Journal of Applied Mathematics

volume V (2012), number III

SOJOURN TIMES IN JACKSON NETWORKS

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Abstract. Jackson queuing networks have a lot of practical applications, mainly in the modelling of computation and telecommunications networks. Evidently the time that one customer - a person, a job, a message … – spends in this kind of systems, its sojourn time, is an important measure of its performance. In this work the practical known results about the sojourn time distribution are collected and presented.

Key words: Jackson networks, sojourn time, randomisation procedure.

Mathematics Subject Classification: 60K35.

1 Introduction

In this work it is intended to present some problems and results that arise in the study of the sojourn time in Jackson networks of queues. These networks have many applications, namely in the modelling of computation and telecommunications networks. And a customer sojourn time, in this kind of system, is evidently an important element to be considered in its performance evaluation.

The model of network to be considered in this paper is briefly described in section 2. The main objective of section 3 is the presentation of formula (10) that, in some situations allows the sojourn times moments exact computation. In section 4 it is given a numerical method for the sojourn times distribution function and any order moments computation, adequate to any Jackson network.

2 General Results and Examples

Along this work the sojourn times in a class of Markovian networks of queues, introduced initially by Jackson, see (Jackson, 1957-1963), will be studied. They are called Jackson networks and have only one class of customers. They are composed of J nodes numbered $1,2,...,J$. It is usual to put $U=\{1,2,...,J\}$.

In each node there is

- Only one server,
- A queue discipline "first-come-first-served" (FCFS),
- An infinite waiting capacity.

They are open networks: any customer may enter and leave.

The exogenous arrivals process at node j is a Poisson Process at rate $\nu_j, j \in U$, independent of the exogenous arrivals processes to the other nodes: $\nu = \sum_{j=1}^{J} \nu_j$.

The service times at node j are independent and identically distributed, having exponential distribution with parameter $\mu_j, j \in U$, and independent from the other nodes service times.

After the completion of a service at node j, a customer is immediately directed to node l with probability p_{jl}, or abandons the network with probability $q_j = 1 - \sum_{l=1}^{J} p_{jl}, j \in U$.

These probabilities are not influenced by the movements of the other customers in the network. The p_{jl} matrix is called P.

The total arrivals rate, exogenous and endogenous, at node j, θ_j satisfies the network traffic equations:

$$\theta_j = \nu_j + \sum_{l=1}^{J} \theta_l p_{lj}, j = 1,2,...,J \qquad (1).$$

The state of the network at instant t is given by $N(t) = [N_1(t),..., N_J(t)]$, where $N_j(t)$ is the number of customers at node j in instant t, $j = 1,2,...,J$.

If $\rho_j = \frac{\theta_j}{\mu_j} < 1, j = 1,2,...,J$ the process $N = \{N(t)\}$ has stationary, or equilibrium, distribution, see for instance (Disney and König, 1985),

$$\pi(n_1, n_2,..., n_J) = \prod_{j=1}^{J} (1 - \rho_j) \rho_j^{n_j}, n_j \geq 0, j = 1,2,...,J \qquad (2).$$

Calling S_j, W_j and X_j the sojourn, waiting and service, respectively, times of a customer at node j

$$S_j = W_j + X_j \qquad (3).$$

The Jackson networks sojourn times considered in this paper are those of typical customers that, arriving at the network, find the process in an equilibrium state. Call S the sojourn time in the network, that is: the time that goes between the arrival at the network and the departure of one of those customers. If in its path it traverses the nodes $1,2,...,l$, $S = S_1 + S_2 + ... + S_l$.

To study the sojourn time, the following is important:

- A network has “feedback” if a customer may come back to the same node after the completion of its service, immediately or in a future instant,
- A network without “feedback” is an “acyclic” one.

Then some examples of typical Jackson networks usually considered in the study of sojourn times are presented.

Simple Queues Series

For this Jackson network

$$p_{jl} = \begin{cases} 1, & \text{if } l = j+1, j = 1,2,...,J-1 \\ 0 & \text{otherwise} \end{cases}$$

$\nu_1 = \nu,\ \nu_j = 0, j = 2,..,J$ and $\theta_{\mathrm{j}} = \nu, j = 1,2,...,J$. Figure 1 is a graphical representation of a simple queues series.

