Complex Analysis and Operator Theory (2025) 19:17
https://doi.org/10.1007/s11785-024-01631-0

Complex Analysis and Operator Theory

On the Relation Between S-Spectrum and Right Spectrum

Luís Carvalho[1,2] · Cristina Diogo[1,2] · Sérgio Mendes[1,3] · Helena Soares[1,4]

Received: 29 June 2023 / Accepted: 12 November 2024

Abstract

We use the $\mathbb{R}$-linearity of $I\lambda - T$ to define $\sigma(T)$, the right spectrum of a right $\mathbb{H}$-linear operator T in a right quaternionic Banach space. We show that $\sigma(T)$ coincides with the S-spectrum $\sigma_S(T)$.

Keywords Spectrum · Quaternionic Hilbert spaces

Mathematics Subject Classification 47A10 · 47S05

1 Introduction

In a complex Banach space $X_{\mathbb{C}}$ the spectrum of a bounded operator $T \in \mathcal{B}(X_{\mathbb{C}})$ is the well-known set

$$\sigma_{\mathbb{C}}(T) = \left\{\lambda \in \mathbb{C} : I\lambda - T \text{ is non invertible in } \mathcal{B}(X_{\mathbb{C}})\right\}.$$

When X is a (right) quaternionic Banach space, the spectrum is more elusive, due to the non commutativity of scalar multiplication. Nevertheless, quaternionic Banach spaces have been used with the notion of point spectrum, i.e eigenvalues. It is, however, clear that this notion is not sound when X is infinite dimensional and therefore, some careful adaptations need to be done.

It is well-known that if the complex Banach space $X_{\mathbb{C}}$ is finite dimensional then the notion of spectrum is the set of eigenvalues. But when $X_{\mathbb{C}}$ is infinite dimensional, the spectrum of an operator T is more than just the eigenvalues. More precisely, according to the nature of the failure of the invertibility of $T_\lambda := I\lambda - T \in \mathcal{B}(X_{\mathbb{C}})$, the spectrum is the union of three disjoint sets: the point spectrum $\sigma_{\mathbb{C},p}(T)$, the set of complex numbers λ where the operator T_λ is not injective; the residual spectrum $\sigma_{\mathbb{C},r}(T)$, the

Communicated by David Kimsey.

The first and second authors were partially supported by FCT through CAMGSD, projects UIDB/04459/2020 and UIDP/04459/2020. The third author was partially supported by FCT through CMA-UBI, project UIDB/00212/2020. The fourth author was partially supported by FCT through CIMA, project UIDB/04674/2020.

Extended author information available on the last page of the article

set of $\lambda \in \mathbb{C} \setminus \sigma_{\mathbb{C},p}(T)$ where the range of T_λ is not dense in $X_{\mathbb{C}}$, $\overline{\mathrm{Ran}(T_\lambda)} \neq X$; and the continuous spectrum $\sigma_{\mathbb{C},c}(T)$, the set of $\lambda \in \mathbb{C} \setminus (\sigma_{\mathbb{C},p}(T) \cup \sigma_{\mathbb{C},r}(T))$ where T_λ is not bounded below.

To define the spectrum of a quaternionic right linear operator $T \in \mathcal{B}(X)$ we cannot simply emulate what happens in the complex case. In fact, the operator of right multiplication by a quaternion, $I\lambda : X \to X$ defined by $I\lambda(x) := x\lambda$, is not right $\mathbb{H}$-linear. Thus, T_λ is also not right $\mathbb{H}$-linear, i.e. T_λ is not in $\mathcal{B}(X)$, but on the contrary, right multiplication by a quaternion is clearly $\mathbb{R}$-linear and so is T_λ. Therefore, it is natural to define a notion of spectrum based on the invertibility of the $\mathbb{R}$-linear operator T_λ. In other words, we consider the larger space of bounded $\mathbb{R}$-linear operators on X, denoted by $\mathcal{B}_{\mathbb{R}}(X)$. Accordingly, we define a modified right spectrum to be the subset of $\mathbb{H}$

$$\sigma(T) = \left\{\lambda \in \mathbb{H} : I\lambda - T \text{ is non invertible in } \mathcal{B}_{\mathbb{R}}(X)\right\}, \tag{1}$$

which resembles the usual notion of spectrum in the complex setting. The spectrum defined this way can be decomposed in the usual family of disjoint sets,

$$\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T),$$

where $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ are respectively the point, residual and continuous spectra, defined by

$$\begin{aligned}
\sigma_p(T) &= \{\lambda \in \mathbb{H} : T_\lambda(x) = 0, \text{ for some } x \in X \setminus \{0\}\}, \\
\sigma_r(T) &= \{\lambda \in \mathbb{H} \setminus \sigma_p(T) : \overline{\mathrm{Ran}(T_\lambda)} \neq X\}, \\
\sigma_c(T) &= \{\lambda \in \mathbb{H} \setminus \big(\sigma_c(T) \cup \sigma_p(T)\big) : T_\lambda \text{ is not bounded below}\}.
\end{aligned}$$

