

# Costs Study through a Diffusion Process of Pensions Funds Held with an Outside Financing Effort

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## Abstract

Pensions funds not auto financed and systematically maintained with an outside financing effort are considered in this work. Representing the unrestricted reserves value process of this kind of funds, a time homogeneous diffusion process with finite expected time till the ruin is proposed. It is also admitted a financial tool that regenerates the diffusion, at some level with positive value every time it hits a barrier at the origin. Then the financing effort may be modeled as a renewal-reward process if the regeneration level is kept constant. The perpetual maintenance cost expected values evaluation and of the finite time period maintenance cost are studied. An application of this approach, when the unrestricted reserves value process behaves as a generalized Brownian motion process, is presented.

**Keywords:** Pensions fund, diffusion process, first passage times, renewal equation

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## 1 Introduction

The protection cost present value expectation for a non-autonomous pension's fund is considered in this work. Two contexts are considered:

- The protection effort is perpetual,
- The protection effort happens for a finite time period.

It is admitted that the unrestricted fund reserves behavior may be modeled as a time homogeneous diffusion process. Then a regeneration scheme of the diffusion to include the effect of an external financing effort is used.

In Gerber and Parfumi [2] a similar work is presented. A Brownian motion process conditioned by a particular reflection scheme was considered there. With fewer constraints, but in different conditions, exact solutions were then obtained for both problems.

The work presented in Refait [1], on asset-liability management aspects, motivated the use of the Brownian motion application example in that domain.

Part of this work is considered in Ferreira [9]. Other works on this subject are Figueira [3] and Figueira and Ferreira [4].

## 2 Pensions Fund Reserves Behavior Stochastic Model

Be  $X(t), t \geq 0$  the reserves value process of a pensions fund given by an initial reserve amount  $a, a > 0$  added to the difference between the total amount of contributions received and the total amount of pensions paid both up to time  $t$ . It is assumed that  $X(t)$  is a time homogeneous diffusion process, with  $X(0) = a$ , defined by drift and diffusion coefficients  $\mu(x)$  and  $\sigma^2(x)$ , respectively.

Call  $S_a$  the first passage time of  $X(t)$  by 0, coming from  $a$ . The funds to be considered in this work are non-autonomous funds. So

$$E[S_a] < \infty, \text{ for any } a > 0 \quad (2.1),$$

that is: funds where the pensions paid consume in finite expected time any initial positive reserve and the contributions received. Then other financing resources are needed in order that the fund survives.

The condition (2.1) may be fulfilled for a specific diffusion process using criteria based on the drift and diffusion coefficients. In this context, here the work presented in Bhattacharya and Waymire [13], pg. 418-422, is followed. So accept that  $P(S_a < \infty) = 1$  if the diffusion scale function is

$$q(x) = \int_{x_0}^x e^{-\int_{x_0}^z \frac{2\mu(y)}{\sigma^2(y)} dy} dz,$$

where  $x_0$  is a diffusion state space fixed arbitrary point, fulfilling  $q(\infty) = \infty$ .

Then the condition (2.1) is equivalent to  $p(\infty) < \infty$ , where

$$p(x) = \int_{x_0}^x \frac{2}{\sigma^2(z)} e^{\int_{x_0}^z \frac{2\mu(y)}{\sigma^2(y)} dy} dz,$$

is the diffusion speed function.

It is admitted that whenever the exhaustion of the reserves happens an external source places instantaneously an amount  $\theta, \theta > 0$  of money in the fund so that it may keep on performing its function.

The reserves value process conditioned by this financing scheme is denoted by the modification  $\check{X}(t)$  of  $X(t)$  that restarts at the level  $\theta$  whenever it hits 0. As  $X(t)$  was defined as a time homogeneous diffusion,  $\check{X}(t)$  is a regenerative process. Call  $T_1, T_2, T_3, \dots$  the sequence of random variables where  $T_n$  denotes the  $n^{\text{th}}$   $\check{X}(t)$  passage time by 0. It is obvious that the sequence of time intervals between these hitting times  $D_1 = T_1, D_2 = T_2 - T_1, D_3 = T_3 - T_2, \dots$  is a sequence of independent random variables where  $D_1$  has the same probability distribution as  $S_a$  and  $D_2, D_3, \dots$  the same probability distribution as  $S_\theta$ .

