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BUSINESS
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Finance Department

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I dedicate this work to my family

Statement of Integrity

Here I state that the present work followed the academic codes of conduct and ethics of ISCTE Business School and declare the absence of plagiarism and other sorts of copy right violations, which could disintegrate the integrity of the present dissertation.

Francisco Manuel P. C. de C. Rodrigues

Lisbon, 01/04/2025

United Nations | Department of Economic and Social Affairs
Sustainable Development

9 INDUSTRY, INNOVATION AND INFRASTRUCTURE

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Acronyms

BS – Black-Scholes Model

BSM – Black-Scholes-Merton Model

CDF – Cumulative Distribution Function

GBM – Geometric Brownian Motion

fBSM – Fractional Black-Scholes Model

fBm – Fractional Brownian Motion

H –Hurst Exponent

SDE – Stochastic Differential Equation

R/S – Rescaled Range Analysis

MMAR – Multifractal Model of Asset Returns

ARCH – Autoregressive conditional heteroskedasticity

SD – Stable Distribution

DOTM – Deep Out of the Money

DITM – Deep In the Money

OTM – Out of the Money

ITM – In the Money

OLS – Ordinary Least Squares

MAPE – Mean Absolute Percentage Error

LEAPS – Long-Term Equity Anticipation Securities

S&P500 – Standard and Poor's 500 Index

Resumo

Cada vez mais a valorização de produtos derivados é um tópico que tem atraído atenção dos investidores especialmente nas últimas décadas, quer seja pelos potencias lucros alavancados obtidos pela incorreta valorização destes contratos ou, devido a atividades de cobertura de risco. Os contratos de opções não são exceção, com o acesso liberalizado à negociação de Long-Term Equity Anticipation Securities através de plataformas focadas no pequeno investidor, a correta precificação destes produtos nunca foi tão importante. No entanto, existem poucos estudos efetuados na valorização destes contratos, tornando esta tarefa difícil.

Desta forma, foram desenvolvidos diversos modelos de valorização destes instrumentos, nomeadamente o Black-Scholes Model (1973) e Merton (1973), contudo a maioria destes apresentam fragilidades nas premissas que estabelecem, assumindo a ausência de memória de longo prazo do parâmetro da volatilidade, sobre a qual incide o foco desta dissertação, e a normalidade dos retornos logarítmicos, a qual é inverificável nos mercados (Taleb, 2007). Por sua vez, em 1999 Hu e Øksendal, visando este problema sugeriram introduzir o Fractional Black-Scholes Model, que propõe o uso do Fractional Brownian Motion o qual postula uma nova classe de movimento Browniano que exibe correlação entre as volatilidades passadas e recentes.

Este estudo propõe aplicar o Fractional Black-Scholes Model na valorização de opções europeias de longo-prazo (Long-Term Equity Anticipation Securities). Para isso, recolheu-se uma amostra de opções do S&P 500 com maturidades acima de um ano e diferentes strikes, e comparou-se o impacto da volatilidade fractal na avaliação destes contratos com os valores obtidos através do modelo Black-Scholes.

Palavras-Chave: Valorização, Opções, LEAPS, volatilidade, Black-Scholes-Merton Model, Fractional Black-Scholes Model

Classificação JEL: G00 e G13

Abstract

In recent decades, the valuation of derivative products is a topic that has increasingly caught the interest of investors, whether due to the potential leveraged profits obtained from the incorrect valuation of these contracts or, in some cases, due to risk hedging activities. Options contracts are no exception, with the liberalized access to Long-Term Equity Anticipation Securities trading through platforms focused on individual investors, the correct assessment of the price attributable to these products has never been more important. However, there are still very few studies conducted on the valuation of such contracts, making this task challenging.

Consequently, various valuation models have been developed for these instruments, namely the well-known Black-Scholes Model (1973) and Merton (1973). However, the majority exhibit weaknesses in the inherent assumptions, presuming no long-term memory of the volatility parameter, which is the focus of this dissertation, and accommodating the normality of logarithmic returns, which is unverifiable in real markets (Taleb, 2007). In 1999, Hu and Øksendal suggested addressing this issue by introducing the Fractional Black-Scholes Model, which proposes the use of Fractional Brownian Motion, a new class of Brownian motion that exhibits correlation between past and recent volatilities.

This study proposes applying the Fractional Black-Scholes Model to assess the effects of volatility for the valuation of long-term European options (Long-Term Equity Anticipation Securities). For this purpose, using a sample of the S&P500-based contracts with maturities exceeding one year and different strikes, this dissertation examines the impact of the fractional volatility versus that of the Black-Scholes model.

Key-words: Pricing, Options, LEAPS, volatility, Black-Scholes-Merton Model, Fractional Black-Scholes Model

JEL Classification: G00 and G13

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Chapter 1: Introduction

1. Contextualization

The inception of option pricing theory can be traced back to the work of Fischer Black and Myron Scholes that in 1973 published in “The Journal of Political Economy” a paper with the first steps for what is now known as the Black-Scholes option pricing formula. The work established a model with fundamental assumptions such as no-dividend paying stock, underlying that followed a log-normal distribution relatively to its price, as well as a constant and known short-term interest rate and volatility. In the same year, Robert Merton formalized and extended Black-Scholes theoretical and mathematical foundation, by independently deriving the same option pricing formula using Ito’s Lemma later called Black-Scholes-Merton Model, an enhanced framework with major contributions from the areas of continuous-time and stochastic processes. This adapted model postulated too the introduction of the underlying’s dividend; an assumption disregarded by the previous authors.

An initial concern emerging with Black-Scholes-Merton model were the fixed parameters such as the interest rate and in the core of our subject the standard deviation of stock returns (or volatility). Later on, more specifically in 1997, based on fractal mathematics, Mandelbrot introduced for the first time in finance the concepts of self-similarity (where an object is expanded or compressed by the same degrees without losing its resemblance) and self-affinity (where an object is expanded or compressed in different degrees without losing its resemblance), suggesting some kind of memory in financial markets. The fundamental assumptions of this rescaling effect impacted vastly the conceptualized BSM, since based on a previous paper published in 1968 from the same author titled " Fractional Brownian motions, fractional noises and applications”, Mandelbrot implemented a new series of Gaussian random functions designated as fractional Brownian Motion, these reinterpreted the GBM by using the Hurst exponent (1965), this new adjustment unlocked the uses of resemblance, a so far unexplored characteristic of time-series in the diffusion process of a financial asset, making the volatility of the underlying dependent onto a fractional parameter H.

Based on this new approach, Hu and Øksendal in the year 1999, derived a close form solution for the Black-Scholes formula incorporating the fractional Brownian motion, the result led to the revolutionary model known as the Fractional BSM, which was free from strong arbitrage and demonstrated completeness, though the model was only valid for Hurst parameters between ½ and 1, imposing some limitations as Hurst exponents can vary from [0;1[, nevertheless for the current purpose of this dissertation the previous restriction does not neglect the Hu and Øksendal fBSM application , since has proved Bayraktar et al. 2004 over the period decurrent from 1997 to 2000 S&P 500 exhibited a $H > 1/2$. Further improvements to this initial formulation where made, namely those performed by Rostek and Schöbel, that through a conditional expectation and an equilibrium pricing approach derived in 2010 a formula for the fBSM that admitted $0 < H < 1$, though not discussed in the current dissertation.

The framework to price European options under stochastic fractional volatility was completed.

A few decades before, the Chicago Board Option Exchange (CBOE) introduced the term LEAPS (or Long-Term Equity Anticipation Security) more specifically in 1990 and it was used to describe options with long-term maturities above 1 year (Shirazi, S. and Ismail, I., 2011), depending on the market scenario for the common investor this could represent a potential opportunity to substitute the holding of the underlying stock for a fraction of the price. This notion further ads to the importance of pricing these contracts correctly.

2. Definition of the Research Question and Clarification of its importance

Hence, assuming constant volatilities with no long-term memory for maturities over 1 year, like in the case of LEAPS, is imprudent. Therefore, in this work we propose to study the effects of fractional volatility on LEAPS contracts, through the use of the Fractional Black-Scholes Model on a sample of S&P500 long-maturity index contracts.

3. Research Hypothesis

Following this idea, the succeeding hypothesis were established:

H1: Demonstrate that the fBSM through Hu and Øksendal pricing formula improves the pricing of S&P500 LEAPS

H2: No evidence corroborates that the fBSM through Hu and Øksendal pricing formula leads to improvements in the pricing of S&P500 LEAPS

4. Statement of Research Questions and Objectives

Consequently, pursuant the objective and hypothesis settled for this dissertation, the main focus of this paper is to:

1. Compute S&P500 index option prices under the BS and fBSM for contracts with maturities over 1 year.
2. Plot the BS corresponding Volatility surfaces.
3. Assess the effects of the Hurst exponent on S&P500 LEAPS.
4. Compare the performance of the Models with real market data.

Chapter 2: Literature Review

2.1 The Black-Scholes Model (1973)

The option pricing theory gained its momentum mostly after in 1973 Fischer Black and Myron Scholes published in "The Journal of Political Economy" a paper with the first steps for the derivation of the now known as the Black-Scholes option pricing formula. This model assumed a market with only two assets:

1. DEFENITION 2.1. *A risky asset whose price is represented by St, where it's change can be defined by the following stochastic equation:*

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \sim N\left(\mu S_t \, dt, \sigma^2 {S_t}^2 \, dt\right), \quad (1)$$

for $t \geq 0$ and $\sigma > 0$, where W_t is defined as a Standard Brownian Motion, the price change with mean equal to $\mu S_t \, dt$ and variance equal to $\sigma^2 {S_t}^2 \, dt$.

DEFENITION 2.2. *The process W of the standard Brownian Motion (Barbosa, 2022) with the following properties:*

1. $W_0 = 0$.
2. $W_{t_2} - W_{t_1} \sim \mathrm{N}(0, t_2 - t_1), \forall t_2 > t_1$, *or in other words the process increments follow a normal distribution with mean equal to 0 and variance equal to $t = t_2 - t_1$, for which $W_t = \sqrt{t}\varepsilon$, considering $\varepsilon \sim N(0,1)$.*
3. $W_{t_2} - W_{t_1}$ *and* $W_{t_4} - W_{t_3}$ *are independently distributed.*
4. *The W increments are continuous in time and assume no jumps.*

2. DEFENITION 2.3. *And a riskless asset represented by B_t for which we have the following differential equation (Rodrigues, 2022):*

$$dB_t = rB_t \, dt, \text{ for } r \geq 0. \quad (2)$$

In the case of this dissertation, we assume that S_t is the S&P500 index quotation at moment t, where for a riskless asset $B_0 = 1$ the expected value for any risk-free investment is $B_t = e^{rt}$ (Rodrigues, F. 2022). Applying assumption 2) of the Brownian Motion and considering $\mu = \mathrm{r}$ we get that:

THEOREM 2.1. *Under the standard Brownian motion the price at time t of a risky asset S is given by:*

(*see Appendix A*)

$$S_t = S_0 \exp\left[\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\varepsilon\right], for\ \varepsilon \sim N(0,1) \quad (3)$$

where $\left(r - \frac{1}{2}\sigma^2\right)dt$ *is the drift term and* σdW_t *is the stochastic component.*

As immediately noticeable this stochastic element leads the underlying S_t logarithmic returns to consider a constant volatility, which value for each maturity is computed based on the probability of the Cumulative Distribution Function for the standard normal distribution. Which for equation (7) raises 1 main problem, especially over long maturities, the mispricing of the contract.