On the other hand, the quaternionic functional calculus has seen a recent major breakthrough and the fundamental stepping-stone of that leap is the introduction of the operator $\Delta_\lambda(T) = T^2 - 2Re(\lambda)T + |\lambda|^2 I$ in $\mathcal{B}(X)$, for any $\lambda \in \mathbb{H}$. The non-invertibility of $\Delta_\lambda(T)$, for a given $T \in \mathcal{B}(X)$, defines a new notion of spectrum, the S-spectrum,

$$\sigma_S(T) = \left\{\lambda \in \mathbb{H} : \Delta_\lambda(T) \text{ is non invertible in } \mathcal{B}(X)\right\}.$$

Again, the S-spectrum can be split into three disjoint sets: the point S-spectrum $\sigma_{S,p}(T)$, the residual S-spectrum $\sigma_{S,r}(T)$, and the continuous S-spectrum $\sigma_{S,c}(T)$ (see [5] and references therein).

This similarity raises the question whether these sets are equal to the ones defined for the right spectrum (1). The purpose of this paper is to answer affirmatively to this question. We prove that the notions of right spectrum and S-spectrum coincide. Actually, we show that each component in the partition of the spectrum coincides with the corresponding component in the partition of the S-spectrum, that is

$$\sigma_p(T) = \sigma_{S,p}(T), \quad \sigma_c(T) = \sigma_{S,c}(T), \quad \sigma_r(T) = \sigma_{S,r}(T).$$

Before delving into the gory details, two comments are worth making. First, our result provides a handy way to calculate the spectrum but the introduction of the operator Δ_λ was crucial for further developing quaternionic operator theory. In fact, many achievements were unthinkable without the discovery of the S-spectrum. To name a few, a generalization of the Riesz-Dunford functional calculus for holomorphic functions to quaternionic linear operators [3], the continuous functional calculus for normal operators on a quaternionic Hilbert space [6] and spectral theorems for unitary [2] and for unbounded normal quaternionic linear operators [1], among others. The second observation is that the equality of the two spectra notions in this paper is natural and in a sense easy, but to the best of our knowledge, has passed unnoticed in the literature, except the equality of the right and S-spectrum in the finite dimensional case. There is also one article that mentions the equality of the right spectrum and the S-spectrum in infinite dimension, but only proves the equality for the eigenvalues, that is, it only establishes the equality of the right point spectrum and the point S-spectrum (see [4, Theorem 2.5]).

For convenience of the reader, we recall some basic definitions and results. The division ring of real quaternions $\mathbb{H}$ is an algebra over $\mathbb{R}$ with basis $\{1, i, j, k\}$ and product given by $i^2 = j^2 = k^2 = ijk = -1$. The pure quaternions are denoted by $\mathbb{P} = \text{span}_{\mathbb{R}}\{i, j, k\}$. The real and imaginary parts of a quaternion $q = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ are denoted by $Re(q) = a_0$ and $Im(q) = a_1 i + a_2 j + a_3 k \in \mathbb{P}$, respectively. The conjugate of q is given by $q^* = Re(q) - Im(q)$ and its norm is $|q| = \sqrt{qq^*}$. Two quaternions $q_1, q_2 \in \mathbb{H}$ are called similar, and we write $q_1 \sim q_2$, if there exists $s \in \mathbb{H}$ with $|s| = 1$ such that $s^* q_2 s = q_1$. Similarity is an equivalence relation and the class of q is denoted by $[q]$. A necessary and sufficient condition for the similarity of q_1 and q_2 is that $Re(q_1) = Re(q_2)$ and $|Im(q_1)| = |Im(q_2)|$. A set $\mathcal{A} \subset \mathbb{H}$ is axially symmetric if $\lambda \in \mathcal{A}$, then $[\lambda] \subset \mathcal{A}$.

As usual a right $\mathbb{H}$-module X equipped with a norm $\|x\|$, for every $x \in X$, is called a right quaternionic Banach space if it is complete with respect to this norm.

