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A NOTE ON THE EVALUATION OF A STORAGE SYSTEM USING AN ELEMENTARY RENEWAL EQUATION

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Abstract. We consider a storage system where a stock of k units is replaced instantaneously, when needed. A Poisson type stochastic demand is admitted. It is verified that the expected present value function of the replenishment cost of the storage system satisfies a defective renewal equation for which an asymptotic expansion is proposed.

Keywords. Renewal equation. Stochastic inventory theory. Present value function.

1 Introduction

Suppose that the demand of some article has the common Poisson probability mass function with parameter μ ($\mu > 0$):

$$\mathrm{P}(\text{demand up to time } t = x) = \frac{e^{-\mu t}(\mu t)^x}{x!}; \quad x = 0, 1, \ldots; \quad t > 0 .$$

It is well known that the duration time X of a stock of k units then has the gamma probability distribution

$$F(x) = \int_{-\infty}^{x} f(y)\, dy, \quad f(x) = \frac{\mu^k x^{k-1} e^{-\mu x}}{(k-1)!} 1_{(x>0)}, \tag{1a}$$

with Laplace transform

$$\mathrm{E}\, e^{-sX} = \left(\frac{\mu}{s+\mu}\right)^k = \phi^k(s), \quad s > -\mu . \tag{1b}$$

Consider now the storage system working in the following way. At time $t = 0$ there are available in stock k units. Every time the system goes out of stock an amount of k units is again instantaneously replaced.

We are interested in the evaluation of such system through the stochastic process $W_k(t)$ and the random variable V_k defined below:

$$W_k(t) = \sum_{n=1}^{N(t)} \theta_k \prod_{m=1}^{n} e^{-rX_m}\ , \quad W_k(t) = 0 \text{ if } N(t) = 0\,, \tag{2a}$$

$$V_k = W_k(\infty) = \lim_{t\to\infty} W_k(t)\,. \tag{2b}$$

What is the motivation for these definitions? $W_k(t)$ is the present value of the stock replenishment cost, up to time t, when a payment of θ_k is made each time k units are replaced. The parameter r ($r > 0$) represents a deterministic discount interest rate. $X_1, X_2, \ldots$ is a sequence of iid random variables which have the probability distribution of X. These variables represent the time intervals between successive stock replacements. $N(t)$ is the associated counting process:

$$N(t) = \sup\left\{ n : \sum_{i=1}^{n} X_i \le t \right\}, \quad N(t) = 0 \text{ if } X_1 > t\,.$$

Finally, V_k is the perpetual cost of the storage system.

We shall consider in particular the expectations of (2a) and (2b):

$$w_k(t) = \mathrm{E}\, W_k(t)\,, \quad v_k = \mathrm{E}\, V_k\,. \tag{3}$$

Remark 1. We may consider in the place of θ_k, the cost of k units, some time dependent value with exponential growth of the form $\theta_k = \theta_k(t) = \tilde{\theta}_k\, e^{\tilde{r}\,t}$. In that case, the present value process of the stock replenishment cost would assume the form

$$W_k(t) = \sum_{n=1}^{N(t)} \tilde{\theta}_k \prod_{m=1}^{n} e^{-(r-\tilde{r})X_m}\ , \quad W_k(t) = 0 \text{ if } N(t) = 0\,.$$

This kind of assumption doesn't change very much the analysis of (2a) and (2b), provided that $\tilde{r} < r$. All we have to do is the substitution of r by ($r - \tilde{r}$) in the formulas given above ◊

Remark 2. There is no need to restrict ourselves to the case where θ_k represents the cost of k units. We may consider that it represents, for instance, the global result of the acquisition and commercialisation of k items. As an example, we admit that the form $\theta_k = bk - a$ could represent such result. In this suggestion, a is a fixed cost associated to any operation concerning the reposition of stocks and b represents a unitary margin associated with buying and selling one article. (Obviously, we should have in this case $b > a/k$.) ◊

2 Main results

Proposition 1.

$$v_k = \frac{\theta_k \, \phi^k(r)}{1-\phi^k(r)} . \tag{4}$$

Proof. Note that V_k is a random perpetuity (Embrechts and Goldie, 1994, and Vervaat, 1979) which may be written as the solution of the random equation

$$V_k \overset{d}{=} e^{-rX}(\theta_k + V_k), \quad X \text{ and } V_k \text{ independent,}$$

where the identity is referred to identity in probability distribution. Taking expectations in both sides of this equation produces the result in the proposition ◊

Proposition 2. The expected value function $w_k(t)$ satisfies the defective renewal equation (Feller, 1971)

$$w_k(t) = \theta_k \phi^k(r) F(t) + \int_0^t w_k(t-s)\phi^k(r) f(s)\, ds . \tag{5}$$

