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THE M/G/ ∞ QUEUE BUSY PERIOD DISTRIBUTION EXPONENTIALITY

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Abstract. Infinite servers queuing systems are often used to solve a large number of practical problems, namely in Engineering, Biology, Management, Finance and Sociology. The study of the M/G/∞ systems busy period is very important. Yet it is a very difficult task. This paper presents a service time distributions collection for which the length of the busy period is almost exponential. It is also shown that in heavy-traffic conditions, for a certain class of service time distributions, the length of the busy period of the M/G/∞ systems is approximately exponentially distributed. Thus, in all these cases, the study of this busy period is much simpler. It is also exemplified, with the power function and the Pareto service distributions, that it is possible to have in some particular situations an approximately exponential behaviour for the M/G/∞ queue busy period length.

Key words. M / G / ∞ queue, busy period, exponential distribution, power function distribution, Pareto distribution.

Mathematics Subject Classification: 60K35.

1 Introduction

The M / G / ∞ queue is a system where customers arrive according to a Poisson process at rate λ. Each customer receives a service which length is a positive random variable with mean α and distribution function $G(.)$, and $\alpha = \int_0^\infty [1 - G(t)]dt$. When a customer arrives its service starts at once (i.e., there are infinite servers) and it is independent both from the services of the other customers and from the arrival process. The quantity $\rho - \lambda\alpha$ is the traffic intensity.

Infinite servers queuing systems are often used to solve a large number of practical problems, namely in Engineering, Biology, Management, Finance and Sociology. See, for instance, (Syski, 1960, 1986), (Kleinrock, 1975), (Kelly, 1979), (Hershey, Weiss and Morris, 1981), (Carrillo, 1981), (Figueira and Ferreira, 1999) and (Ferreira, Andrade and Filipe, 2009).

In this paper some more results will be given about the busy period of the M / G / ∞ queue.

For any queuing system, a busy period begins when a customer arrives at the system, finding it empty, ends, when a customer leaves the system, letting it empty, and in it there is always at least one customer present. So, in a queuing system, there is a sequence of idle and busy periods.

The length of an idle period of a M / G / ∞ queue is a random variable, called I, which distribution is exponential with parameter λ, as happens in any queue where the arrival process is Poisson.

For the random variable B, length of the busy period of the M / G / ∞ queue, the situation is not so simple.
Tackács (1962) studied the busy cycle, a busy period followed by an idle period, for the M / G / ∞ queue. Its length is a random variable Z and, evidently, $Z = B + I$. That author showed that B and I are independent and deduced the expression for the Laplace-Stieltjes transform of Z. Using this expression and the fact that B and I are independent, (Stadje, 1985) deduced the expression

$$\overline{B}(s) = 1 + \lambda^{-1}\left(s - \frac{1}{\int_0^\infty e^{-st-\lambda\int_0^t[1-G(v)]dv}\,dt} \right) \tag{1}$$

for the Laplace-Stieltjes transform of B and it is possible to show that

$$E[B] = \frac{e^{\rho} - 1}{\lambda} \tag{2}$$

which is independent of the form of $G(.)$. This is not, however, valid for higher order moments.

The inversion of (1) to obtain $b(t)$, the probability density function of B, is very hard to carry out.

For a particular service distributions collections, see (Ferreira, 1998), it will be shown that it is easy to invert (1). And simple expressions for $B(t)$ related to the exponential distribution will be obtained. Then, it is deduced that for a large class of service distributions, since ρ and α are great enough, $b(t)$ is approximately exponential.

Finally, service distributions cases for which does not occur necessarily $\lim_{\alpha\to\infty} G(t) = 0$, as it happens with power function and Pareto ones, are studied in order to identify situations for which B has a similar behaviour.

2 The M/G/∞ Systems with Almost Exponential Busy Periods

Lemma 1

If

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda},\qquad t\geq 0,\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \tag{3}$$

B is exponentially distributed at rate $e^{-\rho}(\lambda+\beta)$, with an atom at the origin which value is $\dfrac{e^{-\rho}(\lambda+\beta)-\beta}{\lambda}$.

