ISBN: 978-9952-8071-4-1

EMERGING ISSUES IN THE NATURAL AND APPLIED SCIENCES

Academic book

"PROGRESS"
Baku, Azerbaijan-2011

LCC: AY10-29
UDC: 002

Emerging issues in the natural and applied sciences. "Progress" LLC, Baku, 2011, 308 p.

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ISBN: 978-9952-8071-4-1

TABLE OF CONTENTS

DIFFUSION MODEL FOR THE FINANCING OF A FUND THAT RISKS RUIN

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ABSTRACT

It is proposed initially a diffusion process to represent a system subject to systematic ruin, with a stochastic regime of inflows and outflows of capital, establishing an analogy between this process and a pensions fund. Based the system in a periodic reflection scheme, it is valued the additional necessary financing to overcome the ruin. It is used, with this propose the Renewal Theorem for “renewal-reward” processes. That valuation is discussed, when it is applicable a deterministic interest rate, after the first passage times Laplace Transform.

Key words: Pensions funds, diffusion, ruin, first passage times, Renewal, Laplace Transform.

1. THE MODEL

This approach was originally motivated by the search of a stochastic model to represent the reserves of a fund, subject to an operation regime similar to the social security pensioners systems one, where the inflows through the so called social security contributions are regularly insufficient for the compromised pensions payments. In a more general perspective, it was intended to model a stochastic system with expenditures greater than the resources got during its operation, so needing a compensation external source of that unbalance. It was also intended to evaluate those external needs.

The objective is to do so based on diffusion processes, a special class of stochastic processes with continuous path. The work of (Karlin and Taylor, 1981) will be often used in this application.

So be a diffusion process $X_t, t \geq 0$, that represents the system reserves value at time *t*, defined by

$$X_t = X_0 + X_t^+ - X_t^-$$

where X_t^+ is a diffusion that represents the inflows total value, inputs or contributions till time *t,* and X_t^- is another diffusion that represents the outflows total value, outputs or pensions, till time *t*, and be the particular form

$$X_t = \alpha + at + cW_t - \int_0^t bx_s ds = \alpha + \int_0^t (a - bX_s)\, ds + \int_0^t c dW_s \quad (1.1)$$

where W_t is the standard Brownian motion, see (Arnold, 1974) and $\alpha, a, b \text{ and } c$ assume positive values. Here is, $X_0 = \alpha$ and $X_t^+ = at + cW_t$, the Brownian motion begin at the origin with infinitesimal mean and variance given by $a \text{ and } c^2$, respectively. The Brownian motion is acceptable as approximation to the process when *a* is conveniently greater than *c* so that the Brownian motion approximates a process with non negative increments. It is advanced also with $X_t^- = \int_0^t bX_s ds,$ that is with output process with differential at instant *t* given by *b* times its own value X_t. Note that this representation only makes sense while $X_t \geq 0$. Later this question will be analysed. For the instant it will be considered the process X_t with states space in the set of real numbers, with no restrictions.

Emphasise from (1.1) that X_t is a diffusion process with infinitesimal mean and variance, $\mu_X(x) = a - bx \text{ and } \sigma_X^2(x) = c^2$, where may be recognized the Ornstein-Uhlenbeck “mean-reverting” process, see (Arnold, 1974). The stochastic integral equation (1.1) is a linear equation, *strictu sensum*, that admits an explicit solution

$$X_t = \frac{a}{b}\left(1 - e^{-bt}\right) + \alpha e^{-bt} + \int_0^t c e^{-b(t-s)}\, dW_s \qquad (1.2).$$

It is enough to observe that in (1.2) X_t may be given in the form

$$X_t = e^{-bt} Y_t, \text{with } Y_t = \alpha + a \int_0^t e^{-bs} ds + c \int_0^t e^{-bs} dW_s \qquad (1.3)$$

in order to verify, trough Itô's Theorem that the present solution satisfies the initial equation, see again (Arnold, 1974). Although an alternative approach will be presented, in some aspects simpler.