Considering the GBM with its assumption of risk-neutral environment, the underlying asset price at moment t can be described via equation (3). As a result, the present value of a call future conditional cash-flow is deductible.

THEOREM 2.2. *For a European call the present value of its future conditional cashflow under a risk-neutral environment is:*

(Barbosa, 2022)

$$c_0 = e^{-rT}E_Q\left[(S_T - K)I_{S_T>K}|\mathcal{F}_0\right] \text{ for a put is } (K - S_T) \text{ for } S_T < K. \quad (4)$$

This expected value leads therefore to d_2 and d_1, that under the standard normal cumulative distribution function (CDF) represent respectively the probability of the call/put to be exercised at maturity, and the contract delta or its sensitivity to the underlying price changes. Please find the respective derivations in Appendix B (equation (1) & equation (2)). Subsequently steering us to the Fischer Black and Myron Scholes well known d_2 and d_1 formula.

THEOREM 2.3. *Under the Black-Scholes pricing formula the auxiliary variables d_1 and d_2 are defined as:*

(Barbosa, 2022)

$$d_2 = \frac{\ln\frac{S_0}{K} + \left(\mathrm{r} - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{5}$$

$$d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln\frac{S_0}{K} + \left(\mathrm{r} + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{6}$$

where K represents the option strike, S_0 *the spot price at moment 0,* S_T *the future unknown underlying asset price at expiry,* T *is regarded as the time to maturity,* r *is the annual constant risk-free interest rate, and* σ *is the annual constant standard deviation of the underlying return.*

Applying the previous probabilities to equation (4), the Black-Scholes Model for European options pricing appears:

THEOREM 2.4. *The Black-Scholes pricing formula that calculates the theoretical present value of an option is defined as:*

(*see Appendix C*)

(Barbosa, 2022) $$v_0 = \phi S_0 N(\phi d_1) - e^{-rT}\phi K N(\phi d_2) \tag{7}$$

$$where\ \phi\ is\ equal\ to\ 1\ in\ the\ case\ of\ a\ call\ or\ -1\ in\ the\ case\ of\ a\ put$$

The above option pricing formula established in 1973, the following main assumptions:

1. The option was European-style.
2. Constant and known short-term interest rate.
3. No-dividend paying stock.
4. The stock follows a random walk process in continuous time, with a log-normal distribution of stock prices.
5. Constant variance rate of the stock returns.
6. No transaction costs assumed in buying or selling the underlying or the contract.
7. No adverse effects to short selling.
8. Possibility to borrow at the short-term interest rate any fraction of the price of a security.

2.1.1 The Black-Scholes-Merton Model (1973)

The original Black-Scholes model presented major flaws one of them was the non-dividend assumption that in practice represented a significant element of option pricing, specially when considering dividend-paying stocks. In 1973, Merton by independently deriving what would become to be known as the Black-Scholes-Merton formula introduced the stock discount factor e^{-qT} that considered the impact of the underlying's dividends in the stock price.

THEOREM 2.5. *The Black-Scholes and Merton pricing formula that calculates the theoretical present value of an option is defined as:*

$$v_0 = \phi S_0 e^{-qT} N(\phi d_1) - e^{-rT} \phi K N(\phi d_2) \tag{8}$$

with,

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(\mathrm{r} - \mathrm{q} + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \ and \ d_2 = d_1 - \sigma\sqrt{T} \tag{9}$$

$$where \ q \ is \ the \ dividend \ yield$$

2.2 Fractal Mathematics in Finance

In 1962, while analysing cotton prices, Mandelbrot discovered that those didn't follow a Gaussian stationary random walk, he demonstrated that price changes on financial assets followed a Lévy stable distribution (which can be represented by a power law) unlocking a conceptual structure capable to represent objects that exhibited long-term dependence. In his book "The Misbehaviour of Capital Markets" (2004), Mandelbrot compared this distribution with the Cauchy-Lorentz curve, where large data points contribute and dominate the rest, and the Gaussian curve, where no data point dictates the statistical outcome of the rest, and then combined the two approaches by changing the parameters in each (namely alpha) to produce more symmetric, asymmetric, or squatter/taller curves, that could assume the presence of things such as fat tails. In the early 20's Paul Lévy, a French mathematician, introduced stable distributions that largely contributed significantly to fractal Mathematics, since they exhibited meaningful properties such as scale invariance. The three main attributes of Lévy stable distribution (SD) are: (1) Invariance under addition, (2) Possession of their own domain of attraction, and (3) Admitting a canonical characteristic function. Although, Lévy SDs are not central in this work, since the fractal pricing models rely on the standard Gaussian distribution adapted to include fractal properties, the main focus of the current work will reside in the conceptual structure for what it's called a fractal – "a pattern or shape whose parts relate with the whole" (Mandelbort, 2004) - this dependence structure was characterized by four main properties:

-*Self-similarity*: resizing in the same scale for all directions.

- *Self-Affinity*: resizing in different scales for divergent directions.

- *Monofractality*: scaling of fractals in the same way at different points.

- *Multifractality*: scaling of fractals in different ways at different points.

This last attribute was used in the genesis of the Multifractal Model of Asset Returns (MMAR) that falls outside of the current dissertation scope, which accordingly to Mandelbrot, Fisher and Calvet (1996) was an alternative ARCH model for the distribution of prices. This methodology still contains long-tails as in Mandelbrot (1963) and long dependence Mandelbrot and Van Ness (1968), though not implying necessarily the existence of infinite variance. This model like fBm enabled too the return distribution's moments under time-rescaling.

2.3 Fractional Brownian Motion

Until 1968, the core of the Black-Scholes and Merton formula for European options pricing was the Geometric Brownian motion, that encapsulated the standard Brownian process on its stochastic component. This as discussed above lead to major flaws in volatility pricing, since as unveiled by Mandelbrot in 1962, prices and many other phenomena do not follow a standard Gaussian distribution, instead, they are ruled by a Lévy Stable Distribution, a power law distribution (that assumes fat tails, self-similarity or long-term memory, and infinite variance). As a result, the fractional Brownian Motion came as a generalization of the standard Brownian process in the sense of self-similarity. Though in the fractional Brownian Motion there still is a Gaussian distribution, this is an adaption of the Gaussian process that includes a zero-mean self-similar diffusion with stationary increments, guarantying a proper generalization and, therefore, ensuring assumptions such as the path's continuity over time. Subsequently it's gained flexibility through a dependence structure that approximates to the concept of memory and still assures the independence of increments for H=1/2 (Serrano, F. 2016). Consecutively, Mandelbrot and Van Ness (1968) defined the Fractional Brownian motion as:

DEFENITION 2.4. *Considering the following probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with a right-continuous and increasing filtration* $F = (\mathcal{F}_t)_{0 \leq t \leq T}$, *where ω is the set of all values of a random function and belongs to Ω the sample space, for which* $B_H(t, \omega)$ *is the reduced fractional Brownian motion, we've:*

1. $B_H(0, \omega) = b_0 = 0.$
2. $\mathbb{E}[B_H(\omega)] = 0$.
3. *The covariance function is represented as* $\mathbb{E}[B_H(t, \omega), B_H(s, \omega)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$, *for* $0 < s < t$, *where if H=1/2 then the covariance is the same as the standard Brownian motion (see Appendix D).*
4. *The variance of the fBm is* $Var[B_H(t, \omega)] = t^{2H}$ *(see Appendix E).*
5. $B_H(t_2, \omega) - B_H(t_1, \omega)$ *is independent of* $B_H(t_4, \omega) - B_H(t_3, \omega)$, *considering no overlapping of intervals* (t_1, t_2) *and* (t_3, t_4) *for* $t_1 < t_2 \leq t_3 < t_4$, *if and only if H=1/2, otherwise there is dependency (see Appendix. F).*
6. *The stochastic integral for the Fractional Brownian Motion as given in Appendix G.*
7. *Their increments are self-similar* $\{X(t_0 + \tau, \omega) - X(t_0, \omega)\} \triangleq \{h^{-H} X(t_0 + \tau, \omega) - X(t_0, \omega)\}$ *for h >0.*
8. *The increments are stationary with parameter H.*

To define any Gaussian process, it's necessary to address its second-order statistics (the process mean and covariance function). Hence, the following conditions must be fulfilled: (1) its expected value must have a mean of zero, which was already seen in the second condition of a fractional Brownian motion, and (2) the covariance function must be non-negative (see proof in Sottinen, 2003). The contributions of Hurst and Mandelbrot on Fractal Mathematics, then led to the utilization of fBm in Finance redefining the baseline assumptions of stochastic differential equations, and the foundation of option pricing theory. Additionally, Duncan, Hu, Pasik-Duncan (1991), found that fBm was not a semimartingale for $H \neq 1/2$. Therefore, the Itô-based stochastic calculus could no longer be applied, subsequently Ducan et. Al (2000) introduced the fractional Itô Theorem. Moreover, in 1994 Delbaen and Schachermayer proved that if the stochastic process isn't considered a semimartingale, like in fBm, there exists a weak form of arbitrage. To surpass that, Cheridito (2002) demonstrated that by considering arbitrarily small amounts of time between transactions one can exclude arbitrage from the model. As a result, the fBm representation was complete, with its foundation in the fractional white noise depicting its derivative, which will not be reviewed in much detail (please read Duncan, Hu, Pasik-Duncan (1991), Hida, Kuo, Potthoff and Striet (1993) and Grothaus, Kondratiev, Georgi and Dep. of Mech. and Math, Kiev Univ. (1998)), the process enabled to capture the long-term self-similarity in financial products that Mandelbrot and Van Ness defended.

2.3.1 Hurst Exponent

The hurst exponent was developed by the hydrologist Harold Edwin Hurst, that in 1906 while addressing a problem concerning the Nile River floods, found out that in most of the cases when plotting the number of years versus the maximum and minimum range of each data collection, the range widened exhibiting a three fourths-power law (K = 0.72). This differed from the expected squared root increase for the respective length of the time interval, similarly to the Brownian motion, arriving to the exponential relationship between the series length and R.