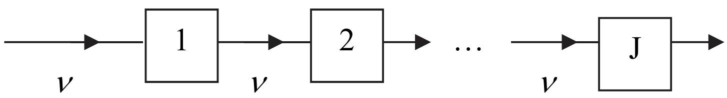

Figure 1: **Simple Queues Series**

Some important results are:

- All customers’ flows, in this network, at stationary state, are Poisson Processes. It is a consequence of it, in stationary state, that the departure process from an M/M/1 queue is a Poisson Process, see for instance (Kelly, 1979),
- The sojourn times in the various nodes are independent random variables. In (Kelly, 1979) it is presented a demonstration of this statement based on the reversibility concept,
- The sojourn time at node j is an exponential random variable with parameter $\mu_j - \nu, j = 1,2,...,J$,
- The waiting times are dependent random variables. See also (Kelly, 1979).

So the sojourn time study in these networks has no difficulty. The same is not true for the waiting time.

M/M/1 Queue with Instantaneous Bernoulli Feedback

It is a network with a single node. $J=1$, $p_{11}=p$, $q_1=1-p$ and $\theta=\frac{\nu}{1-\rho}$, where $\theta=\theta_1$ and $\nu=\nu_1$, see Figure 2.

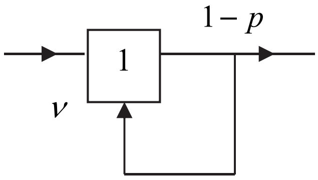

Figure 2: **M/M/1 Queue with Instantaneous Bernoulli Feedback**

Call S_m the m^{th} customer sojourn time in the network. So, if it is served k times

$$S_m=\left(t_{m1}^0-t_{m1}^a\right)+\left(t_{m2}^0-t_{m2}^i\right)+\ldots+\left(t_{mk-1}^0-t_{mk-1}^i\right)+\left(t_{mk}^d-t_{mk}^i\right) \quad (4)$$

where

- $t_{ml}^0-t_{ml}^i$ is the time that the customer spends passing by the service system in the l^{th} time, given by the difference between the l^{th} output (0) instant from the server and the one of the l^{th} junction (*i*) to the queue,
- $t_{m1}^0-t_{m1}^a$ is the time that the customer spends passing by the service system for the first time, given by the difference between the first output (0) instant from the server and the one of the arrival (*a*) to the queue,
- $t_{mk}^d-t_{mk}^i$ is the time that the customer spends passing by the service system for the last time, given by the difference between the departure (*d*) instant from the network and the one of the k^{th} junction (*i*) to the queue.

Note that K, the number of times that the customer passes by the server, is a random variable and $P(K=k)=p(1-p)^{k-1}$, $k=1,2,\ldots$.

$\left\{\left(t_{ml}^0-t_{ml}^i\right): l=2,2,\ldots\right\}$ is not a sequence of independent random variables, see (Disney and König, 1985). So it is not possible to make use of the usual statement to sum independent random variables. But it is possible to get an expression to $P(S_m\le t)$ that requires the k steps transition probabilities for the delayed Markovian renewal process $\left\{\left(N\left(t^i-0\right),\left(t_l^o-t_l^i\right)\right), l=01,2,\ldots\right\}$ conditioning to the number of times that the customer returns to the queue.

Calling that transition probabilities matrix $Q_i^k(t)$, see still (Disney and König, 1985),

$$P(S_m \le t) = \sum_{k=1}^{\infty} \pi\, Q_i^k(t)p(1-p)V \qquad (5)$$

where π is the N^i (embedded version of N in the input instants) stationary distribution, k is the number of times the customer passes by the server and V is a vector which entries are all 1.

So, now, the situation is much more complicated than in the former case owing to the feedback.

The Jackson Three Node Acyclic Network

It is a network with three nodes where $p_{12} = p$, $p_{13} = 1 - p$, $p_{23} = 1$, $p_{jl} = 0$ in the other cases, $\nu_1 = \nu$, $\nu_j = 0, j = 2,3$, $\theta_1 = \nu$, $\theta_2 = p\nu$ and $\theta_3 = \nu$.

In equilibrium, all customers' flows are Poisson Process in this network.

