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Strategic Capacity Investment Under Uncertainty: Mathematical Insights and Practical Applications

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Master in Finance

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BUSINESS SCHOOL

Department of Finance

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To my beloved parents.

For their unwavering support, endless encouragement, and boundless love. This thesis is a testament to your sacrifices and belief in me. Thank you for being my pillars of strength and my greatest inspiration.

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Resumo

O principal objetivo desta tese é replicar os resultados analíticos e numéricos do artigo de Huisman e Kort (2015), bem como detalhar o processo de replicação de figuras e tabelas para orientar os leitores na obtenção de resultados semelhantes. A nossa metodologia utiliza técnicas quantitativas, especificamente cálculo estocástico. Os resultados obtidos são implementados em MATLAB para replicar as figuras e tabelas apresentadas no artigo original. Os dados principais envolvem variáveis do processo geometric Brownian motion, funções de preços (tanto linear como isoelástica) e a raiz positiva de uma solução geral de uma equação diferencial ordinária. A derivação analítica das várias proposições do artigo permitiu concluir que existem algumas imprecisões nas equações, resultados e figuras do artigo. Em particular, as expressões (12), (A22), (A70), (B17) e (B30), bem como a Figura 1(a), são apresentadas erroneamente no artigo original. Além disso, à medida que a incerteza aumenta, as empresas atrasam os seus investimentos, mas investem em capacidades maiores. Por fim, a empresa líder tende a aumentar o seu investimento para impedir que a empresa seguidora invista, embora essa estratégia não possa ser efetuada indefinidamente.

Palavras-chave: Investimento Estratégico em Capacidade, Incerteza, Opções Reais, Decisão de Investimento Ótima, Temporização de Investimento, Cálculo Estocástico

Classificação JEL: C61; D81

Abstract

The primary objective of this thesis is to replicate the analytical and numerical results of the article by Huisman and Kort (2015), as well as to detail the process of replicating figures and tables to guide readers in obtaining similar results. Our methodology employs quantitative techniques, specifically stochastic calculus. The results obtained are implemented in MATLAB to replicate the figures and tables presented in the original article. The primary data involves variables from the geometric Brownian motion process, price functions (both linear and isoelastic), and the positive root of a general solution to an ordinary differential equation. The analytical derivation of the various propositions in the article led to the conclusion that there are some inaccuracies in the equations, results, and figures of the article. In particular, expressions (12), (A22), (A70), (B17), and (B30), as well as Figure 1(a), are erroneously presented in the original article. Furthermore, as uncertainty increases, firms delay their investments but invest in larger capacities. Finally, the leader firm tends to overinvest to deter the follower firm from investing, although this strategy cannot be sustained indefinitely.

Key words: Strategic Capacity Investment, Uncertainty, Real Options, Optimal Investment Decision, Investment Timing, Stochastic Calculus

JEL classification: C61; D81

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CHAPTER 1

Introduction

Investing in capacity, particularly under uncertain conditions, necessitates a robust valuation method and a strategic plan to address future uncertainties and competitive pressures.

Several researchers have posited that the discounted cash flow (DCF) method and the traditional net present value (NPV) analysis may result in suboptimal decisions regarding investment capacity and timing. They suggest that more intricate valuation approaches, particularly those presented in Real Options literature, substantiate these findings. This body of literature emphasises the importance of incorporating flexibility into static NPV analysis, thereby elucidating why firms may opt to invest at a later stage and in greater capacities when faced with heightened uncertainty.

Our primary focus is on the mathematical equations presented in Huisman and Kort (2015) research article, "Strategic capacity investment under uncertainty". We provide detailed, step-by-step mathematical proofs of all the equations and replicate the figures and tables from the article. Additionally, we confirm the interpretations of the analysis made by these authors through our mathematical demonstrations.

We make two significant contributions to the broader literature and the aforementioned research article. First, we address the complexities and uncertainties inherent in investment decisions by offering a detailed, step-by-step mathematical explanation of all the results. This allows readers to gain valuable insights into the methodology and understand how the main conclusions of Huisman and Kort (2015) were derived. Second, we correct the formulas and figures that were mistakenly presented in the original article.

Our analysis encompasses all the equations from the article by Huisman and Kort (2015), which yield intermediate and final results that enable the plotting of graphs and the achievement of results in the tables.

Our methodology is as follows. For each result, we employ quantitative methods using stochastic calculus, followed by applying these results in MATLAB, to replicate the figures and achieve the tables in the article by Huisman and Kort (2015). This approach is chosen as it is the most suitable for describing the mathematical steps of equations involving stochastic variables.

The main results indicate that Huisman and Kort (2015) expressions (12), (A22), (A70), (B17), and (B30), as well as Figure 1(a), are erroneous. The correct versions are presented in this thesis as expressions (4.13), (A4.6), (A6.20), (6.12), and (6.19), respectively, and Figure 4.1.

Additionally, we demonstrate that the leader firm cannot indefinitely prevent the follower from investing and that the leader is typically the largest firm in the market. However, under low uncertainty, the pre-emption effect may result in the second investor becoming the largest firm.

Furthermore, increased uncertainty causes a delay in investments, but when they do occur, they are made at larger capacities. Lastly, the leader tends to overinvest to deter the follower from investing.

The structure of this thesis is as follows. Chapter 2 provides a brief literature review on Real Options, while Chapter 3 presents the methodology employed.

Subsequently, Chapter 4 analyses the monopoly scenario and compares its outcomes with those of the social planner.

Next, in Chapter 5, we introduce competition into the analysis, transitioning the market structure from a monopoly to a duopoly, where firms compete using a Stackelberg strategy. Additionally, we broaden the analysis to consider other scenarios where the cost of investing changes. Finally, we investigate the consequences for total welfare when a monopolist can invest twice.

Chapter 6 examines the transition from a linear price function to an isoelastic price function and the consequent changes in the timing and investment capacity of either the leader or follower firms. This chapter also encompasses both monopoly and duopoly market structures.

Following this, Chapter 7 provides the main conclusions and recommendations.

Lastly, two annexes are provided at the end, offering step-by-step resolutions of all equations presented or omitted in the original research article. The findings presented in the various subchapters should be read in conjunction with the corresponding subchapters in Annexes A or B, as they provide additional details and further explanations.

CHAPTER 2

Literature review

This chapter highlights some of the most significant references in the Real Options literature, essential for analysing a firm's valuation and investment opportunities.

In traditional corporate finance, firms are advised to use a DCF model when evaluating investment projects. This approach involves calculating the present value of the expected cash that the investment is expected to produce and the present value of the expenditures required to support the project. According to Bowman and Moskowitz (2001), Dixit and Pindyck (1995), Keswani and Shackleton (2006), and Ross (1995), if the NPV is positive, the investment should be undertaken. Conversely, if the NPV is negative, the investment should not proceed.

However, this seemingly straightforward approach contains flawed assumptions. For instance, it presumes that the firm adopts a passive commitment, meaning that the project will commence immediately and be operated at a predetermined production scale throughout its useful life, regardless of future market conditions (Brealey et al., 2013; Keswani & Shackleton, 2006; Trigeorgis, 1993). Additionally, it assumes the investment is of a now-or-never nature (Dixit & Pindyck, 1995). Consequently, the NPV rule essentially compares investing today with never investing.

From this, it can be concluded that the traditional DCF model does not account for the value associated with deferring the investment decision (Luehrman, 1998). This omission overlooks the flexibility a project may possess, as managers can adjust the scale of production, reduce funding for research and development (R&D) projects (Bowman & Moskowitz, 2001), or postpone investment decisions in response to uncertain projected cash flows.

Another issue with the conventional NPV approach, as noted by Dixit and Pindyck (1995), is its failure to consider the potential options generated by R&D activities. This oversight can ultimately lead to underinvestment by firms.

Another approach to evaluating investments involves using "rules of thumb" such as hurdle rates, profitability indexes, and payback rules. Although these rules may appear arbitrary, McDonald (2000) demonstrates that they serve as effective proxies for optimal decision-making. Generally, these rules capture between 50% and 90% of a project's option value across various project characteristics. Consequently, they approximate optimal investment timing behaviour, as the value of the timing option is less sensitive to deviations from the optimal rule when it is most valuable. However, these "rules of thumb" are less effective for low-volatility projects.

When considering an investment with the option to delay, it may be optimal to postpone the decision. This is because delaying can lead to a higher payoff as uncertainties are resolved over time (Shackleton et al., 2004). To evaluate such investments, the real options approach is a crucial capital budgeting and strategic decision-making mechanism. It incorporates the value of future flexibility, allowing managers to adjust their strategy over time (Bowman & Moskowitz, 2001).

Other advantages of using real options include: 1) it emphasises shareholder value maximisation, 2) it places greater importance on downstream decisions, such as abandoning uneconomic projects or expanding those that show promise over time, and 3) its conceptual framework simplifies the structuring of investment choices (Triantis, 2005).

When a firm has an option to invest, the traditional NPV analysis can lead to significant errors. This is due to the opportunity cost, as McDonald and Siegel (1986) demonstrate, and the variability in cash flow estimates, as Trigeorgis (1996) argues.

According to Baldwin (1982), firms should require a positive premium over the NPV to offset any potential loss from future opportunity costs. Alternatively, firms should wait until the project's value is twice the NPV before investing, as McDonald and Siegel (1986) suggest. Additionally, Dixit and Pindyck (1995) report that the present value of anticipated cash flows should exceed the project costs by an amount equivalent to the value of maintaining the investment option.

Regarding opportunity cost, Pindyck (1986) argues that a firm should invest when the marginal value of an investment exceeds the total costs by an amount equivalent to keeping the option alive. Moreover, Pindyck (1986) also shows that incorrect computation of opportunity costs can result in overinvestment decisions.

All these different but concurrent interpretations highlight the impact of uncertainty on the value of waiting and, ultimately, on the analysis of the static NPV approach. While these authors offer various perspectives on how to approach investment decisions, they unanimously agree that relying solely on the traditional NPV method is insufficient. Instead, they suggest incorporating an option premium (sometimes referred to as "premium", "opportunity cost" or "flexibility value") to account for potential future cash flow downturns, due to uncertainty.

A different approach seems to have Dixit (1993). This author not only considers the strategic NPV approach, which includes the option value, but also the elasticity of output relative to capital expenditures. If the elasticity is above one, the firm should opt for the largest available project. Conversely, if the elasticity is less than one but increasing, the smallest or the largest projects are optimal choices.

In a competitive environment, Dixit and Pindyck (1994) highlight that the option to invest remains crucial for analysing investments. Although competition may erode the option to wait, pre-emptive investment can lead to better future outcomes, without necessarily reverting to the static NPV approach.

However, when uncertainty is specific to a firm, the value of waiting is not diminished. If a firm faces unique demand uncertainty, it does not need to worry about new entrants, as this demand is exclusive. Consequently, the firm can wait and avoid potential losses if the demand fluctuation is temporary.

Before entering a new market, a prospective entrant must analyse both the scale and timing of its investment, considering the presence of an incumbent firm. The incumbent's pre-entry investment decisions can influence, but not indefinitely prevent, a new competitor from entering the market. It can become active if future conditions allow the entrant to meet the necessary investment threshold. Dixit (1980) and Spence (1977) both discussed this concept. A high initial and irrevocable investment by the incumbent can discourage entry, but if not, various forms of duopoly (Stackelberg, Cournot, or Bertrand) could emerge.

In a Stackelberg duopoly, overinvestment may arise, as Spence (1977) suggests, but this does not occur under the Nash equilibrium, as Dixit (1980) argues.

In the presence of competition, the timing of investments must be adjusted in response to competitors' actions. Shackleton et al. (2004) found that fixed entry costs result in hysteresis in competitors' entry decisions. This hysteresis is positively associated with entry costs and uncertainty but negatively affected by the correlation of competitors' operations. When these fixed costs are eliminated, the hysteresis disappears, leading to exercise strategies based on current yield or profitability criteria.

Additionally, increased volatility reduces the active periods for competing firms, which, in turn, raises their probabilities of entering the market. This dynamic results in short-lived market leadership for any firm (Shackleton et al., 2004).

When producing, firms must choose the type of technology for their production based on the level of uncertainty they face. For instance, Goyal and Netessine (2007) show that under high demand uncertainty, firms (even monopolies) tend to favour flexibility over dedicated technology. This preference is influenced by the low demand correlation between products and the small market size and differential.

However, in competitive markets, the choice of technology depends on the competitor's choice. Flexibility is not always the best response, as firms might mimic their competitors' technology adoption, which holds even for intermediate levels of uncertainty.

From the capacity side, in situations of high uncertainty, Pindyck (1986) indicates that reducing the firm's capacity is optimal. The reasoning is that while the value of the marginal unit increases, the opportunity cost also rises, leading to a net reduction in the firm's capacity.

Pindyck (1986) asserts that firms should maintain lower investment capacity if investments are irreversible or future demand is uncertain. However, since the market value of firms is closely tied to future demand (uncertainty), there seems to be a tendency to overinvest, especially in the oil industry. However, this does not apply to manufacturing firms, as McConnell and Muscarella (1985) evidence. Nonetheless, across sectors, higher uncertainty increases the incentive to delay investment expenditures.

The option for a firm to delay its investments presents two main challenges. On one hand, postponing investments can be costly, as the firm risks losing market share by not investing promptly. On the other hand, delaying investments allows the firm to gather more information about future conditions (Dixit & Pindyck, 1995; Huisman, 2001). This scenario suggests that using the NPV approach alone may lead to suboptimal decisions (Huisman, 2001). Additionally, the benefits of waiting are often substantially greater than exercising the option and investing immediately, especially in scenarios of high uncertainty (Dixit & Pindyck, 1995).

When evaluating strategic options, one crucial aspect is the distribution of the underlying asset price. The Black and Scholes (1973) and Merton (1973) formula assumes a lognormal distribution with constant volatility, which might not be suitable for strategic options. As Bollen (1999) points out, this is especially true in the semiconductor industry, where sales patterns increase and fall abruptly, making them incompatible with a real options model based on a lognormal distribution.

Furthermore, the geometric Brownian motion (GBM) model may not best describe the random movement of projects that contribute to the overall firm value (Alexander et al., 2012). While the market price of limited-liability shares cannot be negative, the value of a division or project within a firm can be. The GBM assumes that the underlying project is always positive, which does not reflect reality. Therefore, this model may not always be realistic or suitable.

To address the limitations of the GBM, Copeland and Antikarov (2001) advocate using arithmetic or additive processes. For example, the arithmetic Brownian motion evaluates the values of underlying projects as an additive process, with its variance remaining constant over time. Additionally, this approach allows the project's value path to be either positive or negative (Alexander et al., 2012).

Despite the many advantages of real options and its potential as a viable alternative to the traditional NPV method, it also comes with certain caveats, such as: 1) it can be complex to use and explain, requiring significant analytical and computational effort, 2) a lack of structure, since the value of an option depends on possible future outcomes, 3) firm's competitors may have investment options affecting our firm's strategy, so their actions must be considered (Brealey et al., 2013), and 4) real options models often assume market perfection rather than reality. For example, it is assumed that managers are always loyal to the firm's shareholders (Triantis, 2005).

In conclusion, while it may seem that real options could replace the DCF method, this is not entirely the case. First, the DCF method and NPV analysis are suitable for projects with stable cash flows and for those whose value is not dependent on options that the underlying business might create or be based on. Secondly, in real options analysis, the present price of the underlying asset is generally determined using the DCF method (Brealey et al., 2013). Additionally, NPV rules can be used when the investment opportunity disappears if not undertaken immediately (Ross, 1995).

Lastly, it is important to emphasise that when uncertainty exists regarding future demand or output prices, it is often optimal to invest in large projects and at a later time. Including the value of flexibility in traditional NPV analysis can result in different investment decisions compared to static NPV analysis.

CHAPTER 3

Methodology

To present all the calculations and results from the article of Huisman and Kort (2015) meticulously, we utilise a methodology involving quantitative methods, specifically stochastic calculus, applied to real options problems in both monopoly and duopoly regimes, followed by their respective numerical implementations in MATLAB.

The quantitative methods include a detailed breakdown of mathematical equations, while the stochastic component is introduced with a variable that adds uncertainty to the output price. This variable, defined as the exogenous shock process, is assumed to follow a GBM process.

Subsequently, the contributions of real options are applied to firms aiming to invest in capacity, determining the optimal levels of capacity and timing. Hypothetical firms are considered in both monopoly and Stackelberg duopoly scenarios.

After obtaining all the results, we implemented them to replicate the figures and tables presented in the original article. The main variables in the Huisman and Kort (2015) article are the parameter values used to compute investment thresholds, capacity levels, firms' value functions, and the welfare implications of investment timing and size.

These variables are 1) μ: representing the drift rate in the GBM process that defines the exogenous shock process X_t, 2) r: denoting the interest rate used to discount the firm's future cash flows, assuming the firm is risk-neutral, 3) η: modelling the price function, 4) σ: indicating the volatility of the exogenous shock process X_t in the GBM process, 5) δ: denoting the unit investment costs, 6) β: corresponding to the positive root of a general solution of the ordinary differential equation, and 7) γ: standing as an elasticity parameter in the isoelastic price function.

Since Huisman and Kort (2015) structured their article around propositions that outline relationships between abstract concepts in the Real Options literature, this master's thesis will follow a similar structure.

The general procedure for processing and analysing the article is as follows. We begin by maximising the firm's value function to determine the optimal capacity size. Then, using real options concepts such as the value matching condition (VMC) and smooth pasting condition (SPC), we identify the optimal investment threshold. Next, we prove all intermediate steps, which are presented in the annexes for each proposition. After obtaining these results, we combine them to derive expressions where the capacity level and the investment threshold are independent of each other, allowing for computation based on fundamental parameter values.

Lastly, if applicable, we recreate the figures and the results in the tables presented in each proposition.

For propositions requiring additional work, such as Propositions 1 and 9, we derive the expressions for the total expected surplus for either the monopolist or the social planner.

CHAPTER 4

Monopoly

Following the structure of Huisman and Kort (2015) article, we begin by analysing the monopoly case and its impact on determining the investment threshold and the capacity level.

Considering $Q(t)$ as the total market output, $\eta > 0$ as a constant, and the demand level $X(t)$ as an exogenous shock process following a GBM

$$dX(t) = \mu X(t)\, dt + \sigma X(t)\, d\omega(t), \tag{4.1}$$

where μ is the drift rate, $d\omega(t)$ is the increment of a Wiener process, and $\sigma > 0$ represents volatility, the market price is given by

$$P(t) = X(t)\big(1 - \eta Q(t)\big). \tag{4.2}$$

It is assumed that $r > \mu$ because it ensures that there is a diminishing return to waiting, which is necessary for determining an optimal investment time. Without this condition, the problem would not have a finite solution, as waiting longer would always seem more advantageous.

Since a unit of capacity costs δ, investing in capacity Q would cost δQ. Additionally, the article by Huisman and Kort (2015) imposes that firms always produce up to capacity.

Our goal in this chapter is to prove mathematically Proposition 1 of Huisman and Kort (2015) and implement it. To accomplish this purpose, we start with the monopolist's optimal investment decision in subchapter 4.1, then analyse the optimal welfare outcome in subchapter 4.2. The detailed proofs of the results presented in subchapters 4.1 and 4.2 are provided in annexes A.1 and A.2, respectively.

4.1. Monopolist's optimal investment decision

Considering that the profit of a single firm at the time t is denoted by $\pi(t)$ and is equal to

$$\pi(t) = P(t) \cdot Q(t), \tag{4.3}$$

we can denote the expected value of the firm at the moment of investment, $V(X, Q)$, as

$$V(X, Q) = \frac{XQ(1 - \eta Q)}{r - \mu} - \delta Q. \tag{4.4}$$

Then, the capacity level of the monopolist is equal to

$$Q^*(X) = \frac{1}{2\eta}\left(1 - \frac{\delta(r - \mu)}{X}\right). \tag{4.5}$$

This equation indicates that at the time of investment, a higher level of X correlates with increased capacity investment by the firm, leading to a greater total profit flow.

The investment threshold can be represented as

$$X^*(Q) = \frac{\beta\delta(r-\mu)}{(\beta-1)(1-\eta Q)}. \tag{4.6}$$

Through substitutions, it is possible to achieve the optimal investment capacity and threshold levels, which are directly independent of each other, as follows

$$Q^* = \frac{1}{(\beta+1)\eta}, \tag{4.7}$$

$$X^* = \frac{\beta\ +1}{\beta\ -1}\delta(r\ -\ \mu). \tag{4.8}$$

When $X < X^*$, the firm is in an idle state, meaning that the value of the monopolist in this region is given by the option value AX^β. However, when X reaches the investment trigger, the value of the monopolist firm is given by equation (4.4), which can be rewritten to give the second branch of expression (4.9):

$$V(X) = \begin{cases} AX^\beta & if\ \ X < X^* \\ \dfrac{\left(X-\delta(r-\mu)\right)^2}{4X\eta(r-\mu)} & if\ X \geq X^* \end{cases}, \tag{4.9}$$

with

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{4.10}$$

and

$$A = \frac{\delta\left(\frac{\beta+1}{\beta-1}\delta(r-\mu)\right)^{-\beta}}{(\beta^2-1)\eta}. \tag{4.11}$$

Differentiating equations (4.8) and (4.7) concerning β gives

$$\frac{\partial X^*}{\partial\beta} = -\frac{2\delta(r-\mu)}{(\beta-1)^2} < 0, \tag{4.12}$$

$$\frac{\partial Q^*}{\partial\beta} = -\frac{1}{\eta\cdot(\beta+1)^2} < 0. \tag{4.13}$$

From expression (4.13), it is important to note that Huisman and Kort (2015) have a typographical error in their expression (12).

From all the previous results, we can conclude that when uncertainty (σ) rises, the optimal investment trigger (X^*) also rises, delaying investments. Additionally, the optimal capacity level (Q^*) rises too (in expression (4.5), the rise is indirectly affected via X), leading the monopolistic firm to adopt larger projects.

From equations (4.7) and (4.8), as the cost per unit (δ) increases, it is evident that the firm invests in the same capacity, but later, since X^* increases while Q^* remains constant.

Lastly, if $\delta = 0$, X^* equals zero according to expression (4.8), and from equation (4.5), $Q^* = \frac{1}{2\eta}$ for a positive X. This result implies that it is optimal for a firm to invest immediately and acquire a production capacity of $\frac{1}{2\eta}$.

4.2. Optimal welfare decision

Since the total expected consumer surplus (CS) is equal to

$$CS(X,Q) = \frac{XQ^2\eta}{2(r-\mu)} \tag{4.14}$$

and the expected producer surplus (PS) is given by the value of the monopolist firm, which is represented by equation (4.4), it can be expressed as

$$PS(X,Q) = \frac{XQ(1-\eta Q)}{r-\mu} - \delta Q. \tag{4.15}$$

Following the formulas above, the total expected surplus (TS), which is the sum of consumer and producer surpluses, can be represented as

$$TS(X,Q) = \frac{XQ(2-\eta Q)}{2(r-\mu)} - \delta Q. \tag{4.16}$$

Incorporating the monopoly decisions, i.e., the optimal capacity level and investment threshold from the previous subchapter, the TS is given by

$$TS(X^*,Q^*) = \frac{3\delta}{2(\beta+1)(\beta-1)\eta}. \tag{4.17}$$

For the case of the social planner, who maximises TS, the investment capacity level and investment trigger are given by

$$Q_w^* = \frac{2}{(\beta+1)\eta} = 2Q^*, \tag{4.18}$$

$$X_w^* = \frac{\beta+1}{\beta-1}\delta(r-\mu) = X^*. \tag{4.19}$$

Based on the results in equations (4.18) and (4.19), it can be concluded that the social planner invests simultaneously with the monopolist (i.e., the investment thresholds are equal), but at twice the capacity level.

From this social planner scenario, the total welfare at the moment of investment is equal to

$$TS_w = TS(X_w^*, Q_w^*) = \frac{2\delta}{(\beta+1)(\beta-1)\eta}. \tag{4.20}$$

Therefore, the welfare loss that exists in a monopoly situation is equal to

$$TS(X_w^*, Q_w^*) - TS(X^*, Q^*) = \frac{\delta}{2(\beta+1)(\beta-1)\eta}. \tag{4.21}$$

However, this expression is not valid when $\delta = 0$. In this scenario, the social planner does not invest at the threshold level because, according to expression (4.19), the threshold would be zero and the total welfare loss is computed using the investment threshold specified in expression (4.19).

Regarding Figure 1(a) of Huisman and Kort (2015), we observed that for the parameter values of $r = 0.1$, $\mu = 0.06$, $\delta = 0.1$, and $\eta = 0.05$, the X^* axis was incorrectly presented. For instance, for approximately $\sigma = 0$, $X^* = 0.01600$, as shown in Table 4.1, and not approximately $X^* = 0.013$ as depicted by Huisman and Kort (2015). Therefore, the correct figure that should have been displayed is Figure 4.1.

Figure 4.1. Optimal investment trigger, X^*, as a function of σ

This figure presents the corrected version of Figure 1(a) from Huisman and Kort (2015). It was generated using expression (4.8) to compute the values for X^*, based on the parameter values: $r = 0.1$, $\mu = 0.06$, $\delta = 0.1$, and $\eta = 0.05$. The figure illustrates that as uncertainty increases, the investment threshold also rises, prompting the monopolist firm to defer the investment decision to a later stage.

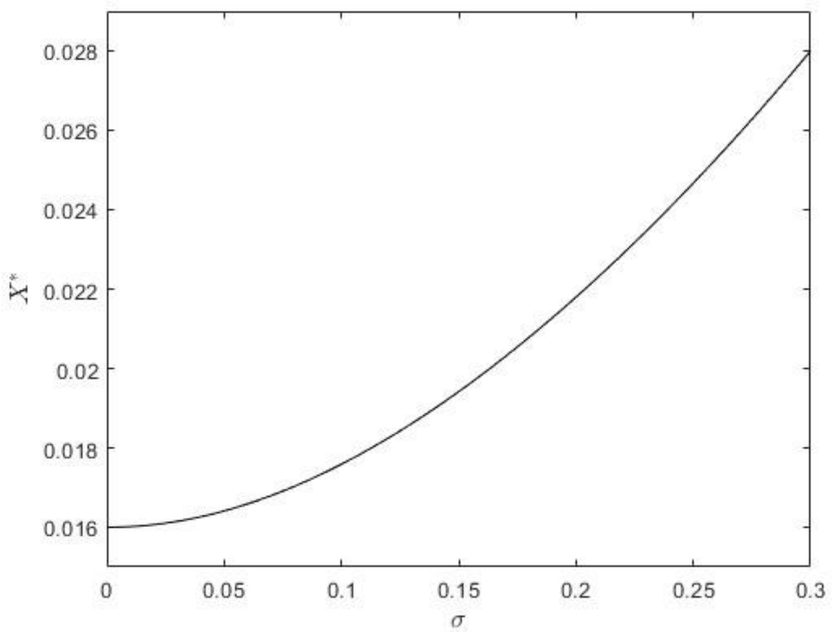


Table 4.1. Comparison of welfare outcomes

The investment threshold X^* and capacity level Q^* for the monopolist were calculated using expressions (4.8) and (4.7), respectively. For the social planner, the corresponding values (X_W^* and Q_W^*) were obtained from expressions (4.19) and (4.18), respectively. All total surplus values represent their present values. The results shown correspond to specific levels of uncertainty (σ), based on the following parameter values: $r = 0.1$, $\mu = 0.06$, $\delta = 0.1$, $\eta = 0.05$, and $X(0) = 0.001$.

	Monopoly			Social Planner			Comparison	
σ	X^*	Q^*	TS	X_W^*	Q_W^*	TS_W	$\frac{TS}{TS_W}$	$\frac{Q^*}{Q_W^*}$
0.00	0.01600	7.500	0.01661	0.01600	15.000	0.02215	0.750	0.500
0.05	0.01641	7.563	0.01766	0.01641	15.125	0.02355	0.750	0.500
0.10	0.01759	7.726	0.02069	0.01759	15.452	0.02759	0.750	0.500
0.15	0.01942	7.940	0.02536	0.01942	15.880	0.03382	0.750	0.500
0.20	0.02180	8.165	0.03128	0.02180	16.330	0.04171	0.750	0.500
0..25	0.02467	8.379	0.03806	0.02467	16.757	0.05075	0.750	0.500
0.30	0.02800	8.571	0.04537	0.02800	17.143	0.06049	0.750	0.500
0.35	0.03178	8.741	0.05291	0.03178	17.483	0.07055	0.750	0.500
0.40	0.03600	8.889	0.06048	0.03600	17.778	0.08064	0.750	0.500

To obtain the values displayed in Table 4.1, we used expressions (4.7) and (4.8) to compute Q^* and X^*, respectively. The investment threshold X_W^* was obtained using expression (4.19), and the optimal investment capacity Q_W^* was derived from equation (4.18).

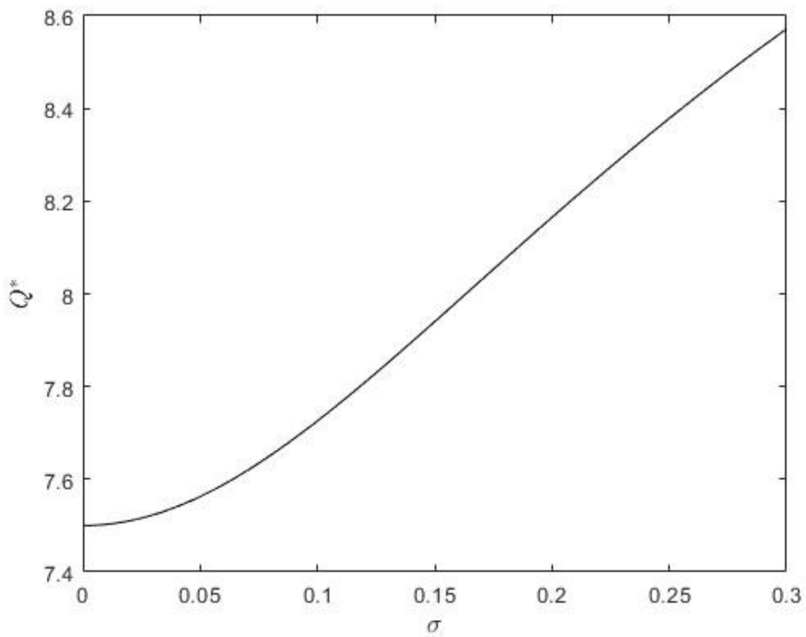
To compute the total surpluses as Huisman and Kort (2015) present them, whether for the monopolist or the social planner, it is necessary to multiply expressions (4.17) and (4.20) by the stochastic discount factor $\left(\frac{X(0)}{X^*}\right)^\beta$ or $\left(\frac{X(0)}{X_W^*}\right)^\beta$, respectively. This adjustment ensures that the results represent the expected present value of future surpluses.

From Figure 4.1, it is evident that X^* is increasing as σ increases. This is because, as uncertainty rises, the firm sets a higher investment threshold to compensate for the increased risk, meaning that the firm will only invest when the expected returns are significantly higher. Overall, firms prefer to wait for clear signals before committing to an investment, leading to a higher optimal investment trigger and delayed investments.

Regarding Figure 1(b) of Huisman and Kort (2015), we obtained the same result as they did, which can be observed in Figure 4.2.

Figure 4.2. Optimal quantity, Q^*, as a function of σ

This figure presents the results obtained using expression (4.7) to compute the optimal capacity level Q^*. The figure closely resembles Figure 1(b) from Huisman and Kort (2015) and uses the same parameter values as those specified in Figure 4.1.



From Figure 4.2, achieved with the same parameter values as Figure 4.1, it is possible to realise that Q^* is increasing as σ increases. This result is achieved indirectly, as from equation (4.5), Q^* increases when X^* rises, which in turn increase with higher σ.

This indicates that when uncertainty rises, firms delay their investments and opt to invest in a higher capacity level. This strategy helps firms to better manage risks, adapt to fluctuating market conditions, strengthen their market position, and avoid the costs and delays associated with incremental capacity expansions in the future. This result also confirms the findings of Dixit (1993).

From Table 4.1, it is possible to conclude that, for the parameters $r = 0.1$, $\mu = 0.06$, $\delta = 0.1$, $\eta = 0.05$, and $X(0) = 0.001$ there is a welfare loss of 25%, and the conclusions from equations (4.18) and (4.19) still hold.

CHAPTER 5

Duopoly

In this chapter, competition is introduced to the model in the previous chapter by incorporating an additional firm.

The first firm, referred to as the leader, invests in a capacity level of Q_L, while the second firm, known as the follower, invests in a capacity level of Q_F. Consequently, the total market quantity is expressed as $Q = Q_L + Q_F$.

Our goal in this chapter is to prove mathematically the results in Propositions 2, 3, 4, 7, and 8 of Huisman and Kort (2015) and implement their results. Propositions 5 and 6 are going to be discussed theoretically. Annexes A.3, A.4, A.5, A.6, and A.7 provide the full proofs of Propositions 2, 3, 4, 7, and 8, respectively.

For that, we start by analysing the situation where there are strong asymmetric investment costs between the two firms. Designating the leader firm as the low-cost firm (firm 1) and the follower firm as firm 2, we assume that $0 < \delta_1 < \delta_2$. Additionally, there exists a cutoff value $\hat{\delta}_1$ such that $\delta_1 < \hat{\delta}_1$. In this situation, the second firm lacks the incentive to invest first, resulting in no competition for leadership. This situation will be further explored in subchapter 5.1.

The second case considered is of symmetric investment costs, meaning that $\delta_1 = \delta_2 = \delta$. Since the investment costs are equal in both firms, it is unknown beforehand which firm will invest first. This scenario is proposed in subchapter 5.2.

The third case involves moderate asymmetry, where $\delta_2 > \delta_1 > \hat{\delta}_1$. This scenario is analysed in subchapter 5.3.

Lastly, the duopoly investment outcome from a welfare perspective is analysed. This scenario is presented in subchapter 5.4.

5.1. Significant leader advantage

As previously mentioned, firm 1 has the lowest investment costs, which means that it will invest first and become the leader. Once the leader has invested, firm 2, the follower, cannot influence the leader's investment decision. This means the follower's decisions lack strategic elements. Given the leader's investment threshold and capacity level, the follower only needs to determine its optimal investment timing $X_F^*(Q_L)$ and the optimal investment capacity $Q_F^*(Q_L)$.

5.1.1. Follower's investment strategy

Given the optimal capacity level of the follower, concerning the current level of X and the leader's capacity level Q_L,

$$Q_F^*(X, Q_L) = \frac{1}{2\eta}\left(1 - \eta Q_L - \frac{\delta_2(r-\mu)}{X}\right), \tag{5.1}$$

the value function of the follower can be represented by

$$V_F^*(X, Q_L) = \begin{cases} A_F(Q_L)X^\beta & if\ X < X_F^*(Q_L) \\ \dfrac{\left(X(1-\eta Q_L) - \delta_2(r-\mu)\right)^2}{4X\eta(r-\mu)} & if\ X \geq X_F^*(Q_L) \end{cases}. \tag{5.2}$$

The rationale behind this is that for $X < X_F^*(Q_L)$, the follower firm is still in the idle state, and, therefore, the value of the firm is given by the option value $A_F(Q_L)X^\beta$ (i.e., the first branch of equation (5.2)). For $X \geq X_F^*(Q_L)$, it is known that the value of the active firm is given by equation (A3.3), which, when rewritten, gives rise to the expression in the second branch of equation (5.2).

Additionally, it can be shown that

$$A_F(Q_L) = \left(\frac{\beta-1}{\beta+1}\frac{1-\eta Q_L}{\delta_2(r-\mu)}\right)^\beta \frac{(1-\eta Q_L)\delta_2}{(\beta-1)(\beta+1)\eta}, \tag{5.3}$$

$$X_F^*(Q_L) = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}, \tag{5.4}$$

and

$$Q_F^*(Q_L) = \frac{1-\eta Q_L}{(\beta+1)\eta}. \tag{5.5}$$

Next, we examine the investment decision of the leader, who considers the strategy of the follower. Since the follower either invests simultaneously with the leader or chooses to invest later, the leader recognises that if he selects a capacity Q_L such that $X_F^*(Q_L) > X$, the follower will opt to invest at a later stage. This type of strategy is referred to as a deterrence strategy.

In the deterrence strategy, the leader maintains a monopoly as long as the demand level X is below $X_F^*(Q_L)$. However, at a certain point, the follower will enter the market and become active. This is because, as shown in equation (4.2), the output price can become arbitrarily large when the demand level X increases sufficiently. The increase in X may eventually reach the optimal investment threshold $X_F^*(Q_L)$.

A deterrence strategy occurs when the leader chooses a capacity level Q_L that exceeds $\hat{Q}_L$ (the minimum capacity required to generate entry deterrence). Conversely, the follower invests simultaneously with the leader when $Q_L \leq \hat{Q}_L$. We can define $\hat{Q}_L$ as

$$\hat{Q}_L(X) = \frac{1}{\eta}\left(1 - \frac{(\beta+1)\delta_2(r-\mu)}{(\beta-1)X}\right). \tag{5.6}$$

Considering the entry deterrence strategy, the optimal leader capacity, denoted as $Q_L^{det}(X)$ $\left(\text{with } Q_L \equiv Q_L^{det}\right)$, is determined by the following condition

$$\frac{X\left(1-2\eta Q_L^{det}\right)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)\left(1-\eta Q_L^{det}\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\left(1-(\beta+1)\eta Q_L^{det}\right)\delta_2}{(\beta-1)\left(1-\eta Q_L^{det}\right)} = 0. \tag{5.7}$$

From this expression, we can conclude that Q_L^{det} increases with X. Furthermore, by setting $Q_L^{det} = 0$, we determine a corresponding value of X. Denoting this value as X_1^{det}, we identify an investment threshold. Below this threshold, the deterrence strategy is not implemented, as the demand level is insufficient to justify the investment.

Conversely, if X is too large, it indicates that the market is very profitable, leading the follower to invest simultaneously with the leader firm, thus preventing any deterrence strategy by the leader. This outcome occurs when $X \geq X_F^*\left(Q_L^{det}(X)\right)$. In other words, the smallest level of X at which entry deterrence is not feasible is denoted by X_2^{det} and it occurs when $X_F^*\left(Q_L^{det}\left(X_2^{det}\right)\right) = X_2^{det}$.

The leader's entry deterrence policy results in the following value function for the leader

$$V_L^{det}(X, Q_L) = \frac{X(1-\eta Q_L)Q_L}{r-\mu} - \delta_1 Q_L - \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L)\eta Q_F^*(Q_L)Q_L}{r-\mu}\right), \tag{5.8}$$

where the first term represents the expected total discounted revenue that the leader gains as a monopolist, producing with capacity Q_L. The second term stands for the initial investment necessary to achieve a production capacity of Q_L. The last term is a negative correction of the first term, as the follower firm will eventually enter the market, transforming the monopoly into a duopoly. When $X(t)$ reaches the investment threshold $X_F^*(Q_L)$, the follower firm invests $Q_F^*(Q_L)$, causing the output price to decrease by $X_F^*(Q_L)\eta Q_F^*(Q_L)$, which in turn reduces the leader's revenue by $X_F^*(Q_L)\eta Q_F^*(Q_L)Q_L$. Additionally, in the last term, $\left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}$ represents the stochastic discount factor and is equal to

$$\left(\frac{X}{X_F^*(Q_L)}\right)^{\beta} = E\left[e^{-r\cdot T}\right], \tag{5.9}$$

where T is the expected first passage time of X reaching $X_F^*(Q_L)$.

5.1.2. Leader's entry deterrence strategy

At level X, the leader's entry deterrence strategy value function is given by

$$V_L^{det}(X) = \frac{XQ_L^{det}(X)\left(1-\eta Q_L^{det}(X)\right)}{r-\mu} - \delta_1 Q_L^{det}(X) - \left(\frac{X(\beta-1)\left(1-\eta Q_L^{det}(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 Q_L^{det}(X)}{\beta-1}. \tag{5.10}$$

Therefore, the optimal capacity level $\left(Q_L^{det}(X)\right)$ during the entry deterrence strategy is determined by

$$\frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0. \tag{5.11}$$

Additionally, the interval of X that the leader will consider using the entry deterrence strategy is $X \in \left]X_1^{det}, X_2^{det}\right[$. The boundaries are defined by

$$\frac{X_1^{det}}{r-\mu} - \delta_1 - \left(\frac{X_1^{det}(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2}{(\beta-1)} = 0 \tag{5.12}$$

and

$$X_2^{det} = \frac{\beta+1}{\beta-1}(\beta(\delta_2-\delta_1)+\delta_1+\delta_2)(r-\mu). \tag{5.13}$$

The optimal capacity level and investment threshold are, respectively, equal to

$$Q_L^{det}\left(X_L^{det}\right) = \frac{1}{(\beta+1)\eta}, \tag{5.14}$$

and

$$X_L^{det} = \frac{\beta+1}{\beta-1}\delta_1(r-\mu). \tag{5.15}$$

From expressions (5.14) and (5.15), it is possible to realise that these investment decisions coincide with the monopolist (see expressions (4.7) and (4.8), respectively).

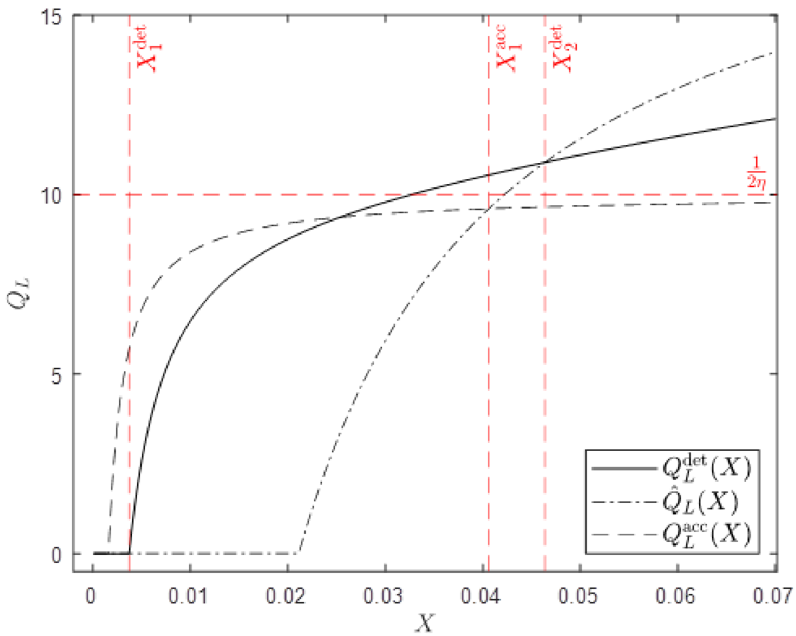
In an alternative scenario where the leader firm invests in a capacity level such that $Q_L < \hat{Q}_L(X)$, the follower firm will invest immediately afterwards. This strategy can be represented as an entry accommodation strategy by the leader.

As shown in Figure 5.1, there is an interval during which the leader firm will consider accommodating the entrance of its competitor. Therefore, there exists a level, denoted by X_1^{acc}, from which the leader firm will consider accommodating the follower firm.

Figure 5.1 shows our results from using expression (5.6) to obtain $\hat{Q}_L(X)$, expression (5.7) to compute $Q_L^{det}(X)$, and expression (5.18) to determine $Q_L^{acc}(X)$. This figure is similar to Figure 2 from Huisman and Kort (2015) and was obtained using $r = 0.1, \mu = 0.06, \sigma = 0.1, \delta_1 = 0.08, \delta_2 = 0.12$, and $\eta = 0.05$ as parameter values.

Figure 5.1. Optimal investment capacities in deterrence strategy $\left(Q_L^{det}\right)$, accommodation strategy (Q_L^{acc}), and minimum capacity for entry deterrence $\left(\hat{Q}_L\right)$ as functions of X

Expressions (5.6), (5.7), and (5.18) are used to determine $\hat{Q}_L(X)$, $Q_L^{det}(X)$, and $Q_L^{acc}(X)$, respectively. The parameter values employed in the analysis are: $r = 0.1$, $\mu = 0.06$, $\sigma = 0.1$, $\delta_1 = 0.08$, $\delta_2 = 0.12$, and $\eta = 0.05$. The figure illustrates the possible strategies that the leader firm may implement for different levels of X.