Proof. Conditioning to $N(t)$ and applying the expectation tower property, we get

$$w_k(t) = \mathrm{E}\,\mathrm{E}(W_k(t) | N(t)) = \theta_k \phi^k(r) \frac{1-\gamma(t,\phi^k(r))}{1-\phi^k(r)}, \tag{6}$$

where $\gamma(t,s)$ is the probability generating function of $N(t)$. Write F^{*n} as the *n*-fold convolution of F with itself and assume $F^{*0}(t)=1$ for all $t \ge 0$. Then we have:

$$\gamma(t,s) = \mathrm{E}\, s^{N(t)} = \sum_{n=0}^{\infty} s^n (F^{*n}(t) - F^{*(n+1)}(t)) = 1-(1-s)\sum_{n=1}^{\infty} s^{n-1} F^{*n}(t).$$

If we now make the substitution in (6) of this result, we arrive to:

$$w_k(t) = \theta_k \, \phi^k(r) \sum_{n=1}^{\infty} \phi^{k(n-1)}(r)\, F^{*n}(t) .$$

Taking Laplace transforms of both sides leads us to

$$\int_0^{\infty} e^{-st} w_k(t) dt = \frac{\theta_k \, \phi^k(r) \phi^k(s)}{s(1-\phi^k(r)\phi^k(s))}, \tag{7}$$

which after rearranging and inversion returns (5) ◊

Now let $J(t) = w_k(\infty) - w_k(t)$. Note that $w_k(\infty) = v_k$. Let

$$j(t) = \frac{\theta_k \, \phi^k(r)}{1-\phi^k(r)} (1-F(t)) .$$

Consider also the positive constant ρ that satisfies

$$\int_0^\infty e^{\rho s} \phi^k(r) f(s) \, ds = \phi^k(r) \, \phi^k(-\rho) = 1 .$$

We mean (recalling (1b)): $\rho = r\mu/(r+\mu)$.

Proposition 3. It's valid for the function $e^{\rho t} J(t)$ the following asymptotic expansion:

$$\lim_{t\to\infty} e^{\rho t} J(t) = \frac{1}{\mu_o} \int_0^\infty e^{\rho s} j(s) \, ds , \tag{8a}$$

$$\mu_o = \int_0^\infty s \, e^{\rho s} \phi^k(r) f(s) \, ds . \tag{8b}$$

Proof. After the computation of $J(t)$, starting from (5), and multiplying the equation obtained by $e^{\rho t}$, we get the following common renewal equation:

$$e^{\rho t} J(t) = e^{\rho t} j(t) + \int_0^t e^{\rho(t-s)} J(t-s) e^{\rho s} \phi^k(r) f(s) \, ds .$$

The proposition is just a consequence of the application of the key renewal theorem (Feller, 1971, or Tijms, 1994) to this equation ◊

Remark 3. Note that the conclusions in the preceding propositions remain valid (with obvious minor adaptations) for many continuous probability distributions of X, at least, if they have support on the nonnegative reals. Next, we specify those results for the special case of the initially defined gamma distribution. Making repeated use of the Laplace transform in (1b), the right side in (8a) may be written as:

$$\frac{1}{\mu_o} \cdot \int_0^\infty e^{\rho s} j(s) \, ds = \frac{1}{\frac{k(r+\mu)}{\mu^2}} \cdot \frac{\theta_k (r+\mu)}{r\mu} = \frac{\theta_k \, \mu}{k\, r} .$$

With this result in hand, we may summarise propositions 1 and 3 by way of the expressions:

$$v_k = \frac{\theta_k}{\alpha^k - 1}, \quad \alpha = \frac{r}{\mu} + 1 , \tag{9a}$$

$$w_k(t) \approx \frac{\theta_k}{\alpha^k - 1} - \frac{\theta_k}{k(\alpha - 1)} e^{-\frac{r}{\alpha} t} . \tag{9b}$$

The approximation in (9b) should be understood with the sense given in (8a). For a more detailed exposition of the different passages and results in this section see Figueira and Ferreira (2001) ◊

3 Exploratory examples

Example 1. Consider the case $k = 1$. Then (9b) is

$$w_1(t) = \frac{\theta_1}{\alpha - 1}(1 - e^{-\frac{r}{\alpha}t}).$$

The right side has Laplace transform:

$$\int_0^\infty e^{-st} w_1(t)\, dt = \frac{\theta_1 r}{s(s\alpha + r)(\alpha - 1)}.$$

Computing the right side in (7) we get the same result. As a matter of fact, in this case, the given expression is exact ◊

Example 2. We shall consider now the suggestion in remark 2: $\theta_k = bk - a$. In particular, for the set of parameters $\mu = a = b = 1$ and $r = 0{,}02$, (9a) is:

$$v_k = \frac{k - 1}{1{,}02^k - 1}.$$

The most interesting value of k is then $\max_k \{v_k\} = 10$. For this specialization we obtain:

$$w_{10}(t) \approx \frac{9}{1{,}02^{10} - 1} - 45\, e^{-\frac{t}{101}}.$$

The following table illustrates some of the values of this function.

t	10	20	50	100	200	500	∞
$w_{10}(t)$	0,339	4,181	13,668	24,378	34,885	40,778	41,097

◊

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