### 3 First Passage Times Laplace Transforms

Call  $f_a(s)$  the probability density function of  $S_a$  (related to  $D_1$ ). The corresponding probability distribution function is denoted by  $F_a(s)$ . The Laplace transform of  $S_a$  is denoted  $\varphi_a(\lambda)$ .

Consequently, the density, distribution and transform of  $S_\theta$  (related to  $D_2, D_3, \dots$ ) will be denoted by  $f_\theta(s), F_\theta(s)$  and  $\varphi_\theta(\lambda)$ , respectively.

The transform  $\varphi_a(\lambda)$  satisfies the second order differential equation

$$\frac{1}{2}\sigma^2(a)u_\lambda''(a) + \mu(a)u_\lambda'(a) = \lambda u_\lambda(a), \quad (3.1),$$

$$u_\lambda(a) = \varphi_a(\lambda), u_\lambda(0) = 1, u_\lambda(\infty) = 0,$$

see Bass [12], pg. 89, Karlin and Taylor [14], pg. 243, and Feller [15], pg. 478.

### 4 Perpetual Maintenance Cost Present Value

Consider the perpetual maintenance cost present value of the pension's fund given by the random variable

$$V(r, a, \theta) = \sum_{n=1}^{\infty} \theta e^{-rT_n}, r > 0,$$

where  $r$  represents the appropriate discount rate. Note that  $V(r, a, \theta)$  is a random perpetuity. What matters is its expected value which is easy to calculate using Laplace transforms. Since the  $T_n$  Laplace transform is

$$E[e^{-\lambda T_n}] = \varphi_a(\lambda)\varphi_\theta^{n-1}(\lambda),$$

$$v_r(a, \theta) = E[V(r, a, \theta)] = \frac{\theta\varphi_a(r)}{1 - \varphi_\theta(r)} \quad (4.1).$$

It is relevant to note<sup>2</sup> that

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<sup>2</sup> Using the alternative notation, that seems more convenient now.

$$\lim_{\theta \rightarrow 0} v_r(a, \theta) = \frac{u_r(a)}{-u_r'(0)} \quad (4.2).$$

## 5 Finite Time Period Maintenance Cost Present Value

Define the renewal process  $N(t) = \sup\{n: T_n \leq t\}$ , generated by the extended sequence  $T_0 = 0, T_1, T_2, \dots$ . The present value of the pensions fund maintenance cost up to time  $t$  is represented by the stochastic process

$$W(t; r, a, \theta) = \sum_{n=1}^{N(t)} \theta e^{-rT_n}, \quad W(t; r, a, \theta) = 0 \text{ if } N(t) = 0.$$

To calculate the expected value function of the process evaluation:  $w_r(t; a, \theta) = E[W(t; r, a, \theta)]$ , begin to note that it may be expressed as a numerical series. In fact, evaluating the expected value function conditioned by  $N(t) = n$ , it is obtained

$$E[W(t; r, a, \theta) | N(t) = n] = \theta \varphi_a(r) \frac{1 - \varphi_\theta^n(r)}{1 - \varphi_\theta(r)}.$$

Repeating the expectation:

$$w_r(t; a, \theta) = E[E[W(t; r, a, \theta) | N(t)]] = \theta \varphi_a(r) \frac{1 - \gamma(t, \varphi_\theta(r))}{1 - \varphi_\theta(r)} \quad (5.1)$$

where  $\gamma(t, \xi)$  is the probability generating function of  $N(t)$ .

Denote now the  $T_n$  probability distribution function by  $G_n(s)$  and assume  $G_0(s) = 1$ , for  $s \geq 0$ . Recalling that  $P(N(t) = n) = G_n(t) - G_{n+1}(t)$ , the above mentioned probability generating function is

$$\gamma(t, \xi) = \sum_{n=0}^{\infty} \xi^n P(N(t) = n) = 1 - (1 - \xi) \sum_{n=1}^{\infty} \xi^{n-1} G_n(t) \quad (5.2).$$

Substituting (5.2) in (5.1),  $w_r(t; a, \theta)$  is expressed in the form of the series

$$w_r(t; a, \theta) = \theta \varphi_a(r) \sum_{n=1}^{\infty} \varphi_\theta^{n-1}(r) G_n(t) \quad (5.3).$$

