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# M|G|∞ SYSTEM PARAMETERS FOR A PARTICULAR COLLECTION OF SERVICE TIME DISTRIBUTIONS

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**Abstract.** In this paper we present the problems that arise when we compute the moments of service time distributions, for which the M|G|∞ busy period and busy cycle become very easy to study. We show how to overcome them. We also compute the busy cycle renewal function and the "peak" and the "modified peak" for the M|G|∞ busy period and busy cycle in the case of those service time distributions.

**Key words.** Probability Distributions, M|G|∞, Busy Period, Busy Cycle.

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## 1 Introduction

When, in the M|G|∞ queue systems, the service time is a random variable with distribution function belonging to the collection

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda},t\geq 0,-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1 \qquad (1.1)$$

the busy period duration is exponential with an atom at the origin. And the busy cycle one is the mixture of two exponential distributions (see Ferreira (2005)). But although it is so easy to study the busy period and the busy cycle in this situation it is very difficult to compute the service time moments.

We present some results, precisely, about the moments computation of the random variables with distribution functions given by that collection that generalize those from Ferreira (1998).

We end presenting formulae that give the busy cycle renewal function and the "peak" and the "modified peak" to the busy period and the busy cycle of the M|G|∞ system for those service time distributions, generalizing results from Ferreira (2004), Ferreira (1997) and Ferreira (1999).

## 2 Moments Computation

Be $G(t), t \geq 0$ a distribution functional and $g(t) = \frac{dG(t)}{dt}$. The differential equation in $G(.)$, $(1-p)\frac{g(t)}{1-G(t)} - \lambda p - \lambda(1-p)G(t) = \beta$, where $\lambda > 0$ and $-\lambda \leq \beta \leq \frac{\lambda(1-pe^{\rho})}{e^{\rho}-1}$, $0 \leq p < 1$ ($\rho = \lambda\alpha$, being $\alpha$ the mean of $G(t)$) is verified by (1.1) (see Ferreira (2005)).
If, in (1.1), $G_i(t)$ is the solution associated to $\rho_i$, $i = 1,2,3,4$ it is easy to see that

$$\frac{G_4(t)-G_2(t)}{G_4(t)-G_1(t)} \cdot \frac{G_3(t)-G_1(t)}{G_3(t)-G_2(t)} = \frac{e^{-\rho_4}-e^{-\rho_2}}{e^{-\rho_4}-e^{-\rho_1}} \cdot \frac{e^{-\rho_3}-e^{-\rho_1}}{e^{-\rho_3}-e^{-\rho_2}} \tag{2.1}$$

as it had to happen since the differential equation considered is a Ricatti one.
And computing,

$$\int_0^{\infty}[1-G(t)]dt = \int_0^{\infty} \frac{(1-e^{-\rho})\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+\lambda} dt =$$

$$= \frac{(1-e^{-\rho})\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda} \int_0^{\infty} \frac{1}{e^{-\rho}\left(e^{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}-1\right)+1} dt =$$

$$= \frac{(1-e^{-\rho})\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda} \int_0^{\infty} \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho} - e^{-\rho}e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t} + e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}} dt =$$

$$= \frac{(1-e^{-\rho})\left(\lambda + \frac{\lambda p+\beta}{1-p}\right)}{\lambda} \int_0^{\infty} \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{e^{-\rho} + (1-e^{-\rho})e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}} dt =$$

$$= \frac{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)}{\lambda} \cdot \frac{-1}{\left(1-e^{-\rho}\right)\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)} \cdot \left[\log\left(e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right)\right]_0^{\infty} =$$

$$= -\frac{1}{\lambda}\left(\log e^{-\rho} - \log 1\right) = \frac{-\rho}{-\lambda} = \alpha$$

as it had to be because we are dealing with a positive random variable.
The density associated to $G(t)$ given by (1.1) is

$$g(t) = \frac{\left(1-e^{-\rho}\right)e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2 e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\lambda\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2}, t>0, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{-\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \quad (2.2).$$