5.1.3. Leader's entry accommodation strategy

In the context of an entry accommodation strategy, the value function when the leader invests at the level X is given by

$$V_L^{acc}(X) = \frac{\left(X - (2\delta_1 - \delta_2)(r - \mu)\right)^2}{8X\eta(r - \mu)}. \tag{5.16}$$

The leader firm will consider this strategy when $X \geq X_1^{acc}$, where the investment threshold is

$$X_1^{acc} = \frac{(2 - 2\beta)\delta_1 + (1 + 3\beta)\delta_2}{\beta - 1}(r - \mu). \tag{5.17}$$

In this strategy, the leader will invest at the optimal capacity level

$$Q_L^{acc}(X) = \frac{1}{2\eta}\left(1 - \frac{(2\delta_1 - \delta_2)(r - \mu)}{X}\right). \tag{5.18}$$

Nevertheless, the optimal investment threshold for the leader, X_L^{acc}, is given by

$$X_L^{acc} = \frac{\beta + 1}{\beta - 1}(2\delta_1 - \delta_2)(r - \mu). \tag{5.19}$$

However, as $X_L^{acc} < X_1^{acc}$, the investment threshold X_L^{acc} does not hold significant meaning since the demand level X must be at least equal to X_1^{acc} for the follower to invest simultaneously with the leader.

At the threshold X_L^{acc}, the optimal capacity level is

$$Q_L^{acc}(X_L^{acc}) = \frac{1}{(\beta + 1)\eta}. \tag{5.20}$$

In Figure 5.1, the functions Q_L^{det}, $\hat{Q}_L$, and Q_L^{acc} are shown as a function of X. For levels of X lower than X_1^{acc}, the leader will apply an entry deterrence strategy, since the optimal capacity level of the leader associated with the entry accommodation strategy, Q_L^{acc}, is higher than the minimum capacity level needed to generate entry deterrence, $\hat{Q}_L$.

For X levels higher than X_2^{det}, which can be visualised by the intersection point of $\hat{Q}_L$ and Q_L^{det}, the leader can only choose the entry accommodation strategy because the optimal capacity level of the leader corresponding to the entry deterrence strategy, Q_L^{det}, is lower than $\hat{Q}_L$, implying that entry accommodation will occur.

In the interval $X \in \left]X_1^{acc}, X_2^{det}\right[$ either the entry deterrence or the entry accommodation strategy can occur because both maximise the leader's value.

5.1.4. Strategic regime switching in leader investment behaviour

Given the results in subchapter 5.1.3, it is possible to construct the capacity level of the leader as follows

$$Q_L^*(X) = \begin{cases} Q_L^{det}\left(X_L^{det}\right) & if\ X \in \left[0, X_L^{det}\right[\\ Q_L^{det}(X) & if\ X \in \left[\,X_L^{det}, \hat{X}\right[, \\ Q_L^{acc}(X) & if\ X \in \left[\hat{X}, \infty\right[\end{cases} \tag{5.21}$$

where $\hat{X}$ is

$$\hat{X} = \min\ \left\{X \in \left]X_1^{acc}, X_2^{det}\right[\ \middle|\ V_L^{acc}(X) = V_L^{det}(X)\right\}. \tag{5.22}$$

In the first branch of the system of equations in (5.21), the leader commits to investing in a fixed capacity level $Q_L^{det}\left(X_L^{det}\right)$. This is because the leader aims to deter entry by committing to a capacity level that is optimal for entry deterrence at the threshold X_L^{det}. By maintaining this fixed capacity level, the leader signals to the follower that entry would not be profitable, thereby discouraging the follower from entering the market.

In the second interval, the leader adjusts its capacity level dynamically based on the current level of X. The leader invests in a capacity level $Q_L^{det}(X)$ that is optimal for entry deterrence given the current X. This dynamic adjustment allows the leader to respond to changes in market conditions and maintain its entry deterrence strategy effectively. This strategy will last until X reaches $\hat{X}$.

Lastly, from subchapter 5.1.2, it is known that there exists a level X_1^{acc} from which the leader considers using the accommodation strategy. The third branch represents the optimal capacity level, under the accommodation strategy, that the leader will produce when X reaches the minimum value of the interval $\hat{X}$.

The leader's value function can be represented as

$$V_L^*(X) = \begin{cases} \left(\dfrac{X}{X_L^{det}}\right)^{\beta} V_L^{det}\left(X_L^{det}\right) & if\ X \in \left[0, X_L^{det}\right[\\ V_L^{det}(X) & if\ X \in \left[\,X_L^{det}, \hat{X}\right[\\ V_L^{acc}(X) & if\ X \in \left[\hat{X}, \infty\right[\end{cases}. \tag{5.23}$$

The first branch represents the present value of the leader's firm when it commits to producing $Q_L^{det}\left(X_L^{det}\right)$. The initial term represents the stochastic discount factor of the leader's firm. This expression reflects the value of the firm that the leader holds but has not yet realised, as the demand level is currently too low to justify an investment.

The second branch represents the leader's value function as described in equation (5.10), applicable when the demand level X falls within the range where the leader can only employ the entry deterrence strategy.

Finally, the third branch illustrates the leader's value function when the entry accommodation strategy is implemented. This occurs when the demand level X is sufficiently high $\left(\text{at least } X \geq \hat{X}\right)$. In this region, the leader's value function can be represented as expressed in expression (5.16).

5.1.5. Leader's investment threshold

Based on the results in subchapter 5.1.4, the investment threshold for the leader with a significant cost advantage can be defined as

$$X_L^* = \begin{cases} X_L^{det} & if\ X \in \left[0, X_L^{det}\right[\\ X & if\ X \in \left[X_L^{det}, \infty\right[\end{cases} \tag{5.24}$$

From Figure 5.2, it is evident that at X_L^{det}, the leader invests in a capacity level higher than the follower. At levels below $\hat{X}$, $Q_L^{det}(X)$ increases with X, as reasoned through expression (5.11).

To obtain Figure 5.2 (whose results are similar to Huisman and Kort (2015)), it is important to highlight that for values of $X \in \left[X_L^{det}, \hat{X}\right[$, we use expression (5.11) to determine the leader's capacity. For values higher than or equal to $\hat{X}$, we use expression (5.18).

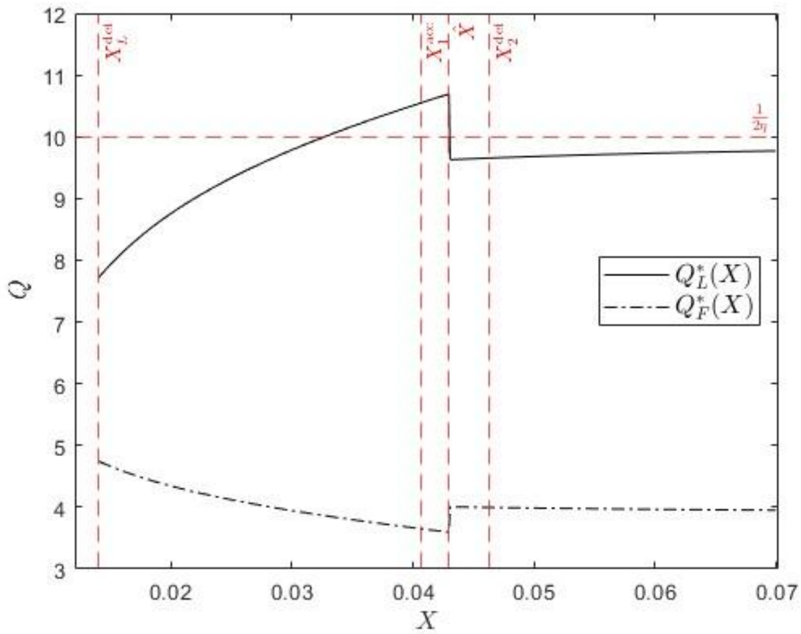
The follower's capacity is computed using the optimal investment capacity in expression (5.5), but selecting the leader's capacity. This means that if X is below $\hat{X}$, we use the leader's capacity obtained from expression (5.11). For values higher than or equal to $\hat{X}$, we use the leader's capacity from expression (5.18).

The threshold $\hat{X}$ is determined by solving a system of two equations. In the first equation, we set $V_L^{acc}(\hat{X}) - V_L^{det}(\hat{X}) = 0$. In the second equation, we solve expression (5.11) where $Q_L^{det}(X \equiv \hat{X})$.

The parameter values used to plot both functions are: $r = 0.1$, $\mu = 0.06$, $\sigma = 0.1$, $\delta_1 = 0.08$, $\delta_2 = 0.12$ and $\eta = 0.05$.

Figure 5.2. Optimal investment capacities as functions of X for the leader (Q_L^*) and follower (Q_F^*) firms

Leader and follower capacity levels as functions of X, shown for values where $X \geq X_L^{det}$. The leader's capacity is determined by expression (5.11) when $X < \hat{X}$, and by expression (5.18) when $X > \hat{X}$. The follower's capacity follows expression (5.5). The parameter values used are: $r = 0.1,\ \mu = 0.06,\ \sigma = 0.1,\ \delta_1 = 0.08,\ \delta_2 = 0.12$, and $\eta = 0.05$. The leader consistently invests in a higher capacity than the follower, reflecting a tendency to overinvest.



In the entry deterrence region, the leader deliberately overinvests in capacity to discourage the follower from investing. This overinvestment is evident when comparing the capacity levels between entry deterrence and entry accommodation strategies at $\hat{X}$.

Overinvestment results from the leader firm's investment. As reasoned in expression (5.5), the higher the leader's capacity Q_L, the lower the follower's capacity. Additionally, from the investment threshold in expression (5.4), the higher the leader's capacity, the later the follower invests.

5.2. No cost advantage for the leader

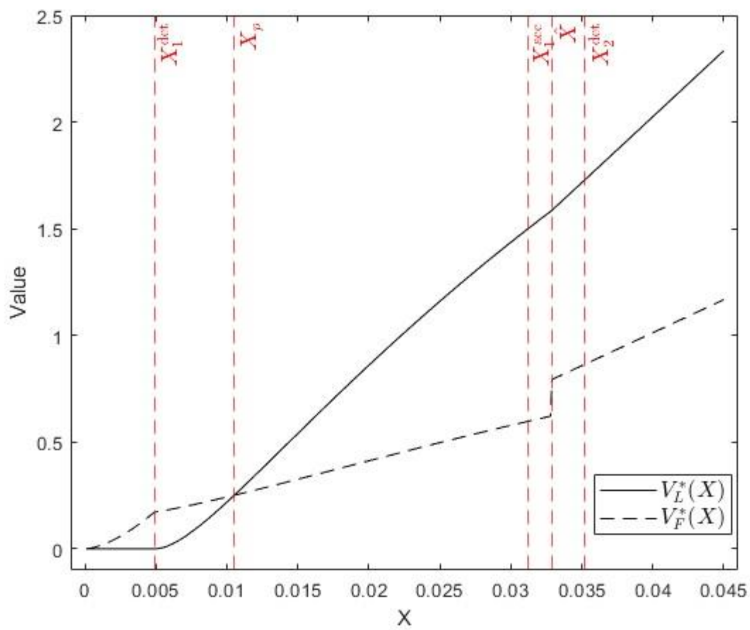
In this subchapter, it is assumed that the leader firm and the follower have identical unit capacity costs, with $\delta_1 = \delta_2 = \delta$. Consequently, as the firms are now symmetric, the investment costs are represented by δQ.

In this scenario, both firms are vying to be the first investor, as the initial investor will enjoy a temporary monopoly until the other firm invests. This creates an incentive for each firm to pre-empt the other. Once the first firm has invested, the strategic considerations for the second firm's investment are no longer relevant. Consequently, the timing and capacity level of the second firm's investment mirror those of the follower in the asymmetric firms' case. Therefore, subchapter 5.1.1 (with δ replacing δ_2) applies to this framework.

Figure 5.3 illustrates that the leader's value, representing the payoff after immediate investment, is lower than the follower's value for X values below the pre-emption trigger, X_P, due to insufficient demand for immediate investment. When the leader shifts from an entry deterrence strategy to an accommodation strategy at $X = \hat{X}$, it reduces its capacity level (as depicted in Figure 5.2). This reduction increases the output price for the follower, causing a noticeable jump in the follower's value at $X = \hat{X}$.

Figure 5.3. Optimal value functions for the leader (V_L^*) and follower (V_F^*) as a function of the investment point X for the leader.

Leader and follower value functions in the symmetric-cost case. The leader's value function is given by expression (5.10) for $X < \hat{X}$, and by expression (5.16) for $X \geq \hat{X}$. The follower's value function is defined by expression (5.2), where $\hat{X}$ is the investment threshold that divides the idle region from the active state. The parameter values used are: $r = 0.1, \mu = 0.06, \sigma = 0.1, \delta = 0.1$, and $\eta = 0.05$.



To achieve the figure above, it is necessary to follow the following steps.

For the interval $X \in [0\,,\ \hat{X}[$, the leader's optimal capacity level $Q_L^*\,(X)$ is simply the capacity level that deters entry, $Q_L^{det}\,(X)$. Since the costs are identical, the leader's entry deterrence strategy remains consistent across this interval. This means the leader does not need to maintain a fixed capacity level to signal to the follower that entry would not be profitable, as is the case in the interval of $X \in [0\,,\ X_L^{det}[$ when $\delta_1 \neq \delta_2$ (see expression (5.21)).

The value function for the interval of $X \in [0\,,\ \hat{X}[$ is outlined in expression (5.10), where $Q_L^{det}\,(X)$ is obtained from expression (5.11). For the interval of $X \in [\hat{X}\,, \infty[$, where the leader accommodates the entry of the follower firm, the value considered is presented in expression (5.16), with the capacity level expressed in expression (5.18).

The point $\hat{X}$ was obtained in the same manner as in Figure 5.2.

Since the follower firm invests at the $\hat{X}$ threshold, we can assume $\hat{X}$ as the $X_F^*\,(Q_L)$ trigger that divides the idle region from the active state in the value function of expression (5.2). Therefore, for $X \in [0\,,\ \hat{X}[$, the value function for the follower firm is given by the first branch of expression (5.2), where $Q_L \equiv Q_L^{det}\,(X)$ from expression (5.11). For the interval $X \in [\hat{X}\,, \infty[$, the value function is given by the second branch of expression (5.2), where $Q_L \equiv Q_L^{acc}\,(X)$ from expression (5.18).

The parameter values used to construct Figure 5.3 are $r = 0.1, \mu = 0.06, \sigma = 0.1, \delta = 0.1$, and $\eta = 0.05$.

5.2.1. Impact of uncertainty on entry deterrence and accommodation strategies

The discussion now centres on the effect of uncertainty on entry deterrence and entry accommodation strategies, following a scenario in which there is no cost advantage for the leader.

As X_1^{acc} increases with uncertainty, the X interval where only deterrence occurs also expands. Consequently, it becomes more appealing for the leader to use an entry deterrence strategy, as they will enjoy a longer period of being a monopolist, while the follower is incentivised to invest later.

However, Maskin (1999) finds the opposite. He argues that increased uncertainty diminishes the feasibility of employing an entry deterrence strategy. This is because uncertainty prompts the incumbent to invest in higher capacity levels, which subsequently reduces the profitability of such a strategy.

The differing results between Huisman and Kort (2015) and Maskin (1999) stem from their methodological approaches. Huisman and Kort (2015) utilise a dynamic framework that accounts for the value of waiting for the entrant, while Maskin (1999) employs a static model.

From the results in expressions (A6.1), (A6.3), and (A6.5), it is possible to realise that the interval where the leader can choose between entry deterrence and entry accommodation strategies, i.e., $X \in]X_1^{acc}, X_2^{det}[$, is independent of uncertainty. This is because X_1^{acc} and X_2^{det} are affected to the same extent by it. Nevertheless, this interval decreases with the drift rate, as expression (A6.21) suggests.

In Figure 5.3, the first investor invests at the pre-emption threshold X_P because at time zero $X(0) < X_P$. The pre-emption threshold X_P is the solution of

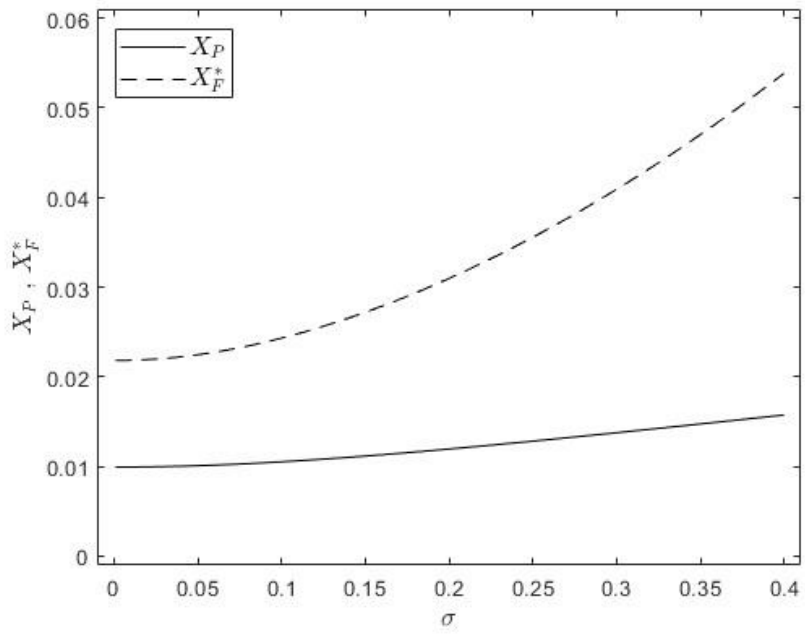
$$V_L^*(X_P) = V_F^*\big(X_P, Q_L^*(X_P)\big). \tag{5.25}$$

The intuition is that when $X < X_P$, the demand level is too low to justify an investment, and the payoff for the second investor is higher than for the first. When $X > X_P$, the payoff for the first investor surpasses that of the second. Knowing in advance that firm 1 will invest at this X level, it is optimal for firm 2 to pre-empt by investing at $X - \varepsilon$. This reaction will induce firm 1 to pre-empt firm 2, and firm 2 will then pre-empt firm 1. As the process continues, a race of pre-emptions occurs until $X - n\varepsilon = X_P$, where an investment will be made by one of the two firms. In this scenario, the first investor, who becomes the leader, invests at the threshold X_P with a capacity level of $Q_L^*(X_P)$, while the second firm, which becomes the follower, invests at the threshold $X_F^*\big(Q_L^*(X_P)\big)$ with a capacity level of $Q_F^*\big(Q_L^*(X_P)\big)$.

In the case of asymmetric cost firms, it is evident that the leader will always invest in a larger capacity level than the follower firm. However, since in this subchapter there is no explicit expression for X_P it becomes impossible to derive precise analytical results about the equilibrium capacity levels when considering the roles of firms that are determined within the system itself. Essentially, the lack of an explicit expression for X_P complicates the analysis and prevents straightforward calculations or conclusions about the capacity level at equilibrium.

Nevertheless, some experiments, such as the one presented in Figure 5.4, concluded that the investment thresholds for the leader, X_P, and the follower, X_F^*, increase with uncertainty.

Figure 5.4. Investment thresholds for the leader (X_P) and follower (X_F^*) firms as functions of σ
Investment thresholds for the leader X_P and follower X_F^* firms under a scenario with no cost advantage. The leader's threshold is derived from expression (5.25), while the follower's threshold is obtained using equation (5.4). These thresholds represent the critical demand levels at which each firm chooses to invest. The parameters used in the calculations are: $r = 0.1$, $\mu = 0.06$, $\delta = 0.1$, and $\eta = 0.05$.



Some steps are necessary to obtain the figure above. First, we need to solve equation (5.25) to determine the threshold X_P, where the value function $V_L^*\ (X_P)$ is represented by expression (5.10), where $X \equiv X_P$ and $Q_L^{det}\ (X)$ is given by expression (5.11). Additionally, the value function $V_F^*\big(X_P, Q_L^*(X_P)\big)$ is represented by the first branch of expression (5.2), where $Q_L \equiv Q_L^{det}$ from expression (5.11). These assumptions for the value functions are based on Figure 5.3, which shows that $X_P \in \left]0\,, \hat{X}\right[$. Therefore, we can disregard the value functions for the interval $\left[\hat{X}\,, \infty\right[$ when computing X_P.

It is also important to note that β should vary according to σ (via expression (4.10)) within the interval σ.

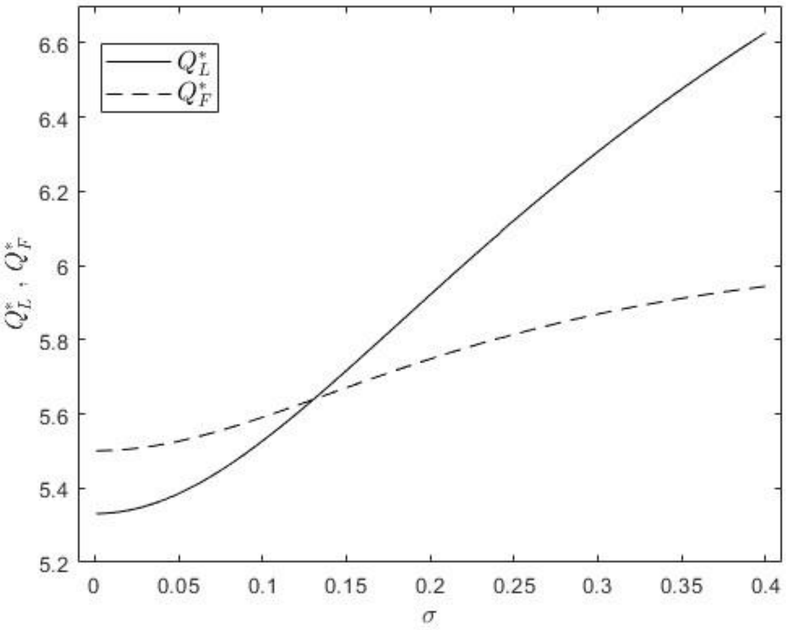
To obtain X_F^*, we need to solve the optimal investment threshold of the follower, represented in the equation (5.4), where $Q_L \equiv Q_L^{det}(X \equiv X_P)$ as given in expression (5.11).

The parameters used in the calculations are: $r = 0.1$, $\mu = 0.06$, $\delta = 0.1$, and $\eta = 0.05$.

Figure 5.5 shows that due to the pre-emption threat, the leader can be forced to invest early when the market is too small to justify a large capacity investment. As a consequence, the leader deters the follower until X is large enough, making $Q_F > Q_L$ possible, meaning that the follower's capacity will be larger in equilibrium, but only for low uncertainty values. For high levels of uncertainty, the value of waiting increases, delaying investments. This implies that when the leader finally invests, the market is large enough to justify a significant capacity investment, resulting in the leader's capacity level being higher than the follower's capacity.

Figure 5.5. Optimal investment capacities for the leader (Q_L^*) and follower (Q_F^*) firms as functions of σ

Optimal investment capacities for the leader Q_L^* and follower Q_F^* firms under a scenario with no cost advantage. The leader's capacity is derived from expression (5.11) and the follower's capacity from equation (5.1). The parameter values are consistent with those used in Figure 5.4. Notably, in low-uncertainty environments, the follower may emerge as the larger firm.



To obtain Figure 5.5, Q_L^* is determined by solving expression (5.11), where $X \equiv X_P$ and β varies according to σ. To obtain Q_F^* we need to solve equation (5.1), where $X \equiv X_F^*$ and $Q_L \equiv Q_L^*$. The parameter values are the same as those used in Figure 5.4.

However, Figures 5.6 and 5.7, computed with the same parameter values as Figure 5.4 but with $\mu = 0.09$, demonstrate that the conclusions in the two previous figures do not hold. In this case, the leader is always the larger firm, regardless of whether the uncertainty is low or high.

Figure 5.6. Investment thresholds for the leader (X_P) and follower (X_F^*) firms as functions of σ Investment thresholds for the leader X_P and follower X_F^* firms under a scenario with no cost advantage. The thresholds are derived using the same expressions as in Figure 5.4. The parameter values are also consistent with Figure 5.4, except for the drift, which is set to $\mu = 0.09$. These thresholds represent the critical demand levels at which each firm chooses to invest.

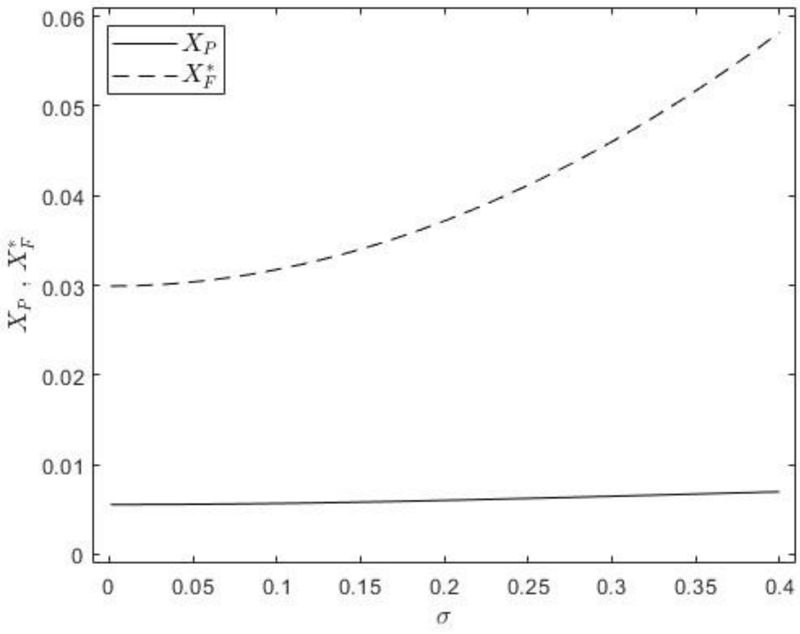
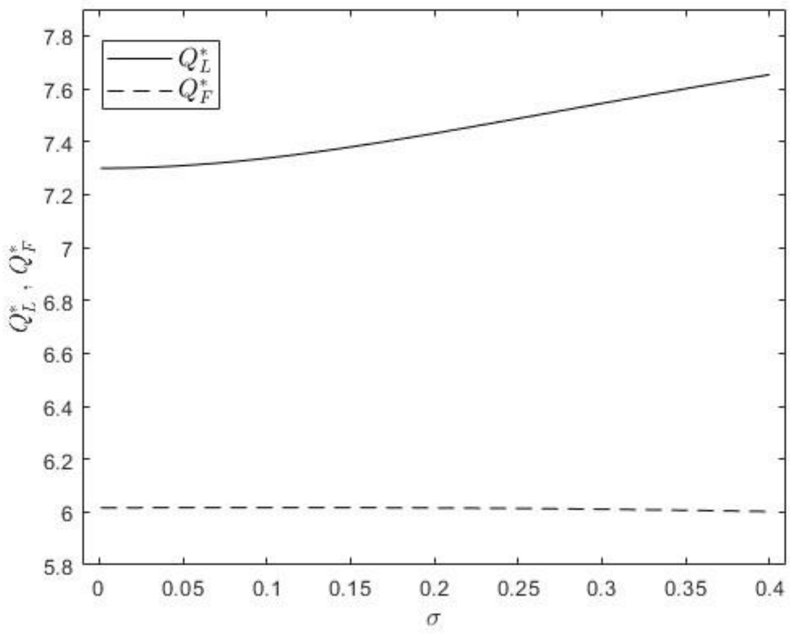


Figure 5.7. Optimal investment capacities for the leader (Q_L^*) and follower (Q_F^*) firms as functions of σ

Optimal investment capacities for the leader Q_L^* and follower Q_F^* firms under a scenario with no cost advantage. The capacities are derived using the same expressions as in Figure 5.5. The parameter values are consistent with those in Figure 5.4, except for the drift, which is set to $\mu = 0.09$. In this scenario, the leader firm consistently emerges as the larger firm.



Finally, to generate Figures 5.6 and 5.7, we follow the same steps outlined for Figures 5.4 and 5.5, respectively, with the drift parameter set to $\mu = 0.09$.

5.3. Moderate asymmetry

In the case of moderate asymmetry, meaning that $\hat{\delta}_1 < \delta_1 < \delta_2$, firm 1 will invest at the pre-emption point of firm 2, X_{P_2}, to prevent firm 2 from investing first. Given the difference in investment costs, it results in $X_{P_2} > X_{P_1}$, implying that firm 2 will invest later than firm 1. Firm 1 has a strong preference to be the first to invest at the pre-emption point of firm 2, X_{P_2}, because it aims to secure a more advantageous position in the market, avoiding the increased competition and potential reduction in profits that would occur if firm 2 were to invest first. Consequently, in this scenario $V_L^*(X_P) > V_F^*\big(X_P, Q_L^*(X_P)\big)$.

As the asymmetry between the two firms increases, X_{P_2} also increases, meaning that firm 1 needs to invest later to pre-empt firm 2. As the asymmetry increases, the equilibrium converges to the leader's cost advantage.

5.4. Welfare

Regarding the welfare perspective of the duopoly investment, Huisman and Kort (2015) considered a social planner who can invest twice. Since the investment strategy that maximises welfare is determined backwards in time, firstly, the investment trigger and capacity level of the second investment are determined conditionally on the capacity level of the first investment. Then, the threshold and capacity level of the first investment are computed.

Before analysing the social planner case, Huisman and Kort (2015) examine the scenario where a monopolist invests twice. This investment scenario is also solved backwards, meaning that to analyse the first investment, the optimal investment behaviour for the second investment is studied first, conditionally on the first investment. To differentiate between the first and second investments, the results and functions for the first investment are represented by the number one, while those for the second investment are denoted by the number two.

For the social planner scenario, however, the first investment is represented by L, W and the second investment by F, W.

5.4.1. Monopolist and social planner with two investment opportunities

In this subchapter, the objective is to compare the outcomes of a monopolist who invests twice over time with those of a social planner who has the same opportunity to invest twice.

For the monopolist, the first investment threshold and optimal investment capacity are given by the following expressions

$$X_1^*(Q_1) = \frac{\beta\delta(r-\mu)}{(\beta-1)(1-\eta Q_1)}, \tag{5.26}$$

$$1 - \frac{\beta\eta Q_1^*}{1-\eta Q_1^*} - 2\left[\frac{\beta(1-2\eta Q_1^*)}{(\beta+1)(1-\eta Q_1^*)}\right]^\beta = 0. \tag{5.27}$$

The second investment's threshold and investment capacity for the monopolist are given by

$$X_2^*(Q_1) = \frac{(\beta+1)\delta(r-\mu)}{(\beta-1)(1-2\eta Q_1)}, \tag{5.28}$$

$$Q_2^*(Q_1) = \frac{1-2\eta Q_1}{(\beta+1)\eta}. \tag{5.29}$$

The initial investment by the social planner, aimed at maximising welfare, is characterised by the following investment threshold and capacity level

$$X_{L,W}^{*}(Q_L) = \frac{\beta(r-\mu)\delta}{(\beta-1)\left(1-\frac{1}{2}\eta Q_L\right)}, \tag{5.30}$$

$$1 - \frac{\beta\frac{1}{2}\eta Q_{L,W}^{*}}{1-\frac{1}{2}\eta Q_{L,W}^{*}} - 2\left(\frac{\beta\left(1-\eta Q_{L,W}^{*}\right)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_{L,W}^{*}\right)}\right)^{\beta} = 0. \tag{5.31}$$

For the second investment, the social planner invests at the following threshold with the capacity level

$$X_{F,W}^{*}(Q_L) = \frac{\beta+1}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_L}, \tag{5.32}$$

$$Q_{F,W}^{*}(Q_L) = \frac{2(1-\eta Q_L)}{(\beta+1)\eta}. \tag{5.33}$$

Comparing the results from the social planner and the monopolist who can invest twice, if the capacity levels of the first investment in the welfare-maximising policy double the capacity level of the monopolist, meaning that $Q_{L,W}^{*} = 2Q_1^{*}$, the investment thresholds (expressions (5.30) and (5.32) for the social planner, and (5.26) and (5.28) for the monopolist who invests twice) will be equal.

Furthermore, the social planner's investment will occur later than that of the follower. This is because, according to expressions (5.32) and (5.4) for the social planner and follower firm, respectively, the investment trigger for the social planner is higher than that for the follower firm. However, the social planner's capacity level (expression (5.33)) will be less than twice that of the follower (expression (5.5)).

In a different scenario, if we focus solely on the capacity levels between the social planner's second welfare investment (expression (5.33)) and the follower firm's capacity (expression (5.5)), it becomes clear that the capacity level under the welfare-maximising policy will be twice as high as the capacity chosen by the follower, assuming the capacity level of the first investment is the same in both cases.

The TS in this market is defined in expression (A7.25), where $X_1 \equiv X_L$, $Q_1 \equiv Q_L$, $X_2 \equiv X_F$, and $Q_2 \equiv Q_F$. However, the TS presented by Huisman and Kort (2015) in their expression (55) represents the present value of (A7.25), considering the timing of each investment. Following their nomenclature, this can be defined as

$$TS(X_L, Q_L, X_F, Q_F) = \left(\frac{X}{X_L}\right)^{\beta}\left(\frac{X_L Q_L(2-\eta Q_L)}{2(r-\mu)} - \delta Q_L\right) + \left(\frac{X}{X_F}\right)^{\beta}\left(\frac{X_F(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)} - \delta Q_F - \frac{X_F Q_L(2-\eta Q_L)}{2(r-\mu)}\right). \tag{5.34}$$

Comparing the monopolist and duopoly cases (from Tables 4.1 and 5.1, respectively), it is evident that the leader firm invests before the monopolist. After the follower invests, the total market capacity exceeds that of the monopolist.

When considering the optimal investment capacity chosen by the social planner for the second investment $(Q^*_{F,W})$, it becomes evident that this capacity initially increases and then decreases with uncertainty (σ). This scenario involves two conflicting effects. Initially, as uncertainty rises, the value of waiting also increases, leading to the first investment occurring at a later stage. As a result, demand will be higher at that time, making it optimal to invest in a higher capacity. However, this higher capacity reduces the profitability of the second investment, resulting in a lower capacity for the second investment. Conversely, the second effect is that increased uncertainty delays the second investment as well, which occurs when demand is higher, thereby positively influencing the capacity level of the second investment.

When comparing welfare loss (see Table 5.2), it becomes evident that the duopoly with symmetric costs incurs less welfare loss than the monopoly, approximately 12% versus 25% (see Tables 5.2 and 4.1, respectively). In the case of asymmetric costs, the welfare loss is even lower, around 9%, likely due to the pre-emption effect, which leads to earlier investments.

For symmetric costs, welfare loss initially decreases slightly with uncertainty and then increases. The minor differences between our values and those presented by Huisman and Kort (2015) in their Table 2 are due to rounding. Uncertainty (σ) affects total surplus via β, which directly and indirectly influences total surplus through the determination of thresholds and investment capacities (as seen in the computations that led to expression (A7.26)).

Comparing the investment capacities of the duopoly with symmetric costs and the monopolist from Table 4.1, it is evident that in the duopoly scenario, capacity increases with uncertainty and better aligns with what the social planner would invest. This means that in the duopoly scenario, there is less uninvested capacity compared to what the social planner would invest.

Table 5.1. Investment characteristics and total surpluses in duopoly and social planner scenarios with firms' symmetric and asymmetric costs

This table summarises the investment thresholds and capacity levels for a duopoly and a social planner who invests twice, assuming symmetric investment costs across firms or investment stages. For both the duopoly and the social planner cases, the final two columns report the present values of total surplus under symmetric and asymmetric cost structures. Table 2 in Huisman and Kort (2015) omits the total surplus values for the social planner under asymmetric cost conditions. All values are computed for specific levels of uncertainty (σ). The parameters used are: $r = 0.1$, $\mu = 0.06$, $\eta = 0.05$, $\delta = 0.1$, and $X(0) = 0.001$. In the case of asymmetric costs, $\delta_1 = 0.08$ and $\delta_2 = 0.12$.

	Duopoly						Social Planner					
σ	X_P	Q_L^*	X_F^*	Q_F^*	TS_{sym}	TS_{asym}	$X_{L,W}^*$	$Q_{L,W}^*$	$X_{F,W}^*$	$Q_{F,W}^*$	$TS_{W_{sym}}$	$TS_{W_{asym}}$
0.00	0.00993	5.331	0.02182	5.501	0.02065	0.02347	0.01291	9.010	0.02912	8.242	0.02352	0.02605
0.05	0.01009	5.385	0.02246	5.526	0.02193	0.02487	0.01321	9.086	0.03007	8.254	0.02497	0.02758
0.10	0.01052	5.527	0.02431	5.591	0.02561	0.02885	0.01406	9.284	0.03283	8.279	0.02915	0.03192
0.15	0.01116	5.716	0.02719	5.671	0.03125	0.03493	0.01538	9.544	0.03714	8.302	0.03557	0.03853
0.20	0.01195	5.921	0.03096	5.748	0.03835	0.04256	0.01709	9.817	0.04281	8.315	0.04365	0.04678
0.25	0.01282	6.121	0.03555	5.814	0.04641	0.05120	0.01916	10.076	0.04972	8.315	0.05285	0.05611
0.30	0.01376	6.306	0.04089	5.869	0.05504	0.06044	0.02156	10.310	0.05779	8.306	0.06271	0.06603
0.35	0.01474	6.475	0.04699	5.911	0.06389	0.06991	0.02427	10.516	0.06701	8.290	0.07285	0.07618
0.40	0.01574	6.627	0.05384	5.944	0.07271	0.07935	0.02730	10.695	0.07738	8.271	0.08297	0.08626

Table 5.2. Comparison of optimal investment capacities and total surpluses in duopoly and social planner scenarios

This table compares the total surpluses reported in Table 5.1 for both symmetric and asymmetric cost scenarios, along with the corresponding optimal investment capacities (under symmetric cost conditions) for a duopoly and a social planner who invests twice. All parameters used are consistent with those presented in Table 5.1.

σ	Comparison		
	$\frac{TS_{sym}}{TS_{W_{sym}}}$	$\frac{TS_{asym}}{TS_{W_{asym}}}$	$\frac{Q_L^* + Q_F^*}{Q_{L,W}^* + Q_{F,W}^*}$
0.00	0.87813	0.90099	0.62785
0.05	0.87826	0.90178	0.62924
0.10	0.87840	0.90385	0.63304
0.15	0.87848	0.90663	0.63807
0.20	0.87850	0.90963	0.64358
0.25	0.87813	0.91254	0.64896
0.30	0.87768	0.91524	0.65402
0.35	0.87706	0.91768	0.65861
0.40	0.87633	0.91983	0.66281

Table 5.1 pertains to a numerical example using the parameter values $r = 0.1$, $\mu = 0.06$, $\eta = 0.05$, $\delta = 0.1$, and $X(0) = 0.001$ for columns two to six and eight to twelve. These columns represent the investment thresholds and capacities determined under the assumption of equal costs for both the leader and follower firms, as well as the two investments made by the social planner and corresponding total surpluses. The values for columns two to five correspond to those depicted in Figures 5.4 and 5.5, but here we focus on specific values of uncertainty (σ).

Columns six, seven, twelve, and thirteen show the present value of the expected total surplus when firms or the social planner invest at their respective thresholds, as suggested by expression (5.34). The formula for computing the total surplus of the social planner is analogous to TS in expression (5.34), but with the optimal investment threshold and capacity values of the social planner substituting the values of the duopoly.

For the case of asymmetrical costs (columns seven and thirteen), the same inputs are used as in the other columns, except $\delta = 0.1$ is replaced by $\delta_1 = 0.08$ for the leader (or first investment of the social planner) and $\delta_2 = 0.12$ for the follower (or second investment of the social planner).

To calculate the total surplus of the duopoly with asymmetrical costs, we use X_L^{det} (see expression (5.15)) as the leader threshold and $X_F^*\left(Q_L \equiv Q_L^{det}\left(X_L^{det}\right)\right)$ (see expression (5.4)) as the follower threshold. The investment capacities are $Q_L^{det}\left(X_L^{det}\right)$ for the leader (see expression (5.14)) and $Q_F^*\left(Q_L \equiv Q_L^{det}\left(X_L^{det}\right)\right)$ for the follower (see expression (5.5)). These variables are then substituted into the corresponding places in expression (5.34) for both the leader and follower firms.

For the social planner's total surplus with asymmetrical costs, δ is replaced by δ_1 in expression (5.30) for the first investment and by δ_2 in expression (5.32) for the second investment. The investment capacities remain the same as in the symmetrical costs scenario, as they are not affected by the investment costs (see expressions (5.31) and (5.33)). These values are then substituted into the variables for the leader and follower to compute the present total surplus.

Table 5.2 compares the welfare loss from the abovementioned scenarios and the invested capacity differences.

CHAPTER 6

The constant elasticity demand case

This chapter focuses on the constant elasticity demand case, which will be analysed to understand its nuances and the robustness of the results presented by Huisman and Kort (2015).

In this chapter, the linear demand curve will be replaced by an isoelastic demand curve given by

$$P(t) = X(t)\big(Q(t)\big)^{-\gamma}, \tag{6.1}$$

where $\gamma \in]0,1[$.

Our goal in this chapter is to mathematically prove the results in Propositions 9, 10, 11, 12, and 15 of the Huisman and Kort (2015) article and implement their findings. Propositions 13 and 14 are going to be discussed theoretically. To achieve this, the following subchapters will derive the results of Propositions 1 to 7 using the isoelastic demand curve.

In subchapter 6.1, we will revisit the analysis from Chapter 4, focusing on the monopoly scenario and proving Proposition 9. This will be followed by examining the welfare implications and changes that occur when a constant elasticity of demand is applied.

Subchapter 6.2 is dedicated to analysing the duopoly case from Chapter 5, but with a constant elasticity demand curve. Here, Propositions 10, 11, 12, 13, 14, and 15 resemble Propositions 2, 3, 4, 5, 6, and 7, respectively. The proofs of the propositions are detailed in Annex B.

6.1. Monopoly analysis

In this subchapter, we revisit the monopoly scenario initially discussed in Chapter 4, focusing on a constant elasticity demand curve. To achieve this, our analysis will be divided into two segments. First, we will examine the monopolist's optimal investment decision. Following this, we will analyse the welfare implications of this market structure.

The ultimate goal is to prove Proposition 9 mathematically.

6.1.1. Monopolist's optimal investment decision

The value function of a monopolist firm with an isoelastic demand curve can be represented as

$$V(X) = \begin{cases} AX^{\beta} & if\ X < X^* \\ \frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0 & if\ X \geq X^* \end{cases}. \tag{6.2}$$

Here, the first branch denotes the option value when the firm is idle, and the second branch represents the monopolist's firm value, dependent only on X, when the firm is active.

In the value function of expression (6.2), the variable A can be defined as

$$A = \left(\frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}\right)^{-\beta}\frac{\delta_0}{\beta\gamma-1}. \tag{6.3}$$

The optimal investment threshold for the monopolist is

$$X^* = \frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}. \tag{6.4}$$

This, in turn, corresponds to the optimal capacity level of

$$Q^*(X^*) \equiv Q^* = \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}. \tag{6.5}$$

6.1.2. Optimal welfare decision

Given that the instantaneous consumer surplus is equal to

$$\int_{P(Q)}^{\infty} D(P)\,dP = \frac{\gamma}{1-\gamma}XQ^{1-\gamma}, \tag{6.6}$$

the total expected consumer surplus is given by

$$CS(X,Q) = \frac{\gamma}{1-\gamma}\frac{XQ^{1-\gamma}}{r-\mu}. \tag{6.7}$$

Since the expected producer surplus is equal to the value of the monopolistic firm from equation (B1.3), it can be represented as

$$PS(X,Q) = \frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q. \tag{6.8}$$

Therefore, the total expected surplus is given by

$$TS(X,Q) = \frac{1}{1-\gamma}\frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q. \tag{6.9}$$

Regarding the social planner, it is possible to conclude that its optimal investment trigger decreases as γ increases, when compared with the investment threshold of the monopolist,

$$X_W^* = \delta_1(r-\mu)\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma} = (1-\gamma)X^* \tag{6.10}$$

and the optimal capacity level is equal to

$$Q_W^* = \frac{\delta_0 \beta (1-\gamma)}{\delta_1 (\beta\gamma - 1)} = Q^*. \tag{6.11}$$

Therefore, contrasting expressions (6.4) and (6.10), it is possible to realise that the social planner invests earlier than the monopolist, but they invest with the same capacity level, as seen in expressions (6.5) and (6.11).

Contrasting the monopolist's total surplus with the social planner's, it is possible to conclude that the welfare loss is equal to

$$\left(\frac{X(0)}{X_W^*}\right)^{\beta} TS(X_W^*, Q_W^*) - \left(\frac{X(0)}{X^*}\right)^{\beta} TS(X^*, Q^*) = \left(\frac{X(0)}{X_W^*}\right)^{\beta} \left(\frac{\delta_0 \left[1 - (1-\gamma)^{\beta-1}(1-\gamma+\beta\gamma)\right]}{\beta\gamma - 1}\right). \tag{6.12}$$

It is important to note that expression (B17) in the article by Huisman and Kort (2015) contains an error. Specifically, in their final step, the entire numerator of the second term should be multiplied by the variable δ_0.