We can look at X as a vector space over $\mathbb{H}$ or $\mathbb{R}$. This allows us to introduce two notions of linear operators over the two fields. A right $\mathbb{H}$-linear operator is a map $T : X \to X$ such that

$$T(u\alpha + v\beta) = T(u)\alpha + T(v)\beta, \text{ for any } u, v \in X \text{ and } \alpha, \beta \in \mathbb{H}.$$

Analogously, if the above equality holds for any $\alpha, \beta \in \mathbb{R}$, we say that T is an $\mathbb{R}$-linear operator. Furthermore, recall that an operator $T : X \to X$ is bounded if there exists $K \geq 0$ such that $\|Tx\| \leq K\|x\|$, $x \in X$, and that the norm of T is defined by $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$.

We denote by $\mathcal{B}(X)$ the set of all bounded right $\mathbb{H}$-linear operators on X and by $\mathcal{B}_{\mathbb{R}}(X)$ the set of all bounded $\mathbb{R}$-linear operators on X. Since a $\mathbb{H}$-linear map is an $\mathbb{R}$-linear map, we have $\mathcal{B}(X) \subseteq \mathcal{B}_{\mathbb{R}}(X)$.

Consider the operator $T_\lambda := I\lambda - T : X \to X$, defined before as $T_\lambda(x) = x\lambda - Tx$. Although T_λ is not right $\mathbb{H}$-linear, it is $\mathbb{R}$-linear operator. Therefore, it makes sense to talk about invertibility of T_λ in $\mathcal{B}_{\mathbb{R}}(X)$ and we define the spectrum of $T \in \mathcal{B}(X)$ as follows.

Definition 1 Let $T \in \mathcal{B}(X)$. Then the right spectrum of T is the set

$$\sigma(T) = \{\lambda \in \mathbb{H} : T_\lambda \text{ is not invertible in } \mathcal{B}_{\mathbb{R}}(X)\}.$$

As proved in the next proposition $\sigma(T)$ splits into a disjoint union of three parts

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T),$$

where the point spectrum, the continuous spectrum and residual spectrum of T are, respectively,

$$\begin{aligned}
\sigma_p(T) &= \{\lambda \in \mathbb{H} : T_\lambda(x) = 0, \text{ for some } x \in X \setminus \{0\}\},\\
\sigma_r(T) &= \{\lambda \in \mathbb{H} \setminus \sigma_p(T) : \overline{\mathrm{Ran}(T_\lambda)} \neq X\},\\
\sigma_c(T) &= \{\lambda \in \mathbb{H} \setminus \big(\sigma_r(T) \cup \sigma_p(T)\big) : T_\lambda \text{ is not bounded below}\}.
\end{aligned}$$

Proposition 2 *Let $T \in \mathcal{B}(X)$, then*

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

Proof Clearly $\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T) \subseteq \sigma(T)$, since if λ belongs to $\sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ then T_λ is not invertible in $\mathcal{B}_{\mathbb{R}}(X)$, thus $\lambda \in \sigma(T)$.

The reverse inclusion is straightforward and we will demonstrate it by proving that if $\lambda \in \sigma(T) \setminus (\sigma_r(T) \cup \sigma_p(T))$, then $\lambda \in \sigma_c(T)$. In other words, we will show that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of unit vectors $x_n \in X$ such that $T_\lambda(x_n) \to 0$.

Since T is bounded we have $T_\lambda \in \mathcal{B}_{\mathbb{R}}(X)$, for any $\lambda \in \mathbb{H}$. Furthermore, if $\lambda \in \sigma(T) \setminus (\sigma_r(T) \cup \sigma_p(T))$, meaning T_λ is bijective and thus invertible. But its inverse is not in $\mathcal{B}_{\mathbb{R}}(X)$, then, being an $\mathbb{R}$-linear operator, it must be unbounded. That is, there exists an unbounded $\mathbb{R}$-linear operator Q such that $QT_\lambda(x) = T_\lambda Q(x) = x$ for any $x \in X$.

