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Convex Sets Strict Separation

in the Minimax Theorem

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Abstract

The convex sets strict separation is very useful to obtain mathematical optimization results. The minimax theorem, a key result in Game Theory is an example. It will be outlined in this work.

Keywords: Minimax theorem, game theory, convex sets

1 Introduction

Be a zero-sum two players game. Call $\boldsymbol{W}$ the winner player and $\boldsymbol{L}$ the loser player. The payoff table when $\boldsymbol{W}$ chooses the strategy $i, i = 1,2, \dots, m$ and $\boldsymbol{L}$ chooses the strategy $j, j = 1,2, \dots, n$ is

$$\begin{array}{ccc} & \text{Player } \boldsymbol{L} & \\ \text{Player } \boldsymbol{W} & [g_{ij}] & i = 1,2, \dots, m \text{ and } j = 1,2, \dots, n \end{array}$$

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reading the player ***W*** the values as earnings and the player ***L*** as losses. Evidently, a negative earning is a loss and vice-versa. In terms of von Neumann's minimax theorem [2], the problem may be solved as a linear programming problem, see for instance [3]:

For player ***W***
the target is to maximize $G_{\boldsymbol{w}}$, the value of the game, subject to the constraints $\sum_{i=1}^{m} g_{ij}x_i \geq G_{\boldsymbol{w}},\ j = 1,2,\dots,n,\ \sum_{i=1}^{m} x_i = 1\,, x_i \geq 0,\ i = 1,2,\dots,m,$ being x_i the frequency at which the player chooses its *i* strategy, $i = 1,2,\dots,m$.

For player ***L***
the target is to minimize $G_{\boldsymbol{L}}$, the value of the game, subject to the constraints $\sum_{j=1}^{n} g_{ij}y_j \geq G_{\boldsymbol{L}},\ i = 1,2,\dots,m,\ \sum_{j=1}^{n} y_j = 1\,, y_j \geq 0,\ j = 1,2,\dots,n,$ being y_j the frequency at which the player chooses its *j* strategy, $j = 1,2,\dots,n$.

Whenever there is a solution $G_{\boldsymbol{w}} = G_{\boldsymbol{L}}$. If $\max_i \min_j g_{ij} = \min_j \max_i g_{ij}\,,\ i = 1,2,\dots,m; j = 1,2,\dots,n$ the common value is the game value and the mathematical programs[2] presented above are not useful. Note that in this kind of problems there is a maximization of minimums and a minimization of maximums.

In order to present a mathematical basis for this tool, it will be seen how the convex sets strict separation allows obtaining this key result in game theory.

Consider then the zero-sum two players games formulation:

-Be $\phi(x, y)$ a real function of two real variables $x, y \in H$ (real Hilbert space).

-Be *A* and *B* two convex sets in *H*.

-One of the players chooses strategies (points) in *A*, in order to maximize $\phi(x, y)$ (or to minimize $(-1)\phi(x, y)$): it is the maximizing player.

- The other player chooses strategies (points) in *B*, in order to minimize $\phi(x, y)$ (or to maximize $(-1)\phi(x, y)$): it is the minimizing player.

-$\phi(x, y)$ is the payoff function. $\phi(x_0, y_0)$ represents, simultaneously, the maximizing player earning and the minimizing player loss in a move at which they chose, respectively, the strategies x_0 and y_0.

This game has value *G* if

$$\sup_{x\in A} \inf_{y\in B} \phi(x, y) = G = \inf_{y\in B} \sup_{x\in A} \phi(x, y) \quad (1.1).$$

If for any (x_0, y_0), $\phi(x_0, y_0) = G$, (x_0, y_0) is a pair of optimal strategies. It is also a saddle point if

$$\phi(x, y_0) \leq \phi(x_0, y_0) \leq \phi(x_0, y), x \in A, y \in B \qquad (1.2).$$

[2]Note that these two programs are duals of each other.

2 Strict Separation

The Banach-Sachs theorem, see for instance [5], is very important for the sequence of this work:

Theorem 2.1 (Banach-Saks)

Suppose that x_n converges weakly to *x*. So it is possible to determine a subsequence $\{x_{n_k}\}$, such that the arithmetical means $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converge to *x*.■

Observation:

-An alternative formulation of this theorem is: **A closed convex subset is weakly closed**.