6.2. Duopoly

In this subchapter, we revisit the concept of competition discussed in Chapter 5. The key difference here is that the price function now follows a constant elasticity demand model.

The investment capacity of the leader's firm is denoted by Q_L, while that of the follower's firm is represented by Q_F. The total market quantity is the sum of these two capacities.

Our objective in this subchapter is to mathematically prove the results presented in Propositions 10, 11, 12, and 15. Propositions 13 and 14, however, will be discussed only in theoretical terms.

6.2.1. Follower's optimal decision rule

Considering that the leader firm invests with the capacity level of Q_L and faces a stochastic demand level of X, the optimal capacity level for the follower, $Q_F^*(X, Q_L)$, is determined by

$$\frac{X(Q_F^* + Q_L)^{-\gamma}}{r - \mu}\left(1 - \frac{\gamma Q_F^*}{Q_F^* + Q_L}\right) - \delta_1 = 0. \tag{6.13}$$

The follower's firm value function can be represented as

$$V_F^*(X, Q_L) = \begin{cases} A_F(Q_L)X^{\beta} & if\ \ X < X_F^*(Q_L) \\ \dfrac{\delta_0 + \delta_1 Q_F^*}{\beta - 1} & if\ X \geq X_F^*(Q_L) \end{cases}, \tag{6.14}$$

where

$$A_F(Q_L) = \frac{\left(X_F^*(Q_L)\right)^{1-\beta}}{\beta}\frac{Q_F^*(Q_F^* + Q_L)^{-\gamma}}{r - \mu}, \tag{6.15}$$

$$X_F^*(Q_L) = \frac{\beta}{\beta - 1}\frac{(r - \mu)(\delta_0 + \delta_1 Q_F^*)}{Q_F^*(Q_F^* + Q_L)^{-\gamma}}, \tag{6.16}$$

and

$$Q_F^*(Q_L) = \frac{\beta\delta_0(1-\gamma) + \delta_1 Q_L + \sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\beta\delta_0\delta_1 Q_L(1 - \gamma\beta)}}{2\delta_1(\gamma\beta - 1)}. \tag{6.17}$$

The leader can use the entry deterrence strategy if it has investment capacity $Q > \hat{Q}_L(X)$, where $\hat{Q}_L(X)$ is implicitly defined by

$$\frac{\beta}{\beta - 1}\frac{(r - \mu)\left(\delta_0 + \delta_1 Q_F^*(X, Q_L)\right)}{Q_F^*(X, Q_L)(Q_F^*(X, Q_L) + Q_L)^{-\gamma}} = X. \tag{6.18}$$

Here, $Q_F^*(X, Q_L)$ is obtained from expression (6.13).

6.2.2. Leader's investment policy under the entry deterrence strategy

In the entry deterrence strategy, the leader firm's value function is given by

$$V_L^{det}(X) = \frac{X\left(Q_L^{det}(X)\right)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L^{det}(X)$$
$$+\left(\frac{X}{X_F^*\left(Q_L^{det}(X)\right)}\right)^{\beta}\left(\frac{X_F^*\left(Q_L^{det}(X)\right)Q_L^{det}(X)\left[\left(Q_L^{det}(X)+Q_F^*\left(Q_L^{det}(X)\right)\right)^{-\gamma}-\left(Q_L^{det}(X)\right)^{-\gamma}\right]}{r-\mu}\right). \tag{6.19}$$

Regarding the leader's value function above, it is important to note that expression (B30) from Huisman and Kort (2015) contains an error in the numerator of the last term. Specifically, their expression fails to incorporate the variable $-\left(Q_L^{det}(X)\right)^{-\gamma}$.

The optimal investment capacity of the leader firm in this strategy, depending on the stochastic demand level X, $Q_L^{det}(X)$, can be implicitly determined by solving

$$\frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu} - \delta_1 + \frac{(1-\beta)X^{\beta}(X_F^*)^{-\beta}Q_L(Q_L+Q_F^*)^{-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L} + \frac{X^{\beta}(X_F^*)^{1-\beta}(Q_L+Q_F^*)^{-\gamma-1}}{r-\mu}\left[Q_L+Q_F^*-\gamma Q_L\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right]$$
$$-\frac{X^{\beta}(X_F^*)^{-\beta}Q_L^{-\gamma}}{r-\mu}\left[(1-\beta)Q_L\frac{\partial X_F^*}{\partial Q_L}+(1-\gamma)X_F^*\right]=0. \tag{6.20}$$

Finally, the optimal investment threshold, X_L^{det}, at which the entry deterrence strategy becomes optimal, is determined using

$$\frac{X_L^{det}}{\beta}\frac{\partial V_L^{det}(X)}{\partial X}\bigg|_{X=X_L^{det}} = V_L^{det}\left(X_L^{det}\right). \tag{6.21}$$

From the results outlined here, it can be inferred that the leader can use the entry deterrence strategy for any value of X.

6.2.3. Leader's investment policy under the entry accommodation strategy

Since the entry accommodation strategy can be applied when $X \geq X_1^{acc}$, with this threshold defined as

$$X_1^{acc} = X_F^*\left(Q_L^{acc}(X_1^{acc})\right), \tag{6.22}$$

the value of the entry accommodation strategy, for the leader firm, is given by

$$V_L^{acc}(X) = \frac{XQ_L^{acc}(X)\left(Q_L^{acc}(X) + Q_F^*\left(X, Q_L^{acc}(X)\right)\right)^{-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L^{acc}(X), \tag{6.23}$$

where $Q_L \equiv Q_L^{acc}(X)$. From expression (6.13), we obtain $Q_F^*(X, Q_L)$.

The leader's optimal investment capacity, $Q_L^{acc}(X) \equiv Q_L$, is defined by

$$\frac{X\left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma-1}}{r-\mu}\left[Q_L + Q_F^*(X, Q_L) - \gamma Q_L\left(1 + \frac{\partial Q_F^*(X, Q_L)}{\partial Q_L}\right)\right] - \delta_1 = 0. \tag{6.24}$$

Lastly, the threshold, X_L^{acc}, from which the leader firm can use the accommodation strategy, is implicitly given by

$$\frac{X_L^{acc}}{\beta}\frac{\partial V_L^{acc}(X)}{\partial X}\bigg|_{X=X_L^{acc}} = V_L^{acc}(X_L^{acc}). \tag{6.25}$$

6.2.4. Leader's strategy across investment regimes

Similarly to the results in subchapter 5.1.4, the leader firm's optimal capacity level is given by

$$Q_L^*(X) = \begin{cases} Q_L^{det}\left(X_L^{det}\right) & if\ X \in \left[0, X_L^{det}\right[\\ Q_L^{det}(X) & if\ X \in \left[X_L^{det}, \hat{X}\right[\\ Q_L^{acc}(X) & if\ X \in \left[\hat{X}, \infty\right[\end{cases}, \tag{6.26}$$

where the point at which the leader firm is indifferent between strategies, $\hat{X}$, is defined as

$$\hat{X} = \min\left\{X \geq X_1^{acc} \mid V_L^{acc}(X) = V_L^{det}(X)\right\}. \tag{6.27}$$

In the first branch of the system of equations in expression (6.26), the stochastic demand level X is below the optimal investment trigger X_L^{det}, and thus no investment will occur. However, the leader commits to investing $Q_L^{det}\left(X_L^{det}\right)$ to discourage the follower from investing, should the follower decide to do so.

In the intermediate region, the leader firm will apply the entry deterrence strategy. However, this is only feasible when $X_L^{det} \leq X < X_1^{acc}$. In this interval, the leader's optimal investment capacity is represented by expression (6.20).

Lastly, in the third branch, the leader's optimal investment capacity is represented by the accommodation strategy. At this level, the leader firm can accommodate the entry of the follower firm by producing up to its capacity $Q_L^{acc}(X)$ in expression (6.24).

Similar to subchapter 5.1.4, the leader's value function is given by

$$V_L^*(X) = \begin{cases} \left(\frac{X}{X_L^{det}}\right)^{\beta} V_L^{det}\left(X_L^{det}\right) & if\ X \in \left[0, X_L^{det}\right[\\ V_L^{det}(X) & if\ X \in \left[\, X_L^{det}, \hat{X}\right[\\ V_L^{acc}(X) & if\ X \in \left[\hat{X}, \infty\right[\end{cases}. \tag{6.28}$$

The first branch, similar to expression (5.23), represents the leader's value function that is held but not yet realised, as no investment has been materialised since $X < X_L^{det}$.

In the middle region, where the deterrence strategy is the optimal strategy for the leader firm, the leader's value function is given by expression (6.19).

Finally, in the third branch, the leader's value function is derived from the implementation of the accommodation strategy. This function corresponds to the one expressed in (6.23).

From these results, we can conclude that the leader will adopt an entry deterrence strategy when the stochastic demand level X is low. As X increases, the leader firm will switch to an accommodation strategy.

6.2.5. Investment threshold of the leader

Based on the results presented in subchapter 6.2.4, the investment threshold for the leader firm, similar to the findings in subchapter 5.1.5, can be expressed as follows

$$X_L^* = \begin{cases} X_L^{det} & if\ X \in \left[0, X_L^{det}\right[\\ X & if\ X \in \left[X_L^{det}, \infty\,\right[\end{cases}. \tag{6.29}$$

6.2.6. Impact of uncertainty on the leader's strategic entry boundaries

The threshold at which the leader firm can apply the accommodation strategy, X_1^{acc}, can be determined by solving the system of equations (6.30), (6.31), and (6.32) for X_1^{acc}, that is

$$X_1^{acc} = \frac{\beta}{\beta - 1} \frac{(r-\mu)(\delta_0 + \delta_1 Q_F)}{Q_F (Q_F + Q_L)^{-\gamma}}, \tag{6.30}$$

where it is assumed that $X_F^* \equiv X_1^{acc}$ and $Q_F^* \equiv Q_F$ in expression (6.16).

The following expression defines Q_F^*.

$$\frac{X_1^{acc}(Q_F + Q_L)^{-\gamma}}{r - \mu}\left(1 - \frac{\gamma Q_F}{Q_F + Q_L}\right) - \delta_1 = 0. \tag{6.31}$$

Here, $X \equiv X_1^{acc}$ and $Q_F^* \equiv Q_F$ are replaced in expression (6.13).

Considering the next expression as the definition of Q_L^{acc} (see expression (B6.1)), we have

$$\frac{X_1^{acc}(Q_L + Q_F)^{-\gamma}}{r - \mu}\left[1 - \frac{\gamma Q_L}{Q_L + Q_F}\left(1 + \frac{\gamma Q_F - Q_L}{2Q_L + (1-\gamma)Q_F}\right)\right] - \delta_1 = 0. \tag{6.32}$$

From expressions (6.31) and (6.32), it can be concluded that $\frac{\partial Q_F}{\partial \sigma} = 0$ and $\frac{\partial Q_L}{\partial \sigma} = 0$, since they are not dependent on β. Consequently, from expression (6.30) $\frac{\partial X_1^{acc}}{\partial \sigma} > 0$. Since $\frac{\partial \beta}{\partial \sigma} < 0$, from expression (6.30) implies that $\frac{\beta}{\beta - 1} > 0$ because as the volatility increases, $\beta - 1 < \beta$.

Therefore, the region $X \in]X_1^{acc}, +\infty[$ in which the leader can choose between deterrence and accommodation strategies decreases with increasing uncertainty.

CHAPTER 7

Conclusions and recommendations

We began by meticulously deconstructing the mathematical equations, enabling us to replicate the figures and tables and correct any inaccuracies found in the original research article by Huisman and Kort (2015). This thorough analysis allowed us to present accurate versions of the expressions, figures, and tables, ensuring the integrity and reliability of the results upon which this thesis is based.

The expressions mistakenly presented in Huisman and Kort (2015) article are (12), (A22), (A70), (B17), and (B30). The correct version is found in this thesis in expressions (4.13), (A4.6), (A6.20), (6.12), and (6.19), respectively. Additionally, Figure 1(a) in the Huisman and Kort (2015) article is incorrect. The correct version is presented in Figure 4.1.

This thesis confirms the findings of Huisman and Kort (2015) that entry deterrence cannot be sustained indefinitely. As markets grow, the second (follower) firm will eventually enter. Additionally, the first (leader) investor tends to overinvest to delay the second investor's entry and reduce their capacity. However, under low uncertainty, the pre-emption effect may cause the first investor to invest too soon, resulting in the second investor becoming the larger firm. In conditions of high uncertainty and symmetric firms, the first investor invests relatively late and in a larger capacity, ultimately becoming the larger firm when the second investor enters. This latter scenario is driven by the value of waiting.

For the case of monopoly, this thesis also confirms that higher uncertainty delays investments, but that leads to larger projects. Comparing welfare outcomes, symmetric cost firms incur less welfare loss than a monopoly. However, asymmetric cost firms reduce welfare loss even more compared to both symmetric cost firms and a monopoly.

In the scenario of a constant elasticity demand curve, the finding that increased uncertainty enhances the likelihood of entry deterrence remains consistent. Additionally, the region where the leader can choose between entry deterrence and accommodation strategies decreases with increasing uncertainty. This contrasts with the linear demand curve scenario, where that region remains unaffected by uncertainty.

The finding of this thesis that deterrence cannot be sustained indefinitely corroborates the analyses of Dixit (1980) and Spence (1977). Additionally, as Spence (1977) suggested, we also found that overinvestment exists in a Stackelberg duopoly.

In contrast to Pindyck (1986), this thesis demonstrates that as uncertainty increases, it is optimal for firms to expand their capacities, aligning with Dixit (1993).

Furthermore, the conclusions of Maskin (1999) that the range of parameters for which deterrence occurs decreases with increasing uncertainty are opposite to those of Huisman and Kort (2015), due to Maskin's static model versus Huisman and Kort's dynamic model.

Lastly, the findings of this thesis align with Shackleton et al. (2004), indicating that hysteresis in firms' actions is positively related to costs and uncertainty and negatively affected by competitors' investment capacities.

This thesis makes significant contributions to the field of strategic capacity investment by providing detailed mathematical validation and rectifying inaccuracies in the foundational research. It offers readers a profound understanding of how the conclusions were reached. By presenting corrected formulas and figures, it enhances the reliability of the Huisman and Kort (2015) article, transforming it into a more robust tool for strategic decision-making. This is valuable for both firms and policymakers who design regulations that maximise social welfare by considering the timing and capacity of investments.

While this thesis provides a comprehensive mathematical validation, it is limited by the scope of the original research article. Future studies could expand on this work by exploring additional scenarios and incorporating empirical data to further validate the theoretical findings.

As stated by Huisman and Kort (2015), some limitations of their work involve firms investing only once, meaning that they cannot expand their installed capacity in the future and must produce up to their installed capacity.

Future research could focus on extending the analysis to different market structures, utilising different price functions, and exploring the impact of varying levels of uncertainty on investment decisions. Additional numerical experiments, using either linear or isoelastic demand curves, could be conducted to test theoretical predictions and provide practical insights for firms operating under uncertainty. Furthermore, deriving expressions for the variables $Q_L^{det}\left(X_L^{det}\right)$ and $Q_L^{acc}(X_L^{acc})$ in the isoelastic demand curve scenario would be beneficial.

Lastly, as evidenced by Huisman and Kort (2015), it would be relevant to address the uncertainty concerning the realised capacity produced compared to initial commitments and model the uncertainty (σ) parameter of demand by a Poisson arrival.

Bibliographical references

Alexander, D. R., Mo, M., & Stent, A. F. (2012). Arithmetic Brownian motion and real options. *European Journal of Operational Research, 219*(1), 114-122. https://doi.org/10.1016/j.ejor.2011.12.023

Baldwin, C. Y. (1982). Optimal Sequential Investment When Capital is Not Readily Reversible. *The Journal of Finance, 37*(3), 763-782. https://www.jstor.org/stable/2327707

Björk, T. (2009). *Arbitrage Theory in Continuous Time* (3rd ed.). https://www.academia.edu/38191641/Tomas_Bjork_Arbitrage_Theory_in_Continuous_Time_Oxford_Finance_2009_

Black, F., & Scholes, M. (1973). The Pricing of Options and Corporate Liabilities. *The Journal of Political Economy, 81*(3), 637-654. https://www.jstor.org/stable/1831029

Bollen, N. P. (1999). Real Options and Product Life Cycles. *Management Science, 45*(5), 670-684. https://www.researchgate.net/publication/2453451_Real_Options_And_Product_Life_Cycles

Bowman, E. H., & Moskowitz, G. T. (2001). Real Options Analysis and Strategic Decision Making. *Organization Science, 12*(6), 772-777. https://doi.org/10.1287/orsc.12.6.772.10080

Brealey, R. A., Myers, S. C., & Allen, F. (2013). *Princípios de Finanças Corporativas* (10 ed.). https://www.academia.edu/29435218/Princ%C3%ADpios_de_Finan%C3%A7as_Corporativas_Brealey_Myers_e_Allen?auto=download

Copeland, T. E., & Antikarov, V. (2001). *Real Options, a Practitioner's Guide.* TEXERE LLC.

Dixit, A. K. (1980). The Role of Investment in Entry-Deterrence. *The Economic Journal, 90*(357), 95-106. http://www.jstor.org/stable/2231658

Dixit, A. K. (1993). Choosing among alternative discrete investment projects under uncertainty. *Economics Letters, 41*(3), 265-268. https://doi.org/10.1016/0165-1765(93)90151-2

Dixit, A. K., & Pindyck, R. S. (1994). *Investment Under Uncertainty.* Princeton University Press. https://books.google.pt/books?hl=pt-PT&lr=&id=VahsELa_qC8C&oi=fnd&pg=PR7&dq=+DIXIT,A.AND+PINDYCK,R.Investment+under+Uncertainty.+Princeton,+NJ:+Princeton+University+Press,+1994.&ots=FEAYvI-bkH&sig=WGoFzkSqpeugIvbquZxmSjvDgNw&redir_esc=y#v=onepage&q&f=fals

Dixit, A. K., & Pindyck, R. S. (1995). The Options Approach to Capital Investment. *Harvard Business Review, 73*, 105-115. https://www.academia.edu/36237393/The_Options_Approach_to_Capital_Investment

Goyal, M., & Netessine, S. (2007). Strategic Technology Choice and Capacity Investment Under Demand Uncertainty. *Management Science, 53*(2), 192-207. https://www.jstor.org/stable/20110690

Huisman, K. (2001). *Technology Investment: A Game Theoretic Real Options Approach.* Kluwer Academic Publishers. https://books.google.pt/books?hl=pt-PT&lr=&id=OyflBwAAQBAJ&oi=fnd&pg=PA1&dq=10.1007/978-1-4757-3423-2&ots=jek5OLxWGr&sig=0FFuHkB8o1dDeKJoWEyibjuIOVM&redir_esc=y#v=onepage&q=10.1007%2F978-1-4757-3423-2&f=false

Huisman, K. J., & Kort, P. M. (2015). Strategic capacity investment under uncertainty. *RAND Journal of Economics, 46*(2), 376-408. https://doi.org/10.1111/1756-2171.12089

Keswani, A., & Shackleton, M. B. (2006). How Real Option Disinvestment Flexibility Augments Project Npv. *European Journal of Operations Research, 168*(1), 240-252. https://ssrn.com/abstract=516124

Luehrman, T. A. (1998). Investment Opportunities as Real Options: Getting Started on the Numbers. *Harvard Business Review, 76*, 89-99. https://pages.stern.nyu.edu/~adamodar/pdfiles/articles/InvestmentsasOptions.pdf

Maskin, E. S. (1999). Uncertainty and entry deterrence. *Economic Theory, 14*, 429-437. https://doi.org/10.1007/s001990050302

McConnell, J. J., & Muscarella, C. J. (1985). Corporate Capital Expenditure Decisions and the Market Value of the Firm. *Journal of Financial Economics, 14*, 399-422. https://business.purdue.edu/faculty/mcconnell/publications/CORPORATE-CAPITAL-EXPENDITURE-DECISIONS.pdf

McDonald, R. L. (2000). Real Options and Rules of Thumb in Capital Budgeting. *Project Flexib*, 1-29. https://www.kellogg.northwestern.edu/faculty/mcdonald/htm/realopt.pdf

McDonald, R. L., & Siegel, D. R. (1986). The Value of Waiting to Invest. *The Quarterly Journal of Economics, 101*(4), 707-728. https://www.jstor.org/stable/1884175

Merton, R. C. (1973). Theory of Rational Option Pricing. *The Bell Journal of Economics and Management Science, 4*(1), 141-183. https://www.jstor.org/stable/3003143

Pindyck, R. S. (1986). *Irreversible Investment, Capacity Choice, and the Value of the Firm.* Working Paper, National Bureau of Economic Research. http://www.nber.org/papers/w1980

Ross, S. A. (1995). Uses, Abuses, and Alternatives to the Net-Present-Value Rule. *Financial Management, 24*(3), 96-102. https://doi.org/10.2307/3665561

Shackleton, M. B., Tsekrekos, A. E., & Wojakowski, R. (2004). Strategic entry and market leadership in a two-player real options game. *Journal of Banking & Finance, 28*(1), 179-201. https://doi.org/10.1016/S0378-4266(02)00403-X

Shreve, S. E. (2004). *Stochastic Calculus for Finance II: Continuous-Time Models.* Springer Science+Business Media. https://github.com/yc-liu/readings/blob/master/Steven%20E.%20Shreve%20Stochastic%20Calculus%20for%20Finance%20II-%20Continuous-Time%20Models%20(Springer%20Finance)%20(v.%202).pdf

Spence, A. M. (1977). Entry, Capacity, Investment and Oligopolistic Pricing. *The Bell Journal of Economics, 8*(2), 534-544. https://doi.org/10.2307/3003302

Triantis, A. (2005). Realizing the Potential of Real Options: Does Theory Meet Practice? *Journal of Applied Corporate Finance, 17*(2), 8-16. https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1745-6622.2005.00028.x

Trigeorgis, L. (1993). Real Options and Interactions with Financial Flexibility. *Financial Management, 22*(3), 202-224. https://doi.org/10.2307/3665939

Trigeorgis, L. (1996). *Real options: managerial flexibility and strategy in resource allocation.* The MIT Press. https://books.google.pt/books?id=Z8o20TmBiLcC&printsec=frontcover&hl=pt-PT&source=gbs_ge_summary_r&cad=0#v=onepage&q&f=false

ANNEX A

Linear demand curve

This annex presents the mathematical proofs for Chapters 4 and 5. The mathematical equations developed here focus on a linear demand curve for the price function.

A.1. Mathematical details of the monopolist's optimal investment decision

Given that the current level of X_t is X (i.e., $X_0 = X$) and the firm invests in Q units of capital (which implies that Q is fixed, meaning that $Q_t \equiv Q$). We can rewrite equation (4.3) as

$$\pi(t) = P(t) \cdot Q(t) \qquad \text{(A1.1)}$$

$$\Leftrightarrow \pi(t) = X(t)\big(1 - \eta Q(t)\big) \cdot Q(t)$$

$$\Leftrightarrow \pi(t) = X(t) \cdot Q \cdot (1 - \eta Q).$$

The expected value of the profit is then

$$E[\pi_t] = E[X_t\, Q(1 - \eta Q)] \qquad \text{(A1.2)}$$

$$\Leftrightarrow E[\pi_t] = Q(1 - \eta Q) \cdot E[X_t]$$

$$\Leftrightarrow E[\pi_t] = Q(1 - \eta Q) \cdot X \cdot e^{\mu \cdot t}.$$

Therefore, the monopolist value function, $V(X, Q)$, can be expressed as

$$V(X,Q) = E\left[\int_{t=0}^{\infty} \pi(t) exp(-rt)\, dt - \delta Q\right] \qquad \text{(A1.3)}$$

$$\Leftrightarrow V(X,Q) = \int_{t=0}^{\infty} Q(1 - \eta Q) \cdot X \cdot \mathrm{e}^{\mu \cdot t} \cdot \mathrm{e}^{-r \cdot t}\ dt - \delta Q$$

$$\Leftrightarrow V(X,Q) = XQ(1 - \eta Q) \int_{t=0}^{\infty} \mathrm{e}^{(\mu - r)t}\, dt - \delta Q$$

$$\Leftrightarrow V(X,Q) = XQ(1 - \eta Q) \cdot \lim_{T \to \infty} \left[\frac{1}{\mu - r} \mathrm{e}^{(\mu - r) \cdot t}\right]_0^T - \delta Q$$

$$\Leftrightarrow V(X,Q) = XQ(1 - \eta Q) \cdot \lim_{T \to \infty} \left[\frac{1}{\mu - r} \mathrm{e}^{(\mu - r) \cdot T} - \frac{1}{\mu - r}\right] - \delta Q$$

$$\Leftrightarrow V(X,Q) = XQ(1 - \eta Q) \cdot \frac{1}{r - \mu} - \delta Q$$

$$\Leftrightarrow V(X,Q) = \frac{XQ(1 - \eta Q)}{r - \mu} - \delta Q.$$

Maximising $V(X, Q)$ to Q gives the optimal capacity size Q^*, concerning X, as follows

$$\frac{\partial V(X,Q)}{\partial Q} = 0 \tag{A1.4}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ(1-\eta Q)}{r-\mu} - \delta Q\right)}{\partial Q} = 0$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ - X\eta Q^2}{r-\mu} - \delta Q\right)}{\partial Q} = 0$$

$$\Leftrightarrow \frac{X}{r-\mu} - \frac{2X\eta}{r-\mu}Q - \delta = 0$$

$$\Leftrightarrow \frac{2X\eta}{r-\mu}Q = \frac{X}{r-\mu} - \delta$$

$$\Leftrightarrow 2X\eta Q = (r-\mu)\left[\frac{X}{r-\mu} - \delta\right]$$

$$\Leftrightarrow 2X\eta Q = X - (r-\mu)\delta$$

$$\Leftrightarrow Q = \frac{X - (r-\mu)\delta}{2X\eta}$$

$$\Leftrightarrow Q = \frac{X}{2X\eta} - \frac{(r-\mu)\delta}{2X\eta}$$

$$\Leftrightarrow Q = \frac{1}{2\eta} - \frac{(r-\mu)\delta}{2X\eta}$$

$$\Leftrightarrow Q \equiv Q^*(X) = \frac{1}{2\eta}\left(1 - \frac{\delta(r-\mu)}{X}\right).$$

Next, we compute an expression for β, noting that the investment opportunity value, denoted by $F(X)$ must follow the ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2 X^2 F''(X) + \mu X F'(X) - rF(X) = 0. \tag{A1.5}$$

Trying the trial solution $F(X) = X^\beta$, it follows that

$$\frac{1}{2}\sigma^2 X^2 \beta(\beta-1)X^{\beta-2} + \mu X \beta X^{\beta-1} - rX^\beta = 0 \tag{A1.6}$$

$$\Leftrightarrow \frac{1}{2}\sigma^2 \beta(\beta-1)X^\beta + \mu\beta X^\beta - rX^\beta = 0$$

$$\Leftrightarrow \left[\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r\right]X^\beta = 0$$

$$\Leftrightarrow \frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0 \;\vee\; X^\beta = 0$$

$$\Leftrightarrow \frac{1}{2}\sigma^2\beta^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\beta - r = 0.$$

The last expression corresponds to two roots, where $\beta_1 > 1$ and $\beta_2 < 0$. Therefore, a general solution to the ODE could be $F = AX^{\beta_1} + BX^{\beta_2}$.

As $X \to 0^+$, it is necessary that $F(X) \to 0$ to satisfy the condition $\lim_{X\to 0^+} F(X) = 0$. But since $\beta_2 < 0, X^{\beta_2} \to \infty$, implying that BX^{β_2} is divergent at zero. Therefore, B must be set equal to zero, yielding that $F(X) = AX^{\beta_1}$, or simply $F(X) = AX^{\beta}$.

Consequently, from equation (A1.6), we can derive the expression for β as follows:

$$\frac{1}{2}\sigma^2\beta^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\beta - r = 0 \tag{A1.7}$$

$$\Rightarrow \beta = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 - 4 \cdot \frac{1}{2}\sigma^2 \cdot (-r)}}{2 \cdot \frac{1}{2}\sigma^2}$$

$$\Leftrightarrow \beta = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\mu^2 - \mu\sigma^2 + \frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2}$$

$$\Leftrightarrow \beta = \frac{-\mu + \frac{1}{2}\sigma^2}{\sigma^2} + \frac{\sqrt{\sigma^4\left(\frac{\mu^2}{\sigma^4} - \frac{\mu}{\sigma^2} + \frac{1}{4} + \frac{2r}{\sigma^2}\right)}}{\sigma^2}$$

$$\Leftrightarrow \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \frac{\sigma^2 \cdot \sqrt{\frac{\mu^2}{\sigma^4} - \frac{\mu}{\sigma^2} + \frac{1}{4} + \frac{2r}{\sigma^2}}}{\sigma^2}$$

$$\Leftrightarrow \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} - \frac{\mu}{\sigma^2} + \frac{1}{4} + \frac{2r}{\sigma^2}}$$

$$\Leftrightarrow \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

To determine the threshold level X^*, the VMC

$$F(X^*) = V(X^*, Q), \tag{A1.8}$$

and SPC

$$\left.\frac{\partial F(X)}{\partial X}\right|_{X=X^*} = \left.\frac{\partial V(X,Q)}{\partial X}\right|_{X=X^*} \tag{A1.9}$$

must be employed. Substituting (A1.8) into (A1.9) and solving for X^* gives

$$\left.\frac{\partial F(X)}{\partial X}\right|_{X=X^*} = \left.\frac{\partial V(X,Q)}{\partial X}\right|_{X=X^*} \tag{A1.10}$$

$$\Leftrightarrow \frac{\partial\left(A(X^*)^{\beta}\right)}{\partial X^*} = \frac{\partial\left(\frac{X^*Q(1-\eta Q)}{r-\mu} - \delta Q\right)}{\partial X^*}$$

$$\Leftrightarrow \beta A \frac{(X^*)^\beta}{X^*} = \frac{Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta\left(\frac{X^* Q(1-\eta Q)}{r-\mu} - \delta Q\right)}{X^*} = \frac{Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta}{X^*}\frac{X^* Q(1-\eta Q)}{r-\mu} - \frac{\beta\delta Q}{X^*} = \frac{Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta Q(1-\eta Q)}{r-\mu} - \frac{\beta\delta Q}{X^*} = \frac{Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta\delta Q}{X^*} = \frac{\beta Q(1-\eta Q)}{r-\mu} - \frac{Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta\delta Q}{X^*} = \frac{\beta Q(1-\eta Q) - Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta\delta Q}{X^*} = \frac{(\beta-1)Q(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta\delta}{X^*} = \frac{(\beta-1)(1-\eta Q)}{r-\mu}$$

$$\Leftrightarrow X^* \equiv X^*(Q) = \frac{\beta\delta(r-\mu)}{(\beta-1)(1-\eta Q)}.$$

Since equation (A1.4) can be rewritten as

$$Q\eta = \frac{1}{2}\left(1 - \frac{\delta(r-\mu)}{X}\right) \tag{A1.11}$$

$$\Leftrightarrow Q\eta = \frac{1}{2}\frac{X}{X} - \frac{1}{2}\frac{\delta(r-\mu)}{X},$$

replacing this equation into expression (A1.10) yields

$$X = \frac{\beta\delta(r-\mu)}{(\beta-1)\left(1 - \frac{1}{2}\frac{X}{X} + \frac{1}{2}\frac{\delta(r-\mu)}{X}\right)} \tag{A1.12}$$

$$\Leftrightarrow X = \frac{\beta\delta(r-\mu)}{(\beta-1)\left(\frac{2X - X + \delta(r-\mu)}{2X}\right)}$$

$$\Leftrightarrow X = \frac{\beta\delta(r-\mu)2X}{(\beta-1)\big(X + \delta(r-\mu)\big)}$$

$$\Leftrightarrow (\beta-1)\big(X + \delta(r-\mu)\big) = 2\beta\delta(r-\mu)$$

$$\Leftrightarrow (\beta-1)X + \beta\delta(r-\mu) - \delta(r-\mu) = 2\beta\delta(r-\mu)$$

$$\Leftrightarrow (\beta-1)X = \beta\delta(r-\mu) + \delta(r-\mu)$$

$$\Leftrightarrow (\beta-1)X = (\beta+1)\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X^* = \frac{\beta\ + 1}{\beta\ - 1}\delta(r - \mu).$$

Incorporating this expression in equation (A1.4) yields

$$Q = \frac{1}{2\eta}\left(1 - \frac{\delta(r-\mu)}{\frac{\beta+1}{\beta-1}\delta(r-\mu)}\right) \tag{A1.13}$$

$$\Leftrightarrow Q = \frac{1}{2\eta}\left(1 - \frac{\beta-1}{\beta+1}\right)$$

$$\Leftrightarrow Q = \frac{1}{2\eta}\left(\frac{\beta+1-\beta+1}{\beta+1}\right)$$

$$\Leftrightarrow Q = \frac{2}{2\eta}\frac{1}{\beta+1}$$

$$\Leftrightarrow Q \equiv Q^* \equiv Q^*(X^*) = \frac{1}{(\beta+1)\eta}.$$

Considering that the VMC states that

$$F(X^*) = V(X^*, Q) \tag{A1.14}$$

$$\Leftrightarrow A(X^*)^\beta = \frac{X^*Q(1-\eta Q)}{r-\mu} - \delta Q,$$

replacing X^* and $Q^*(X^*)$ (of expressions (A1.12) and (A1.13), respectively), in the right-hand side of the equation (A1.14), yields

$$\frac{X^*Q(1-\eta Q)}{r-\mu} - \delta Q \tag{A1.15}$$

$$= \frac{\frac{\beta+1}{\beta-1}\delta(r-\mu)Q(1-\eta Q)}{r-\mu} - \delta Q$$

$$= \frac{\beta+1}{\beta-1}\delta\frac{1}{(\beta+1)\eta}\left(1 - \eta\frac{1}{(\beta+1)\eta}\right) - \delta\frac{1}{(\beta+1)\eta}$$

$$= \frac{1}{\beta-1}\frac{\delta}{\eta}\left(\frac{\beta+1-1}{\beta+1}\right) - \delta\frac{1}{(\beta+1)\eta}$$

$$= \frac{\delta\beta}{(\beta-1)(\beta+1)\eta} - \delta\frac{\beta-1}{(\beta-1)(\beta+1)\eta}$$

$$= \frac{\delta\beta - \delta\beta + \delta}{(\beta^2-1)\eta}$$

$$= \frac{\delta}{(\beta^2-1)\eta}.$$

Therefore,

$$A(X^*)^\beta = \frac{\delta}{(\beta^2-1)\eta} \tag{A1.16}$$

$$\Leftrightarrow A = \frac{\delta (X^*)^{-\beta}}{(\beta^2 - 1)\eta}$$

$$\Leftrightarrow A = \frac{\delta \left(\frac{\beta + 1}{\beta - 1} \delta (r - \mu) \right)^{-\beta}}{(\beta^2 - 1)\eta}.$$

Based on the previous results, we can construct the value function for the monopolist as

$$V(X) = \begin{cases} AX^\beta & if \;\; X < X^* \\ \frac{\left(X - \delta(r - \mu)\right)^2}{4X\eta(r - \mu)} & if \; X \geq X^* \end{cases}. \qquad \text{(A1.17)}$$

The rationale behind this is that for $X < X^*$, the firm remains in an idle state. Therefore, the value of the monopolist firm is given by the option value AX^β (i.e., the first branch of equation (A1.17)). For $X \geq X^*$, is known that the value of the active firm is given by equation (A1.3). Rewriting equation (A1.4), we get

$$Q = \frac{1}{2\eta} \left(1 - \frac{\delta(r - \mu)}{X} \right) \qquad \text{(A1.18)}$$

$$\Leftrightarrow Q = \frac{1}{2\eta X} \left(X - \delta(r - \mu) \right).$$

We must notice that

$$XQ = \frac{1}{2\eta} \left(X - \delta(r - \mu) \right) \qquad \text{(A1.19)}$$

and

$$1 - \eta Q = 1 - \frac{1}{2X} \left(X - \delta(r - \mu) \right) \qquad \text{(A1.20)}$$

$$\Leftrightarrow 1 - \eta Q = \frac{2X - X + \delta(r - \mu)}{2X}$$

$$\Leftrightarrow 1 - \eta Q = \frac{X + \delta(r - \mu)}{2X}.$$

Therefore,

$$XQ(1 - \eta Q) = \frac{1}{2\eta} \left(X - \delta(r - \mu) \right) \frac{\left(X + \delta(r - \mu) \right)}{2X} \qquad \text{(A1.21)}$$

$$\Leftrightarrow XQ(1 - \eta Q) = \frac{1}{4\eta X} \left(X - \delta(r - \mu) \right) \left(X + \delta(r - \mu) \right)$$

and the value of the monopolist firm can be represented as

$$\frac{XQ(1 - \eta Q)}{r - \mu} - \delta Q \qquad \text{(A1.22)}$$

$$= \frac{1}{4\eta X(r-\mu)}\big(X-\delta(r-\mu)\big)\big(X+\delta(r-\mu)\big) - \frac{\delta\big(X-\delta(r-\mu)\big)}{2\eta X}$$

$$= \frac{1}{4\eta X(r-\mu)}(X^2-\delta^2(r-\mu)^2-2\delta X(r-\mu)+2\delta^2(r-\mu)^2).$$

Also, it is important to note that

$$X^2-\delta^2(r-\mu)^2-2\delta X(r-\mu)+2\delta^2(r-\mu)^2 \tag{A1.23}$$

$$= X^2-2\delta X(r-\mu)+\delta^2(r-\mu)^2$$

$$= \big(X-\delta(r-\mu)\big)^2.$$

Based on the preceding equations, the value function in expression (A1.3) corresponds to the second branch of equation (A1.17) because

$$\frac{XQ(1-\eta Q)}{r-\mu} - \delta Q = \frac{\big(X-\delta(r-\mu)\big)^2}{4\eta X(r-\mu)}. \tag{A1.24}$$

Differentiating equations (A1.12) and (A1.13) concerning β gives

$$\frac{\partial X^*}{\partial \beta} \tag{A1.25}$$

$$= \delta(r-\mu)\cdot\frac{1\cdot(\beta-1)-(\beta+1)\cdot 1}{(\beta-1)^2}$$

$$= \delta(r-\mu)\cdot\frac{\beta-1-\beta-1}{(\beta-1)^2}$$

$$= -\frac{2\delta(r-\mu)}{(\beta-1)^2} < 0, since\ \beta > 1 \Rightarrow (\beta-1)^2 > 0,$$

$$r > \mu \Leftrightarrow r-\mu > 0,$$

$$and\ \delta > 0,$$

$$then\ 2\delta(r-\mu) > 0,$$

$$\frac{\partial Q^*}{\partial \beta} \tag{A1.26}$$

$$= \frac{1}{\eta}\cdot\frac{0\cdot(\beta+1)-1\cdot 1}{(\beta+1)^2}$$

$$= -\frac{1}{\eta\cdot(\beta+1)^2} < 0, since\ \beta > 1 \Rightarrow (\beta+1)^2 > 0$$

$$and\ \eta > 0, yielding,$$

$$\eta\cdot(\beta+1)^2 > 0.$$

A.2. Mathematical details of the optimal welfare decision

Given that the instantaneous consumer surplus is represented by

$$\int_{P(Q)}^{X} D(P)\, dP, \tag{A2.1}$$

we can rewrite the price function from equation (4.2) as follows

$$P(Q) = X(1 - \eta Q) \tag{A2.2}$$

$$\Leftrightarrow \frac{P}{X} = 1 - \eta Q$$

$$\Leftrightarrow \eta Q = 1 - \frac{P}{X}$$

$$\Leftrightarrow Q = \frac{1}{\eta}\left(1 - \frac{P}{X}\right)$$

$$\Leftrightarrow D(P) = \frac{1}{\eta}\left(1 - \frac{P}{X}\right) \; \left(rewriting\ Q \equiv D(P)\right).$$

Combining expressions (A2.1) and (A2.2) we achieve that the instantaneous CS is equal to

$$\int_{P(Q)}^{X} D(P)\, dP \tag{A2.3}$$

$$= \int_{X(1-\eta Q)}^{X} \frac{1}{\eta}\left(1 - \frac{P}{X}\right) dP$$

$$= \frac{1}{\eta} \int_{X(1-\eta Q)}^{X} 1\, dP - \frac{1}{\eta X} \int_{X(1-\eta Q)}^{X} P\, dP$$

$$= \frac{1}{\eta} [P]_{P=X(1-\eta Q)}^{P=X} - \frac{1}{\eta X} \left[\frac{P^2}{2}\right]_{P=X(1-\eta Q)}^{P=X}$$

$$= \frac{1}{\eta} [X - X(1 - \eta Q)] - \frac{1}{2\eta X} \left[X^2 - \left(X(1 - \eta Q)\right)^2\right]$$

$$= \frac{1}{\eta} (X - X + X\eta Q) - \frac{1}{2\eta X} [X^2 - (X^2(1 - \eta Q)^2)]$$

$$= XQ - \frac{X}{2\eta} + \frac{X}{2\eta}(1 - 2\eta Q + \eta^2 Q^2)$$

$$= XQ - \frac{X}{2\eta} + \frac{X}{2\eta} - XQ + \frac{X}{2}\eta Q^2$$

$$= \frac{1}{2} X Q^2 \eta.$$

From the previous equation, it is possible to compute the total expected CS as

$$CS(X,Q) = E\left[\int_{t=0}^{\infty} \frac{1}{2} X(t) Q^2 \eta e^{-r\cdot t}\, dt \,|X(0) = X\right] \quad \text{(A2.4)}$$

$$\Leftrightarrow CS(X,Q) = \frac{1}{2} Q^2 \eta E\left[\int_{t=0}^{\infty} X_t e^{-r\cdot t}\, dt\right]$$

$$\Leftrightarrow CS(X,Q) = \frac{1}{2} Q^2 \eta \int_{t=0}^{\infty} E(X_t) e^{-r\cdot t}\, dt$$

$$\Leftrightarrow CS(X,Q) = \frac{1}{2} Q^2 \eta \int_{t=0}^{\infty} X e^{\mu\cdot t} e^{-r\cdot t}\, dt \quad (with\ X \equiv X_0)$$

$$\Leftrightarrow CS(X,Q) = \frac{1}{2} Q^2 \eta X \int_{t=0}^{\infty} e^{(\mu - r)t}\, dt$$

$$\Leftrightarrow CS(X,Q) = \frac{XQ^2\eta}{2(r-\mu)}.$$

Since the PS is equal to the value of the monopolist firm in expression (A1.3), the TS is computed as

$$TS(X,Q) = CS(X,Q) + PS(X,Q) \quad \text{(A2.5)}$$

$$\Leftrightarrow TS(X,Q) = \frac{XQ^2\eta}{2(r-\mu)} + \frac{XQ(1-\eta Q)}{r-\mu} - \delta Q$$

$$\Leftrightarrow TS(X,Q) = \frac{XQ\eta Q}{2(r-\mu)} + \frac{XQ(2-2\eta Q)}{2(r-\mu)} - \delta Q$$

$$\Leftrightarrow TS(X,Q) = \frac{XQ(\eta Q + 2 - 2\eta Q)}{2(r-\mu)} - \delta Q$$

$$\Leftrightarrow TS(X,Q) = \frac{XQ(2-\eta Q)}{2(r-\mu)} - \delta Q.$$

By incorporating the monopoly decisions into the previous equation, by substituting X with equation (A1.12) and Q with equation (A1.13), we get

$$TS(X^*,Q^*) = \frac{X^*Q^*(2-\eta Q^*)}{2(r-\mu)} - \delta Q^* \quad \text{(A2.6)}$$

$$\Leftrightarrow TS(X^*,Q^*) = \frac{\frac{\beta+1}{\beta-1}\delta(r-\mu)\frac{1}{(\beta+1)\eta}\left(2-\eta\frac{1}{(\beta+1)\eta}\right)}{2(r-\mu)} - \delta\frac{1}{(\beta+1)\eta}$$

$$\Leftrightarrow TS(X^*,Q^*) = \frac{\frac{1}{\beta-1}\delta\frac{1}{\eta}\left(2-\frac{1}{\beta+1}\right)}{2} - \frac{\delta}{(\beta+1)\eta}$$

$$\Leftrightarrow TS(X^*, Q^*) = \frac{1}{2}\frac{1}{\beta - 1}\delta\frac{1}{\eta}\left(2 - \frac{1}{\beta + 1}\right) - \frac{\delta}{(\beta + 1)\eta}$$

$$\Leftrightarrow TS(X^*, Q^*) = \frac{1}{\beta - 1}\delta\frac{1}{\eta} - \frac{1}{2}\frac{1}{\beta - 1}\delta\frac{1}{\eta}\frac{1}{\beta + 1} - \frac{\delta}{(\beta + 1)\eta}$$

$$\Leftrightarrow TS(X^*, Q^*) = \frac{\delta}{(\beta - 1)\eta} - \frac{\delta}{2(\beta - 1)(\beta + 1)\eta} - \frac{\delta}{(\beta + 1)\eta}$$

$$\Leftrightarrow TS(X^*, Q^*) = \frac{2\delta(\beta + 1)}{2(\beta - 1)(\beta + 1)\eta} - \frac{\delta}{2(\beta + 1)(\beta - 1)\eta} - \frac{2\delta(\beta - 1)}{2(\beta + 1)(\beta - 1)\eta}$$

$$\Leftrightarrow TS(X^*, Q^*) = \frac{2\delta(\beta + 1) - 2\delta(\beta - 1) - \delta}{2(\beta + 1)(\beta - 1)\eta}$$

$$\Leftrightarrow TS(X^*, Q^*) = \frac{3\delta}{2(\beta + 1)(\beta - 1)\eta}.$$

Based on expression (A2.5), we can conclude that the value function for the social planner at the time of investment is represented by

$$V_W(X_W^* \equiv X) = \frac{XQ(2 - \eta Q)}{2(r - \mu)} - \delta Q. \qquad \text{(A2.7)}$$

