $y$ and $y''$ are arbitrary, it results from (1.4.4) that $c'' \geq c'$. Choosing $c$ in order that $c'' \geq c \geq c'$, it is defined the functional $f'$ on $L'$ through the formula...
\[ f'(tz + x) = tc + f_0(x). \]

This functional satisfies the condition (1.4.1). So, any functional \( f_0 \) defined in a subspace \( L_0 \subset L \) and subject in \( L_0 \) to the condition (1.4.1), may be extended to a subspace \( L' \). The extension \( f' \) satisfies the condition
\[ f'(x) \leq p(x), \forall x \in L'. \]

If \( L \) has an algebraic numerable base \((x_1, x_2, \ldots, x_n, \ldots)\) the functional in \( L \) is built by finite induction, considering the increasing sequence of subspaces
\[ L^{(1)} = (L_0, x_1), L^{(2)} = (L^{(1)}, x_2), \ldots \]

designating \((L^{(k)}, x_{k+1})\) the \( L \) subspace generated by \( L^{(k)} \) and \( x_{k+1} \). In the general case, that is, when \( L \) has not an algebraic numerable base, it is mandatory to use a transfinite induction process, for instance the Haussdorf maximal chain theorem.

Call \( \mathcal{F} \) the set of the whole pairs \((L', f')\), at which \( L' \) is a \( L \) subspace that contains \( L_0 \) and \( f' \) is an extension of \( f_0 \) to \( L' \) that fulfills (1.4.1). Order partially \( \mathcal{F} \) so that
\[ (L', f') \leq (L'', f'') \text{ if and only if } L' \subset L'' \text{ and } f'_{|L'} = f''. \]

By the Haussdorf maximal chain theorem, there is a chain, that is: a subset of \( \mathcal{F} \) totally ordered, maximal, that is: not strictly contained in another chain. Call it \( \Omega \). Be \( \Phi \) the family of the whole \( L' \) such that \((L', f') \in \Omega \). \( \Phi \) is totally ordered by the sets inclusion; so, the union \( T \) of the whole elements of \( \Phi \) is a \( L \) subspace. If \( x \in T \) then \( x \in L' \) for some \( L' \in \Phi \); define \( \tilde{f}(x) = f'(x) \), where \( f' \) is the extension of \( f_0 \) that is in the pair \((L', f')\)- the definition of \( \tilde{f} \) is obviously coherent. It is easy to check that \( T = L \) and that \( f = f' \) satisfies the condition (1.4.1).

Now the Hahn-Banach theorem complex case, corresponding to the contribution of Hahn to the theorem, will be presented. But first:
**Definition 1.4.2**
A linear functional $p$, assuming only positive values, defined in a complex vector space $L$, is homogeneous convex if and only if, for any $x, y \in L$ and any complex number $\lambda$,

\[
p(x + y) \leq p(x) + p(y), \quad p(\lambda x) = |\lambda|p(x). \]

**Theorem 1.4.1a (Hahn-Banach)**
Be $p$ an homogeneous convex functional defined in a vector space $L$ and $f_0$ a linear functional, defined in a subspace $L_0 \subset L$, fulfilling the condition

\[
|f_0(x)| \leq p(x), x \in L_0.
\]

Then, there is a linear functional $f$ defined in $L$, satisfying the conditions

\[
|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0.
\]

**Dem:**
Call $L_R$ and $L_{0R}$ the real vector spaces underlying, respectively, the spaces $L$ and $L_0$. As it is evident, $p$ is an homogeneous convex functional in $L_R$ and $f_{0R}(x) = \text{Re} f_0(x)$ a real linear functional in $L_{0R}$ fulfilling the condition $|f_{0R}(x)| \leq p(x)$ and so,

\[
f_{0R}(x) \leq p(x).
\]

Then, owing to Theorem 1.4.1, there is a real linear functional $f_R$, defined in the whole $L_R$ space, that satisfies the conditions

\[
f_R(x) \leq p(x), x \in L_R; f_R(x) = f_{0R}(x), x \in L_{0R}.
\]

But, $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$, and

\[
|f_R(x)| \leq p(x), x \in L_R \quad (1.4.5).
\]

Define in $L$ the functional $f$ making

\[
f(x) = f_R(x) - if_R(ix).
\]

It is immediate that $f$ is a complex linear functional in $L$ such that
\[ f(x) = f_0(x), x \in L_0; \text{Ref}(x) = f_R(x), x \in L. \]

It only misses to show that \( |f(x)| \leq p(x), \forall x \in L. \)

Proceed by absurd. Suppose that there is \( x_0 \in L \) such that \( |f(x_0)| > p(x_0) \). So, \( f(x_0) = \rho e^{i\phi}, \rho > 0 \), and making \( y_0 = e^{-i\phi} x_0 \), it would happen that \( f_R(y_0) = \text{Re}[e^{-i\phi} f(x_0)] = \rho > p(x_0) = p(y_0) \) that is contrary to (1.4.5). ■

2. THE HAHN-BANACH-THEOREM FOR THE NORMED SPACES

2.1. Normed Spaces

Definition 2.1.1
Calling \( L \) a vector space, a norm in \( L \) is a functional \( p \) such that:

- \( p(x) \geq 0 \),
- \( p(x) = 0 \) if and only if \( x = 0 \),
- \( p(x + y) \leq p(x) + p(y) \),
- \( p(\alpha x) = |\alpha| p(x) \), for every \( \alpha \). ■

A vector space \( L \) with a norm is a normed space. It is usual to designate the norm of an element \( x \in L, \|x\| \).