THEOREM 2.6. *The dam required height R derived from the σ the standard deviations of discharges from one year to the other can be defined as:*

(*see Appendix H*)

$$R = \sigma \left(\frac{N}{2}\right)^{K} \tag{10}$$

where N represents the number of years and the K the scaling exponent.

This formula had major impacts not only in hydrology, but also in other fields like finance. Mandelbrot later renamed the Hurst exponent by using the letter H instead, for seeming more appropriate than the initial K, in his book "The Misbehaviour of Capital Markets" (2004) he explains that by replacing the time elapsed square root power by H the underlying process assumes wider possibilities of exhibiting its full long-memory characteristics, being so he added that when H equal to ½ the data is perceived as independent, the stock memory is considered Gaussian following a standard GBM where no fat tails are considered and price increments are detached from one another. Adversely, when H < ½ the stock process it's said to have an antipersistent memory, or in other words the underlying repeatedly reverses its trend. Finally, if H > ½ the stock follows a persistent movement, displaying a reinforcing memory on one certain direction.

2.3.2 Rescaled Range Analysis

In 1969, Mandelbrot and James developed a non-parametric test know as Rescaled Range Analysis (R/S) to assess the long-term dependence in data series. This method does not rely in data normal fitting, resulting in the significant improvement of the suitability for financial products since they do not display in most cases normality. Additionally, this tool aims to validate whether over varying periods of time, the amount by which the data changes from maximum to minimum is greater or smaller than what was expected in case each data point were independent of the previous (its formula is provided in *Appendix I*). Therefore, for the purpose of this dissertation, the R/S statistic is going to be used in the computation of the optimal Hurst exponent for the provided data. Subsequently, similarly to the Hurst exponent interpretation when the ratio equals to ½ the data is perceived as independent, in case H > ½ data is persistent, in contrast, if H < ½ the series are anti-persistent.

2.4 Fractional Black-Scholes Model

2.4.1 Hu and Øksendal fBSM Pricing Formula (1999)

In the year 1999, following the Fractional Brownian Motion as presented by Mandelbrot, Hu and Øksendal derived, based on Wick Itô type integration, what became to be known as the Fractional Black-Scholes option pricing formula. The model made several assumptions:

1. DEFENITION 2.5. *For a fractional Brownian environment, the change of a risky asset whose price is represented by* St *can be defined by the following stochastic differential equation:*

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t^H \text{, with } S_0 = s > 0 \qquad (11)$$

for $t \geq 0$ *and* $\sigma > 0$ *, where* B_t^H *is defined as a Fractional Brownian Motion, the price change with mean equal to* $\mu S_t\, dt$ *and variance equal to* $\sigma^2 {S_t}^2\, dt$.

2. A constant Hurst parameter within the range of $\left[\frac{1}{2}; 1\right]$.
3. Stochastic differentials performed under Wick Itô.
4. The parameters volatility, drift and risk-free interest rate are constant.
5. The portfolio is defined using the Wick product.
6. DEFENITION 2.6. *The portfolio is Wick Itô admissible being self-financed and with:* $\nu \diamond S \in \hat{\mathcal{L}}_{\Phi}^{1,2}(\mathbb{R})$ *where* $\hat{\mathcal{L}}_{\Phi}^{1,2}(\mathbb{R})$ *is the space defined similarly to* $\mathcal{L}_{\Phi}^{1,2}(\mathbb{R})$ *though with* B_t^H *and* μ_Φ *are substituted by* $\hat{B}_t^H$ *and* $\hat{\mu}_\Phi$*, and finally* υ *is the amount that is invested in the stock.*
7. DEFENITION 2.7. *The portfolio is considered Wick self-financed, where for* $t \in [0; T]$*:*

$$\begin{aligned} dZ^{\theta}(t,\omega) &= u_t dA_t + \nu_t \diamond dS_t \\ &= u_t dA_t + \nu_t \diamond (\mu S_t\, dt + \sigma S_t\, dB_t^H) \\ &= u_t dA_t + \mu \nu_t \diamond S_t\, dt + \sigma \nu_t \diamond S_t\, dB_t^H \end{aligned} \tag{12}$$

for which we consider that υ_t *is the amount invested in the stock at time t and* u_t *is the amount invested in the money account.*

8. Short selling can be considered.
9. No-dividend.
10. Environment with no transaction costs or taxes.
11. Trading is continuous.
12. Securities are perfectly divisible.

THEOREM 2.7. *Accordingly, to Hu and Øksendal at time 0 the price of a Fractional Black-Scholes European Call/Put is given by the following formula:*

(Ostaszewicz, 2012)

$$v^H(0, S_0) = \phi S_0 N(\phi d_1^H) \; - \; \phi K e^{-rT} N(\phi d_2^H) \tag{13}$$

$$\text{where } \phi \text{ is equal to } 1 \text{ in the case of a call or } -1 \text{ in the case of a put}$$

and,

$$d_1^H = \frac{ln\left(\frac{S_0}{K}\right) + rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{14}$$

$$d_2^H = \frac{ln\left(\frac{S_0}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{15}$$

2.4.2 Li and Chen fBSM calibration (2014)

In 2014, Li and Chen formulated two new methods to derive the Hurst exponent and the fractional volatility. The first method relied on the fBSM inversely to derive Implied Hurst exponent and Fractional Volatility though this methodology was model dependent. Conversely the second approach by considering the model-free implied volatility as formulated by Britten – Jones & Neuberger (2000), surpassed the initial biases that appeared due to the introduction of Implied Volatility derived from BSM in the Fractal Ordinary Least Squares regression to extract the fractional volatility and the Hurst exponent. However, the model-free approach assumed that randomness followed a classical Brownian motion, which does not hold in the fractional Brownian market as randomness obeys to fBm. Besides the limitations, the authors derived expected variance without depending on any pricing model or any type of randomness by integrating option prices with different strikes (*see Appendix K*), using the fractional Itô Lemma (Bender 2003a). Subsequently, Li and Chen suggested the introduction of the expected variance in the dependent variable of the Ordinary Least Squares regression equation (18), to estimate fractal volatility and hurst exponent.

Chapter 3: Data & Methodology

3.1 Data Extraction

The pricing of an option contract based on the standard BSM approach requires six main variables:

1. Strike
2. Spot
3. Maturity
4. Volatility
5. Interest Rate
6. Dividend Yield

In the case of the fBSM, an additional variable should be considered, the Hurst exponent, computed through the Rescaled Range methodology for its initial guess as previously described, using further the Li and Chen (2014) model calibration procedure to estimate both fractional volatility and Hurst variable based on the different contract maturities. Further, as already mentioned in the literature review, the Hu and Øksendal (1999) fBSM model considers no dividend environment. To make the outputs comparable we've derived the BS model based IV without the dividend effect (the results are displayed in the Table 3.1. & 3.2. as well as in Figures 4.1 & 4.2 of Chapter 4).

Consequently, within the purpose of this dissertation, it was collected through Refinitiv a daily sample of SPX index option prices, for maturities ranging from 1 year to 5 years, the observation period will be of 4 weeks incurred from 1st to the 29th of April 2022, and will feature accordingly different Spots, Strikes and Implied Volatilities.

SPX Filtered Dataset - Main Statistics	
#Number of Data Points	20 124
#Number of Calls	10 062
#Number of Puts	10 062

Moneyness (S/K) - Calls/Puts	
Moneyness < 0,5	1,95%
0,5 < Moneyness < 1	47,77%
1 < Moneyness < 1,5	44,70%
1,5 < Moneyness < 2	5,58%
2 < Moneyness	0,00%

Maturities in years - Calls/Puts	
Avg. Maturity	1.968
1 < Maturity < 1,5	47.16%
1,5 < Maturity < 2	24.71%
2 < Maturity < 2,5	0.00%
2,5 < Maturity < 3	9.32%
3 < Maturity	18.81%

BSM based Implied Volality (w/ dividend)	
[Min; Max] Calls	[0.14816 ; 0.45081]
[Min; Max] Puts	[0.078856 ; 0.5539]
BSM based Implied Volality (w/o dividend)	
[Min; Max] Calls	[0.096204 ; 0.30601]
[Min; Max] Puts	[0.1755 ; 0.43647]

Table 3.1. Key Statistics SPX Filtered Options Dataset (Maturities > 1 year)

The dataset has the same number of calls and puts, with most calls within the slightly OTM (0.5 < Moneyness < 1: 47.77%) and most puts slightly ITM (0.5 < Moneyness < 1: 47.77%). In terms of Maturities most are in the 1 to 1.5 years range (47.16%), with the average being around 1.968 years.

Additionally, the above dataset was filtered in order to ensure the remotion of any contracts that didn't fulfilled the following arbitrage conditions: $C_{market} > max(S_0 - Ke^{-rT}; 0)$ or $P_{market} > max(Ke^{-rT} - S_0; 0)$.

The data above was filtered from the following dataset, containing the full length of maturities:

SPX Full Dataset - Main Statistics	
#Number of Data Points	308 670
#Number of Calls	154 335
#Number of Puts	154 335

Moneyness (S/K) - Calls/Puts	
Moneyness < 0.5	0.13%
0.5 < Moneyness < 1	47.00%
1 < Moneyness < 1.5	51.75%
1.5 < Moneyness < 2	1.04%
2 < Moneyness	0.08%

Maturities in years - Calls/Puts	
Avg. Maturity	0.3522
Maturity < 0.5	82.27%
0.5 < Maturity < 1	11.21%
1 < Maturity < 1.5	3.07%
1.5 < Maturity < 2	1.61%
2 < Maturity < 2.5	0.00%
2.5 < Maturity < 3	0.61%
3 < Maturity	1.23%

BSM based Implied Volality (w/ dividend)	
[Min; Max] Calls	[0.0974 ; 10.8226]
[Min; Max] Puts	[0.0648 ; 12.3124]
BSM based Implied Volality (w/o dividend)	
[Min; Max] Calls	[0.096204 ; 8.1211]
[Min; Max] Puts	[0.06475 ; 7.2186]

Table 3.2. Key Statistics SPX Full Options Dataset

It's worthwhile mentioning its composition as it will be used to calibrate the model. Subsequently, most maturities (or 93.5%) are below 1 year indicating the market clear preference for these. On the other hand, in terms of Moneyness there is a great concentration in the 0.5 to 1.5 range (or 98.8%) slightly tending for the ITM calls and OTM puts with 51.8%.