Therefore, the social planner, who maximises TS, reveals the following capacity level

$$\frac{\partial V_W(X_W^* \equiv X)}{\partial Q} = 0 \qquad \text{(A2.8)}$$

$$\Leftrightarrow \frac{2X}{2(r - \mu)} - \frac{2\eta XQ}{2(r - \mu)} - \delta = 0$$

$$\Leftrightarrow \frac{X - \eta XQ}{r - \mu} = \delta$$

$$\Leftrightarrow X - \eta XQ = \delta(r - \mu)$$

$$\Leftrightarrow \eta XQ = X - \delta(r - \mu)$$

$$\Leftrightarrow Q = \frac{X - \delta(r - \mu)}{\eta X}$$

$$\Leftrightarrow Q \equiv Q_W^*(X) = \frac{1}{\eta} - \frac{\delta(r - \mu)}{\eta X}.$$

Before investing, the social planner possesses an option value, which can be expressed as

$$F_W(X) = AX^{\beta}. \qquad \text{(A2.9)}$$

To achieve the threshold level X_W^* at which the social planner invests, the VMC

$$F_W(X_W^*) = V_W(X_W^*, Q) \qquad \text{(A2.10)}$$

$$\Leftrightarrow A(X_W^*)^{\beta} = \frac{X_W^* Q(2 - \eta Q)}{2(r - \mu)} - \delta Q,$$

and the SPC

$$\left.\frac{\partial F_W(X)}{\partial X}\right|_{X=X_W^*} = \left.\frac{\partial V(X,Q)}{\partial X}\right|_{X=X_W^*} \tag{A2.11}$$

must be employed. Substituting (A2.10) into (A2.11) and solving for X_W^* gives

$$\left.\frac{\partial F_W(X)}{\partial X}\right|_{X=X_W^*} = \left.\frac{\partial V(X,Q)}{\partial X}\right|_{X=X_W^*} \tag{A2.12}$$

$$\Leftrightarrow \frac{\partial\left(A(X_W^*)^\beta\right)}{\partial X_W^*} = \frac{\partial\left(\frac{X_W^* Q(2-\eta Q)}{2(r-\mu)} - \delta Q\right)}{\partial X_W^*}$$

$$\Leftrightarrow \beta A\frac{(X_W^*)^\beta}{X_W^*} = \frac{Q(2-\eta Q)}{2(r-\mu)}$$

$$\Leftrightarrow \frac{Q(2-\eta Q)}{2(r-\mu)} = \beta\frac{\frac{X_W^* Q(2-\eta Q)}{2(r-\mu)} - \delta Q}{X_W^*}$$

$$\Leftrightarrow \frac{Q(2-\eta Q)}{2(r-\mu)} = \frac{\beta Q(2-\eta Q)}{2(r-\mu)} - \frac{\beta\delta Q}{X_W^*}$$

$$\Leftrightarrow 2-\eta Q = \beta(2-\eta Q) - \frac{2\beta\delta(r-\mu)}{X_W^*}$$

$$\Leftrightarrow (2-\eta Q)(-\beta+1) = -\frac{2\beta\delta(r-\mu)}{X_W^*}$$

$$\Leftrightarrow (\beta-1)(2-\eta Q) = \frac{2\beta\delta(r-\mu)}{X_W^*}$$

$$\Leftrightarrow X_W^*(Q) = \frac{2\beta\delta(r-\mu)}{(\beta-1)(2-\eta Q)}.$$

Substituting $X_W^*(Q)$ into expression (A2.8) gives that

$$Q = \frac{1}{\eta} - \frac{\delta(r-\mu)}{\eta\frac{2\beta\delta(r-\mu)}{(\beta-1)(2-\eta Q)}} \tag{A2.13}$$

$$\Leftrightarrow Q = \frac{1}{\eta} - \frac{(\beta-1)(2-\eta Q)}{2\beta\eta}$$

$$\Leftrightarrow Q = \frac{2\beta-(\beta-1)(2-\eta Q)}{2\beta\eta}$$

$$\Leftrightarrow 2\beta\eta Q = 2\beta - 2\beta + \beta\eta Q + 2 - \eta Q$$

$$\Leftrightarrow \beta\eta Q = 2-\eta Q$$

$$\Leftrightarrow (\beta+1)\eta Q = 2$$

$$\Leftrightarrow Q \equiv Q_W^* = \frac{2}{(\beta+1)\eta} = 2Q^*.$$

From one the results in (A2.8), the optimal investment threshold can be computed as follows

$$X - \eta XQ = \delta(r-\mu) \tag{A2.14}$$

$$\Leftrightarrow X(1-\eta Q) = \delta(r-\mu)$$

$$\Leftrightarrow X = \frac{\delta(r-\mu)}{1-\eta Q}$$

$$\Leftrightarrow X = \frac{\delta(r-\mu)}{1-\eta\frac{2}{(\beta+1)\eta}} \quad (replacing\ Q \equiv Q_w^*)$$

$$\Leftrightarrow X = \frac{\delta(r-\mu)}{1-\frac{2}{\beta+1}}$$

$$\Leftrightarrow X = \frac{\delta(r-\mu)}{\frac{\beta+1-2}{\beta+1}}$$

$$\Leftrightarrow X = \frac{\delta(r-\mu)}{\frac{\beta-1}{\beta+1}}$$

$$\Leftrightarrow X = \frac{\beta+1}{\beta-1}\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X_w^* = \frac{\beta+1}{\beta-1}\delta(r-\mu) = X^*.$$

The total welfare for the welfare-maximising policy at the time of the social planner's investment is equal to

$$TS_w = TS(X_w^*, Q_w^*) = \frac{X_w^* Q_w^*(2-\eta Q_w^*)}{2(r-\mu)} - \delta Q_w^* \tag{A2.15}$$

$$\Leftrightarrow TS_w = \frac{\frac{\beta+1}{\beta-1}\delta(r-\mu)\frac{2}{(\beta+1)\eta}\left(2-\eta\frac{2}{(\beta+1)\eta}\right)}{2(r-\mu)} - \delta\frac{2}{(\beta+1)\eta}$$

$$\Leftrightarrow TS_w = \frac{1}{\beta-1}\delta\frac{1}{\eta}\left(2-\frac{2}{\beta+1}\right) - \delta\frac{2}{(\beta+1)\eta}$$

$$\Leftrightarrow TS_w = \frac{2\delta}{(\beta-1)\eta} - \frac{2\delta}{(\beta-1)(\beta+1)\eta} - \frac{2\delta}{(\beta+1)\eta}$$

$$\Leftrightarrow TS_w = \frac{2\delta(\beta+1) - 2\delta - 2\delta(\beta-1)}{(\beta-1)(\beta+1)\eta}$$

$$\Leftrightarrow TS_w = \frac{2\delta}{(\beta+1)(\beta-1)\eta}.$$

The total welfare loss in a monopoly situation, at the moment of investment, is then

$$TS(X_w^*, Q_w^*) - TS(X^*, Q^*) \tag{A2.16}$$
$$= \frac{2\delta}{(\beta+1)(\beta-1)\eta} - \frac{3\delta}{2(\beta+1)(\beta-1)\eta}$$
$$= \frac{4\delta - 3\delta}{2(\beta+1)(\beta-1)\eta}$$
$$= \frac{\delta}{2(\beta+1)(\beta-1)\eta}.$$

A.3. Mathematical details of the follower's investment strategy

Given that in the duopoly scenario the market capacity is $Q = Q_L + Q_F$, it is possible to express the profit function of the follower firm as

$$\pi_F(t) = P(t) \cdot Q_F(t) \tag{A3.1}$$
$$\Leftrightarrow \pi_F(t) = X(t) \cdot \big(1 - \eta Q(t)\big) \cdot Q_F(t)$$
$$\Leftrightarrow \pi_F(t) = X(t) \cdot Q_F(t) \cdot [1 - \eta(Q_L + Q_F)].$$

The expected profit of the follower firm is then

$$E[\pi_F(t)] = E\big[X(t) \cdot Q_F(t) \cdot [1 - \eta(Q_L + Q_F)]\big] \tag{A3.2}$$
$$\Leftrightarrow E[\pi_F(t)] = Q_F[1 - \eta(Q_L + Q_F)] \cdot E[X_t]$$
$$\Leftrightarrow E[\pi_F(t)] = Q_F[1 - \eta(Q_L + Q_F)] \cdot X \cdot e^{\mu \cdot t}.$$

Therefore, the value function of the follower firm at the moment of investment, depending on X, Q_L, and Q_F, can be represented as follows

$$V_F^*(X, Q_L, Q_F) = E\left[\int_{t=0}^{\infty} \pi_F(t) exp(-rt)\, dt - \delta_2 Q_F\right] \tag{A3.3}$$
$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = \int_{t=0}^{\infty} Q_F[1 - \eta(Q_L + Q_F)] \cdot X \cdot e^{\mu \cdot t} \cdot e^{-r \cdot t}\, dt - \delta_2 Q_F$$
$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = XQ_F[1 - \eta(Q_L + Q_F)] \int_{t=0}^{\infty} e^{(\mu - r)t}\, dt - \delta_2 Q_F$$
$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = \frac{XQ_F[1 - \eta(Q_L + Q_F)]}{r - \mu} - \delta_2 Q_F.$$

By maximising this function with respect to Q_F, we obtain the optimal capacity size of the follower, given X and Q_L, as follows

$$\frac{\partial V_F^*(X, Q_L, Q_F)}{\partial Q_F} = 0 \tag{A3.4}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}-\delta_2 Q_F\right)}{\partial Q_F}=0$$

$$\Leftrightarrow \frac{X-X\eta Q_L-2X\eta Q_F}{r-\mu}-\delta_2=0$$

$$\Leftrightarrow X-X\eta Q_L-2X\eta Q_F=\delta_2(r-\mu)$$

$$\Leftrightarrow 2X\eta Q_F=X-X\eta Q_L-\delta_2(r-\mu)$$

$$\Leftrightarrow Q_F=\frac{1}{2\eta}-\frac{Q_L}{2}-\frac{\delta_2(r-\mu)}{2X\eta}$$

$$\Leftrightarrow Q_F\equiv\ Q_F^*(X,Q_L)=\frac{1}{2\eta}\left(1-\eta Q_L-\frac{\delta_2(r-\mu)}{X}\right).$$

Considering that before an investment has been made, the follower firm has an option to invest, represented as

$$F_F(X)=A_F X^{\beta}, \tag{A3.5}$$

we can determine the indifference level X_F^* by combining the VMC

$$F_F(X_F^*)=V_F^*(X_F^*,Q_L,Q_F) \tag{A3.6}$$

$$\Leftrightarrow A_F(X_F^*)^{\beta}=\frac{X_F^*Q_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}-\delta_2 Q_F$$

and the SPC

$$\left.\frac{\partial F_F(X)}{\partial X}\right|_{X=X_F^*}=\left.\frac{\partial V_F^*\left(X,Q_L,Q_F\right)}{\partial X}\right|_{X=X_F^*}. \tag{A3.7}$$

Using the results from equations (A3.6) and (A3.7), we can obtain the value of X_F^* as follows

$$\left.\frac{\partial F_F(X)}{\partial X}\right|_{X=X_F^*}=\left.\frac{\partial V_F^*(X,Q_L,Q_F)}{\partial X}\right|_{X=X_F^*} \tag{A3.8}$$

$$\Leftrightarrow A_F\beta(X_F^*)^{\beta-1}=\frac{Q_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}$$

$$\Leftrightarrow A_F\beta\frac{(X_F^*)^{\beta}}{X_F^*}=\frac{Q_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta}{X_F^*}A_F(X_F^*)^{\beta}=\frac{Q_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta}{X_F^*}\left[\frac{X_F^*Q_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}-\delta_2 Q_F\right]=\frac{Q_F\left(1-\eta(Q_L+Q_F)\right)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta Q_F\big(1-\eta(Q_L+Q_F)\big)}{r-\mu} - \frac{\beta\delta_2 Q_F}{X_F^*} = \frac{Q_F\big(1-\eta(Q_L+Q_F)\big)}{r-\mu}$$

$$\Leftrightarrow \frac{\beta\delta_2 Q_F}{X_F^*} = \frac{Q_F\big(1-\eta(Q_L+Q_F)\big)}{r-\mu}(\beta-1)$$

$$\Leftrightarrow \frac{\beta\delta_2}{X_F^*} = \frac{1-\eta(Q_L+Q_F)}{r-\mu}(\beta-1)$$

$$\Leftrightarrow X_F^* = \frac{\beta\delta_2(r-\mu)}{(\beta-1)\big(1-\eta(Q_L+Q_F)\big)}$$

$$\Leftrightarrow X_F^* \equiv X_F^*(Q_L,Q_F) = \frac{\beta}{\beta-1}\frac{\delta_2(r-\mu)}{\big(1-\eta(Q_L+Q_F)\big)}.$$

To obtain the optimal capacity level (Q_F^*) and the optimal investment threshold (X_F^*) where they only depend on Q_L we must solve the following system of equations

$$\begin{cases} Q_F^*(X,Q_L) = \dfrac{1}{2\eta}\left(1-\eta Q_L - \dfrac{\delta_2(r-\mu)}{X}\right) \\ X_F^*(Q_L,Q_F) = \dfrac{\beta}{\beta-1}\dfrac{\delta_2(r-\mu)}{\big(1-\eta(Q_L+Q_F)\big)} \end{cases} \tag{A3.9}$$

$$\Leftrightarrow \begin{cases} Q_F^*(X,Q_L) = \dfrac{1}{2\eta}\left(1-\eta Q_L - \dfrac{\delta_2(r-\mu)}{\dfrac{\beta}{\beta-1}\dfrac{\delta_2(r-\mu)}{\big(1-\eta(Q_L+Q_F)\big)}}\right)\big(if\ X = X_F^*(Q_L,Q_F)\big) \\ - \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{1}{2\eta}\left(1-\eta Q_L - \dfrac{(\beta-1)\big(1-\eta(Q_L+Q_F)\big)}{\beta}\right) \\ - \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{1}{2\eta}\left(\dfrac{\beta-\beta\eta Q_L-(\beta-1)(1-\eta Q_L-\eta Q_F)}{\beta}\right) \\ - \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{1}{2\eta}\left(\dfrac{\beta-\beta\eta Q_L-\beta+\beta\eta Q_L+\beta\eta Q_F+1-\eta Q_L-\eta Q_F}{\beta}\right) \\ - \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{1}{2\eta}\left(\dfrac{\beta\eta Q_F+1-\eta Q_L-\eta Q_F}{\beta}\right) \\ - \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{1}{2}\left(\dfrac{\beta Q_F + \dfrac{1}{\eta} - Q_L - Q_F}{\beta}\right) \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* - \dfrac{Q_F^*}{2} + \dfrac{Q_F^*}{2\beta} = \dfrac{1}{2\beta\eta} - \dfrac{Q_L}{2\beta} \ (assuming\ that\ Q_F \equiv Q_F^*) \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^*\left(1 - \dfrac{1}{2} + \dfrac{1}{2\beta}\right) = \dfrac{1}{2\beta\eta} - \dfrac{Q_L}{2\beta} \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^*\left(\dfrac{1}{2} + \dfrac{1}{2\beta}\right) = \dfrac{1}{2}\left(\dfrac{1}{\beta\eta} - \dfrac{Q_L}{\beta}\right) \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} \dfrac{1}{2} Q_F^*\left(1 + \dfrac{1}{\beta}\right) = \dfrac{1}{2}\left(\dfrac{1}{\beta\eta} - \dfrac{Q_L}{\beta}\right) \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^*\left(1 + \dfrac{1}{\beta}\right) = \dfrac{1}{\beta\eta} - \dfrac{Q_L}{\beta} \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{\dfrac{1}{\beta\eta} - \dfrac{Q_L}{\beta}}{1 + \dfrac{1}{\beta}} \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{\dfrac{1}{\beta\eta} - \dfrac{Q_L}{\beta}}{\dfrac{\beta + 1}{\beta}} \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{\dfrac{1 - \eta Q_L}{\beta\eta}}{\dfrac{\beta + 1}{\beta}} \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* = \dfrac{\beta(1 - \eta Q_L)}{\beta(\beta + 1)\eta} \\ \_ \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^* \equiv Q_F^*(X_F^*(Q_L), Q_L) = \dfrac{1-\eta Q_L}{(\beta+1)\eta} \\ - \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{\beta}{\beta-1}\dfrac{\delta_2(r-\mu)}{\left(1-\eta\left(Q_L+\dfrac{1-\eta Q_L}{(\beta+1)\eta}\right)\right)} \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{\beta}{\beta-1}\dfrac{\delta_2(r-\mu)}{1-\eta\left(\dfrac{(\beta+1)\eta Q_L+1-\eta Q_L}{(\beta+1)\eta}\right)} \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{\beta}{\beta-1}\dfrac{\delta_2(r-\mu)}{\dfrac{(\beta+1)\eta-\eta\big((\beta+1)\eta Q_L+1-\eta Q_L\big)}{(\beta+1)\eta}} \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{(\beta+1)\beta\eta}{\beta-1}\dfrac{\delta_2(r-\mu)}{(\beta+1)\eta-\eta\big((\beta+1)\eta Q_L+1-\eta Q_L\big)} \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{(\beta+1)\beta}{\beta-1}\dfrac{\delta_2(r-\mu)}{\beta+1-[(\beta+1)\eta Q_L+1-\eta Q_L]} \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{\beta+1}{\beta-1}\dfrac{\beta\delta_2(r-\mu)}{\beta+1-\beta\eta Q_L-\eta Q_L-1+\eta Q_L} \end{cases}$$

$$\Leftrightarrow \begin{cases} - \\ X_F^* = \dfrac{\beta+1}{\beta-1}\dfrac{\beta\delta_2(r-\mu)}{\beta-\beta\eta Q_L} \end{cases}$$

$$\Leftrightarrow \begin{cases} Q_F^*(Q_L) = \dfrac{1-\eta Q_L}{(\beta+1)\eta} \\ X_F^* \equiv X_F^*(Q_L) = \dfrac{\beta+1}{\beta-1}\dfrac{\delta_2(r-\mu)}{1-\eta Q_L} \end{cases}.$$

To derive an expression for the parameter A_F in the investment option, we begin by substituting $X = X_F^*(Q_L)$ into the right-hand side of the VMC. This allows us to obtain

$$\frac{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}Q_F\big(1-\eta(Q_L+Q_F)\big)}{r-\mu}-\delta_2 Q_F \tag{A3.10}$$

$$=\frac{\beta+1}{\beta-1}\frac{\delta_2}{1-\eta Q_L}Q_F\big(1-\eta(Q_L+Q_F)\big)-\delta_2 Q_F$$

$$=\frac{\beta+1}{\beta-1}\frac{\delta_2}{1-\eta Q_L}\frac{1-\eta Q_L}{(\beta+1)\eta}\left(1-\eta\left(Q_L+\frac{1-\eta Q_L}{(\beta+1)\eta}\right)\right)-\delta_2\frac{1-\eta Q_L}{(\beta+1)\eta}\ \left(replacing\ \ Q_F\equiv Q_F^*(Q_L)\right)$$

$$=\frac{\delta_2}{(\beta-1)\eta}\left(1-\eta\left(Q_L+\frac{1-\eta Q_L}{(\beta+1)\eta}\right)\right)-\delta_2\frac{1-\eta Q_L}{(\beta+1)\eta}$$

$$=\frac{\delta_2}{(\beta-1)\eta}\left(1-\eta Q_L-\frac{1-\eta Q_L}{\beta+1}\right)-\frac{\delta_2}{(\beta+1)\eta}+\frac{\delta_2\eta Q_L}{(\beta+1)\eta}$$

$$=\frac{\delta_2}{(\beta-1)\eta}-\frac{\delta_2\eta Q_L}{(\beta-1)\eta}-\frac{\delta_2}{(\beta-1)(\beta+1)\eta}+\frac{\delta_2\eta Q_L}{(\beta-1)(\beta+1)\eta}-\frac{\delta_2}{(\beta+1)\eta}+\frac{\delta_2\eta Q_L}{(\beta+1)\eta}$$

$$=\frac{\delta_2(\beta+1)-\delta_2\eta Q_L(\beta+1)-\delta_2+\delta_2\eta Q_L-\delta_2(\beta-1)+\delta_2\eta Q_L(\beta-1)}{(\beta-1)(\beta+1)\eta}$$

$$=\frac{\delta_2\beta+\delta_2-\delta_2\eta\beta Q_L-\delta_2\eta Q_L-\delta_2+\delta_2\eta Q_L-\delta_2\beta+\delta_2+\delta_2\eta\beta Q_L-\delta_2\eta Q_L}{(\beta-1)(\beta+1)\eta}$$

$$=\frac{\delta_2-\delta_2\eta Q_L}{(\beta-1)(\beta+1)\eta}$$

$$=\frac{(1-\eta Q_L)\delta_2}{(\beta-1)(\beta+1)\eta}.$$

Therefore,

$$A_F X^{\beta} = \frac{(1-\eta Q_L)\delta_2}{(\beta-1)(\beta+1)\eta} \tag{A3.11}$$

$$\Leftrightarrow A_F = \frac{1}{X^{\beta}} \frac{(1-\eta Q_L)\delta_2}{(\beta-1)(\beta+1)\eta}$$

$$\Leftrightarrow A_F = \frac{1}{\left(\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}\right)^{\beta}} \frac{(1-\eta Q_L)\delta_2}{(\beta-1)(\beta+1)\eta} \ \left(replacing\ X = X_F^*(Q_L)\right)$$

$$\Leftrightarrow A_F \equiv A_F(Q_L) = \left(\frac{\beta-1}{\beta+1}\frac{1-\eta Q_L}{\delta_2(r-\mu)}\right)^{\beta} \frac{(1-\eta Q_L)\delta_2}{(\beta-1)(\beta+1)\eta}.$$

Using the earlier findings, we can formulate the value function for the follower firm as

$$V_F^*(X, Q_L) = \begin{cases} A_F(Q_L)X^{\beta} & if\ \ X < X_F^*(Q_L) \\ \frac{\left(X(1-\eta Q_L) - \delta_2(r-\mu)\right)^2}{4X\eta(r-\mu)} & if\ X \geq X_F^*(Q_L) \end{cases} \tag{A3.12}$$

because for $X < X_F^*(Q_L)$, the follower firm is still in the idle state, and, therefore, the value of the firm is given by the option value $A_F(Q_L)X^{\beta}$. For $X \geq X_F^*(Q_L)$, is known that the value of the active firm is given by equation (A3.3) and rewriting equation (A3.4) yields

$$Q_F \equiv Q_F^*(X, Q_L) = \frac{1}{2\eta}\left(1 - \eta Q_L - \frac{\delta_2(r-\mu)}{X}\right) \tag{A3.13}$$

$$\Leftrightarrow Q_F = \frac{1}{2\eta X}\left(X - X\eta Q_L - \delta_2(r-\mu)\right)$$

$$\Leftrightarrow XQ_F = \frac{1}{2\eta}\left(X - X\eta Q_L - \delta_2(r-\mu)\right)$$

$$\Leftrightarrow XQ_F = \frac{1}{2\eta}\left(X(1-\eta Q_L) - \delta_2(r-\mu)\right).$$

Therefore, incorporating this result in equation (A3.3), we obtain the value of the second branch in expression (A3.12)

$$V_F^*(X, Q_L, Q_F) = \frac{\frac{1}{2\eta}\left(X(1-\eta Q_L) - \delta_2(r-\mu)\right)\left(1-\eta(Q_L+Q_F)\right)}{r-\mu} - \delta_2 Q_F \tag{A3.14}$$

$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = \frac{\left(X(1-\eta Q_L) - \delta_2(r-\mu)\right)\left(1-\eta(Q_L+Q_F)\right)}{2\eta(r-\mu)} - \delta_2 Q_F$$

$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = \frac{\left(X(1-\eta Q_L) - \delta_2(r-\mu)\right)(1-\eta Q_L - \eta Q_F)}{2\eta(r-\mu)} - \delta_2 Q_F$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big[1-\eta Q_L-\frac{1}{2X}\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big]}{2\eta(r-\mu)} - \delta_2\frac{1}{2X\eta}\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{X\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big[1-\eta Q_L-\frac{1}{2X}\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big]}{2X\eta(r-\mu)} - \frac{\delta_2(r-\mu)\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)}{2X\eta(r-\mu)}$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{X\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big[1-\eta Q_L-\frac{1}{2X}\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big] - \delta_2(r-\mu)\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)}{2X\eta(r-\mu)}$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big[X-X\eta Q_L-\frac{1}{2}\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)-\delta_2(r-\mu)\Big]}{2X\eta(r-\mu)}$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big[X(1-\eta Q_L)-\frac{1}{2}X(1-\eta Q_L)+\frac{1}{2}\delta_2(r-\mu)-\delta_2(r-\mu)\Big]}{2X\eta(r-\mu)}$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)\Big[\frac{1}{2}[X(1-\eta Q_L)-\delta_2(r-\mu)]\Big]}{2X\eta(r-\mu)}$$

$$\Leftrightarrow V_F^*(X,Q_L) = \frac{\big(X(1-\eta Q_L)-\delta_2(r-\mu)\big)^2}{4X\eta(r-\mu)}.$$

For the deterrence strategy, $\hat{Q}_L$ can be defined using the investment threshold equation in the system presented in expression (A3.9) as follows

$$X_F^*(Q_L) \equiv X = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L} \tag{A3.15}$$

$$\Leftrightarrow 1-\eta Q_L = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{X}$$

$$\Leftrightarrow Q_L \equiv \hat{Q}_L(X) = \frac{1}{\eta}\left(1 - \frac{(\beta+1)\delta_2(r-\mu)}{(\beta-1)X}\right).$$

Since X_t follows a GBM, Itô's formula (Björk, 2009) provides us with

$$X_t = X_0 e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \int_t^T dW_u^Q\right]} \tag{A3.16}$$

$$\Leftrightarrow X_t = X_0 e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^Q\right]}$$

$$\Leftrightarrow X_t = X e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^Q\right]} \; (replacing\ X_0 \equiv X).$$

Considering τ^* as the optimal stopping time, we can represent it as

$$\tau^* = \inf\{t > 0 \colon X_t = X_F^*(Q_L)\} \tag{A3.17}$$

$$\Leftrightarrow \tau^* = \inf\left\{t > 0 \colon X e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^Q\right]} = X_F^*(Q_L)\right\}$$

$$\Leftrightarrow \tau^* = \inf\left\{t > 0 \colon e^{\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^Q\right]} = \frac{X_F^*(Q_L)}{X}\right\}$$

$$\Leftrightarrow \tau^* = \inf\left\{t > 0 \colon \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t^Q = \ln\left(\frac{X_F^*(Q_L)}{X}\right)\right\}$$

$$\Leftrightarrow \tau^* = \inf\left\{t > 0 \colon \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)t + W_t^Q = \frac{1}{\sigma}\ln\left(\frac{X_F^*(Q_L)}{X}\right)\right\}.$$

Hence, $X_t = X_F^*(Q_L)$ if and only if

$$W_t^Q + \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)t = \frac{1}{\sigma}\ln\left(\frac{X_F^*(Q_L)}{X}\right) \tag{A3.18}$$

$$\Leftrightarrow -W_t^Q - \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)t = -\frac{1}{\sigma}\ln\left(\frac{X_F^*(Q_L)}{X}\right)$$

$$\Leftrightarrow -W_t^Q - \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right)t = \frac{1}{\sigma}\ln\left(\frac{X}{X_F^*(Q_L)}\right).$$

Following Shreve (2004) theorem 8.3.2, the Laplace transformation of the first passage time can be written as

$$E_Q\left[e^{-\lambda\tau^*} 1\!\!1_{\{\tau^* < \infty\}} \,|\mathcal{F}_0\right] = e^{-m\left(-M + \sqrt{M^2 + 2\lambda}\right)}, \tag{A3.19}$$

as long as, for the perpetual call case, the variables $\lambda, m,$ and M are defined as follows

$$\lambda = r, \tag{A3.20}$$

$$m = -\frac{1}{\sigma}\ln\left(\frac{X}{X_F^*(Q_L)}\right), \tag{A3.21}$$

and

$$M = \frac{1}{\sigma}\left(\mu - \frac{1}{2}\sigma^2\right). \tag{A3.22}$$

Incorporating expressions (A3.20), (A3.21), and (A3.22) into the right-hand side of expression (A3.19) results in

$$e^{\frac{1}{\sigma}\ln\left(\frac{X}{X_F^*(Q_L)}\right)\left[-\frac{1}{\sigma}\left(\mu-\frac{1}{2}\sigma^2\right)+\sqrt{\frac{1}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)^2+2r}\right]} \tag{A3.23}$$

$$= e^{\ln\left(\frac{X}{X_F^*(Q_L)}\right)\left[-\frac{1}{\sigma^2}\left(\mu-\frac{1}{2}\sigma^2\right)+\sqrt{\frac{1}{\sigma^4}\left(\mu-\frac{1}{2}\sigma^2\right)^2+\frac{2r}{\sigma^2}}\right]}$$

$$= e^{\ln\left(\frac{X}{X_F^*(Q_L)}\right)\left[\frac{1}{2}-\frac{\mu}{\sigma^2}+\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}\right]}$$

$$= e^{\ln\left(\frac{X}{X_F^*(Q_L)}\right)\beta}$$

$$= \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}.$$

Therefore,

$$E_Q\left[e^{-\lambda\tau^*}\,1\!\!1_{\{\tau^*<\infty\}}\,|\mathcal{F}_0\right] = e^{-m\left(-M+\sqrt{M^2+2\lambda}\right)} \tag{A3.24}$$

$$\Leftrightarrow E\left[e^{-r\cdot T}\right] = \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta} \; (replacing\ E_Q \equiv E, \tau^* \equiv T,$$

$$and\ since\ \tau^* < \infty \;\Rightarrow 1\!\!1_{\{\tau^*<\infty\}} = 1)$$

$$\Leftrightarrow \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta} = E\left[e^{-r\cdot T}\right].$$

A.4. Mathematical details of the leader's entry deterrence strategy

Given that in the deterrence strategy the market investment capacity is $Q = Q_L$, the steps to derive the leader's value function are similar to those in the monopolist case, but with $Q \equiv Q_L$.

However, as stated in subchapter 5.1.1, following expression (5.8), the follower firm will eventually enter the market. Therefore, it is necessary to incorporate a negative correction factor. Consequently, the value function of the leader firm is given by

$$V_L^{det}(X, Q_L) = \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \delta_1 Q_L - \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L)Q_L\eta Q_F^*(Q_L)}{r-\mu}\right). \tag{A4.1}$$

Substituting the follower's optimal investment capacity and the optimal investment threshold from expression (A3.9) in the value function of equation (A4.1) yields

$$\Leftrightarrow V_L^{det}(X,Q_L) = \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \delta_1 Q_L - \left(\frac{X}{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}}\right)^{\beta}\left(\frac{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}Q_L\eta\frac{1-\eta Q_L}{(\beta+1)\eta}}{r-\mu}\right) \tag{A4.2}$$

$$\Leftrightarrow V_L^{det}(X,Q_L) = \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \delta_1 Q_L - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2 Q_L}{\beta-1}$$

$$\Leftrightarrow V_L^{det}(X) = \frac{XQ_L^{det}(X)\left(1-\eta Q_L^{det}(X)\right)}{r-\mu} - \delta_1 Q_L^{det}(X) - \left(\frac{X(\beta-1)\left(1-\eta Q_L^{det}(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2 Q_L^{det}(X)}{\beta-1}\ \left(considering\ Q_L \equiv Q_L^{det}(X)\right).$$

Differentiating the previous equation with respect to Q_L and considering $Q_L^{det}(X) \equiv Q_L$, we obtain

$$\frac{\partial V_L^{det}(X,Q_L)}{\partial Q_L} = 0 \tag{A4.3}$$

$$\Leftrightarrow \frac{X-2X\eta Q_L}{r-\mu} - \delta_1 - \frac{\partial\left[\left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2 Q_L}{\beta-1}\right]}{\partial Q_L} = 0$$

$$\Leftrightarrow \frac{X-2X\eta Q_L}{r-\mu} - \delta_1 - \left[\frac{\partial\left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}}{\partial Q_L}\frac{\delta_2 Q_L}{\beta-1} + \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\partial\left(\frac{\delta_2 Q_L}{\beta-1}\right)}{\partial Q_L}\right] = 0$$

$$\Leftrightarrow \frac{X-2X\eta Q_L}{r-\mu} - \delta_1 - \left[\frac{\beta\left(X(\beta-1)(1-\eta Q_L)\right)^{\beta-1}(X\eta - X\beta\eta)}{\left((\beta+1)\delta_2(r-\mu)\right)^{\beta}}\frac{\delta_2 Q_L}{\beta-1} + \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{\beta-1}\right] = 0$$

$$\Leftrightarrow \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left[\frac{\beta X(\eta-\beta\eta)}{X(\beta-1)(1-\eta Q_L)}\frac{\delta_2 Q_L}{\beta-1} + \frac{\delta_2}{\beta-1}\right] = 0$$

$$\Leftrightarrow \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left[\frac{-\beta\eta(\beta-1)}{(\beta-1)(1-\eta Q_L)}\frac{\delta_2 Q_L}{\beta-1} + \frac{\delta_2}{\beta-1}\right] = 0$$

$$\Leftrightarrow \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{-\beta\eta\delta_2 Q_L + \delta_2(1-\eta Q_L)}{(\beta-1)(1-\eta Q_L)} = 0$$

$$\Leftrightarrow \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{-\beta\eta\delta_2 Q_L + \delta_2 - \delta_2\eta Q_L}{(\beta-1)(1-\eta Q_L)} = 0$$

$$\Leftrightarrow \Phi(X, Q_L) \equiv \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0.$$

Setting $Q_L = 0$ in the previous expression, we define

$$\Phi(X, Q_L = 0) \equiv \psi(X) \tag{A4.4}$$

$$\Leftrightarrow \psi(X) = \frac{X(1-2\eta\cdot 0)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta\cdot 0)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{(1-(\beta+1)\eta\cdot 0)\delta_2}{(\beta-1)(1-\eta\cdot 0)}$$

$$\Leftrightarrow \psi(X) = \frac{X}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{\beta-1}.$$

From the previous equation, we know that

$$\psi(X=0) = \frac{0}{r-\mu} - \delta_1 - \left(\frac{0(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{(\beta-1)} \tag{A4.5}$$

$$\Leftrightarrow \psi(0) = -\delta_1 < 0, since\ \delta_1 > 0,$$

and

$$\psi\big(X_F^*(Q_L=0)\big)=\frac{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta\cdot 0}}{r-\mu}-\delta_1-\left(\frac{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta\cdot 0}(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{(\beta-1)} \tag{A4.6}$$

$$\Leftrightarrow \psi\big(X_F^*(0)\big)=\frac{\beta+1}{\beta-1}\delta_2-\delta_1-\frac{\delta_2}{\beta-1}$$

$$\Leftrightarrow \psi\big(X_F^*(0)\big)=\delta_2\left(\frac{\beta+1-1}{\beta-1}\right)-\delta_1$$

$$\Leftrightarrow \psi\big(X_F^*(0)\big)=\frac{\beta\delta_2}{\beta-1}-\delta_1>0, this\ can\ be\ rewriten\ as\ \frac{\beta\delta_2-\beta\delta_1+\delta_1}{\beta-1}.$$

$$Since\ \beta>1\ and\ \delta_1>0, we\ have\ that$$

$$-\beta\delta_1+\delta_1<0.$$

$$However, since\ \delta_2>\delta_1, it\ follows\ that$$

$$\beta\delta_2-\beta\delta_1+\delta_1>0. Therefore,$$

$$\frac{\beta\delta_2}{\beta-1}-\delta_1>0.$$

From this expression, it is crucial to note that the expression (A22) by Huisman and Kort (2015) contains an error. Specifically, the parameter "$-\delta_1$" is missing. However, the conclusion that $\psi\big(X_F^*(0)\big)>0$ remains valid.

From expression (A4.4), we can also derive that

$$\frac{\partial\psi(X)}{\partial X} \tag{A4.7}$$

$$=\frac{\partial\left[\frac{X}{r-\mu}-\delta_1-\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{\beta-1}\right]}{\partial X}$$

$$=\frac{1}{r-\mu}-\frac{\delta_2}{\beta-1}\left[\frac{\partial\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}}{\partial X}\right]$$

$$=\frac{1}{r-\mu}-\frac{\delta_2}{\beta-1}\left[\frac{\beta\big(X(\beta-1)\big)^{\beta-1}(\beta-1)}{\big((\beta+1)\delta_2(r-\mu)\big)^{\beta}}\right]$$

$$=\frac{1}{r-\mu}-\frac{\delta_2}{\beta-1}\left[\frac{\frac{\beta\big(X(\beta-1)\big)^{\beta}}{X(\beta-1)}(\beta-1)}{\big((\beta+1)\delta_2(r-\mu)\big)^{\beta}}\right]$$

$$= \frac{1}{r-\mu} - \frac{\delta_2}{\beta-1}\left[\frac{\beta}{X}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\right]$$

$$= \frac{1}{r-\mu} - \frac{\delta_2\beta}{(\beta-1)X}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}$$

$$= \frac{1}{r-\mu}\left(1 - \frac{(r-\mu)\delta_2\beta}{(\beta-1)X}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\right)$$

$$= \frac{1}{r-\mu}\left(1 - \frac{(r-\mu)\delta_2\beta(\beta+1)}{(\beta-1)X(\beta+1)}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\right)$$

$$= \frac{1}{r-\mu}\left(1 - \frac{\beta}{\beta+1}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{-1}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\right)$$

$$= \frac{1}{r-\mu}\left(1 - \frac{\beta}{\beta+1}\left(\frac{X(\beta-1)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta-1}\right).$$

For values of $X \in \left]0, X_F^*(0)\right[$, it holds that $\frac{\partial\psi(X)}{\partial X} > 0$. This is because since expression (A4.5) is negative and expression (A4.6) is positive, then the slope of the function in expression (A4.4) must be positive. Therefore, a point within that X interval where the function intersects the axis must exist. We can define this point as X_1^{det}, where expression (A4.4) is equal to zero.