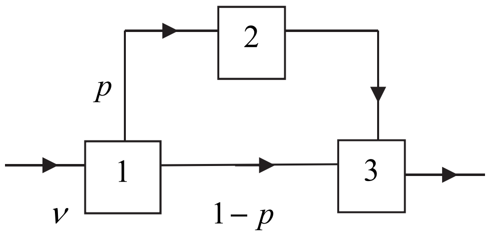

Figure 3: **Jackson Three Node Acyclic Network**

Consequently,

- The sojourn time at node j is a random variable exponentially distributed with parameter $\mu_j - \theta_j, j = 1,2,3$. S_1 and S_2 are independent random variables as well as S_2 and S_3.

This result is valid for any Jackson acyclic network:

- Suppose that a customer follows a path r in a Jackson acyclic network with only one server at each node. If node j belongs to path r, S_j is such that

$$P\left(S_j \le t \middle| \textit{the followed path is } r\right) = 1 - e^{-(\mu_j - \theta_j)t}, t \ge 0 \qquad (6)$$

and, if node j is the next to the server after node l, S_j and S_l are independent random variables.

But,

- S_1 and S_3 are dependent random variables: (Foley and Kiessler, 1989) showed that, in fact, S_1 and S_3 are positively correlated. (Ferreira, 1998) showed that if

$$2(\nu+\mu_1+\mu_2+\mu_3)\left(\frac{1}{\nu+\mu_1+\mu_2+\mu_3}\left(\frac{1}{\mu_1-\nu}+\frac{1}{\mu_3-\nu}\right)+\frac{1}{(\mu_1-\nu)^2}+\frac{1}{(\mu_3-\nu)^2}\right)^{\frac{1}{2}}<1 \quad (7)$$

and if

$$\frac{\mu_2\left(\frac{1}{2(\nu+\mu_1+\mu_2+\mu_3)}-\frac{1}{2}\left(\frac{1}{(\nu+\mu_1+\mu_2+\mu_3)^2}+\frac{4}{\nu+\mu_1+\mu_2+\mu_3}\left(\frac{1}{\mu_1-\nu}+\frac{1}{\mu_3-\nu}\right)-4\left(\frac{1}{(\mu_1-\nu)^2}+\frac{1}{(\mu_3-\nu)^2}\right)\right)^{\frac{1}{2}}\right)}{1-\nu\left(\frac{1}{2(\nu+\mu_1+\mu_2+\mu_3)}-\frac{1}{2}\left(\frac{1}{(\nu+\mu_1+\mu_2+\mu_3)^2}+\frac{4}{\nu+\mu_1+\mu_2+\mu_3}\left(\frac{1}{\mu_1-\nu}+\frac{1}{\mu_3-\nu}\right)-4\left(\frac{1}{(\mu_1-\nu)^2}+\frac{1}{(\mu_3-\nu)^2}\right)\right)^{\frac{1}{2}}\right)}<p<$$

$$<\frac{\mu_2\left(\frac{1}{2(\nu+\mu_1+\mu_2+\mu_3)}+\frac{1}{2}\left(\frac{1}{(\nu+\mu_1+\mu_2+\mu_3)^2}+\frac{4}{\nu+\mu_1+\mu_2+\mu_3}\left(\frac{1}{\mu_1-\nu}+\frac{1}{\mu_3-\nu}\right)-4\left(\frac{1}{(\mu_1-\nu)^2}+\frac{1}{(\mu_3-\nu)^2}\right)\right)^{\frac{1}{2}}\right)}{1-\nu\left(\frac{1}{2(\nu+\mu_1+\mu_2+\mu_3)}+\frac{1}{2}\left(\frac{1}{(\nu+\mu_1+\mu_2+\mu_3)^2}+\frac{4}{\nu+\mu_1+\mu_2+\mu_3}\left(\frac{1}{\mu_1-\nu}+\frac{1}{\mu_3-\nu}\right)-4\left(\frac{1}{(\mu_1-\nu)^2}+\frac{1}{(\mu_3-\nu)^2}\right)\right)^{\frac{1}{2}}\right)} \quad (8)$$

verify both simultaneously it is possible to guarantee that S_1 and S_3 are positively correlated in equilibrium.

There are two alternative paths for a customer to go from node 1 to node 3. And a customer that follows by node 2 may be overtaken by another one that goes directly from node 1 to node 3. So, a customer, when arriving at node 3, may meet there another one that was behind it at node 1 or even that had not arrived when it was there.