For the underlying applying the same tool as the one utilized in the collection of SPX data, daily quotes of the S&P500 Index (Ticket: ^GSPC) in the time interval ranging from the 5th of April 2018 to the 29th of April 2022 will be gathered in order to provide sufficient data for the parameter estimation, capturing the long-term resemblance patterns we are looking for.

S&P500 Dataset - Main Statistics	
#Number of Data Points	1 024
Avg. Quotation	3415.14
[Min; Max] Quotation	[2 246.30 ; 4 796.45]

Table 3.3. Key Statistics S&P500 Index Dataset

3.2 Methodology

To address the hypothesis defined under the introductory part of this work, it's proposed to use the following quantitative/statistic methods:

1. Use the Mandelbrot and James (1969c) Rescaled Range Analysis values and input them in an OLS regression to determine the initial guess for the Hurst parameter.

2. Validate the non-existence of $H \leq 1/2$ during the sample period, for the implementation of the Hu and Øksendal (1999) model.

3. Estimate both fractional volatility and Hurst variable through the use of Li & Chen (2014) calibration methodology for the different contract maturities, this will be achieved by integrating different option prices for different strikes and maturities to obtain the model free expected variance, as previously described see equation (6) in *Appendix K*, and then running the OLS regression using equation (18) to obtain the fractional variance and hurst exponent.

PROOF. *Let* $\sigma^2\, T^{2H}$ *be the unknown fractional time scaled variance, the inverse of the call option price can be written as:*

$$\sigma^2\, T^{2H} = f^{-1}(C; S_0, K, T, r) \tag{16}$$

then,

$$Var\left(ln\left(\frac{S_T}{S_0}\right)\right) = \sigma^2\, T^{2H} \tag{17}$$

Taking logarithm of the variance, Li & Chen (2014) obtained the OLS equation for the application of the model free variance calibration methodology:

$$ln(V) = ln(\,\sigma^2) + \,H\, ln(\,T^2) \tag{18}$$

4. Compute the BS based IV surface during selected days over the observation period.

5. Input the values resultant from the calibration of the sampled SPX data on the Hu and Øksendal (1999) derived fBSM pricing formula and extract the sample period outputs.

6. Compare with BS based market data for the same interval, using the MAPE statistics (Hyndman and Athanasopoulos, 2018):

THEOREM 2.8. *The Mean Absolute Percentage Error between the Forecast and Actual values is given by:*

$$MAPE\; = 100\; \times \frac{1}{n}\sum\left|\frac{(Forecasted\; - \;Actual)}{Actual}\right| \tag{17}$$

where n *represents the sample size.*

7. Compute the time-scaled volatility structure for ITM and OTM options under the fractional and market scenarios.

Chapter 4. Obtained Results

In the following section the exhibited results were computed using the MATLAB R2024b and Microsoft Excel software's version 2411, without any special add-ons for extra tools besides the ones available in each program.

The first step was made by testing the model adequacy. This was achieved through the implementation of the Rescaled Range analysis for the collected underlying data. For this purpose, a 512, 256, 128, 64, 32, 16, 8 and 4 days window was chosen to input the cumulative deviate r returns between the k maximum j shorter periods in the R/S formula as described by Mandelbrot and James, 1969. These time ranges were chosen once the R/S method requires that the number of observations is a logarithm of base 2 (in our case 1024 daily quotes of the S&P 500). This resulted in the following initial guess for Hurst parameter:

R/S Analysis - S&P 500

Sumsamples	2	4	8	16	32	64	128	256
Observations (n)	512	256	128	64	32	16	8	4
Average R/S	30.86	22.95	13.28	9.24	6.4	4.23	2.58	1.43

log (R/S)	3.4295	3.1335	2.5865	2.2234	1.8558	1.4423	0.9485	0.3602
log (n)	6.2383	5.5452	4.8520	4.1589	3.4657	2.7726	2.0794	1.3863

Hurst Exponent	**0.6219**
Standard Error	0.0190

Expected Hurst (GBM)	0.50
t-stat	6.4046
Number of Obs	8
Degrees of freedom	6
p-value	**0.0683%**

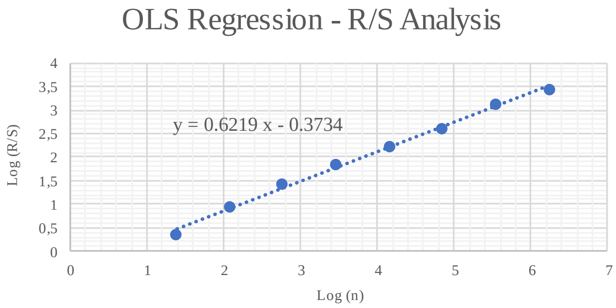


Table 4.1. R/S Analysis – S&P 500

As one can see by the p-value computed through the {=T.DIST.2T} excel function, the Hurst exponent of 0.6219 is significantly different from 0.5 (the baseline GBM) for all significance levels above 1%, thus corroborating the long-term persistence trend of the S&P500 and the necessity of using a time scaled fractional volatility in the option pricing of LEAPS. Additionally, this step provides us with the necessary evidence to implement Hu and Øksendal (1999) option pricing formula, once the underlying exhibits a Hurst exponent superior to 0.5.

On the other hand, as described in the literature review, Hu and Øksendal (1999) option pricing formula (like others please see Necula, 2002) does not allow dividend yield, since the extracted data from Refinitiv provided us IV values computed considering the SPX dividend yield, for comparison reasons we had to compute the no-dividend BS based IV, the volatility surfaces of the first and last trading day of each analysed month are as follows:

Figure 4.1. BS based IV surface - Last trading day (29th of April 2022)

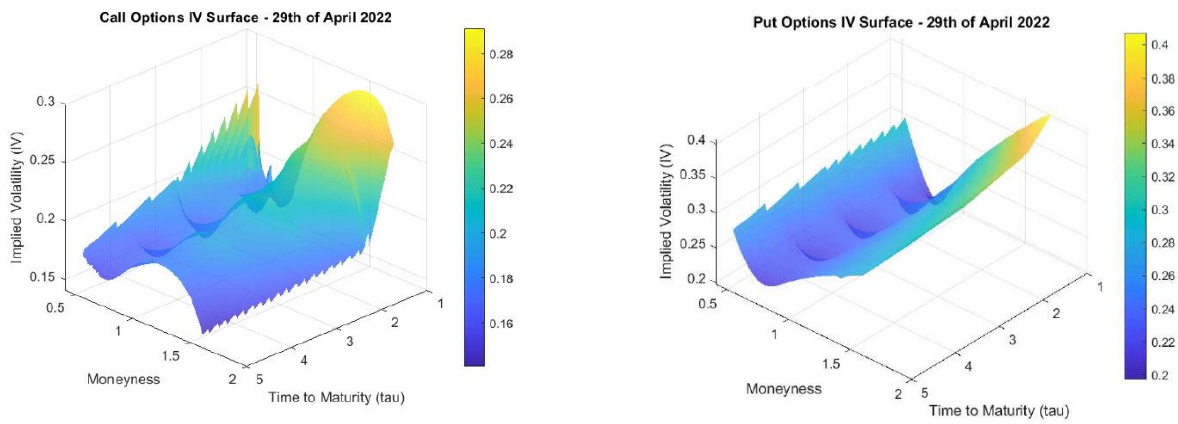
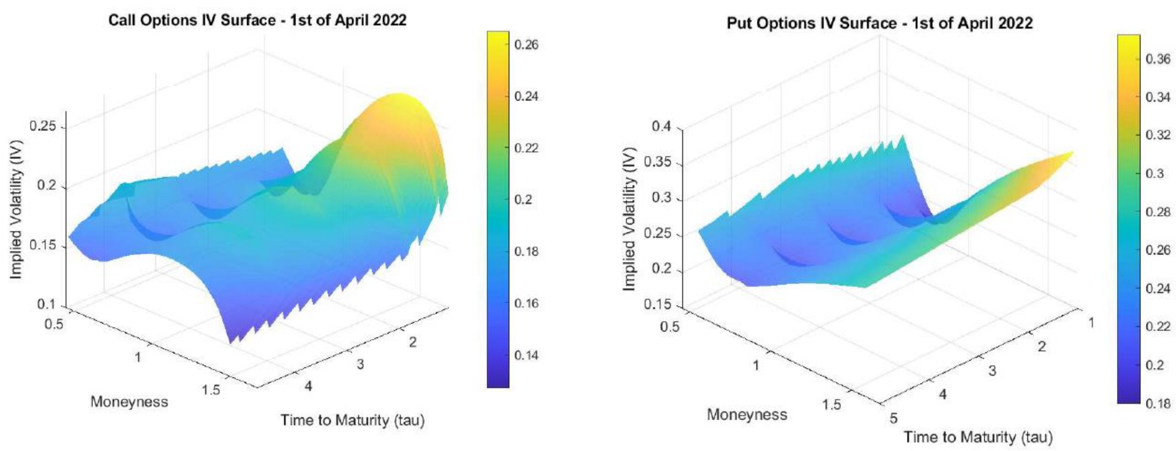


Figure 4.2. BS based IV surface -First trading day (1st of April 2022)



The figures obtained above were generated in MATLAB using the {scatteredInterpolant} function with the natural neighbour interpolation method. As one may notice for the two moments, both surfaces for each option type look very similar with calls presenting greater changes from the 1sr of April to the 29th as volatilities eased. Additionally, both types (calls and puts) have less extreme IVs as time to maturity increases. In terms of moneyness, the OTM in the puts and the DITM calls have highest IV values, with the lowest being in the ITM puts and OTM calls.

In order to compute the most approximate results to the market values, the Hu and Øksendal (1999) calibration is critical. For that purpose using a fractional OLS regression equation (18), on the full length maturities of the option chain for the sample period, the fBSM parameters can be settled, where the model free expected market variance is the dependent variable and the maturity squared is the independent term with fractional variance being the constant. By determining the slope of this line the calibrated hurst parameter is obtained.