Conversely, the leading firm is unable to employ the deterrence strategy if

$$X_F^*\left(Q_L^{det}(X)\right) \leq X. \tag{A4.8}$$

Defining X_2^{det} as

$$X_F^*\left(Q_L^{det}\left(X_2^{det}\right)\right) = X_2^{det}, \tag{A4.9}$$

to determine it, we begin by incorporating the investment threshold from the system of equations in (A3.9) into the equation in (A4.3). This results in the following expression for Q_L

$$\Phi(X,Q_L) \equiv \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0 \tag{A4.10}$$

$$\Leftrightarrow \frac{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{\frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta Q_L}(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0$$

$$\Leftrightarrow \frac{\beta+1}{\beta-1}\frac{1-2\eta Q_L}{1-\eta Q_L}\delta_2 - \delta_1 - 1^{\beta}\cdot\frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0$$

$$\Leftrightarrow \frac{\beta+1}{\beta-1}\frac{1-2\eta Q_L}{1-\eta Q_L}\delta_2 - \delta_1 - \frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0$$

$$\Leftrightarrow \frac{(\beta+1)(1-2\eta Q_L)\delta_2 - (1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = \delta_1$$

$$\Leftrightarrow \beta\delta_2 - 2\beta\eta\delta_2 Q_L + \delta_2 - 2\eta\delta_2 Q_L - \delta_2 + \beta\eta\delta_2 Q_L + \eta\delta_2 Q_L = \delta_1\beta - \eta\beta\delta_1 Q_L - \delta_1 + \eta\delta_1 Q_L$$

$$\Leftrightarrow -\beta\eta\delta_2 Q_L - \eta\delta_2 Q_L + \eta\beta\delta_1 Q_L - \eta\delta_1 Q_L = \delta_1\beta - \delta_1 - \beta\delta_2$$

$$\Leftrightarrow \eta Q_L(-\beta\delta_2 - \delta_2 + \beta\delta_1 - \delta_1) = -\beta\delta_2 + \delta_1(\beta-1)$$

$$\Leftrightarrow -\eta Q_L(\beta(\delta_2-\delta_1) + \delta_1 + \delta_2) = -\beta\delta_2 + \delta_1(\beta-1)$$

$$\Leftrightarrow Q_L = \frac{\beta\delta_2 - (\beta-1)\delta_1}{\eta(\beta(\delta_2-\delta_1)+\delta_1+\delta_2)}.$$

Then, we incorporate this result into the investment threshold from the system of equations in (A3.9), yielding

$$X_F^*(Q_L) \equiv X_2^{det} = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta\frac{\beta\delta_2-(\beta-1)\delta_1}{\eta(\beta(\delta_2-\delta_1)+\delta_1+\delta_2)}} \tag{A4.11}$$

$$\Leftrightarrow X_2^{det} = \frac{\beta+1}{\beta-1} \frac{\delta_2(r-\mu)}{\frac{\beta(\delta_2-\delta_1)+\delta_1+\delta_2-\beta\delta_2+(\beta-1)\delta_1}{\beta(\delta_2-\delta_1)+\delta_1+\delta_2}}$$

$$\Leftrightarrow X_2^{det} = \frac{\beta+1}{\beta-1} \frac{(\beta(\delta_2-\delta_1)+\delta_1+\delta_2)\delta_2(r-\mu)}{\beta\delta_2-\beta\delta_1+\delta_1+\delta_2-\beta\delta_2+\beta\delta_1-\delta_1}$$

$$\Leftrightarrow X_2^{det} = \frac{\beta+1}{\beta-1}(\beta(\delta_2-\delta_1)+\delta_1+\delta_2)(r-\mu).$$

Before the leader has invested (when $X < X_L^{det}$) the firm holds the following option to invest

$$F_L^{det}(X) = A_L^{det} X^{\beta}. \tag{A4.12}$$

We can determine the optimal investment threshold in deterrence policy $\left(X_L^{det}\right)$ combining the results of the VMC

$$F_L^{det}(X) = V_L^{det}(X) \tag{A4.13}$$

$$\Leftrightarrow A_L^{det} X^{\beta} = \frac{XQ_L(X)\left(1-\eta Q_L(X)\right)}{r-\mu} - \delta_1 Q_L(X) - \left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 Q_L(X)}{\beta-1} \left(assuming\ that\ Q_L^{det}(X) \equiv Q_L(X)\right)$$

with the SPC

$$\frac{\partial F_L^{det}(X)}{\partial X} = \frac{\partial V_L^{det}(X)}{\partial X}, \tag{A4.14}$$

it follows that

$$\Leftrightarrow \frac{\partial\left(A_L^{det} X^{\beta}\right)}{\partial X} = \frac{\partial\left(\frac{XQ_L(X)\left(1-\eta Q_L(X)\right)}{r-\mu} - \delta_1 Q_L(X) - \left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 Q_L(X)}{\beta-1}\right)}{\partial X} \tag{A4.15}$$

$$\Leftrightarrow \beta A_L^{det} X^{\beta-1} = \frac{Q_L(X)\left(1-\eta Q_L(X)\right)}{r-\mu} + \frac{X}{r-\mu} \frac{\partial\left(Q_L(X)\left(1-\eta Q_L(X)\right)\right)}{\partial X} - \delta_1 \frac{\partial Q_L(X)}{\partial X}$$

$$-\left[\left[\left[\frac{\partial X^{\beta}}{\partial X}\left[(\beta-1)\big(1-\eta Q_L(X)\big)\right]^{\beta}+[X(\beta-1)]^{\beta}\frac{\partial\big(1-\eta Q_L(X)\big)^{\beta}}{\partial X}\right]\cdot\frac{1}{\big((\beta+1)\delta_2(r-\mu)\big)^{\beta}}\right]\frac{\delta_2 Q_L(X)}{\beta-1}\right.$$

$$\left.+\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\partial\left(\frac{\delta_2 Q_L(X)}{\beta-1}\right)}{\partial X}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu}+\frac{X}{r-\mu}\left[\frac{\partial Q_L(X)}{\partial X}-2\eta Q_L(X)\frac{\partial Q_L(X)}{\partial X}\right]-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left[\left[\left[\beta\frac{X^{\beta}}{X}\left[(\beta-1)\big(1-\eta Q_L(X)\big)\right]^{\beta}+[X(\beta-1)]^{\beta}\beta\frac{\big(1-\eta Q_L(X)\big)^{\beta}}{1-\eta Q_L(X)}\left(-\eta\frac{\partial Q_L(X)}{\partial X}\right)\right]\cdot\frac{1}{\big((\beta+1)\delta_2(r-\mu)\big)^{\beta}}\right]\frac{\delta_2 Q_L(X)}{\beta-1}\right.$$

$$\left.+\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{\beta-1}\frac{\partial Q_L(X)}{\partial X}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)+X\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left[\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left[\frac{\beta}{X}+\frac{\beta}{1-\eta Q_L(X)}\left(-\eta\frac{\partial Q_L(X)}{\partial X}\right)\right]\frac{\delta_2 Q_L(X)}{\beta-1}+\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2}{\beta-1}\frac{\partial Q_L(X)}{\partial X}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)+X\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left[\left[\frac{\beta}{X}+\frac{\beta}{1-\eta Q_L(X)}\left(-\eta\frac{\partial Q_L(X)}{\partial X}\right)\right]\frac{\delta_2 Q_L(X)}{\beta-1}+\frac{\delta_2}{\beta-1}\frac{\partial Q_L(X)}{\partial X}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)+X\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\cdot\left[\frac{\left(\beta\big(1-\eta Q_L(X)\big)-\beta X\eta\frac{\partial Q_L(X)}{\partial X}\right)\delta_2 Q_L(X)+\delta_2\frac{\partial Q_L(X)}{\partial X}X\big(1-\eta Q_L(X)\big)}{X(\beta-1)\big(1-\eta Q_L(X)\big)}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)+X\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\cdot\left[\frac{\delta_2\left(\beta\big(1-\eta Q_L(X)\big)Q_L(X)-\beta X\eta Q_L(X)\frac{\partial Q_L(X)}{\partial X}+\frac{\partial Q_L(X)}{\partial X}X\big(1-\eta Q_L(X)\big)\right)}{X(\beta-1)\big(1-\eta Q_L(X)\big)}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)+X\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\cdot\left[\frac{\delta_2\left(Q_L(X)\left(\beta\big(1-\eta Q_L(X)\big)-\beta X\eta\frac{\partial Q_L(X)}{\partial X}-X\eta\frac{\partial Q_L(X)}{\partial X}\right)+X\frac{\partial Q_L(X)}{\partial X}\right)}{X(\beta-1)\big(1-\eta Q_L(X)\big)}\right]$$

$$\Leftrightarrow \beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\big(1-\eta Q_L(X)\big)+X\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$

$$-\left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2\left(Q_L(X)\left(\beta\left(1-\eta Q_L(X)\right)-(\beta+1)X\eta\frac{\partial Q_L(X)}{\partial X}\right)+X\frac{\partial Q_L(X)}{\partial X}\right)}{X(\beta-1)\left(1-\eta Q_L(X)\right)}.$$

By substituting the result of equation (A4.13) into equation (A4.15), we obtain

$$\beta A_L^{det}X^{\beta-1}=\frac{Q_L(X)\left(1-\eta Q_L(X)\right)+X\frac{\partial Q_L(X)}{\partial X}\left(1-2\eta Q_L(X)\right)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$
$$-\left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2\left(Q_L(X)\left(\beta\left(1-\eta Q_L(X)\right)-(\beta+1)X\eta\frac{\partial Q_L(X)}{\partial X}\right)+X\frac{\partial Q_L(X)}{\partial X}\right)}{X(\beta-1)\left(1-\eta Q_L(X)\right)} \tag{A4.16}$$

$$\Leftrightarrow\frac{\beta}{X}A_L^{det}X^{\beta}=\frac{Q_L(X)\left(1-\eta Q_L(X)\right)+X\frac{\partial Q_L(X)}{\partial X}\left(1-2\eta Q_L(X)\right)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$
$$-\left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2\left(Q_L(X)\left(\beta\left(1-\eta Q_L(X)\right)-(\beta+1)X\eta\frac{\partial Q_L(X)}{\partial X}\right)+X\frac{\partial Q_L(X)}{\partial X}\right)}{X(\beta-1)\left(1-\eta Q_L(X)\right)}$$

$$\Leftrightarrow\frac{\beta}{X}\left(\frac{XQ_L(X)\left(1-\eta Q_L(X)\right)}{r-\mu}-\delta_1Q_L(X)-\left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2Q_L(X)}{\beta-1}\right)$$
$$=\frac{Q_L(X)\left(1-\eta Q_L(X)\right)+X\frac{\partial Q_L(X)}{\partial X}\left(1-2\eta Q_L(X)\right)}{r-\mu}-\delta_1\frac{\partial Q_L(X)}{\partial X}$$
$$-\left(\frac{X(\beta-1)\left(1-\eta Q_L(X)\right)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2\left(Q_L(X)\left(\beta\left(1-\eta Q_L(X)\right)-(\beta+1)X\eta\frac{\partial Q_L(X)}{\partial X}\right)+X\frac{\partial Q_L(X)}{\partial X}\right)}{X(\beta-1)\left(1-\eta Q_L(X)\right)}$$

$$\Leftrightarrow \frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \delta_1 Q_L(X) - \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 Q_L(X)}{\beta-1}$$

$$= \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2 \frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \frac{\delta_1 X}{\beta}\frac{\partial Q_L(X)}{\partial X}$$

$$- \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{X\delta_2\left(Q_L(X)\left(\beta\big(1-\eta Q_L(X)\big) - (\beta+1)X\eta\frac{\partial Q_L(X)}{\partial X}\right) + X\frac{\partial Q_L(X)}{\partial X}\right)}{\beta X(\beta-1)\big(1-\eta Q_L(X)\big)}$$

$$\Leftrightarrow \frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2 \frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \delta_1 Q_L(X) + \frac{\delta_1 X}{\beta}\frac{\partial Q_L(X)}{\partial X}$$

$$- \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 Q_L(X)}{\beta-1}$$

$$+ \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2\beta Q_L(X) - \delta_2\beta\eta Q_L^2(X) - \delta_2\beta\eta X Q_L(X)\frac{\partial Q_L(X)}{\partial X} - \delta_2\eta X Q_L(X)\frac{\partial Q_L(X)}{\partial X} + \delta_2 X\frac{\partial Q_L(X)}{\partial X}}{\beta(\beta-1)\big(1-\eta Q_L(X)\big)} = 0$$

$$\Leftrightarrow \frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2 \frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \delta_1 Q_L(X) + \frac{\delta_1 X}{\beta}\frac{\partial Q_L(X)}{\partial X}$$

$$- \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 Q_L(X)}{\beta-1}$$

$$+ \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} \frac{\delta_2 X\frac{\partial Q_L(X)}{\partial X}\big(1-\eta Q_L(X) - \beta\eta Q_L(X)\big) + \delta_2\beta Q_L(X)\big(1-\eta Q_L(X)\big)}{\beta(\beta-1)\big(1-\eta Q_L(X)\big)} = 0$$

$$\Leftrightarrow \frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \delta_1 Q_L(X) + \frac{\delta_1 X}{\beta}\frac{\partial Q_L(X)}{\partial X}$$

$$-\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{\delta_2 Q_L(X)}{\beta-1} + \left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left(\frac{\delta_2 X\frac{\partial Q_L(X)}{\partial X}\big(1-(\beta+1)\eta Q_L(X)\big)}{\beta(\beta-1)\big(1-\eta Q_L(X)\big)} + \frac{\delta_2 Q_L(X)}{\beta-1}\right) = 0$$

$$\Leftrightarrow \frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2\frac{\partial Q_L(X)}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \delta_1 Q_L(X) + \frac{\delta_1 X}{\beta}\frac{\partial Q_L(X)}{\partial X}$$

$$+\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\cdot\left(\frac{\delta_2 X\frac{\partial Q_L(X)}{\partial X}\big(1-(\beta+1)\eta Q_L(X)\big)}{\beta(\beta-1)\big(1-\eta Q_L(X)\big)} - \frac{\delta_2 Q_L(X)}{\beta-1} + \frac{\delta_2 Q_L(X)}{\beta-1}\right) = 0$$

$$\Leftrightarrow \frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2\frac{\partial Q_L}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \delta_1 Q_L(X) + \frac{\delta_1 X}{\beta}\frac{\partial Q_L}{\partial X}$$

$$+\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left(\frac{\delta_2 X\frac{\partial Q_L}{\partial X}\big(1-(\beta+1)\eta Q_L(X)\big)}{\beta(\beta-1)\big(1-\eta Q_L(X)\big)}\right) = 0\left(defining\ \frac{\partial Q_L(X)}{\partial X} \equiv \frac{\partial Q_L}{\partial X}\right).$$

Rewriting equation (A4.3), we obtain

$$\frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1 - \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = 0 \tag{A4.17}$$

$$\Leftrightarrow \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\frac{(1-(\beta+1)\eta Q_L)\delta_2}{(\beta-1)(1-\eta Q_L)} = \frac{X(1-2\eta Q_L)}{r-\mu} - \delta_1$$

$$\Leftrightarrow \left(\frac{X(\beta-1)(1-\eta Q_L)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta} = \left(\frac{X(1-2\eta Q_L)}{r-\mu}-\delta_1\right)\frac{(\beta-1)(1-\eta Q_L)}{(1-(\beta+1)\eta Q_L)\delta_2}.$$

If we substitute the result of equation (A4.17) into equation (A4.16) and consider that $Q_L(X) \equiv Q_L$, it yields that

(A4.18)

$$\frac{XQ_L(X)\big(1-\eta Q_L(X)\big)}{r-\mu} - \frac{XQ_L(X)\big(1-\eta Q_L(X)\big) + X^2\frac{\partial Q_L}{\partial X}\big(1-2\eta Q_L(X)\big)}{\beta(r-\mu)} - \delta_1 Q_L(X) + \frac{\delta_1 X}{\beta}\frac{\partial Q_L}{\partial X}$$

$$+\left(\frac{X(\beta-1)\big(1-\eta Q_L(X)\big)}{(\beta+1)\delta_2(r-\mu)}\right)^{\beta}\left(\frac{\delta_2 X\frac{\partial Q_L}{\partial X}\big(1-(\beta+1)\eta Q_L(X)\big)}{\beta(\beta-1)\big(1-\eta Q_L(X)\big)}\right) = 0$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \frac{XQ_L(1-\eta Q_L) + X^2\frac{\partial Q_L}{\partial X}(1-2\eta Q_L)}{\beta(r-\mu)} - \delta_1 Q_L + \frac{\delta_1 X}{\beta}\frac{\partial Q_L}{\partial X}$$

$$+\left(\frac{X(1-2\eta Q_L)}{r-\mu}-\delta_1\right)\frac{(\beta-1)(1-\eta Q_L)}{(1-(\beta+1)\eta Q_L)\delta_2}\left(\frac{\delta_2 X\frac{\partial Q_L}{\partial X}(1-(\beta+1)\eta Q_L)}{\beta(\beta-1)(1-\eta Q_L)}\right) = 0$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \frac{XQ_L(1-\eta Q_L) + X^2\frac{\partial Q_L}{\partial X}(1-2\eta Q_L)}{\beta(r-\mu)} - \delta_1 Q_L + \frac{\delta_1 X}{\beta}\frac{\partial Q_L}{\partial X} + \left(\frac{X(1-2\eta Q_L)}{r-\mu}-\delta_1\right)\frac{X}{\beta}\frac{\partial Q_L}{\partial X} = 0$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \frac{XQ_L(1-\eta Q_L)}{\beta(r-\mu)} - \frac{X^2\frac{\partial Q_L}{\partial X}(1-2\eta Q_L)}{\beta(r-\mu)} - \delta_1 Q_L + \frac{\delta_1 X}{\beta}\frac{\partial Q_L}{\partial X} + \frac{X^2\frac{\partial Q_L}{\partial X}(1-2\eta Q_L)}{\beta(r-\mu)} - \frac{\delta_1 X}{\beta}\frac{\partial Q_L}{\partial X} = 0$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{r-\mu} - \frac{XQ_L(1-\eta Q_L)}{\beta(r-\mu)} - \delta_1 Q_L = 0$$

$$\Leftrightarrow \frac{\beta XQ_L(1-\eta Q_L)}{\beta(r-\mu)} - \frac{XQ_L(1-\eta Q_L)}{\beta(r-\mu)} = \frac{\delta_1 Q_L \beta}{\beta}$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{\beta(r-\mu)}(\beta-1) = \frac{\delta_1 Q_L \beta}{\beta}$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{(r-\mu)}(\beta-1) = \frac{\delta_1 Q_L \beta^2}{\beta}$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{(r-\mu)}(\beta-1) - \delta_1 Q_L \beta = 0.$$

Simplifying the previous expression, we obtain

$$\frac{XQ_L(1-\eta Q_L)}{(r-\mu)}(\beta-1) - \delta_1 Q_L \beta = 0 \tag{A4.19}$$

$$\Leftrightarrow \frac{XQ_L(1-\eta Q_L)}{(r-\mu)} = \delta_1 Q_L \beta \frac{1}{\beta-1}$$

$$\Leftrightarrow X(1-\eta Q_L) = \beta \frac{1}{\beta-1} \delta_1 (r-\mu).$$

Considering that during the deterrence strategy, the leader firm acts as a monopolist (since there are no other active firms in the market), we can infer that its investment capacity is the same as that of a monopolist. Therefore, $Q_L \equiv Q_L^{det} = Q^* = \frac{1}{(\beta+1)\eta}$. Consequently, the leader's investment threshold is given by incorporating this investment capacity into expression (A4.19), yielding

$$X\left(1-\eta\frac{1}{(\beta+1)\eta}\right) = \beta\frac{1}{\beta-1}\delta_1(r-\mu) \tag{A4.20}$$

$$\Leftrightarrow X\left(1-\frac{1}{\beta+1}\right) = \beta\frac{1}{\beta-1}\delta_1(r-\mu)$$

$$\Leftrightarrow X\frac{\beta}{\beta+1} = \beta\frac{1}{\beta-1}\delta_1(r-\mu)$$

$$\Leftrightarrow X_L^{det} = \frac{\beta+1}{\beta-1}\delta_1(r-\mu)\ \left(substituting\ X \equiv X_L^{det}\right).$$

A.5. Mathematical details of the leader's entry accommodation strategy

Given that in the accommodation strategy, both firms invest, the market investment capacity is $Q = Q_L + Q_F$. The profit function of the leader is then given by

$$\pi_L(t) = P(t) \cdot Q_L(t) \tag{A5.1}$$

$$\Leftrightarrow \pi_L(t) = X(t) \cdot \left(1-\eta Q(t)\right) \cdot Q_L(t)$$

$$\Leftrightarrow \pi_L(t) = X(t) \cdot [1-\eta(Q_L+Q_F)] \cdot Q_L(t)$$

$$\Leftrightarrow \pi_L(t) = X(t) \cdot Q_L \cdot [1-\eta(Q_L+Q_F)].$$

The expected profit of the leading firm is then

$$E[\pi_L(t)] = E\big[X(t) \cdot Q_L \cdot [1 - \eta(Q_L + Q_F)]\big] \tag{A5.2}$$

$$\Leftrightarrow E[\pi_L(t)] = Q_L\big(1 - \eta(Q_L + Q_F)\big) \cdot E[X_t]$$

$$\Leftrightarrow E[\pi_L(t)] = Q_L\big(1 - \eta(Q_L + Q_F)\big) \cdot X \cdot e^{\mu \cdot t}.$$

Therefore, the leader firm's value function in the accommodation strategy is given by

$$V_L^{acc}(X, Q_L) = E\left[\int_{t=0}^{\infty} \pi_L(t) exp(-rt)\, dt - \delta_1 Q_L\right] \tag{A5.3}$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \int_{t=0}^{\infty} Q_L\big(1 - \eta(Q_L + Q_F)\big) \cdot X \cdot e^{\mu \cdot t} \cdot e^{-r \cdot t}\, dt - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = XQ_L\big(1 - \eta(Q_L + Q_F)\big) \int_{t=0}^{\infty} e^{(\mu - r)t}\, dt - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{XQ_L\big(1 - \eta(Q_L + Q_F)\big)}{r - \mu} - \delta_1 Q_L \; \big(regarding\ that\ Q_F \equiv Q_F^*(Q_L)\big).$$

Substituting Q_F in the previous expression by equation (A3.4) yields

$$V_L^{acc}(X, Q_L) = \frac{XQ_L\left(1 - \eta\left(Q_L + \frac{1}{2\eta}\left(1 - \eta Q_L - \frac{\delta_2(r - \mu)}{X}\right)\right)\right)}{r - \mu} - \delta_1 Q_L \tag{A5.4}$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{XQ_L\left(1 - \eta\left(Q_L + \frac{1}{2\eta} - \frac{1}{2}Q_L - \frac{\delta_2(r - \mu)}{2\eta X}\right)\right)}{r - \mu} - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{XQ_L\left(1 - \eta Q_L - \frac{1}{2} + \frac{\eta Q_L}{2} + \frac{\delta_2(r - \mu)}{2X}\right)}{r - \mu} - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{XQ_L\left(\frac{1}{2} - \frac{\eta Q_L}{2} + \frac{\delta_2(r - \mu)}{2X}\right)}{r - \mu} - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{\frac{1}{2}XQ_L - \frac{1}{2}\eta X Q_L^2 + \frac{1}{2}\delta_2(r - \mu)Q_L}{r - \mu} - \delta_1 Q_L.$$

Maximising the previous equation concerning Q_L yields the leader's capacity level as follows

$$\frac{\partial V_L^{acc}(X, Q_L)}{\partial Q_L} = 0 \tag{A5.5}$$

$$\Leftrightarrow \frac{\partial\left(\frac{\frac{1}{2}XQ_L - \frac{1}{2}\eta XQ_L^2 + \frac{1}{2}\delta_2(r-\mu)Q_L}{r-\mu} - \delta_1 Q_L\right)}{\partial Q_L} = 0$$

$$\Leftrightarrow \left(\frac{1}{2}X - \eta XQ_L + \frac{1}{2}\delta_2(r-\mu)\right)\frac{1}{r-\mu} - \delta_1 = 0$$

$$\Leftrightarrow \frac{1}{2}X - \eta XQ_L + \frac{1}{2}\delta_2(r-\mu) = \delta_1(r-\mu)$$

$$\Leftrightarrow -\eta XQ_L = \delta_1(r-\mu) - \frac{1}{2}\delta_2(r-\mu) - \frac{1}{2}X$$

$$\Leftrightarrow -\eta XQ_L = \frac{1}{2}\big(-X + (2\delta_1 - \delta_2)(r-\mu)\big)$$

$$\Leftrightarrow -\eta Q_L = \frac{1}{2}\left(-1 + \frac{(2\delta_1 - \delta_2)(r-\mu)}{X}\right)$$

$$\Leftrightarrow Q_L \equiv Q_L^{acc}(X) = \frac{1}{2\eta}\left(1 - \frac{(2\delta_1 - \delta_2)(r-\mu)}{X}\right).$$

Combining the results in equations (A5.4) and (A5.5) gives the value of the entry accommodation strategy as

$$V_L^{acc}(X, Q_L) = \frac{\frac{1}{2}XQ_L - \frac{1}{2}\eta XQ_L^2 + \frac{1}{2}\delta_2(r-\mu)Q_L}{r-\mu} - \delta_1 Q_L \tag{A5.6}$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{XQ_L - \eta XQ_L^2 + \delta_2(r-\mu)Q_L}{2(r-\mu)} - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = Q_L\left[\frac{X - \eta XQ_L + \delta_2(r-\mu)}{2(r-\mu)} - \delta_1\right]$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{1}{2\eta}\left(1 - \frac{(2\delta_1 - \delta_2)(r-\mu)}{X}\right)\left[\frac{X - \eta X \frac{1}{2\eta}\left(1 - \frac{(2\delta_1 - \delta_2)(r-\mu)}{X}\right) + \delta_2(r-\mu) - 2\delta_1(r-\mu)}{2(r-\mu)}\right]$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{1}{2\eta X}\big(X - (2\delta_1 - \delta_2)(r-\mu)\big)\left[\frac{X - \frac{X}{2} + \frac{(2\delta_1 - \delta_2)(r-\mu)}{2} - (2\delta_1 - \delta_2)(r-\mu)}{2(r-\mu)}\right]$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{1}{2\eta X}\big(X - (2\delta_1 - \delta_2)(r-\mu)\big)\left[\frac{X - (2\delta_1 - \delta_2)(r-\mu)}{4(r-\mu)}\right]$$

$$\Leftrightarrow V_L^{acc}(X) = \big(X - (2\delta_1 - \delta_2)(r-\mu)\big)\left[\frac{X - (2\delta_1 - \delta_2)(r-\mu)}{8X\eta(r-\mu)}\right]$$

$$\Leftrightarrow V_L^{acc}(X) = \big(X - (2\delta_1 - \delta_2)(r-\mu)\big)\left[\frac{1}{8\eta(r-\mu)} - \frac{2\delta_1 - \delta_2}{8X\eta}\right]$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{X}{8\eta(r-\mu)} - \frac{2\delta_1 - \delta_2}{8\eta} - \frac{2\delta_1 - \delta_2}{8\eta} + \frac{(2\delta_1 - \delta_2)^2(r-\mu)}{8X\eta}$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{X^2 - X(2\delta_1 - \delta_2)(r-\mu) - X(2\delta_1 - \delta_2)(r-\mu) + [(2\delta_1 - \delta_2)(r-\mu)]^2}{8X\eta(r-\mu)}$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{X^2 - 2X(2\delta_1 - \delta_2)(r-\mu) + [(2\delta_1 - \delta_2)(r-\mu)]^2}{8X\eta(r-\mu)}$$

$$\Leftrightarrow V_L^{acc}(X) = \frac{\big(X - (2\delta_1 - \delta_2)(r-\mu)\big)^2}{8X\eta(r-\mu)}.$$

Since the leader only uses its accommodation strategy if the optimal quantity $Q_L^{acc}(X)$ leads to immediate investment of the follower, it follows that

$$X_F^*\big(Q_L^{acc}(X)\big) \leq X. \tag{A5.7}$$

It is possible to define X_1^{acc} as

$$X_1^{acc} = X_F^*\big(Q_L^{acc}(X_1^{acc})\big) \tag{A5.8}$$

and substituting the investment threshold from (A3.9) and the capacity level from equation (A5.5) into equation (A5.8) yields

$$\begin{aligned}
& X_1^{acc} = X_F^*\big(Q_L^{acc}(X_1^{acc})\big) && \text{(A5.9)}\\
\Leftrightarrow\ & X_1^{acc} = X_F^*\left(Q_L^{acc} = \frac{1}{2\eta}\left(1 - \frac{(2\delta_1 - \delta_2)(r-\mu)}{X_1^{acc}}\right)\right)\\
\Leftrightarrow\ & X_1^{acc} = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\eta\frac{1}{2\eta}\left(1-\frac{(2\delta_1-\delta_2)(r-\mu)}{X_1^{acc}}\right)}\\
\Leftrightarrow\ & X_1^{acc} = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{1-\frac{1}{2}+\frac{(2\delta_1-\delta_2)(r-\mu)}{2X_1^{acc}}}\\
\Leftrightarrow\ & X_1^{acc} = \frac{\beta+1}{\beta-1}\frac{\delta_2(r-\mu)}{\frac{X_1^{acc}+(2\delta_1-\delta_2)(r-\mu)}{2X_1^{acc}}}\\
\Leftrightarrow\ & X_1^{acc} = \frac{\beta+1}{\beta-1}\frac{2X_1^{acc}\delta_2(r-\mu)}{X_1^{acc}+(2\delta_1-\delta_2)(r-\mu)}\\
\Leftrightarrow\ & X_1^{acc}(\beta-1)\big(X_1^{acc}+(2\delta_1-\delta_2)(r-\mu)\big) = (\beta+1)2X_1^{acc}\delta_2(r-\mu)\\
\Leftrightarrow\ & \beta X_1^{acc}+\beta(2\delta_1-\delta_2)(r-\mu)-X_1^{acc}-(2\delta_1-\delta_2)(r-\mu) = 2\beta\delta_2(r-\mu)+2\delta_2(r-\mu)\\
\Leftrightarrow\ & X_1^{acc}(\beta-1) = [2\beta\delta_2+2\delta_2-\beta(2\delta_1-\delta_2)+2\delta_1-\delta_2](r-\mu)\\
\Leftrightarrow\ & X_1^{acc}(\beta-1) = (2\beta\delta_2+2\delta_2-2\beta\delta_1+\beta\delta_2+2\delta_1-\delta_2)(r-\mu)\\
\Leftrightarrow\ & X_1^{acc} = \frac{(2-2\beta)\delta_1+(1+3\beta)\delta_2}{\beta-1}(r-\mu).
\end{aligned}$$

For the accommodation strategy, the option value that the leader firm holds is given by

$$F_L^{acc}(X) = A_L^{acc} X^{\beta}. \tag{A5.10}$$

This implies that the VMC is as follows

$$F_L^{acc}(X) = V_L^{acc}(X) \tag{A5.11}$$

$$\Leftrightarrow A_L^{acc} X^{\beta} = \frac{\left(X - (2\delta_1 - \delta_2)(r-\mu)\right)^2}{8X\eta(r-\mu)}.$$

The SPC is then

$$\frac{\partial F_L^{acc}(X)}{\partial X} = \frac{\partial V_L^{acc}(X)}{\partial X} \tag{A5.12}$$

$$\Leftrightarrow \frac{\partial\left(A_L^{acc} X^{\beta}\right)}{\partial X} = \frac{\partial\left(\frac{\left(X - (2\delta_1 - \delta_2)(r-\mu)\right)^2}{8X\eta(r-\mu)}\right)}{\partial X}$$

$$\Leftrightarrow \beta A_L^{acc} X^{\beta-1} = \frac{\partial\left(\frac{X^2 - 2X(2\delta_1 - \delta_2)(r-\mu) + [(2\delta_1 - \delta_2)(r-\mu)]^2}{8X\eta(r-\mu)}\right)}{\partial X}$$

$$\Leftrightarrow \beta A_L^{acc} X^{\beta-1} = \frac{\partial\left(\frac{X}{8\eta(r-\mu)} - \frac{2(2\delta_1 - \delta_2)}{8\eta} + \frac{(2\delta_1 - \delta_2)^2(r-\mu)}{8X\eta}\right)}{\partial X}$$

$$\Leftrightarrow \beta A_L^{acc} X^{\beta-1} = \frac{1}{8\eta(r-\mu)} + \frac{(2\delta_1 - \delta_2)^2(r-\mu)}{8\eta}\left(-\frac{1}{X^2}\right)$$

$$\Leftrightarrow \beta A_L^{acc} X^{\beta-1} = \frac{X^2 - (2\delta_1 - \delta_2)^2(r-\mu)^2}{8X^2\eta(r-\mu)}.$$

To determine the optimal investment threshold for the entry accommodation strategy, we combine the results from equations (A5.11) and (A5.12), yielding

$$\beta A_L^{acc} X^{\beta-1} = \frac{X^2 - (2\delta_1 - \delta_2)^2(r-\mu)^2}{8X^2\eta(r-\mu)} \tag{A5.13}$$

$$\Leftrightarrow \frac{\beta}{X} A_L^{acc} X^{\beta} = \frac{X^2 - (2\delta_1 - \delta_2)^2(r-\mu)^2}{8X^2\eta(r-\mu)}$$

$$\Leftrightarrow A_L^{acc} X^{\beta} = \frac{X(X^2 - (2\delta_1 - \delta_2)^2(r-\mu)^2)}{8\beta X^2\eta(r-\mu)}$$

$$\Leftrightarrow \frac{\left(X - (2\delta_1 - \delta_2)(r-\mu)\right)^2}{8X\eta(r-\mu)} = \frac{X^2 - (2\delta_1 - \delta_2)^2(r-\mu)^2}{8\beta X\eta(r-\mu)}$$

$$\Leftrightarrow \frac{\left(X - (2\delta_1 - \delta_2)(r-\mu)\right)^2}{8X\eta(r-\mu)} - \frac{X^2 - (2\delta_1 - \delta_2)^2(r-\mu)^2}{8\beta X\eta(r-\mu)} = 0$$

$$\Leftrightarrow \frac{\beta\big(X-(2\delta_1-\delta_2)(r-\mu)\big)\big(X-(2\delta_1-\delta_2)(r-\mu)\big)-\big(X-(2\delta_1-\delta_2)(r-\mu)\big)\big(X+(2\delta_1-\delta_2)(r-\mu)\big)}{8\beta X\eta(r-\mu)}=0$$

$$\Leftrightarrow \frac{\big[\beta\big(X-(2\delta_1-\delta_2)(r-\mu)\big)-\big(X+(2\delta_1-\delta_2)(r-\mu)\big)\big]\cdot\big(X-(2\delta_1-\delta_2)(r-\mu)\big)}{8\beta X\eta(r-\mu)}=0$$

$$\Leftrightarrow \frac{\beta\big(X-(2\delta_1-\delta_2)(r-\mu)\big)-\big(X+(2\delta_1-\delta_2)(r-\mu)\big)}{8\beta X\eta(r-\mu)}=0 \;\vee\; X-(2\delta_1-\delta_2)(r-\mu)=0$$

$$\Leftrightarrow \beta\big(X-(2\delta_1-\delta_2)(r-\mu)\big)-\big(X+(2\delta_1-\delta_2)(r-\mu)\big)=0 \;\vee\; X=(2\delta_1-\delta_2)(r-\mu)$$

$$\Leftrightarrow \beta X-\beta(2\delta_1-\delta_2)(r-\mu)-X-(2\delta_1-\delta_2)(r-\mu)=0$$

$$\Leftrightarrow X(\beta-1)=(\beta+1)(2\delta_1-\delta_2)(r-\mu)$$

$$\Leftrightarrow X\equiv X_L^{acc}=\frac{\beta+1}{\beta-1}(2\delta_1-\delta_2)(r-\mu).$$

In the previous equation, the second solution cannot occur; otherwise, the option value that the firm holds during the accommodation strategy, as represented by the VMC (in equation (A5.11)), would be zero.

To determine the optimal capacity level for the entry accommodation strategy, we substitute $X\equiv X_L^{acc}$ into equation (A5.5), yielding

$$Q_L^{acc}(X_L^{acc})=\frac{1}{2\eta}\left(1-\frac{(2\delta_1-\delta_2)(r-\mu)}{X_L^{acc}}\right) \tag{A5.14}$$

$$\Leftrightarrow Q_L^{acc}(X_L^{acc})=\frac{1}{2\eta}\left(1-\frac{(2\delta_1-\delta_2)(r-\mu)}{\frac{\beta+1}{\beta-1}(2\delta_1-\delta_2)(r-\mu)}\right)$$

$$\Leftrightarrow Q_L^{acc}(X_L^{acc})=\frac{1}{2\eta}\left(1-\frac{\beta-1}{\beta+1}\right)$$

$$\Leftrightarrow Q_L^{acc}(X_L^{acc}) = \frac{1}{2\eta}\left(\frac{\beta + 1 - \beta + 1}{\beta + 1}\right)$$

$$\Leftrightarrow Q_L^{acc}(X_L^{acc}) = \frac{1}{2\eta}\frac{2}{\beta + 1}$$

$$\Leftrightarrow Q_L^{acc}(X_L^{acc}) = \frac{1}{(\beta + 1)\eta}.$$

A.6. Mathematical details of the impact of uncertainty on entry deterrence and accommodation strategies

Based on the literature, such as Dixit and Pindyck (1994), it is known that

$$\frac{\partial\beta}{\partial\sigma} < 0. \qquad \text{(A6.1)}$$

Then, we can conclude that X_1^{acc} increases with uncertainty. The proof is as follows: given that $\delta_1 = \delta_2 = \delta$, we can rewrite expression (A5.9) as

$$X_1^{acc} = \frac{(2 - 2\beta)\delta + (1 + 3\beta)\delta}{\beta - 1}(r - \mu) \qquad \text{(A6.2)}$$

$$\Leftrightarrow X_1^{acc} = \frac{2\delta - 2\beta\delta + \delta + 3\beta\delta}{\beta - 1}(r - \mu)$$

$$\Leftrightarrow X_1^{acc} = \frac{3\delta + \beta\delta}{\beta - 1}(r - \mu).$$

This implies that

$$\frac{\partial X_1^{acc}}{\partial\beta} \qquad \text{(A6.3)}$$

$$= \frac{\partial\left(\frac{3\delta + \beta\delta}{\beta - 1}(r - \mu)\right)}{\partial\beta}$$

$$= (r - \mu)\cdot\left[\frac{\frac{\partial(3\delta + \beta\delta)}{\partial\beta}(\beta - 1) - (3\delta + \beta\delta)\frac{\partial(\beta - 1)}{\partial\beta}}{(\beta - 1)^2}\right]$$

$$= (r - \mu)\cdot\left[\frac{\delta(\beta - 1) - 3\delta - \beta\delta}{(\beta - 1)^2}\right]$$

$$= (r - \mu)\cdot\left[\frac{\delta\beta - \delta - 3\delta - \beta\delta}{(\beta - 1)^2}\right]$$

$$= \frac{-4\delta(r - \mu)}{(\beta - 1)^2} < 0, since\ (\beta - 1)^2 > 0\ , \delta > 0, and$$

$$r > \mu \Leftrightarrow r - \mu > 0.$$

Therefore, if uncertainty (σ) increases, β decreases, following the rationale of (A6.1). As β decreases, X_1^{acc} rises, as shown by expression (A6.3).

Also, X_2^{det} increases when uncertainty rises. The proof is as follows: given that $\delta_1 = \delta_2 = \delta$, we can rewrite expression (A4.11) as

$$X_2^{det} = \frac{\beta+1}{\beta-1}(\beta(\delta-\delta)+\delta+\delta)(r-\mu) \tag{A6.4}$$

$$\Leftrightarrow X_2^{det} = \frac{\beta+1}{\beta-1}2\delta(r-\mu).$$

This implies that

$$\begin{aligned} &\frac{\partial X_2^{det}}{\partial \beta} \\ &= \frac{\partial\left(\frac{\beta+1}{\beta-1}2\delta(r-\mu)\right)}{\partial \beta} \\ &= 2\delta(r-\mu)\frac{\partial\left(\frac{\beta+1}{\beta-1}\right)}{\partial \beta} \\ &= 2\delta(r-\mu)\left[\frac{\frac{\partial(\beta+1)}{\partial\beta}(\beta-1)-(\beta+1)\frac{\partial(\beta-1)}{\partial\beta}}{(\beta-1)^2}\right] \\ &= 2\delta(r-\mu)\left[\frac{\beta-1-\beta-1}{(\beta-1)^2}\right] \\ &= \frac{-4\delta(r-\mu)}{(\beta-1)^2} < 0. \end{aligned} \tag{A6.5}$$

Therefore, if uncertainty (σ) increases, β decreases, following the rationale of (A6.1). As β decreases, X_2^{det} rises, as shown by expression (A6.5).

Adapting expression (A4.4) to the scenario in subchapter 5.2, concerning the computation of X_1^{det}, it must hold that

$$\psi\left(X_1^{det} \equiv X, \beta\right) = \frac{X}{r-\mu} - \delta - \left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\frac{\delta}{(\beta-1)} = 0. \tag{A6.6}$$

From the previous expression, it is possible to realise that $X \equiv X_1^{det}$ is dependent on β, so that $X_1^{det}(\beta)$. So, to see how X_1^{det} depends on β we must compute

$$\frac{\partial\psi\left(X_1^{det}(\beta),\beta\right) \equiv \psi\left(X_1^{det},\beta\right)}{\partial X_1^{det}} \cdot \frac{\partial X_1^{det}}{\partial\beta} + \frac{\psi\left(X_1^{det},\beta\right)}{\partial\beta} = 0 \tag{A6.7}$$

$$\Leftrightarrow \frac{\partial X_1^{det}}{\partial \beta} = -\frac{\dfrac{\psi\left(X_1^{det},\beta\right)}{\partial \beta}}{\dfrac{\partial \psi\left(X_1^{det}(\beta),\beta\right) \equiv \psi\left(X_1^{det},\beta\right)}{\partial X_1^{det}}}.$$

The rationale behind this expression is that, since from (A6.6) $X \equiv X_1^{det}$ is dependent on β, its partial derivative concerning β must be done in two steps (as they appear in the first line of (A6.7)). First, we apply the chain rule to differentiate partially $\psi\left(X_1^{det},\beta\right)$ concerning β since X_1^{det} is also dependent on β. Then, after the plus sign, keeping $X \equiv X_1^{det}$ constant, we differentiate partially $\psi\left(X_1^{det},\beta\right)$ with respect to β.

Also, from expressions (A4.5) and (A4.6), we know that $\frac{\partial\psi(X,\beta)}{\partial X}\Big|_{X=X_1^{det}} > 0$. Moreover, it is possible to represent $\frac{\partial\psi(X,\beta)}{\partial X}\Big|_{X=X_1^{det}}$ like

$$\frac{\partial\psi(X,\beta)}{\partial\beta}\Bigg|_{X=X_1^{det}} \tag{A6.8}$$

$$= \frac{\partial\left(\dfrac{X}{r-\mu} - \delta - \left(\dfrac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\dfrac{\delta}{\beta-1}\right)}{\partial\beta}$$

$$= -\frac{\partial\left(\left(\dfrac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\dfrac{\delta}{\beta-1}\right)}{\partial\beta}$$

$$= -\left[\frac{\partial\left(\dfrac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}}{\partial\beta}\cdot\frac{\delta}{\beta-1} + \left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\cdot\frac{\partial\left(\dfrac{\delta}{\beta-1}\right)}{\partial\beta}\right]$$

$$= -\left[\frac{\partial\left(e^{\beta\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)}\right)}{\partial\beta}\cdot\frac{\delta}{\beta-1}+\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\cdot\left(-\frac{\delta}{(\beta-1)^2}\right)\right]$$

$$= -\left[e^{\beta\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)}\frac{\partial\left(\beta\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)\right)}{\partial\beta}\cdot\frac{\delta}{\beta-1}+\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\cdot\left(-\frac{\delta}{(\beta-1)^2}\right)\right]$$

$$= -\left[e^{\beta\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)}\cdot\left(\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)+\frac{2\beta}{(\beta-1)(\beta+1)}\right)\cdot\frac{\delta}{\beta-1}+\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\cdot\left(-\frac{\delta}{(\beta-1)^2}\right)\right]$$

$$= -\left[\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\cdot\left(\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)+\frac{2\beta}{(\beta-1)(\beta+1)}\right)\cdot\frac{\delta}{\beta-1}+\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\cdot\left(-\frac{\delta}{(\beta-1)^2}\right)\right]$$

$$= -\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[\left(\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)+\frac{2\beta}{(\beta-1)(\beta+1)}\right)\cdot\frac{\delta}{\beta-1}-\frac{\delta}{(\beta-1)^2}\right]$$

$$= -\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)\cdot\frac{\delta}{\beta-1}+\frac{2\beta}{(\beta-1)(\beta+1)}\cdot\frac{\delta}{\beta-1}-\frac{\delta}{(\beta-1)^2}\right]$$

$$= -\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)\cdot\frac{\delta(\beta-1)(\beta+1)}{(\beta-1)(\beta-1)(\beta+1)}+\frac{2\beta\delta}{(\beta-1)(\beta-1)(\beta+1)}-\frac{\delta(\beta+1)}{(\beta-1)(\beta-1)(\beta+1)}\right]$$

$$= -\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[\frac{\delta}{\beta^2-1}\left[\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)\cdot\frac{(\beta-1)(\beta+1)}{(\beta-1)}+\frac{2\beta}{\beta-1}-\frac{\beta+1}{\beta-1}\right]\right]$$

$$= -\frac{\delta}{\beta^2-1}\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[(\beta+1)\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)+\frac{2\beta-\beta-1}{\beta-1}\right]$$

$$= -\frac{\delta}{\beta^2 - 1}\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[(\beta+1)\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right) + \frac{\beta-1}{\beta-1}\right]$$

$$= -\frac{\delta}{\beta^2 - 1}\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\left[1 + (\beta+1)\ln\left(\frac{X(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)\right].$$

Regarding expression (A6.8), we use the natural logarithm (ln) instead of the common logarithm (log), as seen in Huisman and Kort (2015).

To ensure that $\left.\frac{\partial\psi(X,\beta)}{\partial X}\right|_{X=X_1^{det}} > 0$ we need $1 + (\beta+1)\ln\left(\frac{X_1^{det}(\beta-1)}{(\beta+1)\delta(r-\mu)}\right) < 0$, as derived from expression (A6.8).