These overtaking customers can delay a certain customer, when it arrives at node 3, for a longer time than that if they were not present. The number of these customers depends, partly, on the number of the customers that arrive while the customer that is being followed is in node 1, partly owing to the supposition of a FCFS discipline. Consequently, the time that a customer waits at node 3 depends on how much time it has waited at node 1.

3 Network Flow Equations

The objective of this section is to present the so called "network flow equations" for the Jackson networks, that allow the deduction of formulae to the computation of sojourn times moments of any order, efficient in some situations.
Following the work of (Lemoine, 1987) call τ_j an arrival instant, endogenous or exogenous, at node j and $\tau_j + T_j$ the departure instant from the network of the customer that arrived in $\tau_j, j = 1, 2, \dots, J$, so

- T_j is the remaining sojourn time, in the network, for the arrival at node j in the instant $\tau_j, j = 1, 2, \dots, J$.

Call h_j the Laplace Transform of the $T_j, j = 1, 2, \dots, J$ distribution. As *N* is a strong Markov Process, and the network state process "seen by the arrivals" is in equilibrium, the $T_j, j = 1, 2, \dots, J$ and its Laplace Transforms are uniquely determined.

Dealing with the sojourn time as the life time of a Markov Process ϑ – as it will be seen in section 4 – it is possible to show that the Laplace Transforms $h_j, j = 1, 2, \dots, J$ satisfy an equations system called the "network flow equations". That is, according with (Lemoine, 1987)

- Being H_j the probability distribution with Laplace Transform h_j, there is a distribution probability with Laplace Transform q_j such as

$$h_j(s) + \frac{s g_j(s)}{\mu_j - \theta_j} = q_j(s) + \sum_{l=1}^{J} p_{jl} h_l(s), s \geq 0 \text{ and } j = 1, 2, \dots, J \qquad (9).$$

In Jackson networks without "overtaking" the Transforms h_j and g_j are identical for each j.
Given $h_j, j = 1, 2, \dots, J$ the Transforms $g_j, j = 1, 2, \dots, J$ are uniquely determined by (9). The converse is also true since *I* – *P*, being *I* the identity matrix, is invertible.

After (9), by successive derivations, (Lemoine, 1987) showed that

- **Network Flow Equations**

  For $j = 1, 2, \dots, J$ and $r = 1, 2, \dots$

$$E[T_j^r] = r!\,(\mu_j - \theta_j)^{-r} + \sum_{l=1}^{J} p_{jl} E[T_l^r] + \sum_{l=1}^{J} p_{jl} \sum_{n=1}^{r-1} \frac{r!}{n!\,(r-n)!} \mu_j^{-n} E\left[T_l^{r-n} \prod_{m=1}^{n} \left(N_j(T_l^-) + m\right)\right] \qquad (10).$$

For *r* = 1, (10) assumes the matrix form

$$\left[E[T_j]\right] = (I - P)^{-1} \left[(\mu_j - \theta_j)^{-1}\right] \qquad (11).$$

For *r* = 2 (10) assumes the form

$$E[T_j^2] = 2(\mu_j - \theta_j)^{-2} + \sum_{l=1}^{J} p_{jl} E[T_l^2] + 2\mu_j^{-1} \sum_{l=1}^{J} p_{jl} E\left[T_l\left(N_j(T_l^-) + 1\right)\right], j = 1, 2, \dots, J \qquad (12).$$

Equality (12) defines a system of J equations and $J^2 + J$ unknowns. In general, when $r \geq 2$, the product terms involving the variables T_l and $N_j(\tau_l^-)$ prevent the exact computation of the sojourn times *r* order moments; there are too many unknowns and too few equations. In these cases other independent equations are needed to complement (12) in order to be possible to obtain exact solutions.

When any pair of nodes in the network is connected by, in the maximum, one oriented path and $p_{jj} = 0, j = 1, 2, \dots, J$, T_l and $N_j(\tau_l^-)$ are independent for $j \neq l$. The computation of $E\left[T_j\left(N_j\left(\tau_j^-\right) + 1\right)\right]$ is irrelevant since $p_{jj} = 0, j = 1, 2, \dots, J$. In this case (10) becomes a compact recursive formula that allows the computation of any order moments of the sojourn times, $T_j, j = 1, 2, \dots, J$. For instance, as, in these conditions,

$$E\left[N_j(\tau_l^-)\right] = \frac{\theta_j}{\mu_j - \theta_j}, j = 1, 2, \dots, J \qquad (13),$$