Figure 4.3. Fractional OLS regression (from 5th to 29th of April 2022) – Full-length maturities

Fractional OLS regression – Full Length

Estimated Coefficients:

	Estimate	SE	tStat	pValue
$\ln(\sigma^2)$ - Intercept	-2.8317	0.001	-2 868.20	-
$\ln(T^2)$ - Coeff.	**0.5208**	0.000	2 360.80	-

Number of observations: 308670. Error degrees of freedom: 308668
Root Mean Squared Error: 0.319
R-squared: 0.948 Adjusted R-Squared: 0.948
F-statistic vs. constant model: 5.57e+06, p-value = 0
Coefficient of log_T_squared: 0.52081
t-statistic: 94.3216
p-value: 0

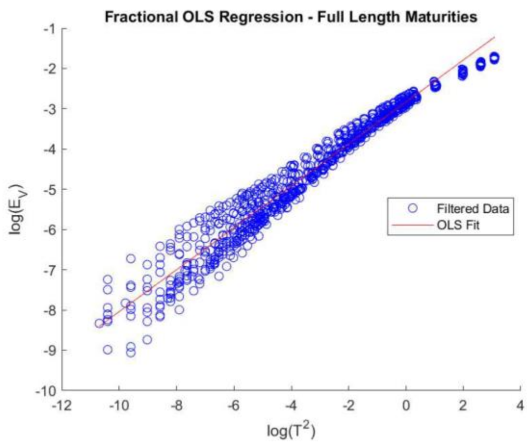


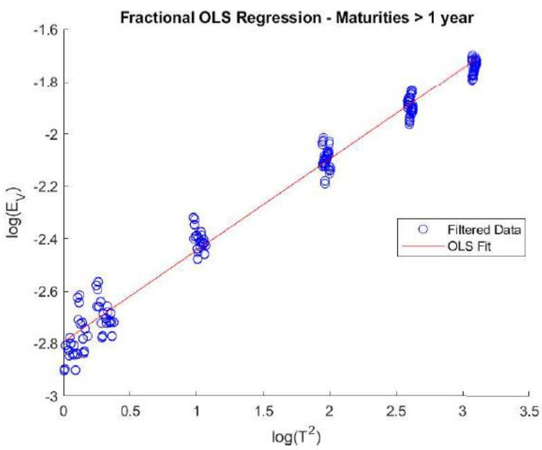
Figure 4.4. Fractional OLS regression (from 5th to 29th of April 2022) - Sample maturities > 1 year

Fractional OLS regression – Maturities > 1

Estimated Coefficients:

	Estimate	SE	tStat	pValue
$\ln(\sigma^2)$ - Intercept	-2.793	0.001	-4 449.50	-
$\ln(T^2)$ - Coeff.	**0.3487**	0.000	820.42	-

Number of observations: 20124. Error degrees of freedom: 20122
Root Mean Squared Error: 0.0619
R-squared: 0.971 Adjusted R-Squared: 0.971
F-statistic vs. constant model: 6.73e+05, p-value = 0
Coefficient of ln (T2): 0.34865
t-statistic: -356.137
p-value: 0



For the obtained results both p-values for the coefficient and Intercept on the Full-length and above 1 year Maturities appear equal to zero indicating their statistical significance for all significance levels above 1%, though in our case the statistical test of the regression outputs should be done versus the baseline scenario of the GBM, for which the time-scaled Implied volatility has and Hurst exponent of 0.5. As a result, the best way to access the significancy of each coefficient is by implementing the following hypothesis:

$$\boldsymbol{H_0}: The\ estimator\ is\ equal\ to\ 0.5$$

$$\boldsymbol{H_a}: The\ estimator\ is\ not\ equal\ to\ 0.5$$

The specified test obtained the following results:

Coefficient of log_T_squared	0.52081
t-statistic:	94.3216
p-value:	**0.000**

Table 4.2 Fractional OLS regression (from 5th to 29th of April 2022) – Full-length maturities

Coefficient of log_T_squared	0.34865
t-statistic:	-356.137
p-value:	**0.000**

Table 4.3. Fractional OLS regression (from 5th to 29th of April 2022) - Sample maturities > 1 year

As represented in both tables, the outputs of the test suggest that the coefficients are statistically different from 0.5, as for a significance level of 1% $>$ *p-value* is smaller lead to the null hypothesis rejection. Additionally, for the full-length maturities there was too a one tail test performed with the following hypothesis:

$$\boldsymbol{H_0}: The\ estimator\ is\ below\ or\ equal\ to\ 0.5$$

$$\boldsymbol{H_a}: The\ estimator\ is\ above\ 0.5$$

In this test the p-value obtained was 0.000 leading to the rejection of the null hypothesis stating that the coefficient is below or equal to 0.5.

These results clearly leave us with a biased looking for the LEAPS scenario as fewer options are traded for longer maturities, resulting in a slope of the full-length maturities with most of its data contribution coming from the range of expiries below 1 year, has options with shorter maturities have higher trading volumes/demand. On the case of LEAPS, the authors Li and Chen (2014) are not clear in the requirement of the full-length maturities to derive the implied fractional volatility and implied Hurst exponent. Furthermore, for the underlying the results of the R/S Analysis point to a Hurst parameter of 0.6219, which goes against the market sentiment for the sampling period. Additionally, the proposed model (fBSM - Hu and Øksendal 1999) points only to Hurst parameters within the persistency range [0.5; 1], clearly leaving us with no choice but to use the parameters derived from the full-length Fractional OLS regression, in order to test the adequacy of Hu and Øksendal (1999) fBSM pricing formula for the LEAPS options.

Subsequently, the Hu and Øksendal (1999) pricing formula given by equation (13) will be used for the calibrated Implied fractional volatility and Hurst parameters $intercept = ln(\sigma_f{}^2) \Leftrightarrow \sigma_f = e^{intercept/2} = e^{-2.8317/2} = 0.2427$ and $H = 0.5208$, assumed to be constant for all strikes and maturities. The following results were obtained comparing the given fBSM pricing formula with the market prices:

fBSM MAPE results for European Calls

Maturity	Moneyness: 0.42 - 0.64	0.64 - 0.86	0.86 - 1.08	1.08 - 1.30	1.30 - 1.53	1.53 - 1.75
1.00 - 1.62	*835.78%*	*412.15%*	*34.13%*	*1.43%*	*0.94%*	*0.23%*
1.62 - 2.24	*492.71%*	*250.82%*	*29.39%*	*2.75%*	*0.50%*	*0.67%*
2.24 - 2.86	*534.02%*	*176.43%*	*30.23%*	*7.38%*	*3.69%*	*2.81%*
2.86 - 3.48	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*
3.48 - 4.10	*468.88%*	*136.29%*	*31.41%*	*11.11%*	*6.53%*	*5.05%*
4.10 - 4.72	*447.45%*	*109.66%*	*32.43%*	*14.31%*	*9.27%*	*NaN*

fBSM MAPE results for European Puts

Maturity	Moneyness: 0.42 - 0.64	0.64 - 0.86	0.86 - 1.08	1.08 - 1.30	1.30 - 1.53	1.53 - 1.75
1.00 - 1.62	*0.55%*	*5.44%*	*8.67%*	*28.32%*	*64.00%*	*85.05%*
1.62 - 2.24	*0.27%*	*6.87%*	*8.02%*	*20.70%*	*52.18%*	*72.20%*
2.24 - 2.86	*0.87%*	*9.80%*	*8.26%*	*13.23%*	*35.65%*	*54.26%*
2.86 - 3.48	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*
3.48 - 4.10	*1.80%*	*11.47%*	*8.77%*	*8.79%*	*27.10%*	*42.30%*
4.10 - 4.72	*2.72%*	*12.23%*	*9.10%*	*5.85%*	*21.17%*	*NaN*

Table 4.4. MAPE statistics for fBSM Hu and Øksendal (1999) – Hurst and fractional vol. calibrated for all maturities

For a threshold of 5%, the MAPE statistics were analysed Table 4.4 indicates that there is a strong relationship between accuracy and Moneyness, since for options more in the DITM region Hu and Øksendal (1999) calibrated fBSM pricing formula exhibits significantly better results than the displayed for the OTM scenarios. On the other hand, the pricing of DITM calls seems to be more precise than the values obtained in the DITM puts.

Though not compatible with the proposed pricing formula, for comparison reasons, the introduction of the calibrated LEAPS only Hurst exponent ($H = 0.3487$) and respective fractional volatility ($\sigma_f = 0.2475$) obtained through the Li and Chen (2014) methodology was made, this less biased parameter clearly enhances the Hu and Øksendal (1999) fBSM model performance with options closer to the ITM area getting smaller MAPE errors:

fBSM MAPE results for European Calls

Maturity \ *Moneyness*	0.42 - 0.64	0.64 - 0.86	0.86 - 1.08	1.08 - 1.30	1.30 - 1.53	1.53 - 1.75
1.00 - 1.62	*859.72%*	*418.54%*	*34.67%*	*1.40%*	*0.91%*	*0.23%*
1.62 - 2.24	*343.20%*	*191.91%*	*20.54%*	*1.07%*	*0.61%*	*0.33%*
2.24 - 2.86	*238.92%*	*103.10%*	*13.37%*	*1.10%*	*0.70%*	*1.26%*
2.86 - 3.48	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*
3.48 - 4.10	*181.77%*	*65.82%*	*10.56%*	*2.13%*	*1.84%*	*2.25%*
4.10 - 4.72	*150.84%*	*44.39%*	*9.18%*	*3.35%*	*3.15%*	*NaN*

fBSM MAPE results for European Puts

Maturity \ *Moneyness*	0.42 - 0.64	0.64 - 0.86	0.86 - 1.08	1.08 - 1.30	1.30 - 1.53	1.53 - 1.75
1.00 - 1.62	*0.54%*	*5.50%*	*8.83%*	*27.85%*	*63.63%*	*84.86%*
1.62 - 2.24	*0.48%*	*4.36%*	*6.09%*	*31.35%*	*62.46%*	*80.18%*
2.24 - 2.86	*1.28%*	*2.26%*	*8.80%*	*36.12%*	*59.38%*	*75.20%*
2.86 - 3.48	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*	*NaN*
3.48 - 4.10	*1.74%*	*1.17%*	*14.55%*	*39.41%*	*59.53%*	*72.59%*
4.10 - 4.72	*2.48%*	*4.41%*	*19.50%*	*42.03%*	*60.11%*	*NaN*

Table 4.5. MAPE statistics for fBSM Hu and Øksendal (1999) – Hurst and fractional vol. calibrated for maturities above 1 year

The MAPE statistics displayed in Table 4.5 clearly suggest that the utilization of the Hurst exponent derived from LEAPS options only (instead of the full-length maturities) enhances the overall ITM strikes for both puts and calls when considering the same threshold as per above. On the other hand, the results for OTM strikes get worse for both scenarios when considering the LEAPS anti-persistency trend.