If Q is unbounded, we can select a sequence of unit vectors $x_n \in X$ such that $\|Qx_n\| \to +\infty$. Then we have

$$T_\lambda\left(\frac{Q(x_n)}{\|Qx_n\|}\right) = \frac{x_n}{\|Qx_n\|} \to 0.$$

In conclusion, if $\lambda \in \sigma(T)$ but not in $\sigma_r(T) \cup \sigma_p(T)$, then T_λ is not bounded below, thus $\lambda \in \sigma_c(T)$. Therefore, the partition is valid for the right spectrum definition introduced. □

Given $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{H}$, we define the operator $\Delta_\lambda(T) : X \longrightarrow X$ by

$$\Delta_\lambda(T) = T^2 - 2Re(\lambda)T + |\lambda|^2 I.$$

Clearly, $\Delta_\lambda(T)$ is a bounded right $\mathbb{H}$-linear operator and we can introduce the definition of the S-spectrum of T.

Definition 3 Let $T \in \mathcal{B}(X)$. Then the S-spectrum of T is the set

$$\sigma_S(T) = \big\{\lambda \in \mathbb{H} : \Delta_\lambda(T) \text{ is not invertible in } \mathcal{B}(X)\big\}.$$

The S-spectrum $\sigma_S(T)$ is a compact nonempty subset of $\mathbb{H}$ and it is always contained in the closed ball of radius $\|T\|$ around origin $\overline{B(0, \|T\|)}$. One can show that $\lambda \in \sigma_S(T)$ is equivalent to $[\lambda] \subseteq \sigma_S(T)$ ([5, Theorem 3.1.8, Theorem 3.1.13]).

Again a classical partition of the spectrum into three disjoint parts, according to the nature of the failure of $\Delta_\lambda(T)$ to be invertible is the following:

$$\sigma_S(T) = \sigma_{S,p}(T) \cup \sigma_{S,c}(T) \cup \sigma_{S,r}(T),$$

where

$$\begin{aligned}
\sigma_{S,p}(T) &= \{\lambda \in \mathbb{H} : \Delta_\lambda(T)(x) = 0, \text{ for some } x \in X \setminus \{0\}\}, \\
\sigma_{S,r}(T) &= \{\lambda \in \mathbb{H} \setminus \sigma_{S,p}(T) : \overline{\mathrm{Ran}(\Delta_\lambda(T)} \neq X\}, \\
\sigma_{S,c}(T) &= \{\lambda \in \mathbb{H} \setminus \big(\sigma_{S,p}(T) \cup \sigma_{S,r}(T)\big) : \Delta_\lambda(T) \text{ is not bounded below}\}.
\end{aligned}$$

The proof that this partition of $\sigma_S(T)$ holds is exactly like the proof of the homologous partition of $\sigma(T)$. If λ is in the spectrum, but not in the point spectrum nor in the residual spectrum, then Δ_λ is bijective, which means that Δ_λ is invertible. However, since Δ_λ^{-1} is not in $\mathcal{B}(X)$, Δ_λ^{-1} must be unbounded. This implies that Δ_λ cannot be bounded below, because if it were, its inverse would be bounded.

Furthermore, notice that if we have $\lambda_1 \sim \lambda_2$ then

$$\Delta_{\lambda_1} = \Delta_{\lambda_2},$$

therefore σ_S and the subsets of its partition, $\sigma_{S,p}(T)$, $\sigma_{S,r}(T)$ and $\sigma_{S,c}(T)$ are obviously axially symmetric. That is if λ belongs to one of these sets, then all the equivalence class $[\lambda]$ is also in that same set.

Proposition 4 *Let $T \in B(X)$, then*

$$\sigma_S(T) = \sigma_{S,p}(T) \cup \sigma_{S,c}(T) \cup \sigma_{S,r}(T).$$

The sets $\sigma_S(T)$, $\sigma_{S,p}(T)$, $\sigma_{S,c}(T)$ and $\sigma_{S,r}(T)$ are axially symmetric.

We have seen $\sigma_S(T)$ is axially symmetric as well as $\sigma_{S,p}$, $\sigma_{S,r}$ and $\sigma_{S,c}$, the same is true for $\sigma(T)$, $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ as shown in the following lemma. Before, observe that when $\lambda = q\mu q^*$, for some unitary quaternion q, we have $\big(T_\lambda x\big)q = T_\mu(xq)$. In fact,

$$\begin{aligned}
T_\lambda x &= x\lambda - Tx = x(q\mu q^*) - T\big(x(qq^*)\big) \\
&= (xq\mu)q^* - T(xq)q^* = \Big((xq)\mu - T(xq)\Big)q^*
\end{aligned}$$

$$= \Big(T_\mu(xq)\Big)q^*. \tag{2}$$

Note that we cannot go further than this since T_μ is not right $\mathbb{H}-$linear, only T is.