Observe that in the solution (1.2) the process X_t results from the transformation

$$X_t = U_t + \int_0^t f(s)\, ds \qquad (1.4)$$

where

$$U_t = \alpha e^{-bt} + \int_0^t c e^{-b(t-s)}\, dW_s \qquad (1.5)$$

is the Ornstein-Uhlenbeck diffusion process with origin at $U_0 = \alpha$ and infinitesimal mean and variance $\mu_U(x) = -bx$ and $\sigma_U^2(x) = c^2$ and

$$\int_0^t f(s) ds = \int_0^t a e^{-bs} ds = \frac{a}{b}\left(1 - e^{-bt}\right) \qquad (1.6)$$

Knowing that the type (1.4) diffusions elementary transformations originate diffusion processes with the coefficients, see (Bhattacharya and Waymire, 1990),

$$\mu_X(t,x) = \mu_U\left(x - \int_0^t f(s)\, ds\right) + f(t) \text{ and } \sigma_X^2(t,x)$$
$$= \sigma_U^2\left(x - \int_0^t f(s)\, ds\right),$$

it is obtained

$$\mu_X(t,x) = -b\left(x - \int_0^t a\, e^{-bs} ds\right) + ae^{-bt} = a - bx \text{ and } \sigma_X^2(t,x) = c^2$$

as it had to be. The present situation is a special case where through a type (1.4) modification, an homogeneous diffusion generates another homogeneous diffusion – time homogeneity – that is, with coefficients μ_X and σ_X^2 independent of *t*.

Note 1.1.

It may be used immediately (1.4) and the known characterristics of the Ornstein-Uhlenbeck process to conclude that X_t is a Gaussian process with, (Arnold, 1974),

$$E(X_t) = E\left(U_t + \int_0^t f(s)\, ds\right) = \alpha e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) \quad (1.7)$$

$$Cov(X_s, X_t) = Cov(U_s, U_t) = e^{-b(s+t)} \frac{c^2}{2b}\left(e^{2bs} - 1\right), 0 \le s \le t \quad (1.8).$$

Observe in particular that $E(X_t) = \frac{a}{b}$ for $\alpha = \frac{a}{b}$ and that in general $\lim_{t\to\infty} E(X_t) = \frac{a}{b}$. ∎

Note 1.2.

In the conditions initially presented, clear that it is obtained for the output process

$$X_t^- = \int_0^t bX_s ds = \alpha + X_t^+ - X_t$$
$$= \alpha\left(1 - e^{-bt}\right)$$
$$+ a\int_0^t \left(1 - e^{-b(t-s)}\right) ds + c\int_0^t \left(1 - e^{-b(t-s)}\right) dW_s$$
$$= \left(\alpha - \frac{a}{b}\right)\left(1 - e^{-bt}\right) + cb\int_0^t e^{-b(t-s)}\, W_s ds \quad (1.9)$$

where, only by reference the last integral is obtained through integration by parts of $c\int_0^t\left(1-e^{-b(t-s)}\right)dW_s$; X_t^- is still a Gaussian process, but not now a Markov process, with expected value and covariance functions that are obtained here with the help of (1.4) and the integrated Ornstein-Uhlenbeck process characteristics, (Arnold, 1974):

$$E(X_t^-)=E\left(\int_0^t bX_s ds\right)=bE\left(\int_0^t U_s ds+\int_0^t\int_0^s f(u)\,duds\right)$$
$$=\left(\alpha-\frac{a}{b}\right)\left(1-e^{-bt}\right)+at \qquad (1.10)$$

$$Cov(X_s^-,X_t^-)=Cov\left(\int_0^s bX_u du,\int_0^t bX_u du\right)$$
$$=b^2Cov\left(\int_0^s U_u du,\int_0^t U_u du\right)$$
$$=c^2s+\frac{c^2}{2b}\left(-2+2e^{-bt}+2e^{-bs}-e^{-b(t-s)}-e^{-b(t+s)}\right),0\le s\le t \qquad (1.11).$$

In particular, $E(X_t^-)=at$ when $\alpha=\frac{a}{b}$, and there is an asymptote $\alpha=\frac{a}{b}+at$ to the function $E(X_t^-)$. ■

2. TIME TO RUIN

Call $T_{\alpha,y}$ the first passage time by $y,-\infty<y<\infty$, of the X_t process, $T_{\alpha,y}=inf\{t\ge 0: X_t=y\}, T_\alpha=T_{\alpha,0}$ and $\rho(\alpha,y)$ the probability of being finite: $\rho(\alpha,y)=P\left(T_{\alpha,y}<\infty\right)$.