Furthermore, once the only parameters that differ between the fBSM and the BS are the fractional volatility/ BS based IV and the Implied Hurst exponent (which is equal to ½ for the standard BS model), to evaluate mispricing the comparison between the two was made. Moreover, since the BS derived market IV is a non-constant parameter that is dependent on the option maturities and moneyness as changing variables comparing it with the calibrated constant fractional volatility would distort the analysis of the fBSM model. Additionally, it would be unfair to do so once the advantage of the fractal environment lies on the time-scaled volatility. Subsequently, contrasting in terms of Moneyness the time-scaled volatility structure of the fBSM (represented by $\sigma_f T^H$) vs. the BS Market derived time-scaled IV (represented by $\sigma_{IV}\sqrt{T}$), can give us a better localized snapshot of the way the non-constant BSM based IVs behave against the constant fractional volatility:

Full length Maturities derived fractional vol. and Implied Hurst - For Moneyness = 0.88

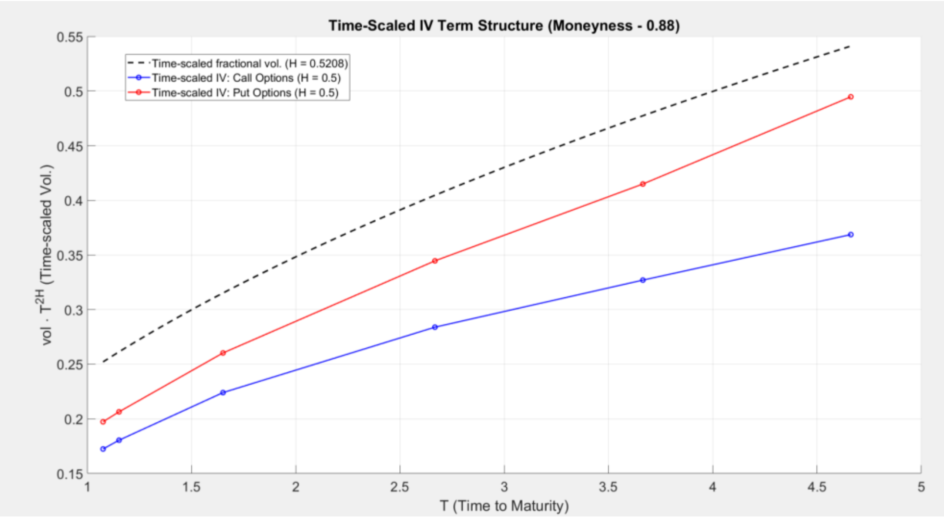


Figure 4.5. Full length Maturities derived fractional vol. and Implied Hurst - Moneyness = 0.88

Full length Maturities derived fractional vol. and Implied Hurst - For Moneyness = 1.13

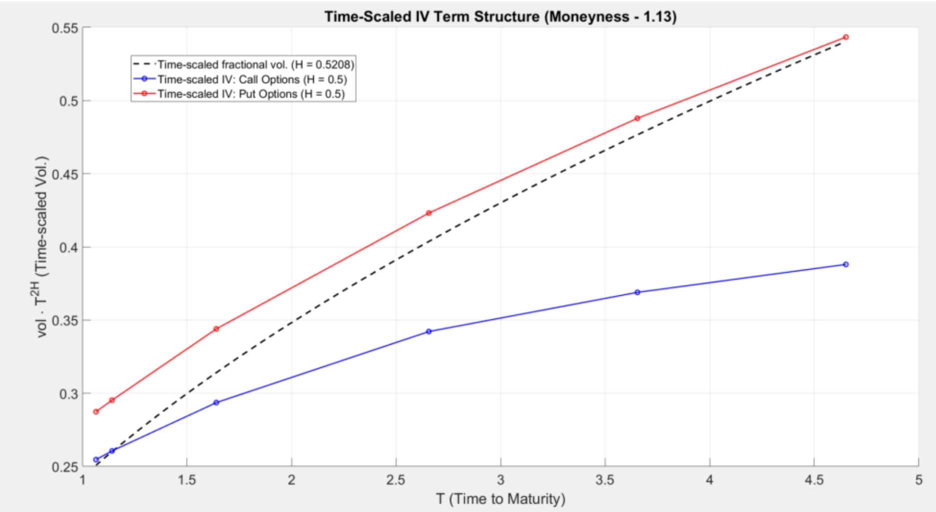


Figure 4.6. Full length Maturities derived fractional vol. and Implied Hurst - Moneyness = 1.13

Please consider that the outputs presented in this section do not encompass the average time-scaled volatility per month, instead they exhibit momentums of the option chain for which a certain moneyness level was observed. Following this notice, though some of the deviations may result from the snapshot not considering the effect of the different momentums with the same moneyness level, pulling/pushing away the average time-scaled Implied volatility closer/further to the calibrated time-scaled fractional volatility, the overall trend is presented where we can see for the Moneyness = 0.88 that puts have a greater accuracy than calls has presented in Table 4.4 with the error for OTM calls appearing to increase as maturities get larger, going against Table 4.4 findings as errors on average terms tend to decrease with the increase in time to maturity. For the Moneyness = 1.13, the put options display overall better pricing (which contradicts the overview obtained through Table 4.4) with larger expiries getting more accurate, on the other hand calls as expected start by having a greater pricing for *T* close to 1, diverging as maturities approach 4.7.

LEAPS derived fractional vol. and Implied Hurst - For Moneyness = 0.88

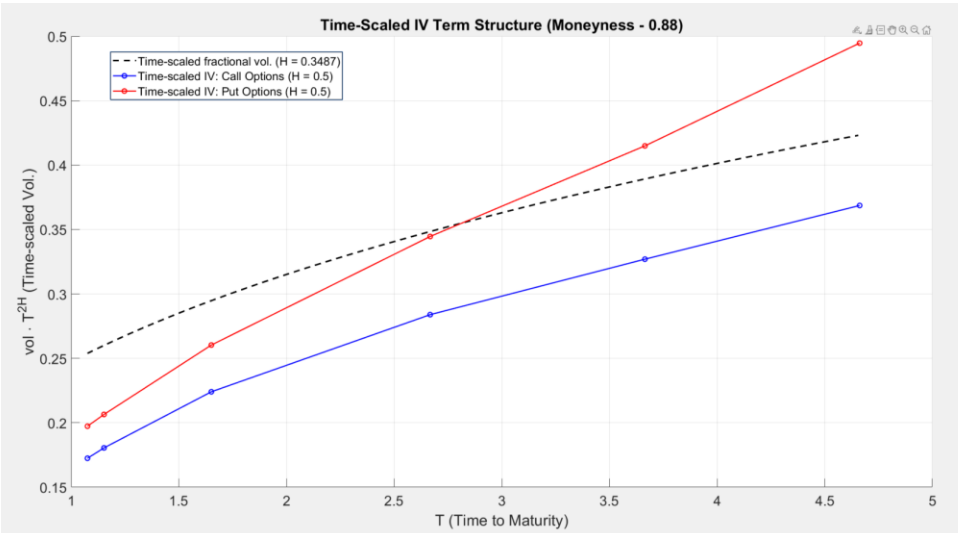


Figure 4.7. LEAPS derived fractional vol. and Implied Hurst - Moneyness = 0.88

LEAPS derived fractional vol. and Implied Hurst - For Moneyness = 1.13

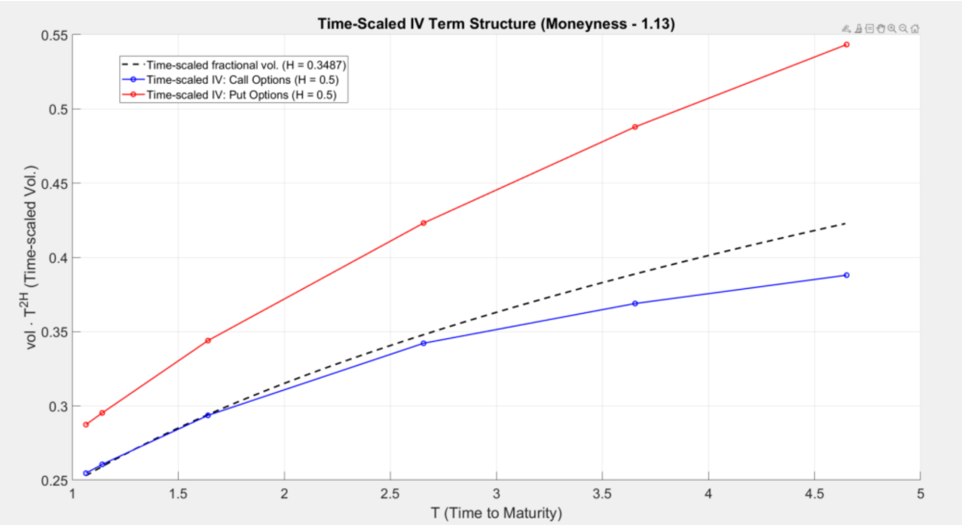


Figure 4.8. LEAPS derived fractional vol. and Implied Hurst - Moneyness = 1.13

For the OTM call and ITM put (with Moneyness = 0.88), both types exhibit errors though calls, as expected by the output obtained in Table 4.5, have greater deviations from the calibrated fractional volatility and Hurst exponent, while puts tended to be closer with greater errors near the maturities of 1 and 4.7 years. In contrast, for the OTM put and ITM call (with Moneyness = 1.13), the market derived fractional volatility and Hurst parameter appeared way more accurate in the ITM calls diverging slightly as maturities increase, with OTM puts showing the same divergent behaviour has *T* gets larger. For the most part, as concluded in the MAPE section the localized results unveil too better accuracy in the LEAPS derived fractional vol and Implied Hurst versus the full-length maturities.

Chapter 5. Conclusion

The present work studied the effects of the Fractional Black-Scholes model on the pricing of SPX LEAPS applying the Hu and Øksendal (1999) methodology. Overall, considering the market derived Hurst parameters and fractional volatility the model derived using only maturities above 1 year exhibited a clear outperformance versus the one obtained through the full spectrum of maturities available for the sampling period in SPX option chain. Despite of that, the obtained Hurst exponent was below the range defined by Hu and Øksendal (1999) (H= [0.5 ; 1]), standing for a clear violation of the model assumptions, using the local tests performed by comparing time-scaled BSM based IV versus time-scaled fractional volatility, the recorded snapshots displayed once again improvements from the usage of calibrated parameters through LEAPS only. Therefore, we can safely conclude that the Implied Hurst parameter for LEAPS might diverge significantly from the other maturities as these contracts have their own dynamics.

In terms of pricing this translates in a biased outlook for the Hu and Øksendal (1999) formula when aiming for all the available maturities, with results expected to be enhanced if a LEAPS only Implied Hurst and fractional volatility are implemented. As a result, for future studies we suggest that the implementation of Li and Chen (2014) calibration methodology in the LEAPS sample should be the first approach, as the persistency of the SPX underlying may not be in line with the Implied Hurst for long-term maturities.

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Appendix

A. BSM – Stochastic Process

PROOF.