Demonstration

Substituting (3) in (1) and inverting the consequent $\overline{B}(s)$, it is obtained the following probability density function:

$$b(t)=\frac{e^{-\rho}(\lambda+\beta)-\beta}{\lambda}\delta(t)+\left(1-\frac{e^{-\rho}(\lambda+\beta)-\beta}{\lambda}\right)e^{-\rho}(\lambda+\beta)e^{-e^{-\rho}(\lambda+\beta)t},\ \ \mathrm{t}\geq 0\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}$$

($\delta(.)$ is the Dirac function).

Notes

- For $\beta=0$, $\overline{B}(s)=e^{-\rho}+\left(1-e^{-\rho}\right)\dfrac{\lambda e^{-\rho}}{\lambda e^{-\rho}+s}$ and then

$$b(t)=e^{-\rho}\delta(t)+\left(1-e^{-\rho}\right)\lambda e^{-\rho}e^{-\lambda e^{-\rho}t}.$$

- Note that for this distribution the Laplace-Stieltjes transform of Z is

$$\overline{Z}(s)=\frac{\lambda}{\lambda+s}\left(e^{-\rho}+\left(1-e^{-\rho}\right)\frac{\lambda e^{-\rho}}{\lambda e^{-\rho}+s}\right)=\frac{\lambda e^{-\rho}}{\lambda e^{-\rho}+s}.$$

That is: Z is exponentially distributed at rate $\lambda e^{-\rho}$ and so the points in time at which begin busy periods occur according to a Poisson process at rate $\lambda e^{-\rho}$.

- For $\beta=\dfrac{\lambda}{e^{\rho}-1}$, $\overline{B}(s)=\dfrac{\lambda\left(e^{-\rho}-1\right)^{-1}}{s+\lambda\left(e^{-\rho}-1\right)^{-1}}$ and B is exponential with mean $\dfrac{e^{\rho}-1}{\lambda}$. Now the situation for Z is not as simple as for $\beta=0$. It is easy to show, following the same procedures as in the precedent case that the probability density function of Z is

$$z(t) = \frac{\lambda}{e^{\rho} - 2}\left(e^{-\frac{\lambda}{e^{\rho}-1}t} - e^{-\lambda t}\right), \quad t \geq 0 .$$

3 Distributions of *B* when the Service Distribution is such that $\lim_{\alpha \to \infty} G(t) = 0$

An example of a distribution for which $\lim_{\alpha \to \infty} G(t) = 0$ is the exponential one.

From (1), see for instance (Stadje, 1985) for the distribution function of B the following expression is obtained

$$B(t) = 1 - \lambda^{-1} \sum_{n=1}^{\infty} \left[\frac{e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1-G(t)dt)}{1-e^{-\rho}}\right]^{*n} .(1-e^{-\rho})^n, t \geq 0 \tag{4}$$

where* is the operator convolution.

Fixing λ, if $\lim_{\alpha\to\infty} G(t) = 0, 1 - G(t) \cong 1$ for α great enough. And, if ρ is great enough, then $1 - e^{-\rho} \cong 1$. Thus

$$\frac{e^{-\lambda \int_0^t [1-G(v)]dv} \lambda(1-G(t))dt}{1-e^{-\rho}} \cong \lambda e^{-\lambda t}$$

and

$$B(t) \cong 1 - \lambda^{-1} \sum_{n=1}^{\infty} \left(\lambda e^{-\lambda t}\right)^{*n} \left(1-e^{-\rho}\right)^n .$$

The Laplace-Stieltjes transform of the second member is

$$\frac{1}{s}\frac{\lambda e^{-\rho}}{s+\lambda e^{-\rho}} + \frac{e^{-\rho}}{s+\lambda e^{-\rho}}$$

and, after its inversion, it is obtained

$$1-\left(1-e^{-\rho}\right)e^{-\lambda e^{-\rho} t} .$$

So it results that in such conditions $B(t) = 1 - e^{-\lambda e^{-\rho} t}$. Then

Lemma 2

For service distributions verifying $\lim_{\alpha\to\infty} G(t) = 0,$ fixing λ, for α and ρ great enough, B is approximately exponentially distributed.

In order to feel the meaning of α and ρ great enough, the coefficient of variation of B, $CV[B]$ the coefficient of symmetry of B, $\gamma_1[B]$ and the kurtosis of B, $\gamma_2[B]$, were computed for the

M/G$_1$/∞ (service distribution given by (3) for $\beta = 0$), M/D/∞ (service distribution constant) and M/M/∞ (service distribution exponential) systems for $\rho = .5, 1, 10, 20, 50, 100$ with $\lambda = 1$ (therefore $\rho = \alpha$).