Consider the ruin problem, that is the evaluation of $\rho(\alpha)=\rho(\alpha,0)$, the probability of the X_t time of first passage by 0 to be finite. The problem may be solved with a support of the scale and velocity functions of the considered diffusion process. This will be done following (Bhattacharya and Waymire, 1990). Attending the coefficients $\mu_X(x) = a - bx$ and $\sigma_X^2(x) = c^2$, it is obtained for the scale function

$$s(x) = \int_u^x exp\left\{-\int_u^y \frac{2\mu_X(z)}{\sigma_X^2(z)}dz\right\}dy$$
$$= \int_{a/_b}^x exp\left\{\frac{(by-a)^2}{bc^2}\right\}dy, \text{any } u \qquad (2.1)$$

and for the velocity function

$$m(x) = \int_u^x \frac{2}{\sigma_X^2(y)} exp\left\{-\int_u^y \frac{2\mu_X(z)}{\sigma_X^2(z)}dz\right\}dy$$
$$= \frac{2\sqrt{\pi}}{c\sqrt{b}}\left[\Phi\left(\frac{\sqrt{2}(bx-a)}{c\sqrt{b}}\right) - \frac{1}{2}\right], \text{any } u, \Phi(x)$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{1}{2}y^2}dy \qquad (2.2).$$

Evaluating the scale function in the bounds of the state space it may be concluded that the process X_t is a recurrent diffusion process, once $s(-\infty) = -\infty$ and $s(\infty) = \infty$ lead to $\rho(\alpha, y) = 1$ for any $-\infty < y < \infty$. Considering additionally the velocity function, the conditions $m(-\infty) = -m(\infty) = -\frac{\sqrt{\pi}}{c\sqrt{b}}, m(-\infty) > -\infty$ and $m(\infty) < \infty$ lead to the positive recurrence for the diffusion process in the states space $(-\infty, \infty)$, that is first passage time with finite expected value: $E(T_{\alpha,y}) < \infty$. In particular, for the case interesting to define $E(T_\alpha)$:

$$E(T_{\alpha,y}) = \int_y^\alpha \frac{2\sqrt{\pi}}{c\sqrt{b}}\Phi\left(\frac{\sqrt{2}(a-bz)}{c\sqrt{b}}\right)exp\left\{\frac{(bz-a)^2}{bc^2}\right\}dz, y < \alpha \qquad (2.3).$$

So for the diffusion process X_t, initially proposed, the ruin is a sure event, $\rho(\alpha) = 1$, in a time with finite expected value, $E(T_\alpha) < \infty$.

Note 2.1.

Pay attention that $T_{\alpha,y}$, the X_t process first passage time by *y*, $\alpha > y$, may be interpreted also as the Ornstein-Uhlenbeck process U_t first passage time by the curve $\left(1 - e^{-bt}\right)\frac{a}{b}$.

It may be also referred that it may be used immediately the scale function in (2.1) to define the function

$$\phi(y) = P\left(T_{\alpha,y} < T_{\alpha,0}\right), 0 \leq \alpha \leq y,$$

corresponding to the probability of not being lesser than *y* the maximum of X_t before the ruin, with value 1 for $\alpha > y$. After (Bhattacharya and Waymire, 1990)

$$\phi(y) = \frac{s(\alpha) - s(0)}{s(y) - s(0)} = \frac{\int_y^\alpha exp\left\{\frac{(bz-a)^2}{bc^2}\right\}dz}{\int_0^y exp\left\{\frac{(bz-a)^2}{bc^2}\right\}dz} \qquad (2.4). \blacksquare$$

Note 2.2.

The formulae (2.3) and (2.4) are relevant results without direct numerical resolution. For the first case, formulae (2.3), some numerical results are presented in Table 1,

Table 1. Numerical results for $E(T_\alpha)$.

α	**a**	**b**	**c**	$E(T_\alpha)$	α	**a**	**b**	**c**	$E(T_\alpha)$
1	10	1	3	32353.20	2	6	2	2	42.42
10	10	1	3	37472.41	3	6	2	2	43.47
1	10	10	3	0.45	2	6	3	2	8.11
10	10	10	3	0.73	3	6	3	2	8.46
1	10	2	3	110.51	5	6	1	2	5117.45
1	10	5	3	2.07	1	6	5	2	1.66
10	10	5	3	3.08	5	6	5	2	2.24

obtained with basis in the approximation by Simpson's rule:

$$E(T_\alpha) = \int_0^\alpha g(x)dx, \text{being } g(x)$$
$$= \frac{2\sqrt{\pi}}{c\sqrt{b}} \Phi\left(\frac{\sqrt{2}(a - bz)}{c\sqrt{b}}\right) exp\left\{\frac{(bz - a)^2}{bc^2}\right\}$$
$$\cong \frac{\alpha}{300}\left(g(0) + 4g\left(\frac{\alpha}{100}\right) + 2g\left(\frac{2\alpha}{100}\right) + 2g\left(\frac{98\alpha}{100}\right)\right.$$
$$\left. + 4g\left(\frac{99\alpha}{100}\right) + g(\alpha)\right). \blacksquare$$

3. FINANCING VALUE

As it was seen, $\rho(\alpha, y) = 1$ and $E(T_{\alpha,y}) < \infty$ with any $y < 0$ holds for the diffusion process X_t defined in section 1.. These conditions are not compatible with the interpretation advanced initially once, as it was mentioned that interpretation only make sense for $X_t \geq 0$.