(Rodrigues, 2022)

If we now consider a process X_t for which:

$(i)\ dX_t = \mu(t, X_t)\mathrm{dt} + \sigma(t, X_t)dW_t$

$(ii)\ And\ g(t, X_t)\ is\ 2\ times\ differentiable$

Using the Itô's lemma we get:

(Rodrigues, 2022)

$$dg = (\frac{\partial g}{\partial t} + \mu(t, X_t)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 g}{\partial x^2})\ dt + \sigma(t, X_t)\frac{\partial g}{\partial x}dW_t \tag{1}$$

Which for $g(x) = \log(x)$ considering S_t means as proven below that $\frac{S_t}{S_0}$ is lognormal with mean $\left(\mu - \frac{1}{2}\sigma^2\right)t$ and variance $\sigma^2 t$:

$$d\log(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \tag{2}$$

$$S_t = S_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right] \tag{3}$$

considering $W_t = \sqrt{t}\varepsilon$

$$S_t = S_0 \exp\left[\left(\mathrm{r} - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\varepsilon\right], for\ \varepsilon \sim N(0,1) \tag{4}$$

B. d1 and d2 – BSM

PROOF.

(Barbosa, 2022)

$$E_Q\left(I_{S_T>K}\middle|\mathcal{F}_0\right) = Q(S_T > K) \tag{1}$$

$$= Q(S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\varepsilon} > K)$$

$$= Q(\ln S_0 + \left(\mathrm{r} - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon > \ln K)$$

$$= Q\left(\varepsilon > -\frac{\ln\frac{S_0}{K} + \left(\mathrm{r} - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$= Q\left(\varepsilon < \frac{\ln\frac{S_0}{K} + \left(\mathrm{r} - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) = N(d_2)$$

$$E_Q\left(S_T I_{S_T>K}\middle|\mathcal{F}_0\right) = E_Q\left(S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\varepsilon} I_{S_T>K}|\mathcal{F}_0\right) \tag{2}$$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} E_Q\left(e^{\sigma\sqrt{T}\varepsilon} I_{S_T>K}|\mathcal{F}_0\right)$$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} E_Q \int_{-\infty}^{\infty} e^{\sigma\sqrt{T}\varepsilon} I_{S_T>K} \phi_{pdf}(\varepsilon) d\varepsilon$$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} E_Q \int_{-d_2}^{\infty} e^{\sigma\sqrt{T}\varepsilon} \phi_{pdf}(\varepsilon) d\varepsilon$$

where $\phi_{pdf}(\varepsilon)$ denotes the standard normal pdf $\phi_{pdf}(\varepsilon) = \frac{e^{-\frac{\varepsilon^2}{2}}}{\sqrt{2\pi}}$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} \int_{-d_2}^{\infty} e^{\sigma\sqrt{T}\varepsilon} \frac{e^{-\frac{\varepsilon^2}{2}}}{\sqrt{2\pi}} d\varepsilon$$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} \int_{-d_2}^{\infty} \frac{e^{-\frac{\varepsilon^2-2\sigma\sqrt{T}\varepsilon}{2}}}{\sqrt{2\pi}} d\varepsilon$$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} \int_{-d_2}^{\infty} \frac{e^{-\frac{\varepsilon^2-2\sigma\sqrt{T}\varepsilon+\sigma^2T-\sigma^2T}{2}}}{\sqrt{2\pi}} d\varepsilon$$

$$= S_0 e^{\left(\mathrm{r}-\frac{1}{2}\sigma^2\right)T} e^{\frac{\sigma^2}{2}T} \int_{-d_2}^{\infty} \frac{e^{-\frac{(\varepsilon-\sigma\sqrt{T})^2}{2}}}{\sqrt{2\pi}} d\varepsilon$$

$$= S_0 e^{rT} \int_{-d_2}^{\infty} \frac{e^{-\frac{(\varepsilon-\sigma\sqrt{T})^2}{2}}}{\sqrt{2\pi}} d\varepsilon$$

replacing $\varepsilon - \sigma\sqrt{T}$ by θ,

$$= S_0 e^{rT} \int_{-d_2-\sigma\sqrt{T}}^{\infty} \frac{e^{-\frac{\theta^2}{2}}}{\sqrt{2\pi}} d\theta$$

$$= S_0 e^{rT} \int_{-\infty}^{d_2+\sigma\sqrt{T}} \phi_{pdf}(\theta)\, d\theta$$

$$= S_0 e^{rT} N(d_1)$$

C. Price of an European Option using BSM

PROOF.

(Barbosa, 2022)

$$v_0 = e^{-rT} E_Q\left[(\phi S_T - \phi K) I_{\phi S_T > \phi K} | \mathcal{F}_0\right] \quad (1)$$

$$= e^{-rT}[E_Q\left(\phi S_T I_{\phi S_T>\phi K} \middle| \mathcal{F}_0\right) - \phi K E_Q\left(I_{\phi S_T>\phi K} \middle| \mathcal{F}_0\right)]$$

$$= e^{-rT}[\phi S_0 e^{rT}\left(S_T I_{\phi S_T>\phi K} \middle| \mathcal{F}_0\right) - \phi K E_Q\left(I_{\phi S_T>\phi K} \middle| \mathcal{F}_0\right)$$

$$= e^{-rT}[\phi S_0 e^{rT} N(\phi d_1) - \phi K N(\phi d_2)$$

$$= \phi S_0 N(\phi d_1) - e^{-rT} \phi K N(\phi d_2)$$

$$where\ \ is\ equal\ to\ 1\ in\ the\ case\ of\ a\ call\ or -1\ in\ the\ case\ of\ a\ put$$

D. Covariance fBm

PROOF.

(Ostaszewicz, 2012)

For H=1/2 consider the following covariance function:

$$\mathbb{E}[B_H(t,\omega), B_H(s,\omega)] = \frac{1}{2}\left(t^{2\frac{1}{2}} + s^{2\frac{1}{2}} - |t-s|^{2\frac{1}{2}}\right) \tag{1}$$

$$= \frac{1}{2}(t + s - |t-s|) = \frac{1}{2}(2s) = s = min(s,t)$$

E. Variance fBm

PROOF.

(Ostaszewicz, 2012)

$$Var[B_H(t,\omega)] = \mathbb{E}[B_H(t,\omega)^2] - \mathbb{E}[B_H(t,\omega)]^2 = \tag{1}$$

$$= \mathbb{E}[B_H(t,\omega) B_H(t,\omega)] = \frac{1}{2}(t^{2H} + t^{2H} - |t-t|^{2H}) = t^{2H}$$

F. Dependence structure fBm

PROOF.

(Serrano, 2016)

Considering the following increments $B_H(t_2,\omega) - B_H(t_1,\omega)$ and $B_H(t_4,\omega) - B_H(t_3,\omega)$, for no overlapping time intervals (t_1,t_2) and (t_3,t_4) for $t_1 < t_2 < t_3 < t_4$, the covariance function between both increments leads to the ensuing expression:

$$\begin{aligned}\mathbb{E}[B_H(t_2,\omega) - B_H(t_1,\omega)\,, B_H(t_4,\omega) - B_H(t_3,\omega)] =& \\ = \frac{1}{2}((t_3-t_2)^{2H} + (t_4-t_1)^{2H}& \\ - (t_4-t_2)^{2H} - (t_3-t_1)^{2H})&\end{aligned} \tag{1}$$

For which, if H=1/2:

$$\begin{aligned}&\mathbb{E}\big[B_{1/2}(t_2,\omega) - B_{1/2}(t_1,\omega)\,, B_{1/2}(t_4,\omega) - B_{1/2}(t_3,\omega)\big] = \\ &= \frac{1}{2}\big((t_3-t_2)^{2*1/2} + (t_4-t_1)^{2*1/2} \\ &- (t_4-t_2)^{2*1/2} - (t_3-t_1)^{2*1/2}\big) \\ &= \frac{1}{2}\big((t_3-t_2) + (t_4-t_1) - (t_4-t_2) - (t_3-t_1)\big) \\ &= \frac{1}{2}\big((t_3-t_2) + (t_4-t_1) - t_4 + t_2 - t_3 + t_1\big) \\ &= 0\end{aligned} \tag{2}$$

Regarding the correlation function

$$COR\big(B_H(t_2,\omega) - B_H(t_1,\omega)\,, B_H(t_4,\omega) - B_H(t_3,\omega)\big) = \tag{3}$$

$$= \frac{COV\big(B_H(t_2,\omega) - B_H(t_1,\omega)\,, B_H(t_4,\omega) - B_H(t_3,\omega)\big)}{SD\big(B_H(t_2,\omega) - B_H(t_1,\omega)\big)SD\big(B_H(t_4,\omega) - B_H(t_3,\omega)\big)}$$

$$= \frac{0}{SD\big(B_H(t_2,\omega) - B_H(t_1,\omega)\big)SD\big(B_H(t_4,\omega) - B_H(t_3,\omega)\big)}$$

$$= 0$$

In which case for H=1/2 the increments are independent, as proved above.

Let's now consider the subsequent increments of $B_H(t,\omega)$, for the instants 0 to 1 from the instants t to $t+1$, for $t \geq 1$, the covariance function $\gamma(\mathrm{n})$ between those increments is given by:

$$\gamma(\mathrm{t}) = \frac{1}{2}((t+1)^{2H} - 2t^{2H} + (t-1)^{2H})\,, \text{considering t} \geq 1 \tag{4}$$

For H=1/2, the increments are still independent $\gamma(\mathrm{t}) = 0$. Though if t tends to infinity the following ratio becomes 1:

$$\lim_{t\to\infty} \frac{\frac{1}{2}((t+1)^{2H} - 2t^{2H} + (t-1)^{2H})}{H(2H-1)t^{2H-2}} = 1, \tag{5}$$

$$for\ H \neq\ 1/2\ in\ the\ interval\ (0,1)\ and\ t > 1$$

Leading to the ensuing conclusion:

$$\gamma(\mathrm{t}) \sim H(2H-1)t^{2H-2}\ as\ t \to \infty\ for\ t > 1 \tag{6}$$

Consequently, from equation (6) one can state that if $H < \frac{1}{2}$ then $\gamma(\mathrm{t}) < 0$ and $\sum_{t=1}^{\infty}|\gamma(\mathrm{t})| < \infty$, converging to a finite value, with increments exhibiting negative correlation and therefore short-range dependence. Adversely, if $H > \frac{1}{2}$ => $\gamma(\mathrm{t}) > 0$

and $\sum_{t=1}^{\infty}|\gamma(t)| = \infty$, diverging with $t \to \infty$, with increments displaying positive correlation and consequently long-range dependence.