According to (Kendall and Stuart, 1979)

$$CV[B] = \frac{\sqrt{E[B^2] - E^2[B]}}{E[B]} \quad (5),$$

$$\lambda_1[B] = \frac{\left(E[B^3] - 3E[B]E[B^2] + 2E^3[B]\right)^2}{\left(E[B^2] - E^2[B]\right)^3} \quad (6)$$

and

$$\lambda_2[B] = \frac{\left(E[B^4] - 4E[B]E[B^3] + 6E^2[B]E[B^2] - 3E^4[B]\right)^2}{\left(E[B^2] - E^2[B]\right)^2} \quad (7).$$

For an exponential distribution, $CV = 1$, $\gamma_2 = 4$ and $\gamma_2 = 9$.

From **Lemma 1** with $\beta = 0$, to the M/G$_1$/∞ queue

$$E[B^n] = \left(1 - e^{-\rho}\right)\frac{n!}{\left(\lambda e^{-\rho}\right)^n}, n = 1,2,... \quad (8).$$

With $n = 1$ (2) results from (8).

For the remaining systems, note that (1) is equivalent to $\left(\overline{B}(s) - 1\right)\left(C(s) - 1\right) = \lambda^{-1} s C(s)$ being $C(s) = \int_0^\infty e^{-st - \lambda \int_0^t [1 - G(v)]dv} \lambda(1 - G(t))dt$. Differentiating n times, using Leibnitz's formula and making $s = 0$, results

$$E[B^n] = (1)^{n+1}\left\{\frac{e^\rho}{\lambda} n C^{(n-1)}(0) - e^\rho \sum_{\rho=1}^{n-1} (-1)^{n-p} \binom{n}{p} . E[B^{n-p}] C^{(p)}(0)\right\}, n = 1,2,... \quad (9)$$

being

$$C^{(n)}(0) = \int_0^\infty (-t)^n e^{-\lambda \int_0^t [1 - G(v)]dv} \lambda(1 - G(t))dt, n = 0,1,2,... \quad (10).$$

The expression (9) gives a recursive method to compute $E[B^n], n = 1,2,...$ as a function of $C^{(n)}(0), n = 0,1,2,...$

Making $n = 1$ (2) results from (9)

- For the M/D/∞ system

$$C^{(0)}(0) = 1 - e^{-\rho}$$

$$C^{(n)}(0) = -e^{-\rho}(-\alpha)^n - \frac{n}{\lambda} C^{n-1}(0), n = 1,2,... \quad (11)$$

and it is possible to compute any $E\left[B^n\right]$ exactly.

- For the M/M/∞ system it is not possible to obtain expressions as simple as (11) to the $C^{(n)}(0)$. It is mandatory to compute numerically integrals with infinite limits and so approximations must be done.

The results are:

System / ρ	$M/G_1/\infty$	$M/D/\infty$	$M/M/\infty$
.5	2.0206405	.40655883	1.1109224
1	1.4710382	.56798436	1.1944614
10	1.0000454	.99959129	1.1227334
20	1.0000000	.99999999	1.0544722
50	1.0000000	.99999999	1.0206393
100	1.0000000	.99999999	1.0101547

Table 1: ***CV*[*B*]**

System / ρ	$M/G_1/\infty$	$M/D/\infty$	$M/M/\infty$
.5	9.5577742	6.0360869	5.0972761
1	5.5867425	4.5899937	5.4821324
10	4.0000000	4.0000000	4.1511831
20	4.0000000	4.0000000	4.0326858
50	4.0000000	4.0000000	4.0049427
100	4.0000000	4.0000000	4.0012250

Table 2: $\gamma_1[B]$

System / ρ	$M/G_1/\infty$	$M/D/\infty$	$M/M/\infty$
.5	15.983720	11.142336	10.454678
1	10.878212	9.6137084	10.923071
10	9.0000000	9.0000000	9.1617573
20	9.0000000	9.0000000	9.0337903
50	9.0000000	9.0000000	9.0550089
100	9.0000000	9.0000000	9.0012250

Table 3: $\gamma_2[B]$

The parameters studied assume values that are typical of an exponential distribution, after $\rho = 10$, for the M/G$_1$/∞ and M/D/∞ queues. For the M/M/∞ system, only after $\rho = 20$ it can be said that those values are the ones of an exponential distribution.