(12) assumes the form

$$E\left[T_j^2\right] = 2\left(\mu_j - \theta_j\right)^{-2} + \sum_{l=1}^{J} p_{jl} E[T_l^2] + 2\left(\mu_j - \theta_j\right)^{-1} \sum_{l=1}^{J} p_{jl} E[T_l], j = 1, 2, \dots, J \qquad (14).$$

Applying (14) to the simple queues series

$$E\left[T_j^2\right] = 2\left(\mu_j - \nu\right)^{-2} + E[T_{l+1}^2] + 2\left(\mu_j - \nu\right)^{-1} E[T_{l+1}], \qquad j = 1, 2, \dots, J - 1$$
$$E\left[T_J^2\right] = 2\left(\mu_J - \nu\right)^{-2} \qquad (15)$$

Putting together (15) and (11) it may be concluded that

$$VAR\left[T_j\right] = \sum_{l=j}^{J} (\mu_l - \nu)^{-2} \qquad (16).$$

For Jackson networks that do not fulfil those conditions, in (Lemoine, 1987) it is suggested to identify adequate Martingale families in *N* as a process to determine independent equations to complement (10). Applying this proceeding to the M/M/1 queue with instantaneous Bernoulli feedback it was obtained

$$VAR[T] = \frac{1}{\left((1 - p)\mu - \nu\right)^2} \frac{(1 - p^2)\mu + \nu p}{(1 - p^2)\mu - \nu p} \qquad (17)$$

and

$$COV[N(\tau^-), T] = \frac{\nu(1 - p)\mu}{(1 - p^2)\mu - \nu p} \qquad (18).$$

4 Sojourn Times Distributions and Moments Numerical Computation

Now it will be described a general method, which key is the proceeding called, in the English language literature "randomisation procedure", to approximate "first passage times" distributions in direct time Markov Processes, being the sojourn times in queue systems a particular case.

Call $\aleph = \{X(t): t \geq 0\}$ a regular Markov Process, in continuous time with a countable states space E and a bounded matrix infinitesimal generator Q.

The elements of Q are designated by $Q(x,y), x, y \in E$ and $Q(x) = \sum_{y \in E-\{x\}} Q(x,y)$. $\psi(t)$ designates the $X(t)$ state probability vector:

$$\psi_t(x) = P\{X(t) = x\}, x \in E \qquad (19).$$

X models the evolution of a queue system during the sojourn of a given, "marked", customer in it.

The states of E have two main components:

i) The queue system state,
ii) The "marked" customer position.

Be

- A the states subset that describes the system till the departure of the "marked" customer, and

- B the states subset that describes the system after the departure of that customer.

Evidently

- $\{A, B\}$ is a partition of E,

- If T is the time that the process $\aleph$ spends in A till attaining B, for the first time, T is precisely the sojourn time of the "marked" customer in the network.

It is supposed that $\aleph$ will remain in B, with probability 1 after having attained it for the first time. In fact, as the evolution of the system after the departure of the "marked" customer is irrelevant, it may be supposed that B is a closed set. That is, the process $\aleph$ cannot come back to A after reaching B. The quantity of interest is the T distribution function, $\tau(t)$. Note that

$$\tau(t) = P\{T \leq t\} = P\{X(t) \in B\} = 1 - P\{X(t) \in A\}, t \geq 0 \qquad (20)$$

since the presented hypotheses guarantee that $\{T \leq t\} = \{X(t) \in B\}$.

After (20) it is concluded that

- The problem of computing $\tau(t)$ is equivalent to the one of the computation of the transient distribution of $X(t)$ in A.

So it is necessary to compute the vector $\psi_t, t \geq 0$. Being $P_t, t \geq 0$, the $\aleph$ n transition matrix,

$$\psi_t = \psi_0 P_t, t \geq 0 \qquad (21)$$

and

$$P_t = exp(Qt) = \sum_{i=0}^{\infty} \frac{t^i}{i!} Q^i, t \geq 0 \qquad (22).$$

The “randomisation procedure” consists in using in (22) an equivalent representation; see (Çinlar, 1975):

$$P_t = exp(-\alpha t) exp\left(\alpha t \left(I + \frac{1}{\alpha} Q\right)\right) = exp(-\alpha t) \sum_{i=0}^{\infty} \frac{\alpha^i t^i}{i!} R^i \qquad (23)$$

where

$$R = I + \frac{1}{\alpha} Q \qquad (24)$$

is called the “randomised matrix” in English language literature,

- I is the identity matrix, and
- α is a positive upper bound for the whole $Q(x), x \in E$.