Lemma 5 *Let $T \in \mathcal{B}(X)$. The sets $\sigma_p(T)$, $\sigma_r(T)$ and $\sigma_c(T)$ are axially symmetric. In particular, $\sigma(T)$ is axially symmetric.*

Proof We will prove this result by showing that when $\lambda \in \mathcal{A}$, then any element of the form $\mu = q\lambda q^*$, with unitary $q \in \mathbb{H}$, is also in $\mathcal{A}$, and therefore $[\lambda] \subseteq \mathcal{A}$, where $\mathcal{A}$ is one of the above spectra.

Let $\lambda \in \sigma_p(T)$. If $x \in X \setminus \{0\}$ is such that $T_\lambda(x) = 0$, taking $y = xq$ and using (2) we have $T_\mu(y) = 0$. Then $\mu \in \sigma_p(T)$ and therefore $\sigma_p(T)$ is axially symmetric.

We now prove that $\sigma_r(T)$ is axially symmetric. Let $\lambda \in \sigma_r(T)$. Note that $\lambda \notin \sigma_p(T)$, $\mu = q\lambda q^*$ and $\sigma_p(T)$ is axially symmetric, hence $\mu \notin \sigma_p(T)$. Since $\mathbb{H}q^* = \mathbb{H}$, using (2) and taking $y = xq$ we have

$$\begin{aligned}
\overline{\mathrm{Ran}(T_\mu)} &= \overline{\{T_\mu(y) : y \in X\}} = \overline{\{T_\mu(xq) : xq \in X\}} \\
&= \overline{\{T_\lambda(x)q : xq \in X\}} = \overline{\{T_\lambda(x)q : x \in Xq^*\}} \\
&= \overline{\{T_\lambda(x)q : x \in X\}} = \overline{\{T_\lambda(x) : x \in X\}q} \\
&= \overline{\mathrm{Ran}(T_\lambda)}q
\end{aligned}$$

Therefore, $\overline{\mathrm{Ran}(T_\lambda)} \neq X$ implies that $\overline{\mathrm{Ran}(T_\lambda)}q = \overline{\mathrm{Ran}(T_\mu)} \neq X$. So $\sigma_r(T)$ is axially symmetric.

It remains to see that $\sigma_c(T)$ is axially symmetric. Let $\lambda \in \sigma_c(T)$. Then $\lambda \notin \sigma_p(T) \cup \sigma_r(T)$, $\mu = q\lambda q^*$ and $\sigma_p(T)$ and $\sigma_r(T)$ are both axially symmetric, so $\mu \notin \sigma_p(T) \cup \sigma_r(T)$. Since T_λ is not bounded below then there is a sequence $x_n \in X$ such that $\|T_\lambda x_n\| \to 0$. Take $y_n = x_n q$, with $q \in \mathbb{H}$ a unitary quaternion, such that $\lambda = q\mu q^*$, then, again by (2), we have $\|T_\mu(y_n)\| = \|\big(T_\lambda(x_n)\big)q\| \to 0$. Then T_μ is not bounded below and so $\sigma_c(T)$ is axially symmetric. □

Before we prove our main theorem we note that the operators $\Delta_\lambda(T)$ and T_λ are related by composition $\Delta_\lambda(T) = T_\lambda \cdot T_{\lambda^*}$. In fact, for all $x \in X$, we have

$$\begin{aligned}
T_\lambda \cdot T_{\lambda^*}(x) &= (I\lambda - T) \cdot (I\lambda^* - T)(x) = (I\lambda - T)(x\lambda^* - Tx) \\
&= (x\lambda^* - Tx)\lambda - T(x\lambda^* - Tx) = x|\lambda|^2 - (Tx)\lambda - (Tx)\lambda^* + T^2(x) \\
&= T^2(x) - (Tx)(\lambda + \lambda^*) + |\lambda|^2 x = \big(T^2 - 2Re(\lambda)T + |\lambda|^2 I\big)x \\
&= \Delta_\lambda(T)(x).
\end{aligned}$$

Likewise we can prove that $T_{\lambda^*} \cdot T_\lambda = \Delta_\lambda(T)$. Summing up we have:

Proposition 6 *Let $T \in \mathcal{B}(X)$. Then*

$$\Delta_\lambda(T) = T_\lambda \cdot T_{\lambda^*} = T_{\lambda^*} \cdot T_\lambda.$$

From this result it is easy to see that when T_λ is not invertible in $\mathcal{B}_{\mathbb{R}}(X)$, $\Delta_\lambda(T)$ is not invertible in $\mathcal{B}_{\mathbb{R}}(X)$ and therefore not invertible in $\mathcal{B}(X)$, which means $\sigma(T) \subset \sigma_S(T)$. We will prove that the converse also holds. This will need some extra work since lemma 5 and proposition 6 only implies a slightly weaker result, if T_λ is invertible in $\mathcal{B}_{\mathbb{R}}(X)$, $\Delta_\lambda(T)$ is invertible in $\mathcal{B}_{\mathbb{R}}(X)$. We can find not only that $\sigma_S(T) = \sigma(T)$, but stronger than that, that the point spectrum of T is the point S-spectrum of T, the continuous spectrum is the continuous S-spectrum; and the residual spectrum is the residual S-spectrum.