So consider the reserves value evolution of the fund, under external financing, stochastic process $\tilde{X}_t$, obtained from X_t under the following financing scheme: whenever the reserves end it is donate to the fund a provision in a value equal to the initial one, that is, with value α. This corresponds to consider a reflecting barrier at 0 that produces an instantaneous reposition effect of the process in α. Having in mind that the origin diffusion is homogeneous in relation with time, the process $\tilde{X}_t$ under the action of this barrier is a regenerative process, as it reinitiate in each reflection instant, that is, regeneration epoch. Clear that will be considered $\tilde{X}_t$ paths righteous continuous, with $\tilde{X}_t = \alpha$ in any regeneration epoch $\tau > 0$.

The sequence of time intervals between regeneration epochs $\Delta_1 = T_\alpha, \Delta_2, \Delta_3, \ldots$ is a sequence of *i.i.d*. random variables and the sequence $\tau_1, \tau_2, \tau_3, \ldots$ of waiting times for the first, the second, the third, …, regeneration is so defined for the sequence of *i.i.d* random variables partial sums $\tau_1 = \Delta_1;\ \tau_n = \tau_{n-1} + \Delta_n, n = 2, 3, \ldots$.

Be now the renewal process N_t defined for the sequence, enlarged, of waiting times $\tau_0 = 0, \tau_1, \tau_2, \ldots, N_t = sup\{n: t_n \leq t\}$ through which it is possible to define in a more formal way the financing process A_t, as it was formerly described, by $A_t = \sum_{n=1}^{N_t} R_n, A_t = 0 \; if \; N_t = 0$, where $R_n = \alpha$ designates the *n-th* provision given to the fund.

The process A_t is a typical “renewal-reward” process for which is valid the Renewal Theorem, see (Tijms, 1994),

$$l_\alpha = \lim_{t\to\infty} \frac{A_t}{t} = \lim_{t\to\infty} \frac{E(A_t)}{t} = \frac{E(R_n)}{E(T_\alpha)} = \frac{\alpha}{\int_0^\alpha \frac{2\sqrt{\pi}}{c\sqrt{b}} \Phi\left(\frac{\sqrt{2}(a-bz)}{c\sqrt{b}}\right) exp\left\{\frac{(bz-a)^2}{bc^2}\right\} dz} \quad (3.1).$$

In particular, only to the protection value against the ruin risk, using L’Hospital’s Rule to determine the following iterated limit

$$l_0 = \lim_{\alpha\to 0}\lim_{t\to\infty} \frac{A_t}{t} = \lim_{\alpha\to 0}\lim_{t\to\infty} \frac{E(A_t)}{t} = \frac{1}{\frac{2\sqrt{\pi}}{c\sqrt{b}} \Phi\left(\frac{a\sqrt{2}}{c\sqrt{b}}\right) exp\left\{\frac{a^2}{bc^2}\right\}} \quad (3.2).$$

It may be useful to consider, in relation with the operation scheme presented at the beginning of this section, the placement of a reflection barrier in a point different from 0, say in $\beta, 0 < \beta < \alpha$. The adaptation of the results (3.1) and (3.2) to this situation is immediate considering $T_{\alpha,\beta}$ instead of T_α and making $R_n = \alpha - \beta$. Consequently,

$$l_{\alpha,\beta} = \lim_{t\to\infty} \frac{A_t}{t} = \lim_{t\to\infty} \frac{E(A_t)}{t} = \frac{E(R_n)}{E(T_{\alpha,\beta})} = \frac{\alpha-\beta}{\int_\beta^\alpha \frac{2\sqrt{\pi}}{c\sqrt{b}} \Phi\left(\frac{\sqrt{2}(a-bz)}{c\sqrt{b}}\right) exp\left\{\frac{(bz-a)^2}{bc^2}\right\} dz} \quad (3.3)$$

and

$$l_{\beta,\beta} = \lim_{\alpha\to\beta}\lim_{t\to\infty}\frac{A_t}{t} = \lim_{\alpha\to\beta}\lim_{t\to\infty}\frac{E(A_t)}{t} = \frac{\alpha-\beta}{\frac{2\sqrt{\pi}}{c\sqrt{b}}\Phi\left(\frac{\sqrt{2}(a-b\beta)}{c\sqrt{b}}\right)exp\left\{\frac{(b\beta-a)^2}{bc^2}\right\}} \qquad (3.4).$$

Note 3.1.