Note that, see (Melamed and Yadin, 1984, 1984a),

- Although the equation (23) seems more complex than (22), it fulfils in fact more favourable computational properties. The most important is that *R* is a stochastic matrix while *Q* is not. Consequently, the computation using (23) is stable and using (22) is not,
- The “randomisation procedure” has an interesting probabilistic meaning, useful to determine bounds for $\tau(t)$. In fact, being *R* a stochastic matrix, it defines a discrete time Markov Process

$$\Im = \{Y_n : n = 0, 1, \dots\} \qquad (25)$$

  if it is assumed $Y_0 = X(0)$. With this procedure, the relation between the processes $\aleph$ and $\Im$ is quite simple as it will be seen next.

Extend the discrete time process $\Im$ to a continuous time Markov Process such that

i) The time intervals between jumps are exponential random variables i.i.d. with mean $1/\alpha$
ii) The jumps are commanded by *R*.

In (Melamed and Yadin, 1984) it is shown that the resulting process is precisely the original process $\aleph$; but when there is a sequence of jumps in $\Im$ from the state $x \in E$ to itself, this will be noticed in $\aleph$ as a long sojourn in state x.

So, the “randomisation procedure” may be interpreted as a sowing in the process $\aleph$ with “fake” random jumps between the true jumps. The resulting process, designated by $\overline{\aleph}$, at which the “fake” jumps are visible, has the same probabilistic structure than $\aleph$ but with an advantage:

- The sequence of the jump instants in $\overline{\aleph}$, “fake” and “true”, is now a Poisson Process. This is not, in general, the case of $\aleph$.

Note that Y_n is the state of $\overline{\aleph}$ in the instant of the n^{th} jump, "fake" or "true".

Suppose that $\overline{\aleph}$ reaches the set B in its n^{th} jump. Consequently the $\overline{\aleph}$ sojourn time, and so also the $\aleph$, in A is the sum of n exponential independent random variables with mean $1/\alpha$. That is, the sojourn time has a n order Erlang distribution with parameter α. Its distribution function will be designated $E_{n,\alpha}(t)$.

Be $h(n)$ the probability that $\overline{\aleph}$ reaches B in its n^{th} jump. Call ϕ_n the state probability vector of Y_n:

$$\phi_n = \psi_0 R^n \qquad (26).$$

The quantities $h(n)$ are given by the equivalent formulae:

$$h(n) = \begin{cases} \sum_{x\in B} \phi_0(x), n = 0 \\ \sum_{x\in A}\sum_{y\in B} \phi_{n-1}(x)R(x,y), n > 0 \end{cases} \qquad (27)$$

or

$$h(n) = \begin{cases} 1 - \sum_{x\in A} \phi_0(x), n = 0 \\ \sum_{x\in A} \phi_{n-1}(x) - \sum_{x\in A} \phi_n(x), n > 0 \end{cases} \qquad (28).$$

Given the probabilities $h(n)$ and, noting that $\sum_{n=0}^{\infty} h(n) = 1$, it is obtained

$$\tau(t) = \sum_{n=0}^{\infty} h(n)E_{n,\alpha}(t), t \geq 0 \qquad (29),$$

$$E[T^m] = \frac{1}{\alpha^m}\sum_{n=0}^{\infty} n(n+1)\dots(n+m-1)h(n), m = 1,2,\dots \qquad (30).$$

The formula (30) for $m = 1$ is

$$E[T] = \frac{1}{\alpha}E[H] \qquad (31)$$

being H the number of $\aleph$ jumps till reaching B. Expression (31) is the Little's Formula in this queues context.