So,

$$\int_0^{\infty} t^n g(t)dt = \frac{\left(1-e^{-\rho}\right)e^{-\rho}\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^2}{\lambda} \cdot \int_0^{\infty} t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2} dt \cdot$$

But,

$$\int_0^{\infty} t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2} dt \ge \int_0^{\infty} t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t} dt =$$

$$= \frac{1}{\lambda+\frac{\lambda p+\beta}{1-p}} \frac{n!}{\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n}, \beta \ne -\lambda \cdot$$

And,

$$\int_0^{\infty} t^n \frac{e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}}{\left[e^{-\rho}+\left(1-e^{-\rho}\right)e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t}\right]^2} dt \le e^{2\rho} \int_0^{\infty} t^n e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t} dt =$$

$$= \frac{e^{2\rho}}{\lambda + \frac{\lambda p + \beta}{1-p}} \frac{n!}{\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)^n}, \beta \neq -\lambda .$$

So, calling T the random variable corresponding to $G(t)$, we have:

$$\frac{\left(1-e^{-\rho}\right)e^{-\rho}}{\lambda} \frac{n!}{\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)^{n-1}} \leq \mathrm{E}\left[\mathrm{T}^n\right] \leq \frac{e^{\rho}-1}{\lambda} \frac{n!}{\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)^{n-1}},$$

$$, -\lambda < \beta \leq \frac{\lambda\left(1-pe^{-\rho}\right)}{e^{\rho}-1}, 0 \leq p < 1, n = 1,2,... \qquad (2.3).$$

**Notes:**

The expression (2.3), giving bounds for $\mathrm{E}\left[\mathrm{T}^n\right]$, guarantees its existence,
For $n=1$ the expression (2.3) is useless because $\mathrm{E}[\mathrm{T}] = \alpha$. Note, curiously, that the upper bound is $\frac{e^{\rho}-1}{\lambda}$, the M|G|∞ system busy period mean value,
For $n=2$, subtracting to both bounds $\alpha^2$, we can get from expression (2.3) bounds for $VAR[\mathrm{T}]$,
For $\beta = -\lambda, \mathrm{E}\left[\mathrm{T}^n\right] = 0, n = 1,2,...,$ evidently.

See, however, that (1.1) can be put like:

$$G(t) = \frac{1 + \frac{\frac{\lambda p + \beta}{1-p}}{\lambda}\left(1-e^{\rho}\right)e^{-\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}}{1-\left(1-e^{\rho}\right)e^{-\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \leq p < 1 \qquad (2.4)$$

And, for $\rho < \log 2$,

$$G(t) = \left(1 + \frac{\frac{\lambda p + \beta}{1-p}}{\lambda}\left(1-e^{\rho}\right)e^{-\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t}\right) \cdot \sum_{k=0}^{\infty}\left(1-e^{\rho}\right)^k e^{-k\left(\lambda + \frac{\lambda p + \beta}{1-p}\right)t},$$

$$, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \leq p < 1 \qquad (2.5).$$

After (2.5) we can derive easily the T Laplace Transform for $\rho < \log 2$. And, so,

- For $\rho < \log 2$

$$\mathrm{E}\left[\mathrm{T}^n\right] = -\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right) n! \sum_{k=1}^{\infty} \frac{\left(1-e^{\rho}\right)^k}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)^n}, -\lambda < \beta \leq \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \leq p < 1,$$
$$, n = 1,2,... \quad (2.6).$$

**Notes:**

$$\mathrm{E}[\mathrm{T}] = -\left(1+\frac{\frac{\lambda p+\beta}{1-p}}{\lambda}\right) \sum_{k=1}^{\infty} \frac{\left(1-e^{\rho}\right)^k}{k\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)} = \frac{1}{\lambda}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\left(1-e^{\rho}\right)}{k} =$$

$$= \frac{1}{\lambda}\log e^{\rho} = \frac{\rho}{\lambda} = \alpha .$$

For $n \geq 2$ we must take only a finite number of parcels in the infinite sum. Calling M this number, to get an error lesser than $\varepsilon$ we must have simultaneously

