BUSY PERIOD OF QUEUEING SYSTEMS WITH INFINITE SERVERS AND LOGISTICS

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1- Introduction

In the M|G|∞ queueing system

- The customers arrive according to a Poisson process at rate l,

- Each of them receives a service whose length is a positive random variable with distribution function G(.) and mean value a. So

$$a = \int_{0}^{\infty} [1 - G(t)]dt \quad (1),$$

-There are infinite servers. So when a customer arrives it always finds a server available,

- The service of a customer is independent of the services of the other customers and of the arrival process.

An important parameter is the traffic intensity that we call ρ, being

$\rho = la$ (2).

It is obvious that in an M|G|∞ queueing system there are neither losses nor waiting. In fact there is no queueing in the formal sense of the word.

For these systems it is not so important to study the populational process as for other systems with losses or waiting.

Generally we are much more interested in the study of other processes as, for instance, the busy period.

The busy period of a queueing system begins when a customer leaves the system letting it empty. During the busy period there is always at least one customer in the system.

Therefore in a queueing system there is a sequence of idle and busy periods.

We will show in the next section that these concepts can be applied in logistics. Namely to the failures that occur in the operation of a fleet of aircraft, of shipping or of trucking.

The results related to the busy period length of the M|G|∞ queueing system, that is a random variable, allow the evaluation of performance measures of the fleet. In consequence it is possible to identify ways of improving the performance of the fleet.
We will illustrate the theory with a very simple and short numerical example.

2 - Results and Applications

Let us call B the busy period length of the M|G|∞ queueing system.
The mean value of B, whatever is G(.), is given by Takács, (1962)

$$\mathrm{E}[B]=\frac{e^{r}-1}{l} \quad (3).$$

But VAR [B], the variance of B, depends largely on the form of B. But Sathe (1985) showed that

$$l^{-2}\max\left[e^{2r}+e^{r}\ r^{2}g_{s}^{2}-2re^{r}-1;0\right]\le VAR[B]\le l^{-2}\left(2e^{r}\left(g_{s}^{2}+1\right)\left(e^{r}-1-r\right)-\left(e^{r}-1\right)^{2}\right)(4),$$

where g_s is the variation coefficient of G(.). And, after (4), we can compute easily bounds to SD [B], the standard deviation of B.
Being R(t) the mean number of busy periods that begin in [0,t] (being t = 0 the beginning of a busy period) we have, Ferreira (1995),

$$e^{-r}(1+lt)\le R(t)\le 1+lt \quad (5).$$

Let us call N_B the mean number of the customers served during a busy period in the M|G|∞ queueing systems. We have, Ferreira (2001),

- If G(.) is exponential

$$N_B^M = e^{r} \qquad (6),$$

- For any other distribution function

$$N_B \cong \frac{e^{r\left(g_s^2+1\right)}\left(r\left(g_s^2+1\right)+1\right)+r\left(g_s^2+1\right)-1}{2r\left(g_s^2+1\right)} \qquad (7).$$

These results can be applied in logistics. For instance to the failures that occur in the operation of a fleet of aircraft, of shipping or of trucking. The customers are the failures. And its service time is the time that goes from the instant in which they occur till the one at which they are completely repaired. For examples of applications of this kind see, for instance, Carrilho (1991). Here

- A busy period is a period, in which there is at least one failure waiting for a reparation or being repaired,

- An idle period is a period in which there are no failures.

Here we gave simple expressions that allow to compute the mean and bounds to the variance of the busy period. And also simple bounds to the mean number of busy periods that begin in a certain length of time. We concluded with expressions to the mean number of failures that occur in a busy period.

These formulas are very simple and of evident application. They only require the knowledge of α, λ, ρ and g_s that are very easy to compute. The only problem is to test the hypothesis of that the failures occur according to a Poisson process.

Only to conclude note that, calling I (t) the distribution function of the idle period of the M | G | ∞ queueing system, we have

$I(t) = 1 - e^{-lt}$ (8),

as it happens with any queueing with arrival Poisson process. In this application it gives the probability of that the length of time with no failures is lesser or equal to t.

3 - Examples

Suppose a fleet where the failures occur at a rate of 20 per year. So l = 20/year. Suppose too that the mean time to repair a failure is 4 days (a = 4 day = (4/365) year). In consequence $r \cong 0.22$.

Possibly r maybe decreased to 0.11. It is enough to make l = 10/year, for instance buying more vehicles and decreasing, in consequence, the intensity of the use of each one. Or making a = 2 day. For instance increasing the teams that repair the failures.

On other side, if nothing is changed, things can get worse and maybe r can increase to 0.44.

If we suppose that the repair services times are exponential (a very frequent supposition for this kind of services), g_s =1, and after (3), (4), (5) and (6) we have with t=1 year (being SD $[B] = \sqrt{Var[B]}$)

r	E[B]	SD [B] (Lower Bound)	SD [B] (Upper Bound)	R (1) (Lower Bound)	R (1) (Upper Bound)	N_B^M
0.11	2.12 day	2.16 day	2.2 day	18	21	1.12
0.22	4.5 day	4.65 day	4.82 day	16	21	1.25
0.44	10 day	10.72 day	11.46 day	13	21	1.60

And it is possible to conclude that when r increases we may have less busy periods in one year, with more failures in each one, of course in mean values.

The mean of the length of the busy period and its dispersion increase with r too.

If we suppose now that the repair service times are constant (D = deterministic), $g_s = 0$, and after (3), (4) (in this case the lower bound is equal to the upper bound and so we get the real value of VAR[B]), (5) and (7) we have with t = 1 year

r	E[B]	SD [B] (Lower Bound)	R (1) (Lower Bound)	R (1) (Upper Bound)	N_B^M
0.11	2.12 day	0.41 day	18	21	1.59
0.22	4.5 day	1.22 day	16	21	1.68
0.44	10 day	3.85 day	13	21	1.90

E [B] and the R (1) bounds are the same that in the former case, evidently. The behaviour of the parameters with the increase of r is similar to the one of the exponential situation. But now the busy period length dispersion is much lesser and the mean value of failures in each busy period is greater.

4 - Conclusion

Of course, in the operation of a fleet, we are interested in big idle periods and in little busy periods. And if these ones occur it is good that they are as rare as possible, with a short number of failures.

Knowing α, λ, ρ and g_s the manager of the fleet can evaluate the conditions of the operation, namely:

-The mean length of a period with failures,

-The mean number of periods with failures that will occur in a certain length if time,

-The mean number of failures that occur in a period with failures.

As the expressions depend only on a few parameters and are very simple they show very simple ways to improve the operation, although they may be hard to implement.

5 - Bibliography

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