And also, see again [5], the

Theorem 2.2 (Convex functionals weak inferior semi continuity**)**

Be $f(\cdot)$ a continuous convex functional in Hilbert space *H*. So if x_n converges weakly to *x*, $\underline{lim}\, f(x_n) \geq f(x)$.■

Theorem 2.3

Consider *A* and *B* closed convex sets in *H*, with *A* bounded. Be $\phi(x,y)$ a real functional defined for *x* in *A* and *y* in B such that:

- $\phi(x,(1-\theta)y_1+\theta y_2) \leq (1-\theta)\,\phi(x,y_1)+\theta\phi(x,y_2)$ for x in A and y_1,y_2 in *B*, 0≤ $\theta \leq 1$,

- $\phi\big((1-\theta)x_1+\theta x_2,y\big) \geq (1-\theta)\,\phi(x_1,y)+\theta\phi(x_2,y)$ for y in B and x_1,x in *A*, 0≤ $\theta \leq 1$,

- $\phi(x,y)$ is continuous in *x* for each *y*, then (1.1) is fulfilled, that is : the game has a value.

Proof: Obviously $\inf_{y\in B}\phi(x,y) \leq \phi(x,y) \leq \sup_{x\in A}\phi(x,y)$ and so $\sup_{x\in A}\inf_{y\in B}\phi(x,y) \leq \inf_{y\in B}\sup_{x\in A}\phi(x,y)$. Then as $\phi(x,y)$ is concave and continuous in $x \in A$, A convex, closed and bounded, it follows that $G = \inf_{y\in B}\sup_{x\in A}\phi(x,y)$. Suppose now that there is $x_0 \in A$ such that $\phi(x_0,y) \geq G$, for each *y* in *B*. If this is the case, $\inf_{y\in B}\phi(x_0,y) \geq G$ or $\sup_{x\in A}\inf_{y\in B}\phi(x,y) \geq G$ as it is convenient. So, in the sequence it will be demonstrated the existence of such a x_0.

For each *y* in *B*, call $A_y = \{x \in A: \phi(x_0,y) \geq G\}$. A is closed, bounded and convex. Suppose that, for a finite set $\{y_1,\ y_2, \ldots y_n\}$, $\bigcap_{i=1}^{n} A_{y_i} = \emptyset$. Consider the transformation from A to E_n defined by $f(x) = (\phi(x,y_1) - G, \phi(x,y_2) - G, \ldots, \phi(x,y_2) - G)$. Call F the $f(A)$ set convex hull closure. Be $\wp$ the closed positive cone in E_n. $\wp \cap F = \emptyset$: in fact, being $\phi(x,y)$ concave in *x*, for any x_k in $k = 1,2,\ldots,n, 0 \leq \theta \leq 1, \sum_{k=1}^{n}\theta_k = 1,\ \ \sum_{k=1}^{n}\theta_k(\phi(x_k,y) - G) \leq$ $\phi(\sum_{k=1}^{n}\theta_k x_k; y) - G$ and so the $f(A)$ convex extension does not intersect $\wp$. Consider now a sequence x_n in *A*, such that $f(x_n)$ converges to $v, v \in E_n$. As *A* is closed, bounded and convex, it is possible to find a subsequence, designated $\{x_m\}$, convergent for an element of *A* (call it x_0). Moreover, for any y_i, as $\phi(x,y_i)$ is concave in x, $\overline{\lim}\,\phi(x_m,y_i) \leq \phi(x_0,y_i)$, that is $f(x_0) \geq \overline{\lim}\, f(x_m) = v$. So $F \cap \wp = \emptyset$. Then *F* and $\wp$ may be **strictly separated,** and it is possible to find a vector in E_n with components a_k, fulfilling

$\sup_{x\in A} \sum_{i=1}^{n} a_i(\phi(x, y_i) - G) < \sum_{i=1}^{n} a_i\rho_i$, with the whole a_i great or equal than zero. Obviously, the whole a_i must be non-negative, not being possible to have the whole of them simultaneously null. So, dividing by $\sum_{i=1}^{n} a_i$, as $\phi(x, y)$ is convex in y, an making the whole ρ_i zero, $\sup_{x\in A} \phi(x, \bar{y}) - G < 0$, with $\bar{y} = \frac{\sum_{i=1}^{n} a_k y_k}{\sum_{i=1}^{n} a_k}$. And $\bar{y} \in B$ that is $\inf_{y\in B} \sup_{x\in A} \phi(x, y) < G$, what is contradictory with the definition of G. So, $\bigcap_{i=1}^{n} A_{y_i} \neq \emptyset$.

