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2012 J. Phys.: Conf. Ser. 394 012036

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SPMCS 2012 IOP Publishing
Journal of Physics: Conference Series **394** (2012) 012036 doi:10.1088/1742-6596/394/1/012036

Representation of Reserves Through a Brownian Motion Model

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Abstract. The Brownian Motion is commonly used as an approximation for some Random Walks and also for the Classic Risk Process. As the Random Walks and the Classic Risk Process are used frequently as stochastic models to represent reserves, it is natural to consider the Brownian Motion with the same purpose. In this study a model, based on the Brownian Motion, is presented to represent reserves. The Brownian Motion is used in this study to estimate the ruin probability of a fund. This kind of models is considered often in the study of pensions funds.

1. Introduction

The Brownian Motion is often used as an approximation for some Random Walks and also for the Classic Risk Process. About the convergence of Random Walks to the Brownian Motion see, for instance, [1], page 23. And on the convergence of the Classic Risk Process to the Brownian Motion see, for instance [2], page16 and [3]. As the Random Walks and the Classic Risk Process are used frequently as stochastic models to represent reserves, it is natural to consider the Brownian Motion with the same purpose.

Consider the following Brownian Motion definition, see [1], page 18:

Definition 1.1

The Brownian Motion with drift μ and diffusion coefficient σ^2 is a stochastic process $X(t)$ with continuous paths and independent Gaussian increments with mean and variance of an increment, $X(t+s)-X(t)$, given by μs and $\sigma^2 s$, respectively.

Note

- If $X(0)=x$, the Brownian Motion is assumed to begin in x,

- A Brownian Motion beginning at the origin, with drift 0 and diffusion coefficient 1, is named *standard* Brownian Motion and it is referred in general by $B(t)$,

- So the general Brownian Motion $X(t)$ with the beginning in x, drift μ and diffusion coefficient σ^2 may be written as

$$X(t)=x+\mu t+\sigma B(t) \qquad (1).$$

The computation of $\rho_k(x)$, the ruin probability of a system which reserves behave in accordance with a general Brownian Motion $X(t)$ will be performed in the next section. It is the probability that $X(t)$, with the beginning in x, reach 0 before arriving at k, $x>0$ and $k>x$. And it is considered also, then, the eventual system ruin probability $\rho(x)$, or the probability that $X(t)$ general Brownian Motion, with the beginning at x, reach anytime 0.

The problem may be approached in different ways. See, for instance, [1], page 24, for an approach of the problem based on the approximation to the Brownian Motion through the simple Random Walk. Or [4], page 360, for an approach to the problem through the application of the Martingales Stopping Theorem. The one presented here stands on the deduction of an ordinary differential equation for $\rho_k(x)$, relatively easy to solve, see, for instance, [5], page 193, or [6], page 373.

2. The Ruin Probability

Suppose initially $\mu \neq 0$. Consider h a time interval small enough such that the probability that $X(t)$ reach 0 before k is negligible. Thus by the Total Probability Theorem:

$$\rho_k(x) = E\left(\rho_k\big(X(h)\big)\right) + o(h) \qquad (2)$$

where $o(h)$ is the usual Landau notation for a function $f(h)$ such that $\lim_{h\to 0}\frac{f(h)}{h} = 0$. Writing now $\Delta X = X(h) - x$ and expanding $\rho_k\big(X(h)\big)$ as a Taylor's Series around x:

$$\rho_k\big(X(h)\big) = \rho_k(x + \Delta\mathrm{X}) = \sum_{n=0}^{\infty} \frac{(\Delta X)^n \rho_k^{(n)}(x)}{n!} \qquad (3).$$

Note that as the increments $X(h) - X(0) = X(h) - x = \Delta X$ are Gaussian Random variables with mean μh and variance $\sigma^2 h$:

- $E(\Delta X) = \mu h$,
- $E((\Delta X)^2) = \sigma^2 h + o(h)$,
- $E((\Delta X)^\nu) = o(h)$, for $\nu > 2$.

With these results in mind, applying operator E to both sides of equation (3) and substituting in (2) it is obtained

$$\mu h \rho_k'(x) + \frac{1}{2}\sigma^2 h \rho_k''(x) = o(h) \qquad (4).$$

Performing the division of both sides by h and making h to converge to 0 it is obtained, finally

$$\mu \rho_k'(x) + \frac{1}{2}\sigma^2 \rho_k''(x) = 0 \qquad (5),$$

with the border conditions $\rho_k(0) = 1$ and $\rho_k(k) = 0$.

The expression (5) is a second order ordinary differential equation that may be reduced to the following first order ordinary differential equation

$$\mu\rho_k(x) + \frac{1}{2}\sigma^2\rho_k'(x) = C_1 \qquad (6).$$

The general solution of (6) is

$$\rho_k(x) = C_1\frac{1}{\mu} + C_2\frac{2}{\sigma^2}e^{-2\mu x/\sigma^2} \qquad (7),$$

resulting from the border conditions $C_1 = \mu - \frac{\mu}{1-e^{-2\mu k/\sigma^2}}$ and $C_2 = \frac{\sigma^2}{2\left(1-e^{-2\mu k/\sigma^2}\right)}$.

So finally

$$\rho_k(x) = \frac{1-e^{-2\mu(k-x)/\sigma^2}}{e^{2\mu x/\sigma^2} - e^{-2\mu(k-x)/\sigma^2}},\ \ \mu \neq 0 \qquad (8).$$

Making μ converge to 0 in (8),

$$\rho_k(x) = \frac{k-x}{k}, \quad \mu = 0 \qquad (9).$$

To estimate now the probability $\rho(x)$, of the eventual ruin of a system which reserves behave in accordance with the general Brownian Motion $X(t)$, make k converge to infinite in (8) and (9) obtaining:

$$\rho_k(x) = \begin{cases} e^{-2\mu x/\sigma^2}, & if\ \mu > 0 \\ 1, & if\ \mu \leq 0. \end{cases}$$

3. Concluding Remarks

It was exposed how to estimate the probability of the eventual ruin of a system which reserves behave in accordance with the general Brownian Motion. It is a very simple and intuitive way of doing it.

But this kind of models, and the consequent appreciations of the stability of the systems, done based on the evaluation of the probability of the exhaustion of the reserves, or ruin, are valid only in scenarios at which are considered constant prices. The integration of factors associated to the temporal depreciation process of the money value, in the modeling of financial reserves, although make eventually more complex the mathematical models involved, seems desirable.

Acknowledgments

This work was financially supported by FCT through the Strategic Project PEst-OE/EGE/UI0315/2011.

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