G. Stochastic Integral for the fBm

LEMMA 2.1. *Let the stochastic integral representation of the fractional Brownian motion* $B^H(t)$ *be defined as:*

(Ostaszewicz, 2012)

$$B^H(t) = B^H(t,\omega) - B^H(0,\omega) = \tag{1}$$
$$= c_H\left(\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right] dB(s)\right.$$
$$\left. + \int_{0}^{t}(t-s)^{H-\frac{1}{2}}\, dB(s)\right)$$

where c_H *is the normalizing constant and it assumes the following expression:*

$$c_H = \sqrt{\frac{2H\,\Gamma(3/2-H)}{\Gamma(H+1/2)\,\Gamma(2-2H)}} \tag{2}$$

For which H ranges between 0 and 1 (non-inclusive of zero), and t and s designate the times of the time interval with $t > s \geq 0$.

It's easily deductible that for $B^{1/2}(t)$ $H = 1/2$ we've the standard Brownian Motion $B(t)$.

H. Hurst - Power law (dam dimensions)

THEOREM 2.8. *The Hurst exponent K for σ standard deviation of river outflows during N number of years under study is given by:*

(Mandelbrot and Hudson, 2004)

$$K = \frac{log\left(\frac{R}{\sigma}\right)}{log\left(\frac{N}{2}\right)} \tag{1}$$

where R represents the required dam dimensions to avoid floods.

I. Rescaled Range Formula

THEOREM 2.9. *The Hurst exponent K for σ standard deviation of river outflows during N number of years under study is given by:*

(Mandelbrot and Hudson, 2004)

$$R/S = \frac{\underset{0\le k\le n}{\text{Max}} \sum_{j=1}^{k}(r_j \; - \bar{r}_n) - \underset{0\le k\le n}{\text{Min}} \sum_{j=1}^{k}(r_j \; - \bar{r}_n)}{\left[\frac{1}{n}\sum_j (r_j \; - \bar{r}_n)^2\right]^{1/2}} \tag{1}$$

where n is the full period under analysis, with k maximum j shorter periods and r_j for the corresponding shorter return, which is deducted from the average stock return for $n \; => \bar{r}_n$. And the denominator represents the standard deviation of the assessed series.

The denominator when compared with the numerator responsible for the range between max and minimum accumulated deviations, results in a long-term dependence metric.

J. Hu and Øksendal Price of a European Call Option using fBSM

PROOF.

(Ostaszewicz, 2012)

Accordingly, to Hu and Øksendal at time T the price of a Fractional Black-Scholes European Call is given by the following formula:

$$F(\omega) = \max\{S(T,\omega) - K, 0\} \qquad (1)$$

Considering, the formula derivation is provided as bellow:

$$C^H(0, S_0) = e^{-rT} E_{\hat{\mu}_\Phi}[F] \qquad (2)$$

$$= e^{-rT} E_{\hat{\mu}_\Phi}[max\{S(T,\omega) - K), 0\}]$$

$$= e^{-rT} E_{\hat{\mu}_\Phi}\left[max\{S_0 exp(\sigma B_T^H + \mu T - \frac{1}{2}\sigma^2 T^{2H}) - K), 0\}\right]$$

$$= e^{-rT} E_{\hat{\mu}_\Phi}\left[max\{S_0 exp(\sigma \hat{B}_T^H + rT - \frac{1}{2}\sigma^2 T^{2H}) - K), 0\}\right]$$

$$= e^{-rT} E_{\mu_\Phi}\left[max\{S_0 exp(\sigma B_T^H + rT - \frac{1}{2}\sigma^2 T^{2H}) - K), 0\}\right]$$

$$= e^{-rT} E_{\mu_\Phi}\left[S_T 1_{S_T > K}\right] - e^{-rT} K\, E_{\mu_\Phi}\left[1_{S_T > K}\right]$$

Solving the boundary we've:

$$S_0 exp(\sigma z + rT - \frac{1}{2}\sigma^2 T^{2H}) > K \Leftrightarrow$$

$$\Leftrightarrow z > \frac{ln\left(\frac{S_0}{K}\right) - rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma} =$$

$$= - \frac{ln\left(\frac{S_0}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma}$$

$$\hat{d}_1 = \frac{ln\left(\frac{S_0}{K}\right) + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma}$$

Setting the first expectation as:

$$E_{\mu_\Phi}\left[S_T 1_{S_T>K}\right] = \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{y^2}{2T^{2H}}\right) x\ exp(\sigma y + rT - \frac{1}{2}\sigma^2 T^{2H}) dy \tag{3}$$

Since the variance of B_t^H is T^{2H} and the mean is 0, using the Gaussian character of the fractional Brownian motion we've:

$$E_{\mu_\Phi}\left[S_T 1_{S_T>K}\right] = e^{rT} \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{y^2}{2T^{2H}} + \sigma y - \frac{1}{2}\sigma^2 T^{2H}\right) x\ dy$$

$$= xe^{rT} \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{1}{2T^{2H}}(y^2 - 2\sigma y T^{2H} + \sigma^2 T^{4H})\right) dy$$

$$= xe^{rT} \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{y - \sigma T^{2H}}{T^H}\right)^2\right) dy$$

Let

$$z = \frac{y - \sigma T^{2H}}{T^H} \Longrightarrow y = zT^H + \sigma T^{2H} \tag{4}$$

differentiating we get

$$dy = T^H dz$$

subsquently,

$$E_{\mu_\Phi}\left[S_T 1_{S_T>K}\right] = xe^{rT} \int_{\frac{-\hat{d}_1 - \sigma T^{2H}}{T^H}}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{1}{2}(z)^2\right) T^H\ dz \tag{5}$$

$$= xe^{rT} \int_{-\infty}^{\frac{\hat{d}_1 + \sigma T^{2H}}{T^H}} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}(z)^2\right) dz$$

$$= xe^{rT} N\left(\frac{\hat{d}_1 + \sigma T^{2H}}{T^H}\right)$$

$$= xe^{rT}N(d_1^H)$$

where,

$$d_1^H = \frac{\hat{d}_1 \ + \ \sigma T^{2H}}{T^H} \tag{6}$$

$$= \frac{\dfrac{ln\left(\dfrac{S_0}{K}\right) + \ rT \ - \ \dfrac{1}{2}\sigma^2 T^{2H}}{\sigma} \ + \ \sigma T^{2H}}{T^H}$$

$$= \frac{ln\left(\dfrac{S_0}{K}\right) + \ rT \ + \ \dfrac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}$$

Regarding the second expectation:

$$E_{\mu_\Phi}\left[1_{S_T > K}\right] = \int_{-\hat{d}_1}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{y^2}{2T^{2H}}\right) \ dy \tag{7}$$

If

$$\omega = \ \frac{y}{T^H} \Longrightarrow y = \ \omega T^H$$

Then

$$dy = \ T^H dz$$

Resulting in

$$E_{\mu_\Phi}\left[1_{S_T > K}\right] = \int_{\frac{-\hat{d}_1}{T^H}}^{\infty} \frac{1}{T^H\sqrt{2\pi}} exp\left(-\frac{1}{2}\omega^2\right) T^H \ d\omega \tag{8}$$

$$= \int_{-\infty}^{\frac{\hat{d}_1}{T^H}} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\omega^2\right) \ d\omega$$

$$= N\left(\frac{\hat{d}_1}{T^H}\right) = N(d_2^H)$$

For which we've

$$d_2^H = \frac{ln\left(\frac{S_0}{K}\right) + \ rT \ - \ \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{9}$$

The price of a European call is then:

$$C^H(0, S_0) = S_0 N(d_1^H) \ - \ Ke^{-rT} N(d_2^H) \tag{10}$$

For which, when H = 1/2 it's obtained the classical Black-Scholes formula.

In other words,

$$v^H(0, S_0) = \phi S_0 N(\phi d_1^H) \ - \ \phi K e^{-rT} N(\phi d_2^H) \tag{11}$$

$$where\ \phi\ is\ equal\ to\ 1\ in\ the\ case\ of\ a\ call\ or -1\ in\ the\ case\ of\ a\ put$$

K. Li and Chen fBSM calibration methodology

PROOF. Let $F(T, S_t)$ be a twice differentiable function under the fractional itô lemma, applied to a Stochastic process S_t:

(Li and Chen, 2014)

$$\begin{aligned} F(T, S_t) = F(t, S_t) &+ \int_t^T \frac{\partial}{\partial u} F(u, S_u) du \\ &+ \mu \int_t^T \frac{\partial}{\partial S} F(u, S_u) S_u du \\ &+ \sigma \int_t^T \frac{\partial}{\partial S} F(u, S_u) S_u dB_u^H \\ &+ H\sigma^2 \int_t^T \frac{\partial^2}{\partial S^2} F(u, S_u) S_u^2 u^{2H-1} du \end{aligned} \tag{1}$$

Taking derivatives,

$$dF(T,S_t) = \frac{\partial}{\partial t}F(t,S_t)dt + \frac{\partial}{\partial S}F(t,S_t)\mu S_t dt + \frac{\partial}{\partial S}F(t,S_t)\sigma S_t dB_t^H + \frac{1}{2}\frac{\partial^2}{\partial S^2}F(t,S_t)\sigma^2 S_t^2 dt^{2H} \quad (2)$$

Introducing $F = ln(S_t)$:

$$dln(S_t) = \mu dt + \sigma dB_t^H - \frac{1}{2}\sigma^2 dt^{2H} \quad (3)$$

Moving for the S_t left side of the fBm SDE, we obtain:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t^H \quad (4)$$

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma dB_t^H$$

Subtracting equation (4) by equation (3) we have

$$\frac{dS_t}{S_t} - dln(S_t) = \frac{1}{2}\sigma^2 dt^{2H}$$

If we take the expectations of both sides from 0 to T, we get:

$$2\, E\left[\int_0^T \frac{dS_t}{S_t}\, dt - ln\left(\frac{S_t}{S_0}\right)\right] = E\left[\int_0^T \sigma^2 dt^{2H}\right] \quad (5)$$

Then through the insertion of the results from a replication strategy discussed by Demeterfi et al. (1999) on the first term of equation (5), the authors arrived to a formula where they can derive expected variance (second term of the previous equation) for different strikes without depending on any specific model:

$$E(V) = 2\left[rT - \frac{S_0 e^{rT} - S_*}{S_*} - ln\left(\frac{S_*}{S_0}\right) + e^{rT}\int_0^{S_*}\frac{1}{K^2}P(K)dK + e^{rT}\int_{S_*}^{\infty}\frac{1}{K^2}C(K)dK\right] \tag{6}$$

where S_* is ATM strike with P(K) and C(K) being the prices of put and call options for strike K.

Subsequently, in this work it will be used the above integration of different option prices for different strikes to obtain the model free expected variance as per equation (6) and by running the fractional OLS regression (see equation (18) - Chapter 3) obtain the fractional variance and hurst exponent.