Defining $\overline{X} = \frac{\beta}{\beta-1}\delta(r-\mu)$, then $X_1^{det} < \overline{X}$, as it holds that

$$\overline{X} < \frac{\beta+1}{\beta-1}\delta(r-\mu), \tag{A6.9}$$

$$\psi\left(X_1^{det}, \beta\right) = 0, \tag{A6.10}$$

$$\frac{\partial\psi(X)}{\partial X} > 0, when\ X \in \left]0, \frac{\beta+1}{\beta-1}\delta(r-\mu)\right[, \tag{A6.11}$$

and

$$\psi\left(\overline{X}, \beta\right) = \frac{\overline{X}}{r-\mu} - \delta - \left(\frac{\overline{X}(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\frac{\delta}{\beta-1} \tag{A6.12}$$

$$\Leftrightarrow \psi\left(\overline{X}, \beta\right) = \frac{\frac{\beta}{\beta-1}\delta(r-\mu)}{r-\mu} - \delta - \left(\frac{\frac{\beta}{\beta-1}\delta(r-\mu)(\beta-1)}{(\beta+1)\delta(r-\mu)}\right)^{\beta}\frac{\delta}{\beta-1}$$

$$\Leftrightarrow \psi\left(\overline{X}, \beta\right) = \frac{\beta\delta}{\beta-1} - \delta - \left(\frac{\beta}{\beta+1}\right)^{\beta}\frac{\delta}{\beta-1}$$

$$\Leftrightarrow \psi\left(\overline{X}, \beta\right) = \frac{\beta\delta - \beta\delta + \delta}{\beta-1} - \left(\frac{\beta}{\beta+1}\right)^{\beta}\frac{\delta}{\beta-1}$$

$$\Leftrightarrow \psi\left(\overline{X}, \beta\right) = \frac{\delta}{\beta-1} - \left(\frac{\beta}{\beta+1}\right)^{\beta}\frac{\delta}{\beta-1}$$

$$\Leftrightarrow \psi\left(\overline{X}, \beta\right) = \frac{\delta}{\beta-1}\left(1 - \left(\frac{\beta}{\beta+1}\right)^{\beta}\right) > 0, since\ \delta > 0,$$

$$\beta > 1 \Leftrightarrow \beta - 1 > 0.$$

$$and\ \frac{\delta}{\beta-1} > 0.$$

$$Therefore, \frac{\beta}{\beta+1} < 1$$

$$\Leftrightarrow \left(\frac{\beta}{\beta+1}\right)^{\beta} < 1$$

$$\Leftrightarrow 1 - \left(\frac{\beta}{\beta+1}\right)^{\beta} > 0.$$

Then, $1 + (\beta + 1)\ln\left(\frac{X_1^{det}(\beta-1)}{(\beta+1)\delta(r-\mu)}\right) < 0$ holds if

$$1 + (\beta + 1)\ln\left(\frac{\overline{X}(\beta-1)}{(\beta+1)\delta(r-\mu)}\right) < 0 \tag{A6.13}$$

$$\Leftrightarrow 1 + (\beta + 1)\ln\left(\frac{\frac{\beta}{\beta-1}\delta(r-\mu)(\beta-1)}{(\beta+1)\delta(r-\mu)}\right) < 0$$

$$\Leftrightarrow 1 + (\beta + 1)\ln\left(\frac{\beta}{\beta+1}\right) < 0.$$

Assuming that the function $\gamma(\beta)$ is

$$\gamma(\beta) = 1 + (\beta + 1)\ln\left(\frac{\beta}{\beta+1}\right), \tag{A6.14}$$

we can conclude that

$$\gamma(1) = 1 + (1 + 1)\ln\left(\frac{1}{1+1}\right) \tag{A6.15}$$

$$\Leftrightarrow \gamma(1) = 1 + 2\ln\left(\frac{1}{2}\right) < 0 \; because \; 1 < \left|2\ln\left(\frac{1}{2}\right)\right|,$$

$$\lim_{\beta\to+\infty} \gamma(\beta) \tag{A6.16}$$

$$= \lim_{\beta\to+\infty}\left(1 + (\beta+1)\ln\left(\frac{\beta}{\beta+1}\right)\right)$$

$$= \lim_{\beta\to+\infty}(1) + \lim_{\beta\to+\infty}\left(\beta\ln\left(\frac{\beta}{\beta+1}\right)\right) + \lim_{\beta\to+\infty}\left(\ln\left(\frac{\beta}{\beta+1}\right)\right)$$

$$= 1 + \lim_{\beta\to+\infty}\left(\frac{\ln\left(\frac{\beta}{\beta+1}\right)}{\frac{1}{\beta}}\right) + \lim_{\beta\to+\infty}\left(\ln\left(\frac{1}{1+\frac{1}{\beta}}\right)\right)$$

$$= 1 + \lim_{\beta\to+\infty}\left(\frac{\frac{\frac{\beta+1-\beta}{(\beta+1)^2}}{\frac{\beta}{\beta+1}}}{-\frac{1}{\beta^2}}\right) + \ln\left(\lim_{\beta\to+\infty}\left(\frac{1}{1+\frac{1}{\beta}}\right)\right)$$

$$= 1 + \lim_{\beta \to +\infty} \left(\frac{\frac{1}{\beta(\beta + 1)}}{-\frac{1}{\beta^2}} \right) + \ln \left(\frac{1}{1 + \frac{1}{+\infty}} \right)$$

$$= 1 + \lim_{\beta \to +\infty} \left(-\frac{\beta}{\beta + 1} \right) + \ln \left(\frac{1}{1 + 0} \right)$$

$$= 1 + \lim_{\beta \to +\infty} \left(-\frac{1}{1 + \frac{1}{\beta}} \right) + \ln(1)$$

$$= 1 - \frac{1}{1 + \frac{1}{+\infty}} + 0$$

$$= 1 - \frac{1}{1 + 0}$$

$$= 1 - 1$$

$$= 0,$$

and

$$\frac{\partial \gamma(\beta)}{\partial \beta} \tag{A6.17}$$

$$= \frac{\partial \left(1 + (\beta + 1) \ln \left(\frac{\beta}{\beta + 1} \right) \right)}{\partial \beta}$$

$$= \frac{\partial (\beta + 1)}{\partial \beta} \cdot \ln \left(\frac{\beta}{\beta + 1} \right) + (\beta + 1) \cdot \frac{\partial \left(\ln \left(\frac{\beta}{\beta + 1} \right) \right)}{\partial \beta}$$

$$= \ln \left(\frac{\beta}{\beta + 1} \right) + (\beta + 1) \cdot \left(\frac{\frac{1}{(\beta + 1)^2}}{\frac{\beta}{\beta + 1}} \right)$$

$$= \ln \left(\frac{\beta}{\beta + 1} \right) + (\beta + 1) \cdot \frac{1}{\beta(\beta + 1)}$$

$$= \frac{1}{\beta} + \ln \left(\frac{\beta}{\beta + 1} \right) > 0, because\ \beta > 1 \Leftrightarrow \frac{1}{\beta} > 0.$$

$$However, \ln \left(\frac{\beta}{\beta + 1} \right) < 0,$$

$$but\, \frac{1}{\beta} > \left| \ln \left(\frac{\beta}{\beta + 1} \right) \right|.$$

The previous expression holds as $\beta > 1$ and

$$\left.\frac{\partial\gamma(\beta)}{\partial\beta}\right|_{\beta=1} \tag{A6.18}$$

$$= \frac{1}{1} + \ln\left(\frac{1}{1+1}\right)$$

$$= 1 + \ln\left(\frac{1}{2}\right) > 0 \; because \; 1 > \left|\ln\left(\frac{1}{2}\right)\right|,$$

$$\lim_{\beta\to+\infty}\left(\frac{\partial\gamma(\beta)}{\partial\beta}\right) \tag{A6.19}$$

$$= \lim_{\beta\to+\infty}\left(\frac{1}{\beta} + \ln\left(\frac{\beta}{\beta+1}\right)\right)$$

$$= \lim_{\beta\to+\infty}\left(\frac{1}{\beta}\right) + \lim_{\beta\to+\infty}\left(\ln\left(\frac{\beta}{\beta+1}\right)\right)$$

$$= \frac{1}{+\infty} + \ln\left(\lim_{\beta\to+\infty}\left(\frac{\beta}{\beta+1}\right)\right)$$

$$= 0 + \ln\left(\lim_{\beta\to+\infty}\left(\frac{1}{1+\frac{1}{\beta}}\right)\right)$$

$$= \ln\left(\frac{1}{1+\frac{1}{+\infty}}\right)$$

$$= \ln\left(\frac{1}{1+0}\right)$$

$$= \ln(1)$$

$$= 0,$$

$$\frac{\partial^2\gamma(\beta)}{\partial\beta^2} \tag{A6.20}$$

$$= \frac{\partial\left(\frac{1}{\beta} + \ln\left(\frac{\beta}{\beta+1}\right)\right)}{\partial\beta}$$

$$= \frac{\partial\left(\frac{1}{\beta}\right)}{\partial\beta} + \frac{\partial\left(\ln\left(\frac{\beta}{\beta+1}\right)\right)}{\partial\beta}$$

$$= -\frac{1}{\beta^2} + \frac{1}{\beta(\beta+1)}$$

$$= -\frac{1}{\beta^2} + \frac{1}{\beta^2+\beta}$$

$$= \frac{-\beta^2 - \beta + \beta^2}{\beta^4 + \beta^3}$$

$$= \frac{-\beta}{\beta^4 + \beta^3}$$

$$= -\frac{1}{\beta^3 + \beta^2} < 0, since\ \beta > 0.$$

From expression (A6.20), it is important to note that expression (A70) by Huisman and Kort (2015) contains an error. The correct denominator should be $\beta^3 + \beta^2$, not $\beta^2 + \beta^2$ as they presented.

It can be concluded that $\frac{\partial \psi(X,\beta)}{\partial \beta}\Big|_{X=X_1^{det}} > 0$ and therefore, $\frac{\partial X_1^{det}}{\partial \beta} < 0$. The region where the leader can choose between the entry deterrence and accommodation strategies is equal to

$$X_2^{det} - X_1^{acc} \tag{A6.21}$$

$$= \frac{\beta + 1}{\beta - 1} 2\delta(r - \mu) + \frac{3\delta + \beta\delta}{\beta - 1}(r - \mu)$$

$$= \frac{2(\beta + 1)}{\beta - 1}\delta(r - \mu) - \frac{\beta + 3}{\beta - 1}\delta(r - \mu)$$

$$= \delta(r - \mu).$$

From this expression, it can be concluded that the interval $X \in]X_1^{acc}, X_2^{det}[$ is unaffected by uncertainty (σ) and decreases with the drift rate (μ).

A.7. Mathematical details of the monopolist and social planner with two investment opportunities

At the time of the second investment, the total market capacity is given by $Q = Q_1 + Q_2$. Therefore, the profit function of the monopolist is

$$\pi_2(t) = P(t) \cdot Q(t) \tag{A7.1}$$

$$\Leftrightarrow \pi_2(t) = X(t)\big(1 - \eta Q(t)\big) \cdot Q(t)$$

$$\Leftrightarrow \pi_2(t) = X(t)(Q_1 + Q_2)[1 - \eta(Q_1 + Q_2)].$$

The expected profit of the monopolist firm is then

$$E[\pi_2(t)] = E\big[X(t)(Q_1 + Q_2)[1 - \eta(Q_1 + Q_2)]\big] \tag{A7.2}$$

$$\Leftrightarrow E[\pi_2(t)] = (Q_1 + Q_2)[1 - \eta(Q_1 + Q_2)] \cdot E[X_t]$$

$$\Leftrightarrow E[\pi_2(t)] = (Q_1 + Q_2)[1 - \eta(Q_1 + Q_2)] \cdot X \cdot e^{\mu \cdot t}.$$

Therefore, the value function of the monopolist firm at the time of the second investment is

$$V_2(X, Q_1, Q_2) = E\left[\int_{t=0}^{\infty} \pi_2(t) exp(-rt)\, dt - \delta Q_2\right] \tag{A7.3}$$

$$\Leftrightarrow V_2(X, Q_1, Q_2) = \int_{t=0}^{\infty} (Q_1 + Q_2)[1 - \eta(Q_1 + Q_2)] \cdot X \cdot \mathrm{e}^{\mu \cdot t} \cdot \mathrm{e}^{-r \cdot t}\, dt - \delta Q_2$$

$$\Leftrightarrow V_2(X, Q_1, Q_2) = X(Q_1 + Q_2)[1 - \eta(Q_1 + Q_2)] \int_{t=0}^{\infty} \mathrm{e}^{(\mu - r)t}\, dt - \delta Q_2$$

$$\Leftrightarrow V_2(X, Q_1, Q_2) = \frac{X(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu} - \delta Q_2.$$

Prior to undertaking this investment, the option value is equivalent to the sum of the expected profit from the first investment and the option value of the second investment. This relationship can be expressed as follows

$$F_2(X, Q_1) = \frac{XQ_1(1 - \eta Q_1)}{r - \mu} + A_2 X^{\beta}. \tag{A7.4}$$

From the VMC, it holds that

$$F_2(X_2 \equiv X, Q_1) = V_2(X_2 \equiv X, Q_1, Q_2) \tag{A7.5}$$

$$\Leftrightarrow \frac{XQ_1(1 - \eta Q_1)}{r - \mu} + A_2 X^{\beta} = \frac{X(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu} - \delta Q_2.$$

According to the SPC, it holds that

$$\frac{\partial F_2(X_2 \equiv X, Q_1)}{\partial X} = \frac{\partial V_2(X_2 \equiv X, Q_1, Q_2)}{\partial X} \tag{A7.6}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_1(1 - \eta Q_1)}{r - \mu} + A_2 X^{\beta}\right)}{\partial X} = \frac{\partial\left(\frac{X(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu} - \delta Q_2\right)}{\partial X}$$

$$\Leftrightarrow \frac{Q_1(1 - \eta Q_1)}{r - \mu} + \beta A_2 X^{\beta - 1} = \frac{(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu}.$$

Combining the results from the SPC and VMC, we find that the investment threshold for the second investment is defined by

$$\frac{Q_1(1 - \eta Q_1)}{r - \mu} + \beta A_2 X^{\beta - 1} = \frac{(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu} \tag{A7.7}$$

$$\Leftrightarrow \frac{Q_1(1 - \eta Q_1)}{r - \mu} + \frac{\beta}{X} A_2 X^{\beta} = \frac{(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu}$$

$$\Leftrightarrow \frac{Q_1(1-\eta Q_1)}{r-\mu} + \frac{\beta}{X}\left[\frac{X(Q_1+Q_2)\big(1-\eta(Q_1+Q_2)\big)}{r-\mu} - \delta Q_2 - \frac{XQ_1(1-\eta Q_1)}{r-\mu}\right] = \frac{(Q_1+Q_2)\big(1-\eta(Q_1+Q_2)\big)}{r-\mu}$$

$$\Leftrightarrow \frac{Q_1(1-\eta Q_1)}{r-\mu} + \frac{\beta(Q_1+Q_2)\big(1-\eta(Q_1+Q_2)\big)}{r-\mu} - \frac{\beta\delta Q_2}{X} - \frac{\beta Q_1(1-\eta Q_1)}{r-\mu} = \frac{(Q_1+Q_2)\big(1-\eta(Q_1+Q_2)\big)}{r-\mu}$$

$$\Leftrightarrow \frac{XQ_1 - X\eta Q_1^2}{X(r-\mu)} + \frac{X\beta Q_1 - X\beta\eta Q_1^2 - X\beta\eta Q_1 Q_2 + X\beta Q_2 - X\beta\eta Q_1 Q_2 - X\beta\eta Q_2^2}{X(r-\mu)} - \frac{\beta\delta Q_2(r-\mu)}{X(r-\mu)} - \frac{X\beta Q_1 - X\beta\eta Q_1^2}{X(r-\mu)}$$

$$= \frac{XQ_1 - X\eta Q_1^2 - X\eta Q_1 Q_2 + XQ_2 - X\eta Q_1 Q_2 - X\eta Q_2^2}{X(r-\mu)}$$

$$\Leftrightarrow XQ_1 - X\eta Q_1^2 + X\beta Q_1 - X\beta\eta Q_1^2 - X\beta\eta Q_1 Q_2 + X\beta Q_2 - X\beta\eta Q_1 Q_2 - X\beta\eta Q_2^2 - \beta\delta Q_2(r-\mu) - X\beta Q_1 + X\beta\eta Q_1^2 = XQ_1 - X\eta Q_1^2$$

$$-X\eta Q_1 Q_2 + XQ_2 - X\eta Q_1 Q_2 - X\eta Q_2^2$$

$$\Leftrightarrow X\beta Q_2 - X\beta\eta Q_2^2 - XQ_2 + X\eta Q_2^2 - 2X\beta\eta Q_1 Q_2 + 2X\eta Q_1 Q_2 = \beta\delta Q_2(r-\mu)$$

$$\Leftrightarrow X[\beta Q_2 - \beta\eta Q_2^2 - Q_2 + \eta Q_2^2 - 2\beta\eta Q_1 Q_2 + 2\eta Q_1 Q_2] = \beta\delta Q_2(r-\mu)$$

$$\Leftrightarrow X\big[\eta(-\beta Q_2^2 + Q_2^2) + Q_2(\beta-1) - \eta\big(2Q_1Q_2(\beta-1)\big)\big] = \beta\delta Q_2(r-\mu)$$

$$\Leftrightarrow X[-\eta Q_2^2(\beta-1) + Q_2(\beta-1) - \eta(\beta-1)2Q_1Q_2] = \beta\delta Q_2(r-\mu)$$

$$\Leftrightarrow X[-\eta Q_2(\beta-1) + \beta - 1 - \eta(\beta-1)2Q_1] = \beta\delta(r-\mu)$$

$$\Leftrightarrow X[(\beta-1)(-\eta Q_2 + 1 - 2\eta Q_1)] = \beta\delta(r-\mu)$$

$$\Leftrightarrow X\big[(\beta-1)\big(1-\eta(2Q_1+Q_2)\big)\big] = \beta\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X_2^*(Q_1,Q_2) = \frac{\beta}{\beta-1}\frac{(r-\mu)\delta}{1-\eta(2Q_1+Q_2)}.$$

From expression (A7.5)

$$\frac{XQ_1(1-\eta Q_1)}{r-\mu} + A_2X^\beta = \frac{X(Q_1+Q_2)\big(1-\eta(Q_1+Q_2)\big)}{r-\mu} - \delta Q_2 \quad \text{(A7.8)}$$

$$\Leftrightarrow A_2X^{\beta} = \frac{X(Q_1+Q_2)\big(1-\eta(Q_1+Q_2)\big)}{r-\mu} - \delta Q_2 - \frac{XQ_1(1-\eta Q_1)}{r-\mu}$$

$$\Leftrightarrow A_2X^{\beta} = \frac{XQ_1 - X\eta Q_1^2 - X\eta Q_1Q_2 + XQ_2 - X\eta Q_1Q_2 - X\eta Q_2^2 - XQ_1 + X\eta Q_1^2}{r-\mu} - \delta Q_2$$

$$\Leftrightarrow A_2X^{\beta} = \frac{-2X\eta Q_1Q_2 + XQ_2 - X\eta Q_2^2}{r-\mu} - \delta Q_2$$

$$\Leftrightarrow A_2X^{\beta} = \frac{X(-2\eta Q_1Q_2 + Q_2 - \eta Q_2^2)}{r-\mu} - \delta Q_2$$

$$\Leftrightarrow A_2X^{\beta} = \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)\delta}{1-\eta(2Q_1+Q_2)}(-2\eta Q_1Q_2 + Q_2 - \eta Q_2^2)}{r-\mu} - \delta Q_2$$

$$\Leftrightarrow A_2X^{\beta} = \frac{\beta\delta}{\beta-1}\frac{Q_2\big(1-\eta(2Q_1+Q_2)\big)}{1-\eta(2Q_1+Q_2)} - \delta Q_2$$

$$\Leftrightarrow A_2X^{\beta} = \frac{\beta\delta Q_2}{\beta-1} - \delta Q_2$$

$$\Leftrightarrow A_2X^{\beta} = \frac{\beta\delta Q_2 - \beta\delta Q_2 + \delta Q_2}{\beta-1}$$

$$\Leftrightarrow A_2X^{\beta} = \frac{\delta Q_2}{\beta-1}$$

$$\Leftrightarrow A_2 = X_2^{*-\beta}\frac{\delta Q_2}{\beta-1}\ (substituting\ X \equiv X_2^*).$$

The optimal investment capacity level for this investment is achieved by maximising the value of the firm in this second investment, yielding

$$\frac{\partial V_2(X_2 \equiv X, Q_1, Q_2)}{\partial Q_2} = 0 \tag{A7.9}$$

$$\Leftrightarrow \frac{\partial \left(\frac{X(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu} - \delta Q_2 \right)}{\partial Q_2} = 0$$

$$\Leftrightarrow \frac{\partial \left(\frac{X(Q_1 + Q_2)\big(1 - \eta(Q_1 + Q_2)\big)}{r - \mu} \right)}{\partial Q_2} - \frac{\partial(\delta Q_2)}{\partial Q_2} = 0$$

$$\Leftrightarrow \frac{\partial \left(\frac{XQ_1 - X\eta Q_1^2 - X\eta Q_1 Q_2 + XQ_2 - X\eta Q_1 Q_2 - X\eta Q_2^2}{r - \mu} \right)}{\partial Q_2} - \delta = 0$$

$$\Leftrightarrow \frac{-2X\eta Q_1 + X - 2X\eta Q_2}{r - \mu} - \delta = 0$$

$$\Leftrightarrow \frac{X\big(1 - 2\eta(Q_1 + Q_2)\big)}{r - \mu} - \delta = 0$$

$$\Leftrightarrow X\big(1 - 2\eta(Q_1 + Q_2)\big) = (r - \mu)\delta$$

$$\Leftrightarrow 1 - 2\eta(Q_1 + Q_2) = \frac{(r - \mu)\delta}{X}$$

$$\Leftrightarrow 2\eta(Q_1 + Q_2) = 1 - \frac{(r - \mu)\delta}{X}$$

$$\Leftrightarrow 2\eta Q_2 = 1 - \frac{(r - \mu)\delta}{X} - 2\eta Q_1$$

$$\Leftrightarrow Q_2 \equiv Q_2^*(X) = \frac{1}{2\eta}\left(1 - \frac{(r - \mu)\delta}{X} - 2\eta Q_1\right).$$

By substituting the previous result into expression (A7.7), we find that the investment threshold for the second investment, depending only on Q_1, is equal to

$$X_2^* = \frac{\beta}{\beta - 1}\frac{(r - \mu)\delta}{1 - \eta(2Q_1 + Q_2)} \tag{A7.10}$$

$$\Leftrightarrow X = \frac{\beta}{\beta - 1}\frac{(r - \mu)\delta}{1 - 2\eta Q_1 - \eta Q_2} \ (since\ X_2^* \equiv X)$$

$$\Leftrightarrow X = \frac{\beta}{\beta - 1}\frac{(r - \mu)\delta}{1 - 2\eta Q_1 - \frac{1}{2}\left(1 - \frac{(r - \mu)\delta}{X} - 2\eta Q_1\right)}$$

$$\Leftrightarrow X = \frac{\beta}{\beta - 1}\frac{(r - \mu)\delta}{1 - 2\eta Q_1 - \frac{1}{2} + \frac{(r - \mu)\delta}{2X} + \eta Q_1}$$

$$\Leftrightarrow X = \frac{\beta}{\beta - 1} \frac{(r-\mu)\delta}{\frac{1}{2} - \eta Q_1 + \frac{(r-\mu)\delta}{2X}}$$

$$\Leftrightarrow X = \frac{\beta}{\beta - 1} \frac{2X(r-\mu)\delta}{X - 2X\eta Q_1 + (r-\mu)\delta}$$

$$\Leftrightarrow X(\beta - 1)(X - 2X\eta Q_1 + (r-\mu)\delta) = 2\beta X(r-\mu)\delta$$

$$\Leftrightarrow X^2\beta - 2X^2\beta\eta Q_1 + \beta X(r-\mu)\delta - X^2 + 2X^2\eta Q_1 - (r-\mu)\delta X = 2\beta X(r-\mu)\delta$$

$$\Leftrightarrow X^2\beta - 2X^2\beta\eta Q_1 - X^2 + 2X^2\eta Q_1 - (r-\mu)\delta X = \beta X(r-\mu)\delta$$

$$\Leftrightarrow X\beta - 2X\beta\eta Q_1 - X + 2X\eta Q_1 - (r-\mu)\delta = \beta(r-\mu)\delta$$

$$\Leftrightarrow X(\beta - 2\beta\eta Q_1 - 1 + 2\eta Q_1) = (\beta + 1)\delta(r-\mu)$$

$$\Leftrightarrow X(\beta - 1)(1 - 2\eta Q_1) = (\beta + 1)\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X_2^*(Q_1) = \frac{(\beta + 1)\delta(r-\mu)}{(\beta - 1)(1 - 2\eta Q_1)}.$$

By substituting the previous result into expression (A7.9), we find that the capacity level for the second investment, depending on Q_1, is

$$Q_2^*(X) = \frac{1}{2\eta}\left(1 - \frac{(r-\mu)\delta}{X} - 2\eta Q_1\right) \tag{A7.11}$$

$$\Leftrightarrow Q_2 = \frac{1}{2\eta}\left(1 - \frac{(r-\mu)\delta}{\frac{(\beta+1)\delta(r-\mu)}{(\beta-1)(1-2\eta Q_1)}} - 2\eta Q_1\right) \; (since\ Q_2^*(X) \equiv Q_2)$$

$$\Leftrightarrow Q_2 = \frac{1}{2\eta}\left(1 - \frac{(\beta - 1)(1 - 2\eta Q_1)}{\beta + 1} - 2\eta Q_1\right)$$

$$\Leftrightarrow Q_2 = \frac{1}{2\eta} - \frac{(\beta - 1)(1 - 2\eta Q_1)}{2\eta(\beta + 1)} - Q_1$$

$$\Leftrightarrow Q_2 = \frac{\beta + 1 - \beta + 2\eta\beta Q_1 + 1 - 2\eta Q_1 - 2\eta\beta Q_1 - 2\eta Q_1}{2\eta(\beta + 1)}$$

$$\Leftrightarrow Q_2 = \frac{2 - 4\eta Q_1}{2\eta(\beta + 1)}$$

$$\Leftrightarrow Q_2 \equiv Q_2^*(Q_1) = \frac{1 - 2\eta Q_1}{(\beta + 1)\eta}.$$

Regarding the first investment, the total market capacity is given by $Q = Q_1$. Therefore, the profit function of the monopolist is

$$\pi_1(t) = P(t) \cdot Q(t) \tag{A7.12}$$

$$\Leftrightarrow \pi_1(t) = X(t)\big(1 - \eta Q(t)\big) \cdot Q(t)$$

$$\Leftrightarrow \pi_1(t) = X(t)Q_1(1 - \eta Q_1).$$

The expected profit of the monopolist firm is then

$$E[\pi_1(t)] = E[X(t)Q_1(1 - \eta Q_1)] \tag{A7.13}$$

$$\Leftrightarrow E[\pi_1(t)] = Q_1(1 - \eta Q_1) \cdot E[X_t]$$

$$\Leftrightarrow E[\pi_1(t)] = Q_1(1 - \eta Q_1) \cdot X \cdot e^{\mu \cdot t}.$$

Consequently, the value function of the monopolist at the time of the first investment is

$$V_1(X, Q_1) = E\left[\int_{t=0}^{\infty} \pi_1(t) exp(-rt)\, dt - \delta Q_1\right] + A_2 X^\beta \tag{A7.14}$$

$$\Leftrightarrow V_1(X, Q_1) = \int_{t=0}^{\infty} Q_1(1 - \eta Q_1) \cdot X \cdot e^{\mu \cdot t} \cdot \mathrm{e}^{-r \cdot t}\, dt - \delta Q_1 + A_2 X^\beta$$

$$\Leftrightarrow V_1(X, Q_1) = X Q_1(1 - \eta Q_1) \int_{t=0}^{\infty} \mathrm{e}^{(\mu - r)t}\, dt - \delta Q_1 + A_2 X^\beta$$

$$\Leftrightarrow V_1(X, Q_1) = \frac{X Q_1(1 - \eta Q_1)}{r - \mu} - \delta Q_1 + A_2 X^\beta.$$

Here, $A_2 X^\beta$ represents the option value of the second investment.

Before the firm invests, the option value equals

$$F_1(X) = A_1 X^\beta. \tag{A7.15}$$

From the VMC, it holds that

$$F_1(X_1 \equiv X) = V_1(X_1 \equiv X, Q_1) \tag{A7.16}$$

$$\Leftrightarrow A_1 X^\beta = \frac{X Q_1(1 - \eta Q_1)}{r - \mu} - \delta Q_1 + A_2 X^\beta.$$

According to the SPC, it holds that

$$\frac{\partial F_1(X_1 \equiv X)}{\partial X} = \frac{\partial V_1(X_1 \equiv X, Q_1)}{\partial X} \tag{A7.17}$$

$$\Leftrightarrow \frac{\partial\left(A_1 X^\beta\right)}{\partial X} = \frac{\partial\left(\frac{XQ_1(1-\eta Q_1)}{r-\mu} - \delta Q_1 + A_2 X^\beta\right)}{\partial X}$$

$$\Leftrightarrow \beta A_1 X^{\beta-1} = \frac{Q_1(1-\eta Q_1)}{r-\mu} + \beta A_2 X^{\beta-1}.$$

Combining the results from the SPC and VMC, for this first investment, reveals that the investment threshold is equal to

$$\beta A_1 X^{\beta-1} = \frac{Q_1(1-\eta Q_1)}{r-\mu} + \beta A_2 X^{\beta-1} \tag{A7.18}$$

$$\Leftrightarrow \frac{\beta}{X} A_1 X^\beta = \frac{Q_1(1-\eta Q_1)}{r-\mu} + \frac{\beta}{X} A_2 X^\beta$$

$$\Leftrightarrow \frac{\beta}{X}\left(\frac{XQ_1(1-\eta Q_1)}{r-\mu} - \delta Q_1 + A_2 X^\beta\right) = \frac{Q_1(1-\eta Q_1)}{r-\mu} + \frac{\beta}{X} A_2 X^\beta$$

$$\Leftrightarrow \frac{\beta Q_1(1-\eta Q_1)}{r-\mu} - \frac{\beta\delta Q_1}{X} + \frac{\beta}{X} A_2 X^\beta = \frac{Q_1(1-\eta Q_1)}{r-\mu} + \frac{\beta}{X} A_2 X^\beta$$

$$\Leftrightarrow \frac{X\beta Q_1(1-\eta Q_1)}{X(r-\mu)} - \frac{\beta\delta Q_1(r-\mu)}{X(r-\mu)} = \frac{XQ_1(1-\eta Q_1)}{X(r-\mu)}$$

$$\Leftrightarrow X\beta Q_1 - X\beta\eta Q_1^2 - \beta\delta Q_1(r-\mu) = XQ_1 - X\eta Q_1^2$$

$$\Leftrightarrow X\beta - X\beta\eta Q_1 - X + X\eta Q_1 = \beta\delta(r-\mu)$$

$$\Leftrightarrow X(\beta - \beta\eta Q_1 - 1 + \eta Q_1) = \beta\delta(r-\mu)$$

$$\Leftrightarrow X(\beta-1)(1-\eta Q_1) = \beta\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X_1^*(Q_1) = \frac{\beta\delta(r-\mu)}{(\beta-1)(1-\eta Q_1)}.$$

Rewriting expression (A7.17), we obtain

$$\beta A_1 X^{\beta-1} = \frac{Q_1(1-\eta Q_1)}{r-\mu} + \beta A_2 X^{\beta-1} \tag{A7.19}$$

$$\Leftrightarrow \beta A_1 X^{\beta-1} - \beta A_2 X^{\beta-1} = \frac{Q_1(1-\eta Q_1)}{r-\mu}$$

$$\Leftrightarrow \beta X^{\beta-1}(A_1 - A_2) = \frac{Q_1(1-\eta Q_1)}{r-\mu}$$

$$\Leftrightarrow A_1 - A_2 = \frac{Q_1(1-\eta Q_1)}{\beta X^{\beta-1}(r-\mu)}$$

$$\Leftrightarrow A_1 = A_2 + \frac{X^{1-\beta}}{\beta}\frac{Q_1(1-\eta Q_1)}{r-\mu}$$

$$\Leftrightarrow A_1 = A_2 + \frac{(X_1^*)^{1-\beta}}{\beta}\frac{Q_1(1-\eta Q_1)}{r-\mu}\ (considering\ X \equiv X_1^*).$$

The optimal investment capacity level for this investment is determined by maximising the firm's value function at the first investment stage, resulting in

$$\frac{\partial V_1(X_1 \equiv X, Q_1)}{\partial Q_1} = 0 \tag{A7.20}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_1(1-\eta Q_1)}{r-\mu} - \delta Q_1 + A_2(Q_1)X^\beta\right)}{\partial Q_1} = 0$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_1 - X\eta Q_1^2}{r-\mu} - \delta Q_1 + A_2(Q_1)X^\beta\right)}{\partial Q_1} = 0$$

$$\Leftrightarrow \frac{X - 2X\eta Q_1}{r-\mu} - \delta + \frac{\partial A_2(Q_1)}{\partial Q_1}X^\beta = 0$$

$$\Leftrightarrow \frac{X - X\eta Q_1 - X\eta Q_1}{r-\mu} - \delta + \frac{\partial A_2(Q_1)}{\partial Q_1}X^\beta = 0$$

$$\Leftrightarrow \frac{X(1-\eta Q_1)}{r-\mu} - \frac{XQ_1\eta}{r-\mu} - \delta + \frac{\partial A_2(Q_1)}{\partial Q_1}X^\beta = 0.$$

Meanwhile, we can represent the expression (A7.8) as follows

$$A_2(Q_1) = \left(X_2^*(Q_1)\right)^{-\beta}\frac{\delta Q_2}{\beta-1} \tag{A7.21}$$

$$\Leftrightarrow A_2(Q_1) = \frac{\delta(1-2\eta Q_1)\left(X_2^*(Q_1)\right)^{-\beta}}{(\beta-1)(\beta+1)\eta}$$

$$\Leftrightarrow A_2(Q_1) = \frac{\delta(1-2\eta Q_1)}{(\beta-1)(\beta+1)\eta}\left(\frac{\beta+1}{\beta-1}\frac{(r-\mu)\delta}{1-2\eta Q_1}\right)^{-\beta}$$

$$\Leftrightarrow A_2(Q_1) = \frac{\delta}{(\beta-1)(\beta+1)\eta}\left(\frac{(\beta+1)(r-\mu)\delta}{\beta-1}\right)^{-\beta}(1-2\eta Q_1)^{\beta+1},$$

so that

$$\frac{\partial A_2(Q_1)}{\partial Q_1} \tag{A7.22}$$

$$= \frac{\delta}{(\beta-1)(\beta+1)\eta}\left(\frac{(\beta+1)(r-\mu)\delta}{\beta-1}\right)^{-\beta}(\beta+1)(1-2\eta Q_1)^\beta \cdot (-2\eta)$$

$$= -\frac{2\delta}{(\beta-1)}\left(\frac{(\beta+1)(r-\mu)\delta}{(\beta-1)(1-2\eta Q_1)}\right)^{-\beta}$$

$$= -\frac{2\delta\left(X_2^*(Q_1)\right)^{-\beta}}{\beta - 1}.$$

Incorporating the previous result into expression (A7.20), we obtain

$$\frac{X(1-\eta Q_1)}{r-\mu} - \frac{XQ_1\eta}{r-\mu} - \delta + \frac{\partial A_2(Q_1)}{\partial Q_1}X^\beta = 0 \tag{A7.23}$$

$$\Leftrightarrow \frac{X(1-2\eta Q_1)}{r-\mu} - \delta - \frac{2\delta\left(X_2^*(Q_1)\right)^{-\beta}}{\beta-1}X^\beta = 0$$

$$\Leftrightarrow \frac{X(1-2\eta Q_1)}{r-\mu} - \delta - \frac{2\delta}{\beta-1}\left(\frac{X}{X_2^*(Q_1)}\right)^\beta = 0.$$

By incorporating the investment threshold of the second investment from expression (A7.10) and the investment threshold of the first investment from expression (A7.18) into the previous expression, we obtain that the capacity level of the first investment is given by

$$\frac{X(1-2\eta Q_1)}{r-\mu} - \delta - \frac{2\delta}{\beta-1}\left(\frac{X}{X_2^*(Q_1)}\right)^\beta = 0 \tag{A7.24}$$

$$\Leftrightarrow \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_1}(1-2\eta Q_1)}{r-\mu} - \delta - \frac{2\delta}{\beta-1}\left[\frac{\frac{\beta}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_1}}{\frac{(\beta+1)\delta(r-\mu)}{(\beta-1)(1-2\eta Q_1)}}\right]^\beta = 0$$

$$\Leftrightarrow \frac{\beta\delta}{\beta-1}\frac{1-2\eta Q_1}{1-\eta Q_1} - \frac{\delta(\beta-1)}{\beta-1} - \frac{2\delta}{\beta-1}\left[\frac{\beta(1-2\eta Q_1)}{(\beta+1)(1-\eta Q_1)}\right]^\beta = 0$$

$$\Leftrightarrow \frac{\beta - 2\beta\eta Q_1}{1-\eta Q_1} - \beta + 1 - 2\left[\frac{\beta(1-2\eta Q_1)}{(\beta+1)(1-\eta Q_1)}\right]^\beta = 0$$

$$\Leftrightarrow 1 + \frac{\beta - 2\beta\eta Q_1 - \beta + \beta\eta Q_1}{1-\eta Q_1} - 2\left[\frac{\beta(1-2\eta Q_1)}{(\beta+1)(1-\eta Q_1)}\right]^\beta = 0$$

$$\Leftrightarrow 1 - \frac{\beta\eta Q_1^*}{1-\eta Q_1^*} - 2\left[\frac{\beta(1-2\eta Q_1^*)}{(\beta+1)(1-\eta Q_1^*)}\right]^\beta = 0 \; (replacing\ Q_1 \equiv Q_1^*).$$

The total surplus in this monopolistic market, where $X_1 \equiv X_L, Q_1 \equiv Q_L, X_2 \equiv X_F$, and $Q_2 \equiv Q_F$, and considering that the consumer surpluses are calculated similarly as in expression (A2.4), and the producer surpluses are the value functions at each time the monopolist invests, is then

$$TS(X_L, Q_L, X_F, Q_F) = CS + PS \tag{A7.25}$$

$$\Leftrightarrow TS(X_L, Q_L, X_F, Q_F) = \frac{X_L (Q_L)^2 \eta}{2(r-\mu)} + \frac{X_F (Q_L + Q_F)^2 \eta}{2(r-\mu)} + \frac{X_L Q_L (1 - \eta Q_L)}{r-\mu} - \delta Q_L + \frac{X_F (Q_L + Q_F)\big(1 - \eta (Q_L + Q_F)\big)}{r-\mu} - \delta Q_F - \frac{X_F (Q_L)^2 \eta}{2(r-\mu)} - \frac{X_F Q_L (1 - \eta Q_L)}{r-\mu}$$

$$\Leftrightarrow TS(X_L, Q_L, X_F, Q_F) = \frac{X_L Q_L \eta Q_L}{2(r-\mu)} + \frac{X_L Q_L (2 - 2\eta Q_L)}{2(r-\mu)} - \delta Q_L + \frac{X_F (Q_L + Q_F)\eta (Q_L + Q_F)}{2(r-\mu)} + \frac{X_F (Q_L + Q_F)\big(2 - 2\eta (Q_L + Q_F)\big)}{2(r-\mu)} - \delta Q_F - \frac{X_F Q_L \eta Q_L}{2(r-\mu)} - \frac{X_F Q_L (2 - 2\eta Q_L)}{2(r-\mu)}$$

$$\Leftrightarrow TS(X_L, Q_L, X_F, Q_F) = \frac{X_L Q_L (2 + \eta Q_L - 2\eta Q_L)}{2(r-\mu)} - \delta Q_L + \frac{X_F (Q_L + Q_F)\big(2 + \eta (Q_L + Q_F) - 2\eta (Q_L + Q_F)\big)}{2(r-\mu)} - \delta Q_F - \frac{X_F Q_L (2 + \eta Q_L - 2\eta Q_L)}{2(r-\mu)}$$

$$\Leftrightarrow TS(X_L, Q_L, X_F, Q_F) = \frac{X_L Q_L (2 - \eta Q_L)}{2(r-\mu)} - \delta Q_L + \frac{X_F (Q_L + Q_F)\big(2 - \eta (Q_L + Q_F)\big)}{2(r-\mu)} - \delta Q_F - \frac{X_F Q_L (2 - \eta Q_L)}{2(r-\mu)}.$$

Huisman and Kort (2015), in their expression (55), represent the present value (PV) of the TS achieved above. Therefore, to the previous expression, we should incorporate the stochastic discount factor at each time the investments are undertaken. Thus, maintaining Huisman and Kort (2015) nomenclature, we have

$$PV\_TS(X_L, Q_L, X_F, Q_F) \equiv TS(X_L, Q_L, X_F, Q_F) = \left(\frac{X}{X_L}\right)^{\beta} \left(\frac{X_L Q_L (2 - \eta Q_L)}{2(r-\mu)} - \delta Q_L\right) + \left(\frac{X}{X_F}\right)^{\beta} \left(\frac{X_F (Q_L + Q_F)\big(2 - \eta (Q_L + Q_F)\big)}{2(r-\mu)} - \delta Q_F - \frac{X_F Q_L (2 - \eta Q_L)}{2(r-\mu)}\right). \tag{A7.26}$$

From expression (A7.25), we can conclude that the value function of the social planner at the moment of the second investment is given by

$$V_{F,W}\left(X_{F,W} \equiv X, Q_L, Q_F\right) = \frac{X(Q_L + Q_F)\big(2 - \eta(Q_L + Q_F)\big)}{2(r - \mu)} - \delta Q_F \tag{A7.27}$$

and the option value, before the investment, is equal to the expected profit of the first social planner's investment and the option value of the second investment. This can be represented as

$$F_{F,W}(X, Q_L) = \frac{X Q_L(2 - \eta Q_L)}{2(r - \mu)} + A_2 X^{\beta}. \tag{A7.28}$$

From the VMC, it holds that

$$V_{F,W}(X, Q_L, Q_F) = F_{F,W}(X, Q_L) \tag{A7.29}$$

$$\Leftrightarrow \frac{X(Q_L + Q_F)\big(2 - \eta(Q_L + Q_F)\big)}{2(r - \mu)} - \delta Q_F = \frac{X Q_L(2 - \eta Q_L)}{2(r - \mu)} + A_2 X^{\beta}.$$

According to the SPC, it yields that

$$\frac{\partial V_{F,W}(X, Q_L, Q_F)}{\partial X} = \frac{\partial F_F(X, Q_L)}{\partial X} \tag{A7.30}$$

$$\Leftrightarrow \frac{\partial \left(\frac{X(Q_L + Q_F)\big(2 - \eta(Q_L + Q_F)\big)}{2(r - \mu)} - \delta Q_F\right)}{\partial X} = \frac{\partial \left(\frac{X Q_L(2 - \eta Q_L)}{2(r - \mu)} + A_2 X^{\beta}\right)}{\partial X}$$

$$\Leftrightarrow \frac{(Q_L + Q_F)\big(2 - \eta(Q_L + Q_F)\big)}{2(r - \mu)} = \frac{Q_L(2 - \eta Q_L)}{2(r - \mu)} + \frac{\beta}{X} A_2 X^{\beta}.$$

Combining the results from the VMC and SPC, we find that the investment threshold for the second investment in this social planner scenario is denoted by

$$\frac{(Q_L + Q_F)\big(2 - \eta(Q_L + Q_F)\big)}{2(r - \mu)} = \frac{Q_L(2 - \eta Q_L)}{2(r - \mu)} + \frac{\beta}{X} A_2 X^{\beta} \tag{A7.31}$$

$$\Leftrightarrow \frac{(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)} = \frac{Q_L(2-\eta Q_L)}{2(r-\mu)} + \frac{\beta}{X}\left[\frac{X(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)} - \delta Q_F - \frac{XQ_L(2-\eta Q_L)}{2(r-\mu)}\right]$$

$$\Leftrightarrow \frac{(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)} = \frac{Q_L(2-\eta Q_L)}{2(r-\mu)} + \frac{\beta(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)} - \frac{\beta\delta Q_F}{X} - \frac{\beta Q_L(2-\eta Q_L)}{2(r-\mu)}$$

$$\Leftrightarrow X(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big) = XQ_L(2-\eta Q_L) + X\beta(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big) - 2\beta\delta Q_F(r-\mu) - X\beta Q_L(2-\eta Q_L)$$

$$\Leftrightarrow 2XQ_L - X\eta Q_L^2 - X\eta Q_L Q_F + 2XQ_F - X\eta Q_L Q_F - X\eta Q_F^2 = 2XQ_L - X\eta Q_L^2 + 2X\beta Q_L - X\beta\eta Q_L^2 - X\beta\eta Q_L Q_F + 2X\beta Q_F - X\beta\eta Q_L Q_F$$
$$-X\beta\eta Q_F^2 - 2\beta\delta Q_F(r-\mu) - 2X\beta Q_L + X\beta\eta Q_L^2$$

$$\Leftrightarrow -2X\eta Q_L Q_F + 2XQ_F - X\eta Q_F^2 = -2X\beta\eta Q_L Q_F + 2X\beta Q_F - X\beta\eta Q_F^2 - 2\beta\delta Q_F(r-\mu)$$

$$\Leftrightarrow -2X\eta Q_L + 2X - X\eta Q_F = -2X\beta\eta Q_L + 2X\beta - X\beta\eta Q_F - 2\beta\delta(r-\mu)$$

$$\Leftrightarrow 2X\eta Q_L + 2X - X\eta Q_F + 2X\beta\eta Q_L - 2X\beta + X\beta\eta Q_F = -2\beta\delta(r-\mu)$$

$$\Leftrightarrow X[2\eta Q_L + 2 - \eta Q_F + 2\beta\eta Q_L - 2\beta + \beta\eta Q_F] = -2\beta\delta(r-\mu)$$

$$\Leftrightarrow X[\eta(2Q_L - Q_F + 2\beta Q_L + \beta Q_F) - 2(\beta-1)] = -2\beta\delta(r-\mu)$$

$$\Leftrightarrow X[\eta(\beta-1)(2Q_L+Q_F) - 2(\beta-1)] = -2\beta\delta(r-\mu)$$

$$\Leftrightarrow X(\beta-1)[2-\eta(2Q_L+Q_F)] = 2\beta\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X_{F,W}^*(Q_L,Q_F) = \frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{2-\eta(2Q_L+Q_F)}.$$