- $\mathrm{M} > \dfrac{1}{\lambda+\frac{\lambda p+\beta}{1-p}} - 1,$

- $\mathrm{M} > \log_{\left(e^{\rho}-1\right)} \dfrac{\varepsilon e^{\rho}\lambda}{n!\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)} - 1 .$

So it is evident now that this distributions collection moments computation is a complex task. This was already true for the study of Ferreira (1998) where the results presented are a particular situation of these ones for $p = 0$.

The consideration of the approximation

$$\mathrm{E}_m^n = \sum_{k=1}^{\infty}\left(\frac{k}{m}\right)^n\left[G\left(\frac{k}{m}\right)-G\left(\frac{k-1}{m}\right)\right], -\lambda < \beta \leq \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \leq p < 1, n = 1,2,... \quad (2.7)$$

may be helpful since $\lim_{m\to\infty}\mathrm{E}_m^n = \mathrm{E}\left[\mathrm{T}^n\right], n = 1,2,...$ (see Ferreira (1987)) that allow the numerical computation of the moments.

## 3 Busy Cycle Renewal Function Computation

The busy cycle (an idle period followed by a busy period) renewal function value of the M|G|∞ queue, at $t$, gives the mean number of busy periods that begin in $[0,t]$ (see Ferreira (2004)). If the service is a random variable with distribution function given by a member of the collection (1.1), calling the value of the renewal function at $t$ $R(t)$, we have

$$R(t) = e^{-\rho}(1+\lambda t) + \left(1-e^{-\rho}\right)\frac{\lambda p+\beta}{\lambda+\beta} e^{-\left(\lambda+\frac{\lambda p+\beta}{1-p}\right)t} + \left(1-e^{-\rho}\right)\frac{\lambda p+\beta}{\lambda+\beta}, -\lambda \le$$

$$\le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \quad (3.1).$$

For $p=0$ we get the result presented in Ferreira (2004).

## 4 The "Peak" and the "Modified Peak" for the Busy Period and for the Busy Cycle

The M|G|∞ queue busy period "peak" is the Laplace Transform of its duration at $1/\alpha$ (Ferreira (1997)). It is a parameter that characterizes the busy period distribution duration and contains information about all its moments. For the collection of service distributions (1.1) the "peak", named $pi$, is

$$pi = \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{\lambda\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \quad (4.1).$$

In Ferreira (1997) we also introduce another measure, the "modified peak" got after the "peak" taking out the terms that are permanent for the busy period in different service distributions and putting over the common part. Calling it $qi$ we have $qi = pi\frac{\rho}{e^{\rho}-\rho-1}+1$ and, so, for the distributions given by collection (1.1) we have

$$qi = \frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p+\beta}{\lambda\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}\frac{\rho}{e^{\rho}-\rho-1}+1, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},$$

$$0 \le p < 1 \quad (4.2).$$

For the busy cycle of the M|G|∞ queue we have also in this manner the "peak" (Ferreira (1999)), that we call now $pi'$, and for the service distributions given by the collection (1.1) it is

$$pi' = \alpha\frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{(\rho+1)\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}, -\lambda \le \beta \le \frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1}, 0 \le p < 1 \quad (4.3)$$

and the "modified peak", that we call now $qi'$, given by $pi'\frac{\rho}{e^{\rho}-\rho}+1$, and for the service distributions given by the collection (1.1) it is

$$qi'=\alpha\frac{e^{-\rho}(\lambda+\beta)(\rho+1)-\lambda p-\beta}{(\rho+1)\left(e^{-\rho}(\rho+\alpha\beta)+1-p\right)}\frac{\rho}{e^{\rho}-\rho}+1,-\lambda\leq\beta\leq\frac{\lambda\left(1-pe^{\rho}\right)}{e^{\rho}-1},0\leq p<1 \qquad (4.4).$$

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