It is used the following numerical example to emphasize the utilization of the financing scheme presented in the formation of the implicit output process rate at $\tilde{X}_t$, say the process$\tilde{X}_t^-$. Remembering the observations presented ate section 1., in particular to the case $\alpha = \frac{a}{b}$, that rate will be

$$a^*(t) = \frac{dE(\tilde{X}_t^-)}{dt} \cong a + l_{a/b}$$

$$= a + \frac{a}{b\int_0^{a/b}\frac{2\sqrt{\pi}}{c\sqrt{b}}\Phi\left(\frac{\sqrt{2}(a-bz)}{c\sqrt{b}}\right)exp\left\{\frac{(bz-a)^2}{bc^2}\right\}dz} \qquad (3.5).$$

Choosing, for instance, *a* = 6, *c* = 2 and $a^*(t) \cong 1.1a = 6.6$, the expression (3.5) is fulfilled to the value *b* = 4.61.■

Note 3.2.

In this note it is presented briefly the formulation of the financing actual value when this proceeding is applicable, namely when the process described runs under the influence of a capitalization function, and the problems found in that valorisation. So when it is appropriate the application of a deterministic actualization rate, say with value $r, r > 0$, the perpetual financing actual value is the random variable

$$B_t = \sum_{n=1}^{\infty} \alpha e^{-rt_n} \qquad (3.6)$$

with expected value

$$E(B_t) = E\left(\sum_{n=1}^{\infty} \alpha e^{-rt_n}\right) = \frac{\alpha E(e^{-rT_\alpha})}{1 - E(e^{-rT_\alpha})} \qquad (3.7).$$

The resolution of (3.7) demands the knowledge of $E(e^{-rT_\alpha})$ that while function of *r* is the Laplace Transform of the first passage time T_α. Instead, consider that function dependent of α, that is, $\nu(\alpha) = E(e^{-rT_\alpha})$. It is known, see (Karlin and Taylor, 1981) and (Feller, 1971) that ν satisfies the 2nd order differential equation $\frac{1}{2}\sigma_X^2 \nu'' + \mu_X \nu' = r\nu, \text{with } \nu(0) = 1$, that is

$$\frac{c^2}{2}\nu'' + (a - b\alpha)\nu' = r\nu, \text{with } \nu(0) = 1 \qquad (3.8).$$

The equation (3.8) has a general solution expressed as a power series, see (Apostol, 1969), $\sum_{n=0}^{\infty} k_n \alpha^n$, in the form

$$\nu(\alpha) = k_0 \nu_0(\alpha) + k_1 \nu_1(\alpha) = k_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} k_{0,n} + k_1 \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} k_{1,n}$$

where the coefficients $k_{0,n}$ and $k_{1,n}$ may be obtained through the following recursive scheme

$$\begin{aligned} k_{0,0} = 1,\ k_{0,1} = 0, \ldots, k_{0,n} &= \frac{-2a}{c^2} k_{0,n-1} + \frac{2(r + (n-2)b)}{c^2} k_{0,n-2}, \\ k_{1,0} = 0,\ k_{1,1} = 1, \ldots, k_{1,n} &= \frac{-2a}{c^2} k_{1,n-1} + \frac{2(r + (n-2)b)}{c^2} k_{1,n-2} \end{aligned} \qquad (3.9).$$

The border condition $\nu(0) = 1$ implies the specialization of this solution $k_0 = 1$, so obtaining $\nu(\alpha) = \nu_0(\alpha) + k_1\nu_1(\alpha)$. To get a particular solution depends only on

$$k_1 = \nu'(0) = \lim_{\varepsilon \to 0} \frac{\nu(\varepsilon) - \nu(0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{E(e^{-rT_\varepsilon}) - 1}{\varepsilon}.$$

Important for this subject are the papers (Gerber and Pafumi, 1998) and (Figueira and Ferreira, 2000) that developed the solution of this problem in the case where the Brownian motion is the support model.■

ACKNOWLEDGMENT

The authors gratefully thank to Professor J. Figueira from ISCTE – IUL his cooperation in this paper and permission to use results from (Figueira and Ferreira, 2000) and (Figueira, 2003).

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