Call the  $w_r(t; a, \theta)$  ordinary Laplace transform  $\psi(\lambda)$ . The probability distribution function  $G_n(s)$ , of  $T_n$ , ordinary Laplace transform is given  $\varphi_a(\lambda) \frac{\varphi_a^{n-1}(\lambda)}{\lambda}$  and performing the Laplace transforms in both sides of (5.3) it is obtained

$$\psi(\lambda) = \frac{\theta \varphi_a(r) \varphi_a(\lambda)}{\lambda (1 - \varphi_\theta(r) \varphi_\theta(\lambda))}$$

or

$$\psi(\lambda) = \theta \varphi_a(r) \frac{\varphi_a(\lambda)}{\lambda} + \psi(\lambda) \varphi_\theta(r) \varphi_\theta(\lambda) \quad (5.4).$$

By Laplace transforms inversion in both sides of (5.4) the following defective renewal equation results:

$$w_r(t; a, \theta) = \theta \varphi_a(r) F_a(t) + \int_0^t w_r(t-s; a, \theta) \varphi_\theta(r) f_\theta(s) ds \quad (5.5).$$

Now an asymptotic approximation of  $w_r(t; a, \theta)$  will be obtained through

the key renewal theorem, following Feller [15], pg. 376.

If in (5.5)  $t \rightarrow \infty$

$$w_r(\infty; a, \theta) = \theta\varphi_a(r) + w_r(\infty; a, \theta)\varphi_\theta(r) \quad (5.6)$$

or

$$w_r(\infty; a, \theta) = \frac{\theta\varphi_a(r)}{1 - \varphi_\theta(r)} = v_r(a, \theta).$$

This is the expression (4.1) for  $v_r(a, \theta)$ . Subtracting each side of (5.6) from each side of (5.5), and performing some elementary calculations the following, still defective, renewal equation

$$J(t) = j(t) + \int_0^t J(t-s)\varphi_\theta(r)f_\theta(s)ds \quad (5.7)$$

where

$$J(t) = w_r(\infty; a, \theta) - w_r(t; a, \theta) \text{ and } j(t) = \theta\varphi_a(r)(1 - F_a(t)) + \frac{\theta\varphi_a(r)\varphi_\theta(r)}{1 - \varphi_\theta(r)}(1 - F_\theta(t)).$$

Now, to obtain a common renewal equation from (5.7), it must be admitted the existence of a value  $k > 0$  such that

$$\int_0^\infty e^{ks}\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta(-k) = 1.$$

This imposes that the transform  $\varphi_\theta(\lambda)$  is defined in a domain different from the one initially considered, that is a domain including a convenient subset of the negative real numbers.

Multiplying both sides of (5.7) by  $e^{kt}$  the common renewal equation desired is finally obtained:

$$e^{kt}J(t) = e^{kt}j(t) + \int_0^t e^{k(t-s)}J(t-s)e^{ks}\varphi_\theta(r)f_\theta(s)ds$$

from which, through the application of the key renewal theorem, it results

$$\lim_{t \rightarrow \infty} e^{kt}J(t) = \frac{1}{k_0} \int_0^\infty e^{ks}j(s)ds \quad (5.8)$$

with  $k_0 = \int_0^\infty se^{ks}\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta'(-k)$ , since  $e^{kt}j(t)$  is directly Riemann integrable. The integral in (5.8) may expressed in terms of transforms as

$$\int_0^\infty e^{ks}j(s)ds = \frac{\theta\varphi_a(r)\varphi_a(-k)}{k}.$$

So, in this section:

An asymptotic approximation, in the sense of (5.8) was obtained:

$$w_r(t; a, \theta) \approx v_r(a, \theta) - c_r(a, \theta)e^{-k_r(\theta)t} \quad (5.9)$$

where  $k_r(\theta)$  is the positive value of  $k$  that satisfies

$$\varphi_\theta(r)\varphi_\theta(-k) = 1 \quad (5.10)$$

and

$$c_r(a, \theta) = \frac{\theta\varphi_a(r)\varphi_a(-k_r(\theta))}{-k_r(\theta)\varphi_\theta(r)\varphi_\theta(-k_r(\theta))} \quad (5.11).$$

## 6 The Brownian Motion Example

Suppose that the diffusion process  $X(t)$ , underlying the reserves value behavior of the pensions fund, is a generalized Brownian motion process, with drift  $\mu(x) = \mu, \mu < 0$  and diffusion coefficient  $\sigma^2(x) = \sigma^2, \sigma > 0$ . Observe that the setting satisfies the conditions that were assumed above in this work. Namely  $\mu < 0$  implies condition (2.1). Everything else remaining as previously stated, it will be proceeded to present the consequences of this particularization. In general it will be added a  $(*)$  to the notation used before because it is intended to use these specific results later.