The corresponding capacity level for this second investment is determined by maximising the social planner's value function at the time of this second investment, which implies that

$$\frac{\partial V_{F,W}(X_F \equiv X, Q_L, Q_F)}{\partial Q_F} = 0 \tag{A7.32}$$

$$\Leftrightarrow \frac{\partial\left(\frac{X(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)}-\delta Q_F\right)}{\partial Q_F}=0$$

$$\Leftrightarrow \frac{\partial\left(\frac{2XQ_L-X\eta Q_L^2-X\eta Q_LQ_F+2XQ_F-X\eta Q_LQ_F-X\eta Q_F^2}{2(r-\mu)}\right)}{\partial Q_F}-\delta=0$$

$$\Leftrightarrow \frac{-X\eta Q_L+2X-X\eta Q_L-2X\eta Q_F}{2(r-\mu)}=\delta$$

$$\Leftrightarrow -2X\eta Q_L+2X-2X\eta Q_F=2\delta(r-\mu)$$

$$\Leftrightarrow X\eta Q_L-X+X\eta Q_F=-\delta(r-\mu)$$

$$\Leftrightarrow X\eta Q_F=X-\delta(r-\mu)-X\eta Q_L$$

$$\Leftrightarrow \eta Q_F=1-\frac{\delta(r-\mu)}{X}-\eta Q_L$$

$$\Leftrightarrow Q_F\equiv Q_{F,W}^*(X,Q_L)=\frac{1}{\eta}\left[1-\frac{\delta(r-\mu)}{X}-\eta Q_L\right].$$

Combining the results from expressions (A7.31) and (A7.32), we obtain that the investment threshold, depending only on Q_L, can be expressed as

$$X=\frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{2-\eta(2Q_L+Q_F)} \tag{A7.33}$$

$$\Leftrightarrow X=\frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{2-2\eta Q_L-\eta Q_F}$$

$$\Leftrightarrow X=\frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{2-2\eta Q_L-\left(1-\frac{\delta(r-\mu)}{X}-\eta Q_L\right)}$$

$$\Leftrightarrow X=\frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{2-2\eta Q_L-1+\frac{\delta(r-\mu)}{X}+\eta Q_L}$$

$$\Leftrightarrow X=\frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{1-\eta Q_L+\frac{\delta(r-\mu)}{X}}$$

$$\Leftrightarrow X=\frac{\beta}{\beta-1}\frac{2X\delta(r-\mu)}{X-X\eta Q_L+\delta(r-\mu)}$$

$$\Leftrightarrow X-X\eta Q_L+\delta(r-\mu)=\frac{\beta}{\beta-1}2\delta(r-\mu)$$

$$\Leftrightarrow X(1-\eta Q_L)=\frac{2\beta\delta(r-\mu)-\beta\delta(r-\mu)+\delta(r-\mu)}{\beta-1}$$

$$\Leftrightarrow X(1-\eta Q_L)=\frac{\beta\delta(r-\mu)+\delta(r-\mu)}{\beta-1}$$

$$\Leftrightarrow X(1-\eta Q_L) = \frac{(\beta+1)(r-\mu)\delta}{\beta-1}$$

$$\Leftrightarrow X \equiv X^*_{F,W}(Q_L) = \frac{\beta+1}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_L}.$$

Incorporating the previous result into expression (A7.32) demonstrates that the optimal investment capacity level, depending only on Q_L, is equal to

$$Q_F = \frac{1}{\eta}\left[1 - \frac{\delta(r-\mu)}{X} - \eta Q_L\right] \tag{A7.34}$$

$$\Leftrightarrow Q_F = \frac{1}{\eta}\left[1 - \frac{\delta(r-\mu)}{\frac{\beta+1}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_L}} - \eta Q_L\right]$$

$$\Leftrightarrow Q_F = \frac{1}{\eta}\left[1 - \frac{(\beta-1)(1-\eta Q_L)}{\beta+1} - \eta Q_L\right]$$

$$\Leftrightarrow Q_F = \frac{1}{\eta}\left[\frac{\beta+1-\beta+\beta\eta Q_L+1-\eta Q_L-\beta\eta Q_L-\eta Q_L}{\beta+1}\right]$$

$$\Leftrightarrow Q_F = \frac{1}{\eta}\frac{2-2\eta Q_L}{\beta+1}$$

$$\Leftrightarrow Q_F \equiv Q^*_{F,W}(Q_L) = \frac{2(1-\eta Q_L)}{(\beta+1)\eta}.$$

From expression (A7.25), we can derive the value function of the active investment resulting from the first investment made by the social planner. However, we must incorporate the option value A_2X^β to account for the possibility of undertaking the second investment. Therefore, the value function of the social planner for the first investment is given by

$$V_{L,W}(X_{L,W} \equiv X, Q_L) = \frac{XQ_L(2-\eta Q_L)}{2(r-\mu)} - \delta Q_L + A_2X^\beta. \tag{A7.35}$$

The option value of the first investment is given by

$$F_{L,W}(X, Q_L) = A_1X^\beta. \tag{A7.36}$$

From the VMC, it holds that

$$V_{L,W}(X, Q_L) = F_{L,W}(X, Q_L) \tag{A7.37}$$

$$\Leftrightarrow \frac{XQ_L(2-\eta Q_L)}{2(r-\mu)} - \delta Q_L + A_2X^\beta = A_1X^\beta.$$

From the SPC, it yields that

$$\frac{\partial V_{L,W}(X, Q_L)}{\partial X} = \frac{\partial F_{L,W}(X, Q_L)}{\partial X} \tag{A7.38}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_L(2-\eta Q_L)}{2(r-\mu)} - \delta Q_L + A_2X^\beta\right)}{\partial X} = \frac{\partial\left(A_1X^\beta\right)}{\partial X}$$

$$\Leftrightarrow \frac{Q_L(2-\eta Q_L)}{2(r-\mu)} + \frac{\beta}{X}A_2X^\beta = \frac{\beta}{X}A_1X^\beta.$$

Combining the results from the VMC and SPC, we find that the investment threshold, depending only on Q_L, for the first investment can be represented as

$$\frac{Q_L(2-\eta Q_L)}{2(r-\mu)} + \frac{\beta}{X}A_2X^\beta = \frac{\beta}{X}A_1X^\beta \tag{A7.39}$$

$$\Leftrightarrow \frac{Q_L(2-\eta Q_L)}{2(r-\mu)} + \frac{\beta}{X}A_2X^\beta = \frac{\beta}{X}\left[\frac{XQ_L(2-\eta Q_L)}{2(r-\mu)} - \delta Q_L + A_2X^\beta\right]$$

$$\Leftrightarrow \frac{Q_L(2-\eta Q_L)}{2(r-\mu)} + \frac{\beta}{X}A_2X^\beta = \frac{\beta Q_L(2-\eta Q_L)}{2(r-\mu)} - \frac{\beta\delta Q_L}{X} + \frac{\beta}{X}A_2X^\beta$$

$$\Leftrightarrow XQ_L(2-\eta Q_L) = X\beta Q_L(2-\eta Q_L) - 2\beta\delta Q_L(r-\mu)$$

$$\Leftrightarrow 2X - X\eta Q_L = 2X\beta - X\beta\eta Q_L - 2\beta\delta(r-\mu)$$

$$\Leftrightarrow 2X - 2X\beta - X\eta Q_L + X\beta\eta Q_L = -2\beta\delta(r-\mu)$$

$$\Leftrightarrow -2X + 2X\beta + X\eta Q_L - X\beta\eta Q_L = 2\beta\delta(r-\mu)$$

$$\Leftrightarrow 2X\left[-1 + \beta + \frac{1}{2}\eta Q_L - \frac{1}{2}\beta\eta Q_L\right] = 2\beta\delta(r-\mu)$$

$$\Leftrightarrow X(\beta-1)\left(1 - \frac{1}{2}\eta Q_L\right) = \beta\delta(r-\mu)$$

$$\Leftrightarrow X \equiv X^*_{L,W}(Q_L) = \frac{\beta(r-\mu)\delta}{(\beta-1)\left(1-\frac{1}{2}\eta Q_L\right)}.$$

By maximising the firm's value function concerning the capacity level of the first investment, we derive the capacity level function as follows

$$\frac{\partial V_{L,W}(X_L \equiv X, Q_L)}{\partial Q_L} = 0 \tag{A7.40}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_L(2-\eta Q_L)}{2(r-\mu)} - \delta Q_L + A_2(Q_L)X^\beta\right)}{\partial Q_L} = 0$$

$$\Leftrightarrow \frac{\partial\left(\frac{2XQ_L - X\eta Q_L^2}{2(r-\mu)} - \delta Q_L + A_2(Q_L)X^\beta\right)}{\partial Q_L} = 0$$

$$\Leftrightarrow \frac{2X - 2X\eta Q_L}{2(r-\mu)} - \delta + \frac{\partial A_2(Q_L)}{\partial Q_L}X^\beta = 0$$

$$\Leftrightarrow \frac{X(1-\eta Q_L)}{r-\mu} - \delta + \frac{\partial A_2(Q_L)}{\partial Q_L}X^\beta = 0.$$

From the VMC of the second investment in expression (A7.29), we know that

$$\frac{X(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)}-\delta Q_F=\frac{XQ_L(2-\eta Q_L)}{2(r-\mu)}+A_2X^\beta \tag{A7.41}$$

$$\Leftrightarrow A_2X^\beta=\frac{X(Q_L+Q_F)\big(2-\eta(Q_L+Q_F)\big)}{2(r-\mu)}-\delta Q_F-\frac{XQ_L(2-\eta Q_L)}{2(r-\mu)}$$

$$\Leftrightarrow A_2X^\beta=\frac{2XQ_L-X\eta Q_L^2-X\eta Q_LQ_F+2XQ_F-X\eta Q_LQ_F-X\eta Q_F^2-2XQ_L+X\eta Q_L^2}{2(r-\mu)}-\delta Q_F$$

$$\Leftrightarrow A_2X^\beta=\frac{-2X\eta Q_LQ_F+2XQ_F-X\eta Q_F^2}{2(r-\mu)}-\delta Q_F$$

$$\Leftrightarrow A_2X^\beta=\frac{X(-2\eta Q_LQ_F+2Q_F-\eta Q_F^2)}{2(r-\mu)}-\delta Q_F$$

$$\Leftrightarrow A_2X^\beta=\frac{\frac{\beta}{\beta-1}\frac{2\delta(r-\mu)}{2-\eta(2Q_L+Q_F)}\left[Q_F\big(2-\eta(2Q_L+Q_F)\big)\right]}{2(r-\mu)}-\delta Q_F\ \left(incorporating\ X=X^*_{F,W}(Q_L,Q_F)\right)$$

$$\Leftrightarrow A_2X^\beta=\frac{\beta\delta Q_F}{\beta-1}-\delta Q_F$$

$$\Leftrightarrow A_2X^\beta=\frac{\beta\delta Q_F-\beta\delta Q_F+\delta Q_F}{\beta-1}$$

$$\Leftrightarrow A_2X^\beta=\frac{\delta Q_F}{\beta-1}$$

$$\Leftrightarrow A_2=\left(X^*_{F,W}(Q_L)\right)^{-\beta}\frac{\delta Q_F}{\beta-1}\ \left(substituting\ X^\beta\equiv\left(X^*_{F,W}(Q_L)\right)^\beta\right)$$

$$\Leftrightarrow A_2 = \frac{2\delta(1-\eta Q_L)\left(X_{F,W}^*(Q_L)\right)^{-\beta}}{(\beta-1)(\beta+1)\eta} \left(replacing\ Q_F \equiv Q_{F,W}^*(Q_L)\right)$$

$$\Leftrightarrow A_2 = \frac{2\delta(1-\eta Q_L)}{(\beta-1)(\beta+1)\eta}\left(\frac{\beta+1}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_L}\right)^{-\beta}$$

$$\Leftrightarrow A_2(Q_L) = \frac{2\delta}{(\beta-1)(\beta+1)\eta}\left(\frac{(\beta+1)(r-\mu)\delta}{\beta-1}\right)^{-\beta}(1-\eta Q_L)^{\beta+1}.$$

Differentiating the previous expression in order to Q_L, we obtain

$$\frac{\partial A_2(Q_L)}{\partial Q_L} \tag{A7.42}$$

$$= \frac{\partial\left(\frac{2\delta}{(\beta-1)(\beta+1)\eta}\left(\frac{(\beta+1)(r-\mu)\delta}{\beta-1}\right)^{-\beta}(1-\eta Q_L)^{\beta+1}\right)}{\partial Q_L}$$

$$= \frac{2\delta}{(\beta-1)(\beta+1)\eta}\left(\frac{(\beta+1)(r-\mu)\delta}{\beta-1}\right)^{-\beta}(\beta+1)(1-\eta Q_L)^{\beta}(-\eta)$$

$$= -\frac{2\delta}{\beta-1}\left(\frac{(\beta+1)(r-\mu)\delta}{(\beta-1)(1-\eta Q_L)}\right)^{-\beta}$$

$$= -\frac{2\delta}{\beta-1}\left(X_{F,W}^*(Q_L)\right)^{-\beta}.$$

Recalling expression (A7.40) and incorporating the result of expression (A7.42), we can simplify the expression for the optimal capacity level for this first investment as follows

$$\frac{X(1-\eta Q_L)}{r-\mu} - \delta + \frac{\partial A_2(Q_L)}{\partial Q_L}X^{\beta} = 0 \tag{A7.43}$$

$$\Leftrightarrow \frac{X(1-\eta Q_L)}{r-\mu} - \delta - \frac{2\delta}{\beta-1}\left(X_{F,W}^*(Q_L)\right)^{-\beta} X^{\beta} = 0$$

$$\Leftrightarrow \frac{X(1-\eta Q_L)}{r-\mu} - \delta - \frac{2\delta}{\beta-1}\left(\frac{X}{X_{F,W}^*(Q_L)}\right)^{\beta} = 0$$

$$\Leftrightarrow \frac{\frac{\beta\delta(r-\mu)}{(\beta-1)\left(1-\frac{1}{2}\eta Q_L\right)}(1-\eta Q_L)}{r-\mu} - \delta - \frac{2\delta}{\beta-1}\left(\frac{\frac{\beta\delta(r-\mu)}{(\beta-1)\left(1-\frac{1}{2}\eta Q_L\right)}}{\frac{\beta+1}{\beta-1}\frac{(r-\mu)\delta}{1-\eta Q_L}}\right)^{\beta} = 0\left(considering\ X \equiv X_{L,W}^*(Q_L)\right)$$

$$\Leftrightarrow \frac{\beta\delta(1-\eta Q_L)}{(\beta-1)\left(1-\frac{1}{2}\eta Q_L\right)} - \delta - \frac{2\delta}{\beta-1}\left(\frac{\beta(1-\eta Q_L)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_L\right)}\right)^{\beta} = 0$$

$$\Leftrightarrow \frac{\beta\delta(1-\eta Q_L)}{(\beta-1)\left(1-\frac{1}{2}\eta Q_L\right)} - \frac{\delta(\beta-1)}{\beta-1} - \frac{2\delta}{\beta-1}\left(\frac{\beta(1-\eta Q_L)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_L\right)}\right)^{\beta} = 0$$

$$\Leftrightarrow \frac{\beta\delta(1-\eta Q_L)}{1-\frac{1}{2}\eta Q_L} - \delta(\beta-1) - 2\delta\left(\frac{\beta(1-\eta Q_L)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_L\right)}\right)^{\beta} = 0$$

$$\Leftrightarrow \frac{\beta(1-\eta Q_L)}{1-\frac{1}{2}\eta Q_L} - \beta + 1 - 2\left(\frac{\beta(1-\eta Q_L)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_L\right)}\right)^{\beta} = 0$$

$$\Leftrightarrow 1+\frac{\beta-\beta\eta Q_L-\beta+\frac{1}{2}\beta\eta Q_L}{1-\frac{1}{2}\eta Q_L}-2\left(\frac{\beta(1-\eta Q_L)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_L\right)}\right)^{\beta}=0$$

$$\Leftrightarrow 1-\frac{\beta\frac{1}{2}\eta Q_{L,W}^{*}}{1-\frac{1}{2}\eta Q_{L,W}^{*}}-2\left(\frac{\beta\left(1-\eta Q_{L,W}^{*}\right)}{(\beta+1)\left(1-\frac{1}{2}\eta Q_{L,W}^{*}\right)}\right)^{\beta}=0\ \left(replacing Q_L\equiv Q_{L,W}^{*}\right).$$

ANNEX B

Isoelastic demand curve

This annex presents the detailed mathematical derivations for Propositions 9, 10, 11, 12, and 15, as well as the welfare analysis. The equations below are derived with a focus on an isoelastic demand curve for the price function.

It is important to note that β remains defined by expression (A1.7) and that $\beta\gamma > 1$ must hold. Regarding investment costs, they are now expressed as $\delta_0 + \delta_1 Q$. If $\delta_0 = 0$, firms will opt to invest right at the beginning of the investment game.

B.1. Mathematical details of the monopolist's optimal investment decision

Incorporating the new price function from expression (6.1) into equation (4.3) yields that the total expected profit of a monopolistic firm at time t is given by

$$\pi(t) = P(t) \cdot Q(t) \tag{B1.1}$$

$$\Leftrightarrow \pi(t) = X(t)\big(Q(t)\big)^{-\gamma} \cdot Q(t)$$

$$\Leftrightarrow \pi(t) = X(t)\big(Q(t)\big)^{1-\gamma}.$$

Therefore, the expected total profit is

$$E[\pi_t] = E\left[X_t Q_t^{1-\gamma}\right] \tag{B1.2}$$

$$\Leftrightarrow E[\pi_t] = E\left[X_t Q_t^{1-\gamma}\right]$$

$$\Leftrightarrow E[\pi_t] = Q^{1-\gamma} E[X_t]$$

$$\Leftrightarrow E[\pi_t] = Q^{1-\gamma} X e^{\mu \cdot t}.$$

The value of the monopolist firm can be expressed as

$$V(X, Q) = E\left[\int_{t=0}^{\infty} \pi(t) e^{-rt}\, dt - (\delta_0 + \delta_1 Q)\right] \tag{B1.3}$$

$$\Leftrightarrow V(X, Q) = \int_{t=0}^{\infty} Q^{1-\gamma} \cdot X \cdot e^{\mu \cdot t} \cdot e^{-rt}\, dt - \delta_0 - \delta_1 Q$$

$$\Leftrightarrow V(X, Q) = X Q^{1-\gamma} \int_{t=0}^{\infty} e^{(\mu - r)\cdot t}\, dt - \delta_0 - \delta_1 Q$$

$$\Leftrightarrow V(X, Q) = X Q^{1-\gamma} \frac{1}{r - \mu} - \delta_0 - \delta_1 Q$$

$$\Leftrightarrow V(X,Q) = \frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q.$$

Maximising the previous monopolist's value function with respect to Q gives the optimal capacity level concerning X

$$\frac{\partial V(X,Q)}{\partial Q} = 0 \tag{B1.4}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q\right)}{\partial Q} = 0$$

$$\Leftrightarrow \frac{(1-\gamma)XQ^{-\gamma}}{r-\mu} - \delta_1 = 0$$

$$\Leftrightarrow (1-\gamma)XQ^{-\gamma} = \delta_1(r-\mu)$$

$$\Leftrightarrow \frac{1}{Q^{\gamma}} = \frac{\delta_1(r-\mu)}{(1-\gamma)X}$$

$$\Leftrightarrow Q^{\gamma} = \frac{(1-\gamma)X}{\delta_1(r-\mu)}$$

$$\Leftrightarrow Q \equiv Q^*(X) = \left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}.$$

Given that X^* is the optimal investment threshold, for $X < X^*$, the firm remains idle, and the value of the monopolist firm is given by the option value AX^{β}. For $X \geq X^*$, substituting Q from expression (B1.4) into expression (B1.3) results in the value function of the active firm being

$$V(X,Q) = \frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q \tag{B1.5}$$

$$\Leftrightarrow V(X) = \frac{X\left[\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}\right]^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}$$

$$\Leftrightarrow V(X) = \frac{X\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{-1}}{r-\mu} - \delta_0 - \delta_1\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}$$

$$\Leftrightarrow V(X) = \left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}\left[\frac{X}{(r-\mu)\frac{(1-\gamma)X}{\delta_1(r-\mu)}} - \delta_1\right] - \delta_0$$

$$\Leftrightarrow V(X) = \left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} \left(\frac{\delta_1}{1-\gamma} - \delta_1\right) - \delta_0$$

$$\Leftrightarrow V(X) = \left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} \left(\frac{\delta_1 - \delta_1 + \delta_1\gamma}{1-\gamma}\right) - \delta_0$$

$$\Leftrightarrow V(X) = \frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0.$$

Combining the results, the value of the monopolist's firm can be expressed as

$$V(X) = \begin{cases} AX^{\beta} & if\ X < X^* \\ \frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0 & if\ X \geq X^* \end{cases}. \tag{B1.6}$$

Since the option value of the monopolist firm is given by the first branch of expression (B1.6), it can be represented as

$$F(X) = AX^{\beta}. \tag{B1.7}$$

Then, the VMC yields that

$$F(X) = V(X, Q) \tag{B1.8}$$

$$\Leftrightarrow AX^{\beta} = \frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0.$$

Consequently, the SPC yields that

$$\frac{\partial F(X)}{\partial X} = \frac{\partial V(X,Q)}{\partial X} \tag{B1.9}$$

$$\Leftrightarrow \frac{\partial\left(AX^{\beta}\right)}{\partial X} = \frac{\partial\left(\frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0\right)}{\partial X}$$

$$\Leftrightarrow \beta AX^{\beta-1} = \frac{\gamma\delta_1}{1-\gamma}\frac{1}{\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1}\frac{1-\gamma}{\delta_1(r-\mu)}$$

$$\Leftrightarrow \beta AX^{\beta-1} = \frac{\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1}\frac{1-\gamma}{\delta_1(r-\mu)}.$$

Incorporating the result of the VMC in the SPC, we achieve that the optimal investment threshold is

$$\beta AX^{\beta-1} = \frac{\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1}\frac{1-\gamma}{\delta_1(r-\mu)} \tag{B1.10}$$

$$\Leftrightarrow \frac{\beta}{X} A X^{\beta} = \frac{\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1} \frac{1-\gamma}{\delta_1(r-\mu)}$$

$$\Leftrightarrow \frac{\beta}{X}\left[\frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0\right] = \frac{\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1} \frac{1-\gamma}{\delta_1(r-\mu)}$$

$$\Leftrightarrow \frac{\beta}{X}\frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \frac{\beta\delta_0}{X} = \frac{1}{r-\mu}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1}$$

$$\Leftrightarrow \frac{\beta\gamma\delta_1}{1-\gamma}\frac{(1-\gamma)^{\frac{1}{\gamma}} X^{\left(\frac{1}{\gamma}-1\right)}}{\left(\delta_1(r-\mu)\right)^{\frac{1}{\gamma}}} - \frac{\beta\delta_0}{X} = \frac{1}{r-\mu}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1}$$

$$\Leftrightarrow \frac{\beta\gamma\delta_1}{1-\gamma}\frac{(1-\gamma)^{\frac{1}{\gamma}} X^{\left(\frac{1}{\gamma}-1\right)}}{\left(\delta_1(r-\mu)\right)^{\frac{1}{\gamma}}} - \frac{1}{r-\mu}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1} = \frac{\beta\delta_0}{X}$$

$$\Leftrightarrow X^{\left(\frac{1}{\gamma}-1\right)}\left[\frac{\beta\gamma\delta_1}{1-\gamma}\left(\frac{1-\gamma}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \frac{1}{r-\mu}\left(\frac{1-\gamma}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}-1}\right] = \frac{\beta\delta_0}{X}$$

$$\Leftrightarrow X^{\frac{1}{\gamma}}\left(\frac{1-\gamma}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}\left(\frac{\beta\gamma\delta_1}{1-\gamma} - \frac{1}{(r-\mu)\frac{1-\gamma}{\delta_1(r-\mu)}}\right) = \beta\delta_0$$

$$\Leftrightarrow X^{\frac{1}{\gamma}}\left(\frac{1-\gamma}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}\left(\frac{\beta\gamma\delta_1-\delta_1}{1-\gamma}\right) = \beta\delta_0$$

$$\Leftrightarrow X^{\frac{1}{\gamma}}\left(\frac{1-\gamma}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}\left(\frac{\delta_1(\beta\gamma-1)}{1-\gamma}\right) = \beta\delta_0$$

$$\Leftrightarrow X\left(\frac{1-\gamma}{\delta_1(r-\mu)}\right)\left(\frac{\delta_1(\beta\gamma-1)}{1-\gamma}\right)^{\gamma} = (\beta\delta_0)^{\gamma}$$

$$\Leftrightarrow X\left(\frac{\delta_1(\beta\gamma-1)}{1-\gamma}\right)^{\gamma} = \frac{(\beta\delta_0)^{\gamma}}{\frac{1-\gamma}{\delta_1(r-\mu)}}$$

$$\Leftrightarrow X\left(\frac{\delta_1(\beta\gamma-1)}{1-\gamma}\right)^{\gamma} = \frac{\delta_1(r-\mu)}{1-\gamma}(\beta\delta_0)^{\gamma}$$

$$\Leftrightarrow X \equiv X^{*} = \frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}.$$

Combining the results of the VMC and the investment threshold from the previous expression, the variable A of the option value can be calculated as follows

$$AX^{\beta} = \frac{\gamma\delta_1}{1-\gamma}\left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} - \delta_0 \tag{B1.11}$$

$$\Leftrightarrow AX^{\beta} = \frac{\gamma\delta_1}{1-\gamma}\left(\frac{1-\gamma}{\delta_1(r-\mu)}\frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}\right)^{\frac{1}{\gamma}} - \delta_0$$

$$\Leftrightarrow AX^{\beta} = \frac{\gamma\delta_1}{1-\gamma}\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)} - \delta_0$$

$$\Leftrightarrow AX^{\beta} = \frac{\gamma\delta_0\beta}{\beta\gamma-1} - \delta_0$$

$$\Leftrightarrow AX^{\beta} = \frac{\gamma\delta_0\beta - \delta_0\beta\gamma + \delta_0}{\beta\gamma-1}$$

$$\Leftrightarrow AX^{\beta} = \frac{\delta_0}{\beta\gamma-1}$$

$$\Leftrightarrow A = X^{-\beta}\frac{\delta_0}{\beta\gamma-1}$$

$$\Leftrightarrow A = \left(\frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}\right)^{-\beta}\frac{\delta_0}{\beta\gamma-1}.$$

Additionally, the optimal capacity level in expression (B1.4), when the investment threshold from expression (B1.10) is incorporated, can be represented as follows

$$Q^*(X) = \left(\frac{(1-\gamma)X}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}} \tag{B1.12}$$

$$\Leftrightarrow Q^* = \left(\frac{(1-\gamma)\frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}}{\delta_1(r-\mu)}\right)^{\frac{1}{\gamma}}$$

$$\Leftrightarrow Q^* \equiv Q^*(X^*) = \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}.$$

B.2. Mathematical details of the optimal welfare decision

From equation (6.1), the demand concerning the price P can be represented as follows

$$P(t) = X(t)\big(Q(t)\big)^{-\gamma} \tag{B2.1}$$

$$\Leftrightarrow P = XQ^{-\gamma}$$

$$\Leftrightarrow Q^{\gamma} = \frac{X}{P}$$

$$\Leftrightarrow Q = \left(\frac{X}{P}\right)^{\frac{1}{\gamma}}$$

$$\Leftrightarrow D(P) = \left(\frac{X}{P}\right)^{\frac{1}{\gamma}} \ \big(representing\ Q \equiv D(P)\big).$$

Therefore, the instantaneous consumer surplus can be computed as

$$\int_{P(Q)}^{\infty} D(P)\, dP \tag{B2.2}$$

$$= \int_{XQ^{-\gamma}}^{\infty} \left(\frac{X}{P}\right)^{\frac{1}{\gamma}} dP$$

$$= \int_{XQ^{-\gamma}}^{\infty} \frac{X^{\frac{1}{\gamma}}}{P^{\frac{1}{\gamma}}}\, dP$$

$$= X^{\frac{1}{\gamma}} \int_{XQ^{-\gamma}}^{\infty} P^{-\frac{1}{\gamma}}\, dP$$

$$= X^{\frac{1}{\gamma}} \left[\frac{P^{-\frac{1}{\gamma}+1}}{-\frac{1}{\gamma}+1}\right]_{P=XQ^{-\gamma}}^{P=\infty}$$

$$= X^{\frac{1}{\gamma}} \left(\frac{\infty^{-\frac{1}{\gamma}+1}}{-\frac{1}{\gamma}+1} - \frac{(XQ^{-\gamma})^{-\frac{1}{\gamma}+1}}{-\frac{1}{\gamma}+1}\right)$$

$$= X^{\frac{1}{\gamma}} \left(\frac{0}{-\frac{1}{\gamma}+1} - \frac{(XQ^{-\gamma})^{-\frac{1}{\gamma}+1}}{-\frac{1}{\gamma}+1}\right)$$

$$= X^{\frac{1}{\gamma}} \left(-\frac{(XQ^{-\gamma})^{-\frac{1}{\gamma}+1}}{-\frac{1}{\gamma}+1}\right)$$

$$= X^{\frac{1}{\gamma}} \left(\frac{(XQ^{-\gamma})^{-\frac{1}{\gamma}+1}}{\frac{1}{\gamma}-1}\right)$$

$$= \frac{\gamma}{1-\gamma} X^{\frac{1}{\gamma}} (XQ^{-\gamma})^{-\frac{1}{\gamma}+1}$$

$$= \frac{\gamma}{1-\gamma} X^{\frac{1}{\gamma}} X^{-\frac{1}{\gamma}+1} Q^{1-\gamma}$$

$$= \frac{\gamma}{1-\gamma} X Q^{1-\gamma}.$$

Consequently, the total expected consumer surplus is given by

$$CS(X,Q) = E\left[\int_{t=0}^{\infty} \frac{\gamma}{1-\gamma} X(t)\big(Q(t)\big)^{1-\gamma} e^{-r\cdot t}\, dt\right] \tag{B2.3}$$

$$\Leftrightarrow CS(X,Q) = \frac{\gamma}{1-\gamma} Q^{1-\gamma} E\left[\int_{t=0}^{\infty} X(t) e^{-r\cdot t}\, dt\right] \; (Q \; is \; fixed, then \; Q(t) \equiv Q)$$

$$\Leftrightarrow CS(X,Q) = \frac{\gamma}{1-\gamma} Q^{1-\gamma} \int_{t=0}^{\infty} E\big(X(t)\big) e^{-r\cdot t}\, dt$$

$$\Leftrightarrow CS(X,Q) = \frac{\gamma}{1-\gamma} Q^{1-\gamma} \int_{t=0}^{\infty} X e^{\mu\cdot t} e^{-r\cdot t}\, dt$$

$$\Leftrightarrow CS(X,Q) = \frac{\gamma}{1-\gamma} X Q^{1-\gamma} \int_{t=0}^{\infty} e^{(\mu-r)\cdot t}\, dt$$

$$\Leftrightarrow CS(X,Q) = \frac{\gamma}{1-\gamma} X Q^{1-\gamma} \frac{1}{r-\mu}$$

$$\Leftrightarrow CS(X,Q) = \frac{\gamma}{1-\gamma} \frac{X Q^{1-\gamma}}{r-\mu}.$$

Since the expected producer surplus is equal to the value of the firm from equation (B1.3), it can be represented it as

$$PS(X,Q) = \frac{X Q^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q. \tag{B2.4}$$

Thus, the total expected surplus is

$$TS(X,Q) = CS(X,Q) + PS(X,Q) \tag{B2.5}$$

$$\Leftrightarrow TS(X,Q) = \frac{\gamma}{1-\gamma} \frac{X Q^{1-\gamma}}{r-\mu} + \frac{X Q^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q$$

$$\Leftrightarrow TS(X,Q) = \frac{X Q^{1-\gamma}}{r-\mu}\left[\frac{\gamma}{1-\gamma} + 1\right] - \delta_0 - \delta_1 Q$$

$$\Leftrightarrow TS(X,Q) = \frac{X Q^{1-\gamma}}{r-\mu}\left[\frac{\gamma + 1 - \gamma}{1-\gamma}\right] - \delta_0 - \delta_1 Q$$

$$\Leftrightarrow TS(X,Q) = \frac{1}{1-\gamma} \frac{X Q^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q.$$

From the TS, we can conclude that the value function of the social planner at the moment of investment is given by

$$V_W(X_W \equiv X, Q) = \frac{1}{1-\gamma}\frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q. \tag{B2.6}$$

Before the investment is undertaken, the social planner holds an option value that is equal to

$$F_W(X) = AX^{\beta}. \tag{B2.7}$$

The capacity level of the social planner is determined by maximising the social planner's value function at the time of the investment, which implies that

$$\frac{\partial V_W(X_W \equiv X, Q)}{\partial Q} = 0 \tag{B2.8}$$

$$\Leftrightarrow \frac{\partial\left(\frac{1}{1-\gamma}\frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q\right)}{\partial Q} = 0$$

$$\Leftrightarrow \frac{1}{1-\gamma}(1-\gamma)\frac{XQ^{1-\gamma-1}}{r-\mu} - \delta_1 = 0$$

$$\Leftrightarrow \frac{X}{Q^{\gamma}(r-\mu)} = \delta_1$$

$$\Leftrightarrow Q^{\gamma} \equiv (Q_W^*)^{\gamma}(X) = \frac{X}{\delta_1(r-\mu)}.$$

From the VMC, it holds that

$$\mathrm{V_W(X_W \equiv X, Q) = F_W(X_W \equiv X)} \tag{B2.9}$$

$$\Leftrightarrow \frac{1}{1-\gamma}\frac{\mathrm{XQ}^{1-\gamma}}{\mathrm{r}-\mu} - \delta_0 - \delta_1\mathrm{Q} = \mathrm{AX}^{\beta}$$

and together with the SPC below, this yields the social planner's investment threshold as

$$\left.\frac{\partial V_W(X,Q)}{\partial X}\right|_{X=X_W^*} = \left.\frac{\partial F_W(X)}{\partial X}\right|_{X=X_W^*} \tag{B2.10}$$

$$\Leftrightarrow \frac{\partial\left(\frac{1}{1-\gamma}\frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q\right)}{\partial X} = \frac{\partial\left(AX^{\beta}\right)}{\partial X}$$

$$\Leftrightarrow \frac{1}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu} = \beta A\frac{X^{\beta}}{X}$$

$$\Leftrightarrow \frac{1}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu} = \frac{\beta}{X}\left(\frac{1}{1-\gamma}\frac{XQ^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q\right)$$

$$\Leftrightarrow \frac{1}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu} = \frac{\beta}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu} - \frac{\beta}{X}(\delta_0 + \delta_1 Q)$$

$$\Leftrightarrow \frac{\beta}{X}(\delta_0 + \delta_1 Q) = \frac{1}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu}(\beta - 1)$$

$$\Leftrightarrow X \equiv X_W^*(Q) = \frac{\beta(\delta_0 + \delta_1 Q)}{\frac{1}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu}(\beta - 1)}.$$

By replacing $X_W^*(Q)$ into expression (B2.8), we obtain that the social planner's optimal capacity level is equal to

$$Q^\gamma = \frac{\frac{\beta(\delta_0 + \delta_1 Q)}{\frac{1}{1-\gamma}\frac{Q^{1-\gamma}}{r-\mu}(\beta - 1)}}{\delta_1(r-\mu)} \tag{B2.11}$$

$$\Leftrightarrow Q^\gamma = \frac{\frac{\beta(\delta_0 + \delta_1 Q)(1-\gamma)(r-\mu)}{Q^{1-\gamma}(\beta - 1)}}{\delta_1(r-\mu)}$$

$$\Leftrightarrow Q^\gamma = \frac{\beta(\delta_0 + \delta_1 Q)(1-\gamma)(r-\mu)}{Q^{1-\gamma}(\beta - 1)\delta_1(r-\mu)}$$

$$\Leftrightarrow Q(\beta - 1)\delta_1 = \beta(\delta_0 + \delta_1 Q)(1-\gamma)$$

$$\Leftrightarrow \beta\delta_1 Q - Q\delta_1 = \beta\delta_0 - \beta\delta_0\gamma + \beta\delta_1 Q - \beta\delta_1\gamma Q$$

$$\Leftrightarrow \beta\delta_1 Q - Q\delta_1 - \beta\delta_1 Q + \beta\delta_1\gamma Q = \beta\delta_0 - \beta\delta_0\gamma$$

$$\Leftrightarrow -Q\delta_1 + \beta\delta_1\gamma Q = \beta\delta_0(1-\gamma)$$

$$\Leftrightarrow Q(-\delta_1 + \beta\delta_1\gamma) = \delta_0\beta(1-\gamma)$$

$$\Leftrightarrow Q\big(\delta_1(\beta\gamma - 1)\big) = \delta_0\beta(1-\gamma)$$

$$\Leftrightarrow Q \equiv Q_W^* = \frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma - 1)} = Q^*.$$

Replacing Q_W^* into expression (B2.10), we get that the social planner's optimal investment threshold is equal to

$$X = \frac{\beta\left(\delta_0 + \delta_1 \frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)}{\frac{1}{1-\gamma} \frac{\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{1-\gamma}}{r-\mu} (\beta - 1)} \tag{B2.12}$$

$$\Leftrightarrow X = \frac{\beta\delta_0 + \beta\delta_1 \frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}}{\frac{\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{1-\gamma} (\beta - 1)}{(1-\gamma)(r-\mu)}}$$

$$\Leftrightarrow X = \frac{\beta\delta_0(1-\gamma)(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}}{\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)(\beta - 1)} + \frac{\beta\delta_1\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)(1-\gamma)(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}}{\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)(\beta - 1)}$$

$$\Leftrightarrow X = \frac{(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}}{\frac{\beta - 1}{\delta_1(\beta\gamma - 1)}} + \frac{\beta\delta_1(1-\gamma)(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}}{\beta - 1}$$

$$\Leftrightarrow X = \frac{\delta_1(\beta\gamma - 1)(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma} + \beta\delta_1(1-\gamma)(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}}{\beta - 1}$$

$$\Leftrightarrow X = \frac{\delta_1(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}\left(\beta\gamma - 1 + \beta(1-\gamma)\right)}{\beta - 1}$$

$$\Leftrightarrow X = \frac{\delta_1(r-\mu)\left(\frac{\delta_0 \beta(1-\gamma)}{\delta_1(\beta\gamma - 1)}\right)^{\gamma}(\beta\gamma - 1 + \beta - \beta\gamma)}{\beta - 1}$$

$$\Leftrightarrow X = \frac{\delta_1(r-\mu)\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}(\beta-1)}{\beta-1}$$

$$\Leftrightarrow X = \delta_1(r-\mu)\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}$$

$$\Leftrightarrow X \equiv X_W^* = (1-\gamma)X^*.$$

The (present) welfare loss in the monopoly scenario is equal to

$$\left(\frac{X(0)}{X_W^*}\right)^{\beta} TS(X_W^*, Q_W^*) - \left(\frac{X(0)}{X^*}\right)^{\beta} TS(X^*, Q^*) \tag{B2.13}$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{1}{1-\gamma}\frac{X_W^*(Q_W^*)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_W^*\right) - \left(\frac{X(0)}{X^*}\right)^{\beta}\left(\frac{1}{1-\gamma}\frac{X^*(Q^*)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q^*\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{1}{1-\gamma}\frac{X_W^*(Q_W^*)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_W^* - \left(\frac{X_W^*}{X^*}\right)^{\beta}\left(\frac{1}{1-\gamma}\frac{X^*(Q^*)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q^*\right)\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{X^*(Q^*)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)} - \left(\frac{(1-\gamma)X^*}{X^*}\right)^{\beta}\left(\frac{1}{1-\gamma}\frac{X^*(Q^*)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{X^*(Q^*)^{1-\gamma}}{r-\mu} - \delta_0 - \frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1} - (1-\gamma)^{\beta}\left(\frac{1}{1-\gamma}\frac{X^*(Q^*)^{1-\gamma}}{r-\mu} - \delta_0 - \frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\right)\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{1-\gamma}}{r-\mu} - \delta_0 - \frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\right.$$

$$-\left(\frac{X(0)}{X_W^*}\right)^{\beta}(1-\gamma)^{\beta}\left(\frac{1}{1-\gamma}\frac{\frac{\delta_1(r-\mu)}{1-\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{\gamma}\left(\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}\right)^{1-\gamma}}{r-\mu}-\delta_0-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_1}{1-\gamma}\frac{\delta_0\beta(1-\gamma)}{\delta_1(\beta\gamma-1)}-\delta_0-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}-(1-\gamma)^{\beta}\left(\frac{\delta_1\delta_0\beta(1-\gamma)}{(1-\gamma)^2\delta_1(\beta\gamma-1)}-\delta_0-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\right)\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_0\beta}{\beta\gamma-1}-\delta_0-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}-(1-\gamma)^{\beta}\left(\frac{\delta_0\beta}{(1-\gamma)(\beta\gamma-1)}-\delta_0-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\right)\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_0\beta}{\beta\gamma-1}-\delta_0-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}-(1-\gamma)^{\beta-1}\frac{\delta_0\beta}{\beta\gamma-1}+(1-\gamma)^{\beta}\delta_0+(1-\gamma)^{\beta}\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_0\beta}{\beta\gamma-1}\left(1-(1-\gamma)^{\beta-1}\right)-\delta_0\left(1-(1-\gamma)^{\beta}\right)-\frac{\delta_0\beta(1-\gamma)}{\beta\gamma-1}\left(1-(1-\gamma)^{\beta}\right)\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_0\beta\left(1-(1-\gamma)^{\beta-1}\right)-\delta_0(\beta\gamma-1)\left(1-(1-\gamma)^{\beta}\right)-\delta_0\beta(1-\gamma)\left(1-(1-\gamma)^{\beta}\right)}{\beta\gamma-1}\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_0\beta\left(1-(1-\gamma)^{\beta-1}\right)+(-\delta_0\beta\gamma+\delta_0)\left(1-(1-\gamma)^{\beta}\right)+(-\delta_0\beta+\delta_0\beta\gamma)\left(1-(1-\gamma)^{\beta}\right)}{\beta\gamma-1}\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{\delta_0\beta-\delta_0\beta(1-\gamma)^{\beta-1}-\delta_0\beta\gamma+\delta_0\beta\gamma(1-\gamma)^{\beta}+\delta_0-\delta_0(1-\gamma)^{\beta}-\delta_0\beta+\delta_0\beta(1-\gamma)^{\beta}+\delta_0\beta\gamma-\delta_0\beta\gamma(1-\gamma)^{\beta}}{\beta\gamma-1}\right)$$

$$=\left(\frac{X(0)}{X_W^*}\right)^{\beta}\left(\frac{-\delta_0\beta(1-\gamma)^{\beta-1}+\delta_0-\delta_0(1-\gamma)^{\beta}+\delta_0\beta(1-\gamma)^{\beta}}{\beta\gamma-1}\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta} \left(\frac{\delta_0\left[1 - \beta(1-\gamma)^{\beta-1} - (1-\gamma)^{\beta} + \beta(1-\gamma)^{\beta}\right]}{\beta\gamma - 1}\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta} \left(\frac{\delta_0\left[1 - (1-\gamma)^{\beta-1}\big(\beta + (1-\gamma) - \beta(1-\gamma)\big)\right]}{\beta\gamma - 1}\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta} \left(\frac{\delta_0\left[1 - (1-\gamma)^{\beta-1}(\beta + 1 - \gamma - \beta + \beta\gamma)\right]}{\beta\gamma - 1}\right)$$

$$= \left(\frac{X(0)}{X_W^*}\right)^{\beta} \left(\frac{\delta_0\left[1 - (1-\gamma)^{\beta-1}(1 - \gamma + \beta\gamma)\right]}{\beta\gamma - 1}\right).$$

B.3. Mathematical details of the follower's optimal decision rule

Given that in the duopoly scenario the market capacity is $Q = Q_L + Q_F$, the profit function of the follower firm can be expressed as

$$\pi_F(t) = P(t) \cdot Q_F(t) \tag{B3.1}$$

$$\Leftrightarrow \pi_F(t) = X(t) \cdot \big(Q(t)\big)^{-\gamma} \cdot Q_F(t)$$

$$\Leftrightarrow \pi_F(t) = X(t) \cdot Q_F(t) \cdot (Q_L + Q_F)^{-\gamma}.$$

The expected profit of the follower is then

$$E[\pi_F(t)] = E[X(t) \cdot Q_F(t) \cdot (Q_L + Q_F)^{-\gamma}] \tag{B3.2}$$

$$\Leftrightarrow E[\pi_F(t)] = Q_F(t)(Q_L + Q_F)^{-\gamma} \cdot E[X_t]$$

$$\Leftrightarrow E[\pi_F(t)] = Q_F(Q_L + Q_F)^{-\gamma} \cdot X \cdot e^{\mu \cdot t}.$$

Therefore, the follower's firm value function is

$$V_F^*(X, Q_L, Q_F) = E\left[\int_{t=0}^{\infty} \pi_F(t) exp(-rt)\, dt - \delta_0 - \delta_1 Q_F\right] \tag{B3.3}$$