Now it will be shown that, in fact, $\bigcap_{y \in B} A_y \neq \emptyset$, using that result and proceeding by absurd. Note that A_y is closed and convex. So, by Theorem 2.1, is also weakly closed, and, as it is bounded, is compact in the weak topology such as A. Call D_y the A_y complementary. D_y is open in the weak topology. So, if $\bigcap_{y \in B} A_y$ is empty, $\bigcup_{y \in B} D_y \supset H \supset A$. But, being A compact, it is known that a finite number of D_{y_i} is enough to cover A: $\bigcup_{i=1}^{m} D_{y_i} \supset A$,that is $\bigcup_{i=1}^{m} A_{y_i}$ is contained in the complementary of A and it must be $\bigcap_{i=1}^{n} A_{y_i} = \emptyset$, leading to a contradiction.

So suppose that $x_0 \in \bigcap_{y \in B} A_y$, then x_0 satisfies $\phi(x_0, y) \geq G$, as wished.■

3 The Minimax Theorem

The minimax theorem is obtained as a corollary of Theorem 2.3 strengthening its hypothesis. Now it is important to the sequence the following result:

Theorem 3.1

A continuous convex functional in a Hilbert space has minimum in any closed and bounded convex set.

Proof: In a space with finite dimension, obviously the set convexity is not needed. In spaces with infinite dimension, note that if x_n is a minimizing sequence, so, as the sequence is bounded, it is admissible to work with a weakly convergent sequence and, by Theorem 2.2, there is weak inferior semi continuity: $\underline{lim}\, f(x_n) \geq f(x)$, designating $f(\cdot)$ the functional, where x is the weak limit, and so the minimum is $f(x)$. As a closed convex set is weakly closed, *x* belongs to the closed convex set.■

Theorem 3.2 (Minimax**)**

Suppose that the Theorem 2.3 functional $\phi(x, y)$ is continuous in both variables, separately, and is also bounded. Then there is an optimal pair of strategies fulfilling the property of being a saddle point.

Proof: It was seen that there is x_0 satisfying $\phi(x_0, y) \geq G$ for each y. As $\phi(x_0, y)$ is continuous in y and *B* is bounded, $\inf_{y \in B} \phi(x_0, y) = \phi(x_0, y_0) \geq G$, for some y_0 in *B*, by Theorem 3.1. But $\inf_{y \in B} \phi(x_0, y) \leq \sup_{x \in A} \inf_{y \in B} \phi(x, y) = G$ and so $\phi(x_0, y_0) = G$. The saddle point property follows trivially from the above arguments.■

4 Conclusions

The separation concept, see [4], is very important to establish rigorously the Mathematical Convex Programing results, as it was exemplified in this work with the minimax theorem. The Khun-Tucker theorem and the Nash equilibrium are other very interesting examples, see [3] and [6]. Note finally the important role played by the weak convergence in Hilbert spaces concept in the demonstrations presented.

References

[1] A. V. Balakrishnan, Applied Functional Analysis, Springer-Verlag, New York Inc., New York, 1981.

[2] J. von Neuman, O. Morgenstern, Theory of Games and Economic Behavior, John Wiley & Sons Inc., New York, 1967.

[3] M. A. M. Ferreira, M. Andrade, Management optimization problems, International Journal of Academic Research, 3(2011), 2, Part III, 647-654.

[4] M. A. M. Ferreira, M. Andrade, Separation of a vector space convex parts, International Journal of Academic Research, 4(2012), 2, 5-8.

[5] M. A. M. Ferreira, M. Andrade, J. A. Filipe, The concept of weak convergence in Hilbert spaces, International Conference Aplimat, 12th, Bratislava: Slovak University of Technology, 2013.

http://repositorio-iul.iscte.pt/handle/10071/5702

[6] M. C. Matos, M. A. M. Ferreira, Game representation-code form, Lecture Notes in Economics and Mathematical Systems, 567, 321-334, 2006. DOI: 10.1007/3-540-28727-2_22

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