To obtain the first passage time  $S_a$  Laplace transform, remember (3.1), it must be solved the equation:

$$\frac{1}{2}\sigma^2(a)u_\lambda^{*''}(a) + \mu(a)u_\lambda^{*'}(a) = \lambda u_\lambda^*(a), u_\lambda^*(a) = \varphi_a(\lambda), u_\lambda^*(0)=1, u_\lambda^*(\infty) = 0.$$

This is a homogeneous second order differential equation with constant coefficients, which general solution is

$$u_\lambda^*(a) = \beta_1 e^{\alpha_1 a} + \beta_2 e^{\alpha_2 a}, \text{ with } \alpha_1, \alpha_2 = \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}.$$

Condition  $u_\lambda^*(\infty) = 0$  implies  $\beta_1 = 0$  and  $u_\lambda^*(0)=1$  implies  $\beta_2=1$  so that the particular solution is achieved:

$$u_\lambda^*(a) = e^{-K_\lambda a} (= \varphi_a^*(\lambda)), K_\lambda = \frac{\mu + \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2} \quad (6.1).$$

In this case, the perpetual maintenance cost present value of the pensions fund is given by, following (4.1) and using (6.1),

$$v_r^*(a, \theta) = \frac{\theta e^{-K_r a}}{1 - e^{-K_r \theta}} \quad (6.2).$$

Note that  $v_r^*(a, \theta)$  is a decreasing function of the first variable and an increasing function of the second. Proceeding as before, in particular:

$$\lim_{\theta \rightarrow 0} v_r^*(a, \theta) = \frac{e^{-K_r a}}{K_r} \quad (6.3).$$

This expression has been obtained in Gerber and Parfumi [2], in a different context and using different methods but, obviously, with identical significance. In Gerber and Parfumi [2] the authors worked then with a generalized Brownian

motion, with no constraints in what concerns the drift coefficient, conditioned by a reflection scheme at the origin.

A way to reach an expression for the finite time period maintenance cost present value, is starting by the computation of  $k_r^*(\theta)$ , solving (5.10). This means to determine a positive number  $k$  satisfying

$$e^{-K_r\theta}e^{-K_{-\lambda}\theta} = 1 \text{ or } K_r + K_{-\lambda} = 0.$$

This identity is verified for the value of  $k$

$$k_r^*(\theta) = \frac{\mu^2 - (-2\mu - \sqrt{\mu^2 + 2r\sigma^2})^2}{2\sigma^2}, \text{ if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \quad (6.4).$$

Note that the solution is independent of  $\theta$  in these circumstances. A simplified solution, independent from  $a$  and  $\theta$ , for  $c_r^*(a, \theta)$  was also obtained. Using (5.11) the result is

$$c_r^*(a, \theta) = \frac{2\sigma^2(-2\mu - \sqrt{\mu^2 + 2r\sigma^2})}{\mu^2 - (-2\mu - \sqrt{\mu^2 + 2r\sigma^2})^2} \quad (6.5).$$

Merging these results, (6.4) and (6.5), as in (5.9) it is observable that the asymptotic approximation for this particularization reduces to  $w_r^*(t; a, \theta) \approx v_r^*(a, \theta) - \pi_r(t)$ , where the function  $\pi_r(t)$  is, considering (6.4) and (6.5),

$$\begin{aligned} \pi_r(t) &= \frac{2\sigma^2(-2\mu - \sqrt{\mu^2 + 2r\sigma^2})}{\mu^2 - (-2\mu - \sqrt{\mu^2 + 2r\sigma^2})^2} e^{-\frac{\mu^2 - (-2\mu - \sqrt{\mu^2 + 2r\sigma^2})^2}{2\sigma^2}t}, \text{ if } \mu \\ &< -\sqrt{\frac{2r\sigma^2}{3}} \quad (6.6). \end{aligned}$$

## 7 The Assets And Liability Behavior Representation

In this section it is presented an application of the results obtained above to an asset-liability management scheme of a pension's fund. Assume that the assets value process of a pensions fund may be represented by the geometric Brownian motion process

$$A(t) = be^{a+(\rho+\mu)t+\sigma B(t)} \text{ with } \mu < 0 \text{ and } a\rho + \mu\sigma > 0,$$

where  $B(t)$  is a standard Brownian motion process. Suppose also that the fund liabilities value process performs such as the deterministic process  $L(t) = be^{\rho t}$ .