Finally note that the convolution of the exponentials with parameters λ and $\lambda e^{-\rho}$ gives an approximate expression for $z(t)$, distribution function of Z, in the same condition of Lemma 2:

$$z(t) \cong \frac{1-e^{\rho}-e^{-\lambda t}+e^{\rho-\lambda e^{-\rho}t}}{1-e^{p}}, \qquad t \geq 0 \tag{12}$$

4 Power Function Service Distribution

If the service distribution is a power function with parameter $C, C > 0$

$$G(t) = \begin{cases} t^{c} & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases},$$

$$\alpha = \frac{C}{C+1} \text{ and } 0 < \alpha < 1.$$

So $\lim\limits_{C\to\infty} G(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases}$ and $\lim\limits_{C\to\infty} \alpha = 1$

For this service time distribution, since ρ and C are great enough,

$$\frac{e^{-\lambda\int_0^t [1-G(v)]dv} \lambda(1-G(t)}{1-e^{-\rho}} \cong \lambda e^{-\lambda t}, 0 \leq t \leq 1.$$

Computing the Laplace-Stieltjes transform of (4) with this approximation it is obtained first

$$\int_0^1 e^{-st}\lambda e^{-\lambda t}dt = \lambda\int_0^1 e^{-(s+\lambda)t}dt = \lambda\left[-\frac{e^{-(s+\lambda)t}}{s+\lambda}\right]_0^1 = \lambda\frac{1-e^{-(s+\lambda)}}{s+\lambda}.$$

But, for λ great enough, this is approximately $\frac{\lambda}{\lambda+s}$. So, from (4), assuming the Laplace-Stieltjes transform of

$$\frac{e^{-\lambda\int_0^t [1-G(v)]dv} \lambda(1-G(t)}{1-e^{-\rho}}, 0 \leq t \leq 1 \text{ as being } \frac{\lambda}{\lambda+s}$$

the conclusion is that $B(t) \cong 1-e^{-\lambda e^{-\rho}t}$. That is:

Lemma 3

In an M/G/∞ queue where the service distribution is a power function, for α near 1, since α and ρ are great enough B is approximately exponential with mean $\frac{e^{\rho}}{\lambda}$.

For this system the values of $\gamma_1(B)$ and $\gamma_2(B)$ were computed for $\alpha = .25$, .5 and .8 making, in each case, ρ take values from .5 till 100. The results are:

ρ	$\alpha = .25$		$\alpha = .5$		$\alpha = .8$	
	$\gamma_1(B)$	$\gamma_2(B)$	$\gamma_1(B)$	$\gamma_2(B)$	$\gamma_1(B)$	$\gamma_2(B)$
.5	3.0181197	9.5577742	1.5035507	5.9040102	3.8933428	9.3287992
1	4.4211164	9.1402097	2.7111584	7.4994861	3.9854257	9.0702715
1.5	5.3090021	10.433228	3.3711526	8.2784408	3.9749455	8.9969919
2	5.8206150	11.140255	3.7332541	8.6924656	3.9751952	8.9815770
2.5	6.0803833	11.489308	3.9322871	8.9173048	3.9809445	8.9828631
3	6.1786958	11.619970	4.0388433	9.0369125	3.9871351	8.9877124
6	5.7006232	11.020248	4.0969263	9.1024430	3.9996462	3.9996459
7	5.5034253	10.774653	4.0765395	9.0804332	3.9999342	8.9999341
8	5.3382992	10.570298	4.0596336	9.0623268	3.9999992	8.9999992
9	5.2037070	10.404722	4.0467687	9.0486468	4.0000086	9.0000086
10	5.0944599	10.271061	4.0372385	9.0385796	4.0000068	9.0000068
15	4.7702550	9.8790537	4.0152698	9.0156261	4.0000005	9.0000005
20	4.6102777	9.6888601	4.0082556	9.0083980	4.0000000	9.0000000
50	4.3045903	9.3338081	4.0012425	9.0012513	4.0000000	9.0000000
100	4.1715617	9.1842790	4.0003047	9.0003057	4.0000000	9.0000000

Table 4: **$\gamma_1(B)$ and $\gamma_2(B)$ considering power function service distribution**

The analysis of the results shows a strong trend of $\gamma_1(B)$ and $\gamma_2(B)$, to 4 and 9, respectively, after $\rho = 10$. This trend is faster the greatest is the value of α.