$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = \int_{t=0}^{\infty} Q_F(Q_L + Q_F)^{-\gamma} \cdot X \cdot e^{\mu \cdot t} \cdot e^{-r \cdot t}\, dt - \delta_0 - \delta_1 Q_F$$

$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = XQ_F(Q_L + Q_F)^{-\gamma} \int_{t=0}^{\infty} e^{(\mu - r)t}\, dt - \delta_0 - \delta_1 Q_F$$

$$\Leftrightarrow V_F^*(X, Q_L, Q_F) = \frac{XQ_F(Q_L + Q_F)^{-\gamma}}{r - \mu} - \delta_0 - \delta_1 Q_F.$$

Maximising this function with respect to Q_F yields the optimal capacity level of the follower, $Q_F^*(X, Q_L)$. This can be determined by solving

$$\frac{\partial V_F^*(X, Q_L, Q_F)}{\partial Q_F} = 0 \tag{B3.4}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_F(Q_L + Q_F)^{-\gamma}}{r - \mu} - \delta_0 - \delta_1 Q_F\right)}{\partial Q_F} = 0$$

$$\Leftrightarrow \frac{X}{r - \mu} \frac{\partial(Q_F(Q_L + Q_F)^{-\gamma})}{\partial Q_F} - \delta_1 = 0$$

$$\Leftrightarrow \frac{X}{r - \mu}\left[\frac{\partial Q_F}{\partial Q_F}(Q_L + Q_F)^{-\gamma} + Q_F \frac{\partial((Q_L + Q_F)^{-\gamma})}{\partial Q_F}\right] - \delta_1 = 0$$

$$\Leftrightarrow \frac{X}{r - \mu}[(Q_L + Q_F)^{-\gamma} + Q_F(-\gamma(Q_L + Q_F)^{-\gamma - 1})] - \delta_1 = 0$$

$$\Leftrightarrow \frac{X}{r - \mu}\left[(Q_L + Q_F)^{-\gamma}\left(1 - \frac{\gamma Q_F}{Q_L + Q_F}\right)\right] - \delta_1 = 0$$

$$\Leftrightarrow \frac{X(Q_F^* + Q_L)^{-\gamma}}{r - \mu}\left(1 - \frac{\gamma Q_F^*}{Q_F^* + Q_L}\right) - \delta_1 = 0\ (considering\ Q_F^* \equiv Q_F).$$

For values of $X < X_F^*(Q_L)$, the follower firm is in the idle state, and the value of the firm is given by the option value

$$F_F(X) = A_F X^{\beta}. \tag{B3.5}$$

Combining the value function of the follower and the option value, the VMC yields

$$V_F^*(X, Q_L, Q_F) = F_F(X) \tag{B3.6}$$

$$\Leftrightarrow \frac{XQ_F(Q_L + Q_F)^{-\gamma}}{r - \mu} - \delta_0 - \delta_1 Q_F = A_F X^{\beta}.$$

Then, the SPC can be represented as

$$\left.\frac{\partial V_F^*(X,Q_L,Q_F)}{\partial X}\right|_{X=X^*}=\left.\frac{\partial F(X)}{\partial X}\right|_{X=X^*}. \quad \text{(B3.7)}$$

Incorporating the VMC result into the SPC and solving for X^*, we find that the investment threshold of the follower, with respect to Q_L, is given by

$$\left.\frac{\partial V_F^*(X,Q_L,Q_F)}{\partial X}\right|_{X=X^*}=\left.\frac{\partial F(X)}{\partial X}\right|_{X=X^*} \quad \text{(B3.8)}$$

$$\Leftrightarrow \frac{\partial\left(\frac{X^*Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_F\right)}{\partial X^*}=\frac{\partial\left(A_F(X^*)^\beta\right)}{\partial X^*}$$

$$\Leftrightarrow \frac{Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}=\beta A_F(X^*)^{\beta-1}$$

$$\Leftrightarrow \frac{Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}=\frac{\beta}{X^*}A_F(X^*)^\beta$$

$$\Leftrightarrow \frac{Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}=\frac{\beta}{X^*}\left[\frac{X^*Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_F\right]$$

$$\Leftrightarrow \frac{Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}=\frac{\beta Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}-\frac{\beta\delta_0}{X^*}-\frac{\beta\delta_1 Q_F}{X^*}$$

$$\Leftrightarrow (\beta-1)\frac{Q_F(Q_L+Q_F)^{-\gamma}}{r-\mu}=\frac{\beta\delta_0+\beta\delta_1 Q_F}{X^*}$$

$$\Leftrightarrow X_F^*(Q_L)=\frac{\beta}{\beta-1}\frac{(r-\mu)(\delta_0+\delta_1 Q_F^*)}{Q_F^*(Q_F^*+Q_L)^{-\gamma}} \quad (replacing\ Q_F\equiv Q_F^*\ and\ X\equiv X_F^*).$$

For values of $X\geq X_F^*(Q_L)$ and considering $X\equiv X^*(Q_L)$ and $Q_F\equiv Q_F^*$, the value of the follower firm can be rewritten as

$$V_F^*(X,Q_L,Q_F)=\frac{XQ_F(Q_L+Q_F)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_F \quad \text{(B3.9)}$$

$$\Leftrightarrow V_F^*(X,Q_L,Q_F^*)=\frac{\frac{\beta}{\beta-1}\frac{(r-\mu)(\delta_0+\delta_1 Q_F^*)}{Q_F^*(Q_F^*+Q_L)^{-\gamma}}Q_F^*(Q_L+Q_F^*)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_F^*$$

$$\Leftrightarrow V_F^*(Q_F^*)=\frac{\beta}{\beta-1}(\delta_0+\delta_1 Q_F^*)-\delta_0-\delta_1 Q_F^*$$

$$\Leftrightarrow V_F^*(Q_F^*)=\frac{\beta\delta_0+\beta\delta_1 Q_F^*-\beta\delta_0-\beta\delta_1 Q_F^*+\delta_0+\delta_1 Q_F^*}{\beta-1}$$

$$\Leftrightarrow V_F^*(Q_F^*)=\frac{\delta_0+\delta_1 Q_F^*}{\beta-1}.$$

Based on the previous results, it is possible to construct the value function for the follower firm as

$$V_F^*(X,Q_L)=\begin{cases}A_F(Q_L)X^{\beta} & if\ \ X<X_F^*(Q_L)\\ \dfrac{\delta_0+\delta_1 Q_F^*}{\beta-1} & if\ X\geq X_F^*(Q_L)\end{cases}. \tag{B3.10}$$

From the VMC and substituting X for $X_F^*(Q_L)$, it is possible to define $A_F(Q_L)$ as

$$\frac{XQ_F(Q_L+Q_F)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_F=A_F X^{\beta} \tag{B3.11}$$

$$\Leftrightarrow \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)(\delta_0+\delta_1 Q_F^*)}{Q_F^*(Q_F^*+Q_L)^{-\gamma}}Q_F^*(Q_L+Q_F^*)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_F^*=A_F X^{\beta}$$

$$\Leftrightarrow \frac{\beta}{\beta-1}(\delta_0+\delta_1 Q_F^*)-\delta_0-\delta_1 Q_F^*=A_F X^{\beta}$$

$$\Leftrightarrow (\delta_0+\delta_1 Q_F^*)\left(\frac{\beta}{\beta-1}-1\right)=A_F X^{\beta}$$

$$\Leftrightarrow (\delta_0+\delta_1 Q_F^*)\frac{1}{\beta-1}=A_F X^{\beta}$$

$$\Leftrightarrow \frac{(\delta_0+\delta_1 Q_F^*)\beta(r-\mu)}{(\beta-1)Q_F^*(Q_F^*+Q_L)^{-\gamma}}=A_F X^{\beta}\frac{\beta(r-\mu)}{Q_F^*(Q_F^*+Q_L)^{-\gamma}}$$

$$\Leftrightarrow X\equiv X_F^*(Q_L)=A_F X^{\beta}\frac{\beta(r-\mu)}{Q_F^*(Q_F^*+Q_L)^{-\gamma}}$$

$$\Leftrightarrow A_F(Q_L)=\frac{\left(X_F^*(Q_L)\right)^{1-\beta}}{\beta}\frac{Q_F^*(Q_F^*+Q_L)^{-\gamma}}{r-\mu}\ \left(considering\ A_F\equiv A_F(Q_L)\right).$$

By incorporating the investment threshold $X_F^*(Q_L)$ in expression (B3.4), it is possible to obtain the optimal investment capacity of the follower firm as follows

$$\frac{X(Q_F^*+Q_L)^{-\gamma}}{r-\mu}\left(1-\frac{\gamma Q_F^*}{Q_F^*+Q_L}\right)-\delta_1=0 \tag{B3.12}$$

$$\Leftrightarrow \frac{\frac{\beta}{\beta-1}\frac{(r-\mu)(\delta_0+\delta_1 Q_F^*)}{Q_F^*(Q_F^*+Q_L)^{-\gamma}}(Q_F^*+Q_L)^{-\gamma}}{r-\mu}\left(1-\frac{\gamma Q_F^*}{Q_F^*+Q_L}\right)-\delta_1=0$$

$$\Leftrightarrow \frac{\beta}{\beta-1}\frac{\delta_0+\delta_1 Q_F^*}{Q_F^*}\left(1-\frac{\gamma Q_F^*}{Q_F^*+Q_L}\right)-\delta_1=0$$

$$\Leftrightarrow \frac{\beta}{\beta-1}\left[\frac{\delta_0+\delta_1 Q_F^*}{Q_F^*}-\frac{\delta_0+\delta_1 Q_F^*}{Q_F^*}\frac{\gamma Q_F^*}{Q_F^*+Q_L}\right]=\delta_1$$

$$\Leftrightarrow \frac{(\delta_0+\delta_1 Q_F^*)(Q_F^*+Q_L)-(\delta_0+\delta_1 Q_F^*)\gamma Q_F^*}{Q_F^*(Q_F^*+Q_L)}=\frac{\delta_1(\beta-1)}{\beta}$$

$$\Leftrightarrow \frac{\delta_0 Q_F^*+\delta_0 Q_L+\delta_1(Q_F^*)^2+\delta_1 Q_L Q_F^*-\delta_0\gamma Q_F^*-\delta_1\gamma(Q_F^*)^2}{(Q_F^*)^2+Q_L Q_F^*}=\frac{\delta_1(\beta-1)}{\beta}$$

$$\Leftrightarrow \beta\delta_0 Q_F^* + \beta\delta_0 Q_L + \beta\delta_1 (Q_F^*)^2 + \beta\delta_1 Q_L Q_F^* - \beta\delta_0\gamma Q_F^* - \beta\delta_1\gamma (Q_F^*)^2 = \beta\delta_1 (Q_F^*)^2 - \delta_1 (Q_F^*)^2 + \beta\delta_1 Q_L Q_F^* - \delta_1 Q_L Q_F^*$$

$$\Leftrightarrow 0 = -\delta_1 (Q_F^*)^2 + \beta\delta_1\gamma (Q_F^*)^2 - \delta_1 Q_L Q_F^* - \beta\delta_0 Q_F^* + \beta\delta_0\gamma Q_F^* - \beta\delta_0 Q_L$$

$$\Leftrightarrow 0 = (\beta\gamma - 1)\delta_1 (Q_F^*)^2 + (-\beta\delta_0(1-\gamma) - \delta_1 Q_L) Q_F^* - \beta\delta_0 Q_L$$

$$\Leftrightarrow 0 = (\beta\gamma - 1)\delta_1 (Q_F^*)^2 - (\beta\delta_0(1-\gamma) + \delta_1 Q_L) Q_F^* - \beta\delta_0 Q_L$$

$$\Rightarrow Q_F^* = \frac{\beta\delta_0(1-\gamma) + \delta_1 Q_L \pm \sqrt{\left(-(\beta\delta_0(1-\gamma) + \delta_1 Q_L)\right)^2 - 4\cdot(\beta\gamma - 1)\delta_1 \cdot (-\beta\delta_0 Q_L)}}{2(\beta\gamma - 1)\delta_1}$$

$$\Leftrightarrow Q_F^* = \frac{\beta\delta_0(1-\gamma) + \delta_1 Q_L \pm \sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\cdot(-\beta\gamma + 1)\delta_1 \cdot \beta\delta_0 Q_L}}{2(\beta\gamma - 1)\delta_1}$$

$$\Leftrightarrow Q_F^* \equiv Q_F^*(Q_L) = \frac{\beta\delta_0(1-\gamma) + \delta_1 Q_L + \sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}{2\delta_1(\gamma\beta - 1)}.$$

For the investment capacity level in the previous expression, we should ignore the other root, as it yields negative values for Q_F^*.

As in subchapter 5.1.1, $\hat{Q}_L(X)$ represents the capacity level from which entry deterrence is possible if $Q > \hat{Q}_L(X)$. Then the investment threshold that separates the accommodation and entry deterrence strategies can be defined, for a given level X, as $X_F^*\left(\hat{Q}_L(X)\right) = X$.

The capacity level $\hat{Q}_L(X)$ is implicitly determined by the investment threshold in expression (B3.8), where $Q_F^* \equiv Q_F^*(X, Q_L)$ is from equation (B3.4). Consequently, $\hat{Q}_L(X)$ can be indirectly defined by

$$\frac{\beta}{\beta - 1}\frac{(r-\mu)\left(\delta_0 + \delta_1 Q_F^*(X, Q_L)\right)}{Q_F^*(X, Q_L)(Q_F^*(X, Q_L) + Q_L)^{-\gamma}} = X. \tag{B3.13}$$

B.4. Mathematical details of the leader's investment policy under the entry deterrence strategy

Given that in the entry deterrence strategy the market investment capacity is $Q = Q_L$, the steps to derive the leader's value function are similar to those in the monopolist scenario, but with $Q \equiv Q_L$. However, since the follower firm will eventually enter the market, it is necessary to incorporate a correction factor. Consequently, the value function of the leader firm is given by

$$V_L^{det}(X) = \frac{XQ_L^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L + \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L)Q_L\big(Q_L + Q_F^*(Q_L)\big)^{-\gamma}}{r-\mu} - \frac{X_F^*(Q_L)Q_L^{1-\gamma}}{r-\mu}\right). \tag{B4.1}$$

By differentiating this value function concerning Q_L, we can obtain an expression for the optimal capacity level $Q_L^{det}(X)$ as follows

$$\frac{\partial V_L^{det}(X, Q_L)}{\partial Q_L} = 0 \tag{B4.2}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_L^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L + \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L)Q_L\big(Q_L + Q_F^*(Q_L)\big)^{-\gamma}}{r-\mu} - \frac{X_F^*(Q_L)Q_L^{1-\gamma}}{r-\mu}\right)\right)}{\partial Q_L} = 0$$

$$\Leftrightarrow \frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu} - \delta_1 + \beta\left(\frac{X}{X_F^*}\right)^{\beta-1}\frac{-X\frac{\partial X_F^*}{\partial Q_L}}{(X_F^*)^2}\left(\frac{X_F^*Q_L(Q_L+Q_F^*)^{-\gamma}}{r-\mu} - \frac{X_F^*Q_L^{1-\gamma}}{r-\mu}\right)$$

$$+\left(\frac{X}{X_F^*}\right)^{\beta}\left[\frac{1}{r-\mu}\left(\frac{\partial X_F^*}{\partial Q_L}Q_L(Q_L+Q_F^*)^{-\gamma} + X_F^*\left[(Q_L+Q_F^*)^{-\gamma} + Q_L(-\gamma)(Q_L+Q_F^*)^{-\gamma-1}\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right]\right)\right]$$

$$-\left(\frac{X}{X_F^*}\right)^{\beta}\frac{1}{r-\mu}\left[\frac{\partial X_F^*}{\partial Q_L}Q_L^{1-\gamma} + X_F^*(1-\gamma)Q_L^{-\gamma}\right] = 0$$

$$\Leftrightarrow \frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu} - \delta_1 + \frac{-\beta X^{\beta}(X_F^*)^{-\beta}\left[Q_L(Q_L+Q_F^*)^{-\gamma} - Q_L^{1-\gamma}\right]}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}$$

$$+\left(\frac{X}{X_F^*}\right)^{\beta}\left[\frac{1}{r-\mu}\left(\frac{\partial X_F^*}{\partial Q_L}Q_L(Q_L+Q_F^*)^{-\gamma}+X_F^*(Q_L+Q_F^*)^{-\gamma}+\ X_F^*Q_L(-\gamma)(Q_L+Q_F^*)^{-\gamma-1}\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right)\right]$$

$$-\left(\frac{X}{X_F^*}\right)^{\beta}\frac{1}{r-\mu}\left[\frac{\partial X_F^*}{\partial Q_L}Q_L^{1-\gamma}+X_F^*(1-\gamma)Q_L^{-\gamma}\right]=0$$

$$\Leftrightarrow\frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu}-\delta_1+\frac{-\beta X^{\beta}(X_F^*)^{-\beta}\left[Q_L(Q_L+Q_F^*)^{-\gamma}-Q_L^{1-\gamma}\right]}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}+\left(\frac{X}{X_F^*}\right)^{\beta}\frac{1}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}Q_L(Q_L+Q_F^*)^{-\gamma}$$

$$+\left(\frac{X}{X_F^*}\right)^{\beta}\left[\frac{1}{r-\mu}\left(X_F^*(Q_L+Q_F^*)^{-\gamma}+X_F^*Q_L(-\gamma)(Q_L+Q_F^*)^{-\gamma-1}\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right)-\frac{1}{r-\mu}\left[\frac{\partial X_F^*}{\partial Q_L}Q_L^{1-\gamma}+X_F^*(1-\gamma)Q_L^{-\gamma}\right]\right]=0$$

$$\Leftrightarrow\frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu}-\delta_1+\frac{-\beta X^{\beta}(X_F^*)^{-\beta}Q_L(Q_L+Q_F^*)^{-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}+\frac{\beta X^{\beta}(X_F^*)^{-\beta}Q_L^{1-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}+\frac{X^{\beta}(X_F^*)^{-\beta}Q_L(Q_L+Q_F^*)^{-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}$$

$$+\frac{X^{\beta}(X_F^*)^{1-\beta}(Q_L+Q_F^*)^{-\gamma-1}}{r-\mu}\left[Q_L+Q_F^*-\gamma Q_L\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right]-\left(\frac{X}{X_F^*}\right)^{\beta}\frac{1}{r-\mu}\left[\frac{\partial X_F^*}{\partial Q_L}Q_L^{1-\gamma}+X_F^*(1-\gamma)Q_L^{-\gamma}\right]=0$$

$$\Leftrightarrow\frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu}-\delta_1+\frac{(1-\beta)X^{\beta}(X_F^*)^{-\beta}Q_L(Q_L+Q_F^*)^{-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}+\frac{X^{\beta}(X_F^*)^{1-\beta}(Q_L+Q_F^*)^{-\gamma-1}}{r-\mu}\left[Q_L+Q_F^*-\gamma Q_L\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right]$$

$$+\frac{\beta X^{\beta}(X_F^*)^{-\beta}Q_L^{1-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}-\frac{X^{\beta}(X_F^*)^{-\beta}Q_L^{1-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}-\frac{X^{\beta}(X_F^*)^{-\beta}X_F^*(1-\gamma)Q_L^{-\gamma}}{r-\mu}=0$$

$$\Leftrightarrow\frac{(1-\gamma)XQ_L^{-\gamma}}{r-\mu}-\delta_1+\frac{(1-\beta)X^{\beta}(X_F^*)^{-\beta}Q_L(Q_L+Q_F^*)^{-\gamma}}{r-\mu}\frac{\partial X_F^*}{\partial Q_L}+\frac{X^{\beta}(X_F^*)^{1-\beta}(Q_L+Q_F^*)^{-\gamma-1}}{r-\mu}\left[Q_L+Q_F^*-\gamma Q_L\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right)\right]$$

$$-\frac{X^{\beta}(X_F^*)^{-\beta}Q_L^{-\gamma}}{r-\mu}\left[(1-\beta)Q_L\frac{\partial X_F^*}{\partial Q_L}+(1-\gamma)X_F^*\right]=0.$$

Differentiating the investment threshold in expression (B3.8) with respect to Q_L gives

$$\frac{\partial X_F^*}{\partial Q_L} \tag{B4.3}$$

$$= \frac{\partial \left(\frac{\beta}{\beta - 1} \frac{(r-\mu)(\delta_0 + \delta_1 Q_F^*)}{Q_F^*(Q_F^* + Q_L)^{-\gamma}} \right)}{\partial Q_L}$$

$$= \frac{\beta}{\beta - 1} \frac{(r-\mu)\delta_1 \frac{\partial Q_F^*}{\partial Q_L} Q_F^*(Q_F^* + Q_L)^{-\gamma} - (r-\mu)(\delta_0 + \delta_1 Q_F^*) \left[\frac{\partial Q_F^*}{\partial Q_L} (Q_F^* + Q_L)^{-\gamma} + Q_F^*(-\gamma)(Q_F^* + Q_L)^{-\gamma-1} \left(\frac{\partial Q_F^*}{\partial Q_L} + 1 \right) \right]}{(Q_F^*(Q_F^* + Q_L)^{-\gamma})^2}$$

$$= \frac{\beta}{\beta - 1} \frac{(r-\mu) \left[\delta_1 \frac{\partial Q_F^*}{\partial Q_L} Q_F^*(Q_F^* + Q_L)^{-\gamma} - (\delta_0 + \delta_1 Q_F^*)(Q_F^* + Q_L)^{-\gamma} \left(\frac{\partial Q_F^*}{\partial Q_L} - \gamma Q_F^*(Q_F^* + Q_L)^{-1} \left(\frac{\partial Q_F^*}{\partial Q_L} + 1 \right) \right) \right]}{(Q_F^*)^2(Q_F^* + Q_L)^{-2\gamma}}$$

$$= \frac{\beta}{\beta - 1} \frac{(r-\mu) \left[\delta_1 \frac{\partial Q_F^*}{\partial Q_L} Q_F^* - (\delta_0 + \delta_1 Q_F^*) \left(\frac{\partial Q_F^*}{\partial Q_L} - \gamma Q_F^*(Q_F^* + Q_L)^{-1} \left(\frac{\partial Q_F^*}{\partial Q_L} + 1 \right) \right) \right]}{(Q_F^*)^2(Q_F^* + Q_L)^{-\gamma}}$$

$$= \frac{\beta}{\beta - 1} \frac{(r-\mu) \left[\delta_1 \frac{\partial Q_F^*}{\partial Q_L} Q_F^* - \delta_0 \frac{\partial Q_F^*}{\partial Q_L} + \delta_0 \gamma Q_F^*(Q_F^* + Q_L)^{-1} \frac{\partial Q_F^*}{\partial Q_L} + \delta_0 \gamma Q_F^*(Q_F^* + Q_L)^{-1} - \delta_1 Q_F^* \frac{\partial Q_F^*}{\partial Q_L} + \gamma \delta_1 (Q_F^*)^2 (Q_F^* + Q_L)^{-1} \frac{\partial Q_F^*}{\partial Q_L} \right]}{(Q_F^*)^2(Q_F^* + Q_L)^{-\gamma}}$$

$$+ \frac{\beta}{\beta - 1} \frac{(r-\mu)\gamma \delta_1 (Q_F^*)^2 (Q_F^* + Q_L)^{-1}}{(Q_F^*)^2(Q_F^* + Q_L)^{-\gamma}}$$

$$= \frac{\beta}{\beta - 1} \frac{(r-\mu) \left[-\delta_0 \frac{\partial Q_F^*}{\partial Q_L} + \delta_0 \gamma Q_F^*(Q_F^* + Q_L)^{-1} \left(1 + \frac{\partial Q_F^*}{\partial Q_L} \right) + \gamma \delta_1 (Q_F^*)^2 (Q_F^* + Q_L)^{-1} \left(1 + \frac{\partial Q_F^*}{\partial Q_L} \right) \right]}{(Q_F^*)^2(Q_F^* + Q_L)^{-\gamma}}$$

$$= \frac{\beta}{\beta - 1} \frac{(r-\mu)(Q_F^* + Q_L)^{-1} \left[\gamma \delta_1 (Q_F^*)^2 \left(1 + \frac{\partial Q_F^*}{\partial Q_L} \right) - \delta_0 (Q_F^* + Q_L) \frac{\partial Q_F^*}{\partial Q_L} + \delta_0 \gamma Q_F^* \left(1 + \frac{\partial Q_F^*}{\partial Q_L} \right) \right]}{(Q_F^*)^2(Q_F^* + Q_L)^{-\gamma}}$$

$$= \frac{\beta}{\beta-1} \frac{(r-\mu)\left[\gamma\delta_1(Q_F^*)^2\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right) - \delta_0 Q_F^* \frac{\partial Q_F^*}{\partial Q_L} - \delta_0 Q_L \frac{\partial Q_F^*}{\partial Q_L} + \delta_0\gamma Q_F^* + \delta_0\gamma Q_F^* \frac{\partial Q_F^*}{\partial Q_L}\right]}{(Q_F^*)^2(Q_F^*+Q_L)^{1-\gamma}}$$

$$= \frac{\beta}{\beta-1} \frac{(r-\mu)\left[\gamma\delta_1(Q_F^*)^2\left(1+\frac{\partial Q_F^*}{\partial Q_L}\right) - \delta_0 Q_L \frac{\partial Q_F^*}{\partial Q_L} + \delta_0 Q_F^*\left(\gamma - (1-\gamma)\frac{\partial Q_F^*}{\partial Q_L}\right)\right]}{(Q_F^*)^2(Q_F^*+Q_L)^{1-\gamma}}.$$

Differentiating expression (B3.12) with respect to Q_L yields

$$\frac{\partial Q_F^*}{\partial Q_L} \tag{B4.4}$$

$$= \frac{\partial\left(\frac{\beta\delta_0(1-\gamma)+\delta_1 Q_L+\sqrt{(\beta\delta_0(1-\gamma)+\delta_1 Q_L)^2-4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}{2\delta_1(\gamma\beta-1)}\right)}{\partial Q_L}$$

$$= \frac{\left[\delta_1+\frac{1}{2}[(\beta\delta_0(1-\gamma)+\delta_1 Q_L)^2-4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)]^{-\frac{1}{2}}\cdot[2(\beta\delta_0(1-\gamma)+\delta_1 Q_L)\delta_1-4\beta\delta_0\delta_1(1-\gamma\beta)]\right]\cdot 2\delta_1(\gamma\beta-1)}{\left(2\delta_1(\gamma\beta-1)\right)^2}$$

$$= \frac{\delta_1+\frac{1}{2}[(\beta\delta_0(1-\gamma)+\delta_1 Q_L)^2-4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)]^{-\frac{1}{2}}\cdot[2(\beta\delta_0(1-\gamma)+\delta_1 Q_L)\delta_1-4\beta\delta_0\delta_1(1-\gamma\beta)]}{2\delta_1(\gamma\beta-1)}$$

$$= \frac{\delta_1}{2\delta_1(\gamma\beta-1)} + \frac{[(\beta\delta_0(1-\gamma)+\delta_1 Q_L)^2-4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)]^{-\frac{1}{2}}\cdot[(\beta\delta_0(1-\gamma)+\delta_1 Q_L)\delta_1-2\beta\delta_0\delta_1(1-\gamma\beta)]}{2\delta_1(\gamma\beta-1)}$$

$$= \frac{1}{2(\gamma\beta-1)} + \frac{\beta\delta_0(1-\gamma)+\delta_1 Q_L-2\beta\delta_0(1-\gamma\beta)}{2(\gamma\beta-1)\sqrt{(\beta\delta_0(1-\gamma)+\delta_1 Q_L)^2-4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}$$

$$= \frac{1}{2(\gamma\beta-1)}\left(1+\frac{\beta\delta_0-\beta\delta_0\gamma+\delta_1 Q_L-2\beta\delta_0+2\beta^2\delta_0\gamma}{\sqrt{(\beta\delta_0(1-\gamma)+\delta_1 Q_L)^2-4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}\right)$$

$$= \frac{1}{2(\beta\gamma - 1)}\left(1 - \frac{-\beta\delta_0 + \beta\delta_0\gamma - \delta_1 Q_L + 2\beta\delta_0 - 2\beta^2\delta_0\gamma}{\sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}\right)$$

$$= \frac{1}{2(\beta\gamma - 1)}\left(1 - \frac{\beta\delta_0 + \beta\delta_0\gamma - \delta_1 Q_L - 2\beta^2\delta_0\gamma}{\sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}\right)$$

$$= \frac{1}{2(\beta\gamma - 1)}\left(1 - \frac{\beta\delta_0(1 + \gamma - 2\beta\gamma) - Q_L\delta_1}{\sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}\right)$$

$$= \frac{1}{2(\beta\gamma - 1)}\left(1 - \frac{\beta(1 + (1-2\beta)\gamma)\delta_0 - Q_L\delta_1}{\sqrt{(\beta\delta_0(1-\gamma) + \delta_1 Q_L)^2 - 4\beta\delta_0\delta_1 Q_L(1-\gamma\beta)}}\right).$$

It is possible to rewrite the leader's value function from expression (B4.1) as follows

$$V_L^{det}(X, Q_L) = \frac{X Q_L^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L + \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L) Q_L \left(Q_L + Q_F^*(Q_L)\right)^{-\gamma}}{r-\mu} - \frac{X_F^*(Q_L) Q_L^{1-\gamma}}{r-\mu}\right) \tag{B4.5}$$

$$\Leftrightarrow V_L^{det}(X, Q_L) = \frac{X Q_L^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L + \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L) Q_L \left(Q_L + Q_F^*(Q_L)\right)^{-\gamma} - X_F^*(Q_L) Q_L^{1-\gamma}}{r-\mu}\right)$$

$$\Leftrightarrow V_L^{det}(X, Q_L) = \frac{X Q_L^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L + \left(\frac{X}{X_F^*(Q_L)}\right)^{\beta}\left(\frac{X_F^*(Q_L) Q_L \left[\left(Q_L + Q_F^*(Q_L)\right)^{-\gamma} - Q_L^{-\gamma}\right]}{r-\mu}\right)$$

$$\Leftrightarrow V_L^{det}(X) = \frac{X\left(Q_L^{det}(X)\right)^{1-\gamma}}{r-\mu} - \delta_0 - \delta_1 Q_L^{det}(X)$$

$$+ \left(\frac{X}{X_F^*\left(Q_L^{det}(X)\right)}\right)^{\beta}\left(\frac{X_F^*\left(Q_L^{det}(X)\right) Q_L^{det}(X)\left[\left(Q_L^{det}(X) + Q_F^*\left(Q_L^{det}(X)\right)\right)^{-\gamma} - \left(Q_L^{det}(X)\right)^{-\gamma}\right]}{r-\mu}\right)$$

$$\left(considering\ X_F^*(Q_L) \equiv X_F^*\left(Q_L^{det}(X)\right), Q_L \equiv Q_L^{det}(X), and\ Q_F^*(Q_L) \equiv Q_F^*\left(Q_L^{det}(X)\right)\right).$$

When $X < X_L^{det}$, the leader firm is in the idle state and holds an option value equal to

$$F_L(X) = A_L^{det} X^{\beta}. \tag{B4.6}$$

The VMC yields that

$$F_L(X) = V_L^{det}(X) \tag{B4.7}$$
$$\Leftrightarrow A_L^{det} X^{\beta} = V_L^{det}(X).$$

Therefore, the SPC gives that

$$\frac{\partial F_L(X)}{\partial X} = \frac{\partial V_L^{det}(X)}{\partial X} \tag{B4.8}$$
$$\Leftrightarrow \frac{\partial\left(A_L^{det} X^{\beta}\right)}{\partial X} = \frac{\partial V_L^{det}(X)}{\partial X}$$
$$\Leftrightarrow \beta A_L^{det} X^{\beta-1} = \frac{\partial V_L^{det}(X)}{\partial X}.$$

The optimal investment threshold X_L^{det} can be achieved via

$$\beta A_L^{det} X^{\beta-1} = \frac{\partial V_L^{det}(X)}{\partial X} \tag{B4.9}$$
$$\Leftrightarrow \frac{\beta}{X} A_L^{det} X^{\beta} = \frac{\partial V_L^{det}(X)}{\partial X}$$
$$\Leftrightarrow \frac{\beta}{X_L^{det}} V_L^{det}\left(X_L^{det}\right) = \left.\frac{\partial V_L^{det}(X)}{\partial X}\right|_{X=X_L^{det}} \quad \left(considering\ X \equiv X_L^{det}\ and\ incorporating\ the\ VMC\right)$$
$$\Leftrightarrow \frac{X_L^{det}}{\beta} \left.\frac{\partial V_L^{det}(X)}{\partial X}\right|_{X=X_L^{det}} = V_L^{det}\left(X_L^{det}\right).$$

B.5. Mathematical details of the leader's investment policy under the entry accommodation strategy

In the accommodation strategy, both firms invest, but the investment capacity of the follower firm depends on the investment capacity of the leader. Therefore, the total market capacity is given by $Q = Q_L + Q_F^*(X, Q_L)$. The profit function of the leader is then

$$\pi_L(t) = P(t) \cdot Q_L(t) \tag{B5.1}$$

$$\Leftrightarrow \pi_L(t) = X(t) \cdot \left(Q(t)\right)^{-\gamma} \cdot Q_L(t)$$

$$\Leftrightarrow \pi_L(t) = X(t) \cdot Q_L \cdot \left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma}.$$

The expected profit of the leader firm is then

$$E[\pi_L(t)] = E\left[X(t) \cdot Q_L \cdot \left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma}\right] \tag{B5.2}$$

$$\Leftrightarrow E[\pi_L(t)] = Q_L \cdot \left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma} \cdot E[X_t]$$

$$\Leftrightarrow E[\pi_L(t)] = Q_L\left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma} \cdot X \cdot e^{\mu \cdot t}.$$

Therefore, in the accommodation strategy, the value function of the leader is

$$V_L^{acc}(X, Q_L) = E\left[\int_{t=0}^{\infty} \pi_L(t) exp(-rt)\, dt - \delta_0 - \delta_1 Q_L\right] \tag{B5.3}$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \int_{t=0}^{\infty} Q_L\left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma} \cdot X \cdot e^{\mu \cdot t} \cdot e^{-r \cdot t}\, dt - \delta_0 - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = X Q_L\left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma} \int_{t=0}^{\infty} e^{(\mu - r)t}\, dt - \delta_0 - \delta_1 Q_L$$

$$\Leftrightarrow V_L^{acc}(X, Q_L) = \frac{X Q_L\left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma}}{r - \mu} - \delta_0 - \delta_1 Q_L.$$

Differentiating this value function with respect to Q_L, we find that the optimal capacity level of the leader firm, $Q_L^{acc}(X)$, is given by

$$\frac{\partial V_L^{acc}(X,Q_L)}{\partial Q_L}=0 \tag{B5.4}$$

$$\Leftrightarrow \frac{\partial\left(\frac{XQ_L\left(Q_L+Q_F^*(X,Q_L)\right)^{-\gamma}}{r-\mu}-\delta_0-\delta_1 Q_L\right)}{\partial Q_L}=0$$

$$\Leftrightarrow \frac{1}{r-\mu}\left[X\left(Q_L+Q_F^*(X,Q_L)\right)^{-\gamma}+XQ_L(-\gamma)\left(Q_L+Q_F^*(X,Q_L)\right)^{-\gamma-1}\left(1+\frac{\partial Q_F^*(X,Q_L)}{\partial Q_L}\right)\right]-\delta_1=0$$

$$\Leftrightarrow \frac{X\left(Q_L+Q_F^*(X,Q_L)\right)^{-\gamma-1}}{r-\mu}\left[Q_L+Q_F^*(X,Q_L)-\gamma Q_L\left(1+\frac{\partial Q_F^*(X,Q_L)}{\partial Q_L}\right)\right]-\delta_1=0.$$

Differentiating expression (B3.4) with respect to Q_L reveals that $\frac{\partial Q_F^*(X,Q_L)}{\partial Q_L}$ is equal to

$$\frac{\partial\left(\frac{X(Q_F^*+Q_L)^{-\gamma}}{r-\mu}\left(1-\frac{\gamma Q_F^*}{Q_F^*+Q_L}\right)-\delta_1\right)}{\partial Q_L}=0 \tag{B5.5}$$

$$\Leftrightarrow \frac{1}{r-\mu}\left[X(-\gamma)(Q_F+Q_L)^{-\gamma-1}\left(\frac{\partial Q_F}{\partial Q_L}+1\right)\right]\left(1-\frac{\gamma Q_F}{Q_F+Q_L}\right)+\frac{X(Q_F+Q_L)^{-\gamma}}{r-\mu}\left(-\frac{\gamma\frac{\partial Q_F}{\partial Q_L}(Q_F+Q_L)-\gamma Q_F\left(\frac{\partial Q_F}{\partial Q_L}+1\right)}{(Q_F+Q_L)^2}\right)=0$$

$$(substituting\ Q_F^*\equiv Q_F)$$

$$\Leftrightarrow \frac{X(Q_F+Q_L)^{-\gamma}}{r-\mu}\left[-\gamma(Q_F+Q_L)^{-1}\left(\frac{\partial Q_F}{\partial Q_L}+1\right)\right]\left(1-\frac{\gamma Q_F}{Q_F+Q_L}\right)+\frac{X(Q_F+Q_L)^{-\gamma}}{r-\mu}\left(-\frac{\gamma\frac{\partial Q_F}{\partial Q_L}(Q_F+Q_L)-\gamma Q_F\left(\frac{\partial Q_F}{\partial Q_L}+1\right)}{(Q_F+Q_L)^2}\right)=0$$

$$\Leftrightarrow \left[-\gamma(Q_F+Q_L)^{-1}\left(\frac{\partial Q_F}{\partial Q_L}+1\right)\right]\left(1-\frac{\gamma Q_F}{Q_F+Q_L}\right)-\frac{\gamma\frac{\partial Q_F}{\partial Q_L}(Q_F+Q_L)-\gamma Q_F\left(\frac{\partial Q_F}{\partial Q_L}+1\right)}{(Q_F+Q_L)^2}=0$$

$$\Leftrightarrow -\gamma(Q_F+Q_L)\left(\frac{\partial Q_F}{\partial Q_L}+1\right)\left(1-\frac{\gamma Q_F}{Q_F+Q_L}\right)-\gamma\frac{\partial Q_F}{\partial Q_L}(Q_F+Q_L)+\gamma Q_F\left(\frac{\partial Q_F}{\partial Q_L}+1\right)=0$$

$$\Leftrightarrow -\gamma(Q_F+Q_L)\frac{\partial Q_F}{\partial Q_L}+\gamma(Q_F+Q_L)\frac{\partial Q_F}{\partial Q_L}\frac{\gamma Q_F}{Q_F+Q_L}-\gamma(Q_F+Q_L)+\gamma(Q_F+Q_L)\frac{\gamma Q_F}{Q_F+Q_L}-\gamma\frac{\partial Q_F}{\partial Q_L}(Q_F+Q_L)+\gamma Q_F\left(\frac{\partial Q_F}{\partial Q_L}+1\right)=0$$

$$\Leftrightarrow -2\gamma(Q_F+Q_L)\frac{\partial Q_F}{\partial Q_L}+\gamma^2 Q_F\frac{\partial Q_F}{\partial Q_L}-\gamma(Q_F+Q_L)+\gamma^2 Q_F+\gamma Q_F\frac{\partial Q_F}{\partial Q_L}+\gamma Q_F=0$$

$$\Leftrightarrow -2(Q_F+Q_L)\frac{\partial Q_F}{\partial Q_L}+\gamma Q_F\frac{\partial Q_F}{\partial Q_L}-Q_F-Q_L+\gamma Q_F+Q_F\frac{\partial Q_F}{\partial Q_L}+Q_F=0$$

$$\Leftrightarrow 2(Q_F+Q_L)\frac{\partial Q_F}{\partial Q_L}-\gamma Q_F\frac{\partial Q_F}{\partial Q_L}+Q_L-\gamma Q_F-Q_F\frac{\partial Q_F}{\partial Q_L}=0$$

$$\Leftrightarrow \frac{\partial Q_F}{\partial Q_L}[2(Q_F+Q_L)-\gamma Q_F-Q_F]=\gamma Q_F-Q_L$$

$$\Leftrightarrow \frac{\partial Q_F}{\partial Q_L}[2Q_F+2Q_L-\gamma Q_F-Q_F]=\gamma Q_F-Q_L$$

$$\Leftrightarrow \frac{\partial Q_F}{\partial Q_L}[2Q_L+Q_F-\gamma Q_F]=\gamma Q_F-Q_L$$

$$\Leftrightarrow \frac{\partial Q_F}{\partial Q_L}[2Q_L+(1-\gamma)Q_F]=\gamma Q_F-Q_L$$

$$\Leftrightarrow \frac{\partial Q_F^*(X,Q_L)}{\partial Q_L}=\frac{\gamma Q_F-Q_L}{2Q_L+(1-\gamma)Q_F}\left(substituting\ \frac{\partial Q_F}{\partial Q_L}\equiv\frac{\partial Q_F^*(X,Q_L)}{\partial Q_L}\right).$$

When the stochastic demand level $X<X_L^{acc}$, the leader holds an option value equal to

$$F_L(X)=A_L^{acc}X^{\beta}. \tag{B5.6}$$

Therefore, the VMC yields

$$F_L(X) = V_L^{acc}(X) \quad \text{(B5.7)}$$
$$\Leftrightarrow A_L^{acc} X^{\beta} = V_L^{acc}(X).$$

Therefore, from the SPC and combining it with the VMC, the optimal investment threshold X_L^{acc} is defined by

$$\frac{\partial F_L(X)}{\partial X} = \frac{\partial V_L^{acc}(X)}{\partial X} \quad \text{(B5.8)}$$
$$\Leftrightarrow \frac{\partial\left(A_L^{acc} X^{\beta}\right)}{\partial X} = \frac{\partial V_L^{acc}(X)}{\partial X}$$
$$\Leftrightarrow \beta A_L^{acc} X^{\beta-1} = \frac{\partial V_L^{acc}(X)}{\partial X}$$
$$\Leftrightarrow \frac{\beta}{X} A_L^{acc} X^{\beta} = \frac{\partial V_L^{acc}(X)}{\partial X}$$
$$\Leftrightarrow \frac{\beta}{X} V_L^{acc}(X) = \frac{\partial V_L^{acc}(X)}{\partial X}$$
$$\Leftrightarrow \frac{\beta}{X_L^{acc}} V_L^{acc}(X_L^{acc}) = \left.\frac{\partial V_L^{acc}(X)}{\partial X}\right|_{X=X_L^{acc}} \quad (considering\ X \equiv X_L^{acc})$$
$$\Leftrightarrow \frac{X_L^{acc}}{\beta} \left.\frac{\partial V_L^{acc}(X)}{\partial X}\right|_{X=X_L^{acc}} = V_L^{acc}(X_L^{acc}).$$

B.6. Mathematical details of the impact of uncertainty on the leader's strategic entry boundaries

From expression (B5.4) and considering that $X \equiv X_1^{acc}$ and $Q_F^*(X, Q_L) \equiv Q_F$, and incorporating the result form expression (B5.5), we obtain

$$\frac{X\left(Q_L + Q_F^*(X, Q_L)\right)^{-\gamma-1}}{r-\mu}\left[Q_L + Q_F^*(X, Q_L) - \gamma Q_L\left(1 + \frac{\partial Q_F^*(X, Q_L)}{\partial Q_L}\right)\right] - \delta_1 = 0 \quad \text{(B6.1)}$$
$$\Leftrightarrow \frac{X_1^{acc}(Q_L + Q_F)^{-\gamma}}{(r-\mu)(Q_L + Q_F)}\left[Q_L + Q_F - \gamma Q_L\left(1 + \frac{\gamma Q_F - Q_L}{2Q_L + (1-\gamma)Q_F}\right)\right] - \delta_1 = 0$$
$$\Leftrightarrow \frac{X_1^{acc}(Q_L + Q_F)^{-\gamma}}{r-\mu}\left[\frac{Q_L + Q_F - \gamma Q_L\left(1 + \frac{\gamma Q_F - Q_L}{2Q_L + (1-\gamma)Q_F}\right)}{Q_L + Q_F}\right] - \delta_1 = 0$$
$$\Leftrightarrow \frac{X_1^{acc}(Q_L + Q_F)^{-\gamma}}{r-\mu}\left[1 - \frac{\gamma Q_L}{Q_L + Q_F}\left(1 + \frac{\gamma Q_F - Q_L}{2Q_L + (1-\gamma)Q_F}\right)\right] - \delta_1 = 0.$$