Consider now the stochastic process  $Y(t)$  obtained by the elementary transformation of  $A(t)$

$$Y(t) = \ln \frac{A(t)}{L(t)} = a + \mu t + \sigma B(t).$$

This is a generalized Brownian motion process exactly as the one studied before, starting at  $a$ , with drift  $\mu$  and diffusion coefficient  $\sigma^2$ . Note also that the first passage time of the assets process  $A(t)$  by the mobile barrier  $T_n$ , the

liabilities process, is the first passage time of  $Y(t)$  by 0-with finite expected time under the condition, stated before,  $\mu < 0$ .

Consider also the pensions fund management scheme that raises the assets value by some positive constant  $\theta_n$ , when the assets value falls equal to the liabilities process by the  $n^{\text{th}}$  time. This corresponds to consider the modification  $\bar{A}(t)$  of the process  $A(t)$  that restarts at times  $T_n$  when  $A(t)$  hits the barrier  $L(t)$  by the  $n^{\text{th}}$  time at the level  $L(T_n) + \theta_n$ . For purposes of later computations, it is a convenient choice the management policy where

$$\theta_n = L(T_n)(e^\theta - 1), \text{ for some } \theta > 0 \quad (7.1).$$

The corresponding modification  $\tilde{Y}(t)$  of  $Y(t)$  will behave as a generalized Brownian motion process that restarts at the level  $\ln \frac{L(T_n) + \theta_n}{L(T_n)} = \theta$  when it hits 0 (at times  $T_n$ ).

Proceeding this way, it is reproduced via  $\tilde{Y}(t)$  the situation observed before when the Brownian motion example was treated. In particular the Laplace transform in (6.1) is still valid.

Similarly to former proceedings, the results for the present case will be distinguished with the symbol (#). It is considered the pensions fund perpetual maintenance cost present value, as a consequence of the proposed asset-liability management scheme, given by the random variable:

$$V^\#(r, a, \theta) = \sum_{n=1}^{\infty} \theta_n e^{-rT_n} = \sum_{n=1}^{\infty} b(e^\theta - 1)e^{-(r-\rho)T_n}, r > \rho$$

where  $r$  represents the appropriate discount interest rate. To obtain the above expression it was only made use of the  $L(t)$  definition and (7.1). Note that it is possible to express the expected value of the above random variable with the help of (6.2) as

$$v_r^\#(a, \theta) = \frac{b(e^\theta - 1)}{\theta} v_{r-\rho}^*(a, \theta) = \frac{b(e^\theta - 1)e^{-K_{r-\rho}a}}{1 - e^{-K_{r-\rho}\theta}} \quad (7.2).$$

As  $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} v_r^\#(a, \theta) = \frac{be^{-K_{r-\rho}a}}{K_{r-\rho}} \quad (7.3),$$

another expression that may be found in Gerber and Parfumi [2].

In a similar way, the maintenance cost up to time  $t$  in the above mentioned management scheme, is the stochastic process

$W^\#(t; r, a, \theta) = \sum_{n=1}^{N(t)} b(e^\theta - 1)e^{-(r-\rho)T_n}$ ,  $W^\#(t; r, a, \theta) = 0$  if  $N(t) = 0$ , with expected value function

$$w_r^\#(t; a, \theta) = \frac{b(e^\theta - 1)}{\theta} w_{r-\rho}^*(t; a, \theta) \quad (7.4).$$

The results of section 6 with  $r$  replaced by  $r - \rho$  may be combined as in (7.4) to obtain an asymptotic approximation.



## 8 Concluding Remarks

In the general diffusion setting, the main results are formulae (4.1) and (5.9). The whole work depends on the equation (3.1) solvability, in order to obtain the first passage times Laplace transforms. But the known solutions happen only in very rare cases. An obvious case, for which the equation solution is available, is the Brownian motion diffusion process. The main results concerning this particularization are formulae (6.2) and (6.6). Certain Brownian motion process transformations, that allowed making use of the available Laplace transform, may be explored as it was done in section 7. Formulae (7.2) and (7.4) are this case most relevant results.

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