Every normed space is a metric space, with the distance

\[ d(x, y) = \|x - y\|. \]

2.2. Continuous Linear Functionals
Be \( E \) a normed vector space.

Definition 2.2.1
A linear functional \( f \), defined in \( E \), is continuous in \( x_0 \in E \) if and only if, for any \( \varepsilon < 0 \), there is a neighboring \( U \) of \( x_0 \) such that

\[ |f(x) - f(x_0)| < \varepsilon \text{ for } x \in U. \]
Definition 2.2.2
A linear functional $f$, defined in $E$, is continuous if it is continuous in all $x_0 \in E$. ■

Follow some important results on the continuity of linear functionals defined in normed vector spaces.

Proposition 2.2.1
Be $E$ a normed vector space and $f$ a linear functional in $E$. So

\( i) \) If $E$ has finite dimension, $f$ is continuous,

\( ii) \) $f$ is continuous if and only if $f$ is continuous at the origin,

\( iii) \) $f$ is continuous if and only if $f$ is bounded over the unitary ball.■

Definition 2.2.3
Be $f$ a continuous linear functional in a normed space $E$. It is called $f$ norm, and designated $\|f\|$,

\[
\|f\| = \sup_{\|x\| \leq 1} |f(x)|
\]

that is: the supreme of the values that $|f(x)|$ assumes in the $E$ unitary ball.■

Observation:
- The class of continuous linear functionals so defined, is a vector normed space, called the $E$ dual space, designated $E^\ast$.

2.3. The Hahn-Banach Theorem Version in Normed Spaces
The Theorem 1.4.1 is as follows, in normed spaces:

Theorem 2.3.1 (Hahn-Banach)
Name $L$ a subspace of a real normed space $E$ and $f_0$ a bounded linear functional in $L$. So, there is a linear functional defined in $E$, extension of $f_0$, such that

\[
\|f_0\|_L = \|f\|_E.
\]

Dem:
It is enough to think in the functional $K\|x\|$ at which $K = \|f_0\|_L$. As it is convex and positively homogeneous, it is possible to put $p(x) = K\|x\|$ and to apply Theorem 1.4.1.■
Observation:
- To see an interesting geometric interpretation of this theorem, consider the equation $\|f_0(x)\| = 1$. It defines, in $L$, an hiperplane at distance $\frac{1}{\|f_0\|}$ of 0. Considering the extension $f$ of $f_0$, with norm conservation, it is obtained an hiperplane in $E$, that contains the hiperplane considered behind in $L$, and that at the same distance from the origin.

The version for normed spaces of Theorem 1.4.1a is:

**Theorem 2.3.1a (Hahn-Banach)**
Be $E$ a complex normed space and $f_0$ a bounded linear functional defined in a subspace $L \subseteq E$. So, there is a bounded linear functional $f$, defined in $E$, such that

$$f(x) = f_0(x), x \in L; \|f\|_E = \|f_0\|_L.$$

2.4. Separation Theorems
In this section, two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, will be presented.

Observation:
- It was seen that a convex body, in a vector space, is a convex set with non-empty nucleus. It may be stated that:
  i) In a normed space, the nucleus of a set is coincident with the total of its interior points, and so
  ii) In a normed space, a convex body is a convex set the has, at least, one interior point.

**Theorem 2.4.1 (Separation)**
Consider two convex sets $A$ and $B$ in a normed space $E$. If one of them, for instance $A$, has at least on interior point and $(intA) \cap B = \emptyset$, there is a continuous linear functional non-null that separates the sets $A$ and $B$.■
Theorem 2.4.2 (Separation)
Consider a closed convex set \( A \), in a normed space \( E \), and a point \( x_0 \in E \), not belonging to \( A \). So, there is a continuous linear functional, non-null, that separates strictly \( \{x_0\} \) and \( A \). ■

3. CONCLUSIONS

After a review on convex sets and bodies, homogeneous convex functionals, Minkowsky functionals and continuous convex functionals, the Hahn-Banach theorem for the normed spaces is presented, of course base on its general version. In addition two important separation theorems consequences of the Hahn-Banach theorem for the normed spaces are enounced. These last results are important in the optimization of functionals.

REFERENCES