Equation (29) allows obtaining simple bounds for $\tau(t)$ that may, in principle, to become arbitrarily close. Equation (30) allows obtaining $E[T^k]$, in principle, so close of $E[T^k]$ as wished. So, given any integer $k \geq 0$

$$L_k(t) \leq \tau(t) \leq U_k(t) \qquad (32)$$

where

$$L_k(t) = \sum_{n=0}^{k} h(n)E_{n,\alpha}(t), t \geq 0 \qquad (33),$$

$$U_k(t) = 1 - \sum_{n=0}^{k} h(n)\bar{E}_{n,\alpha}(t), t \geq 0 \tag{34}$$

and

$$E[T^m]_{L,k} \leq E[T^m], \;\; m = 1, 2, \ldots. \tag{35}$$

where

$$E[T^m]_{L,k} = \frac{1}{\alpha^m}\sum_{n=0}^{k} n(n+1)\ldots(n+m-1)h(n), m = 1, 2, \ldots \tag{36}.$$

It is easy to prove that

Proposition

If, for any $\varepsilon > 0$, k is chosen in accordance with the rule

$$k = min\left\{n \geq 0: \sum_{i=0}^{n} h(i) \geq 1 - \varepsilon\right\} = k(\varepsilon), \tag{37}$$

or equivalently

$$J = min\left\{n \geq 0: \sum_{x \in A} \phi_n(x) \leq \varepsilon\right\} = J(\varepsilon), \tag{38}$$

$$\left|\tau(t) - L_{J(\varepsilon)}\right| \leq \varepsilon \text{ and } \left|\tau(t) - U_{J(\varepsilon)}\right| \leq \varepsilon, \text{uniformely in } t \geq 0. \blacksquare$$

The main problem in the application of the method presented, that in principle would solve any computation problems related to the distribution of sojourn times, stays in the difficulty of the $h(n)$ computation. In fact, for it, it is necessary to compute the vectors ϕ_n but only in the subset *A* of the states space. When states space *E* is finite, as it happens in the case of closed networks, both $h(n)$ and ϕ_n can, at first glance, be computed exactly, apart the mistakes brought by the approximations.

In practice the states space is often infinite or, although finite, prohibitively great. In this situations it is mandatory to truncate *E*. So, it must be considered a new level of approximation since the $h(n)$, ϕ_n, etc. must also be approximated now.
In fact, what are viable to obtain is $h(n)$ lower bounds because the *E* truncation is translated in probability loss (Melamed and Yadin, 1984a). So, with these $h(n)$ approximate values, (32) and (35) go on being valid but

- The uniform convergence property seen above is lost,

- The rules analogous to (37) and (38) are not equivalent. The one generated by (37) may be even unviable and in practice it is used only the one generated by (38) (Melamed and Yadin, 1984a).

Using this method (Kiessler *et al.*, 1988) achieved to show that, in a Jackson three node acyclic network, the total sojourn time distribution function for a customer that follows the path integrated by the nodes 1, 2, and 3 is not the same obtained considering that S_1, S_2 and S_3 are independent although this one, designated by $F(t)$, is a "good" approximation of that one. They show that in some particular cases it was not true that

$$F^L(t) \leq F(t) \leq F^U(t), t \geq 0 \qquad (39)$$

being $F^L(t)$ and $F^U(t)$ the lower bound and the upper bound, respectively, of that customer sojourn time distribution function, obtained through the described method.

This conclusion is important because, in spite of the dependence between S_1 and S_3, $F(t)$ could be the S distribution function. In fact, (Feller, 1966) presents an example of dependent random variables which sum has the same distribution as if the random variables were independent.

Finally note that the formula (30), apparently new, seems to be more efficient than (10), although only allows to obtain moments lower bounds, because its field of application is much greater.

5 Conclusions

The sojourn time has an evident practice interest. And is and has been intensively studied. Evidently the problem of the computation of the sojourn times in networks of queues is one of the most difficult in these networks study. In fact, analytic solutions are the exception and not the rule. And, when existing, are quite rough.

The most of the known works only present results on sojourn time distributions for only one customer in paths without overtaking with FCFS disciplines in the nodes. It seems that still there are not results for simultaneous distributions of various customers sojourn times.

It follows, from the examples seen in section 2, that the sojourn times, at Jackson networks computations, difficulties occur when there are feedback and overtaking. In the first case the input server process is not a Poisson Process, becoming everything more complex. In the second case dependencies exist among a customer sojourn times in the various nodes, simultaneously complicated and subtle, that make the total sojourn time computation difficult even if the sojourn times in each node are easy to compute.

From all this it results the interest of the methods presented in sections 3 and 4 to compute exactly and approximately the quantities related with the Jackson networks sojourn times.

Acknowledgement

This work was financially supported by FCT through the Strategic Project PEst-OE/EGE/UI0315/2011.

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