5 Pareto Service Distribution

Through this section only some examples will be presented. So consider a Pareto distributions such that

$$1 - G(t) = \begin{cases} 1, & t < k \\ \left(\frac{k}{t}\right)^3, & t \geq k \end{cases}, k > 0 \tag{13}$$

Then, $\alpha = \frac{3}{2}k$ (see, for instance, (Murteira, 1979)).

The values calculated for $\gamma_1(B)$ and $\gamma_2(B)$ with $\lambda = 1$ and, so, $\rho = \alpha$ were

$\alpha = \rho$	**$\gamma_1(B)$**	**$\gamma_2(B)$**
.5	1028.5443	1373.4466
1	1474.7159	1969.0197
10	38.879220	54.896896
20	4.0048588	9.0049233
50	4.0000000	9.0000000
100	4.0000000	9.0000000

Table 5: **$\gamma_1(B)$ and $\gamma_2(B)$ considering the Pareto service distribution given by (13)**

and show a strong trend from $\gamma_1(B)$ and $\gamma_2(B)$ to 4 and 9, respectively, after $\rho = 20$. This is natural because, in this case, the convergence of α to infinite imposes the same behaviour to k. And so, after (13), it results $\lim_{\alpha \to \infty} G(t) = 0$.

But, considering now a Pareto distribution, such that

$$1 - G(t) = \begin{cases} 1, & t < .4 \\ \left(\frac{.4}{t}\right)^{\theta} & t \geq .4 \end{cases}, \qquad \theta < 1 \tag{14},$$

$\alpha = \frac{.4\theta}{\theta - 1}$ (See, again, Murteira, 1979) and the values got for $\gamma_1(B)$ and $\gamma_2(B)$ in the same conditions of the previous case are

$\alpha = \rho$	**$\gamma_1(B)$**	**$\gamma_2(B)$**
.5	10.993704	16.675733
1	6.8553306	12.010791
10	4.5112470	9.5724605
20	4.4832270	9.5397410
50	4.4669879	9.5208253
100	4.4616718	9.5146406

Table 6: **$\gamma_1(B)$ and $\gamma_2(B)$ considering the Pareto service distribution given by (14)**

and do not go against the hypothesis of the existence of a trend from $\gamma_1(B)$ and $\gamma_2(B)$ to 4 and 9, respectively, although much slower than in the previous case. But, now, the convergence of α to infinite implies the convergence of θ to 1. So

$$\lim_{\alpha \to \infty} G(t) = \begin{cases} 0, & t < .4 \\ 1 - \frac{.4}{t} & t \geq .4 \end{cases}$$

and it is not possible to guarantee at all that for α great enough $1 - G(t) \cong 1$.

6 Conclusions

The exponential distribution is very simple and quite useful from a practical point of view. It has been frequently used to study queuing systems. Among its various properties, it is remarkable its lack of memory, i.e.: $P[T > t + y|T > y] = P[T > t]$, where T is an exponential random variable.

The determination of $b(t)$ is very fastidious for any kind of queuing system and not only for the M/G/ ∞ queue.

Conditions under which B is exponential or approximately exponential for the M/G/ ∞ queue were derived.

Many quantities of interest in queues are insensible. This means that they depend on the service distribution only by its mean. Thus it is indifferent which service distribution is being considered. But using those given by (3), result quasi-exponential or exponential busy periods. And, for these service distributions, all distributions related to the busy cycle have simple forms and are related to the exponential distribution.

In section 3, for a large class of distributions under conditions of heavy- traffic, it was proved that B is approximately exponential irrespectively of the service distribution.

But, for instance, if the service distribution is a power function, as it was seen, such conditions can not hold, at least in the same way. However, for α near 1 and λ and ρ great enough, it is possible to guarantee that B is approximately exponentially distributed.

And in the case of the Pareto distribution, where $G(t) \cong 0$ for α great enough does not necessary hold. Although it is not possible to give identical guarantees to those of the power function service, the results got for $\gamma_1(B)$ and $\gamma_2(B)$ are not against that, for ρ great enough, B is approximately exponentially distributed.

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