Theorem 7 *Let $T \in \mathcal{B}(X)$. We have the following equalities*

$$\sigma_p(T) = \sigma_{S,p}(T), \quad \sigma_r(T) = \sigma_{S,r}(T) \quad \textit{and} \quad \sigma_c(T) = \sigma_{S,c}(T).$$

Proof If $\lambda \in \sigma_{S,p}(T)$, then $\Delta_\lambda(T)x = 0$ for some $x \in X \setminus \{0\}$. Taking $y = T_{\lambda^*}(x)$ and using proposition 6, we have $T_\lambda y = 0$. Either $\lambda \in \sigma_p(T)$, and we are done, or $T_{\lambda^*}(x) = y = 0$, in which case $\lambda^* \in \sigma_p(T)$. Since $\sigma_p(T)$ is axially symmetric, $\lambda \in \sigma_p(T)$. We conclude that $\sigma_{S,p}(T) \subseteq \sigma_p(T)$. For the converse inclusion, if $T_\lambda(x) = 0$ for some $x \in X \setminus \{0\}$, using proposition 6, it follows that $\Delta_\lambda(T)x = 0$ for some $x \in X \setminus \{0\}$, that is, $\lambda \in \sigma_{S,p}(T)$.

Let us now prove that $\sigma_r(T) = \sigma_{S,r}(T)$. Since $\overline{\operatorname{Ran}(T_\lambda \cdot T_{\lambda^*})} \subseteq \overline{\operatorname{Ran}(T_\lambda)} \neq X$ we have $\sigma_r(T) \subseteq \sigma_{S,r}(T)$. To prove the converse inclusion, we will use the contrapositive: $\overline{\operatorname{Ran}(T_\lambda)} = X$ implies $\overline{\operatorname{Ran}(\Delta_\lambda(T))} = X$. From (2) we have $T_\lambda(x)q = T_{\lambda^*}(xq)$, where $\lambda^* = q^*\lambda q$, with q unitary. Since $T_{\lambda^*}(\cdot\ q) \in \mathcal{B}_{\mathbb{R}}(X)$ with $x \mapsto T_{\lambda^*}(xq)$, we have that

$$X = Xq = \overline{\operatorname{Ran}(T_\lambda)}\, q = \overline{\operatorname{Ran}(T_\lambda)\, q} = \overline{\operatorname{Ran}(T_{\lambda^*}(\cdot\ q))} = \overline{\operatorname{Ran}(T_{\lambda^*})}.$$

It follows that $\overline{\operatorname{Ran}(T_\lambda)} = X$ implies $\overline{\operatorname{Ran}(T_{\lambda^*})} = X$ (and vice-versa). So assume that $\overline{\operatorname{Ran}(T_\lambda)} = \overline{\operatorname{Ran}(T_{\lambda^*})} = X$. To prove that $\overline{\operatorname{Ran}(\Delta_\lambda(T))} = X$ we will find that, for any $x \in X$, there is a sequence $x_n \in X$ such that $\Delta_\lambda(T)(x_n) = T_{\lambda^*} \cdot T_\lambda(x_n) \xrightarrow{n} x$. Since the range of T_{λ^*} is dense there is a sequence $y_n \in X$ such that $T_{\lambda^*}(y_n) \xrightarrow{n} x$; and since $\operatorname{Ran}(T_\lambda)$ is also dense, for each n there is a sequence $y_{n,k}$ such that $T_\lambda(y_{n,k}) \xrightarrow{k} y_n$. For any $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\|T_{\lambda^*}(y_n) - x\| < \epsilon$, when $n \geq N$. For any of these n pick $k(n) \in \mathbb{N}$ satisfying $\|T_\lambda(y_{n,k(n)}) - y_n\| < \epsilon$. Then,

$$\begin{aligned}
\|T_{\lambda^*} \cdot T_\lambda(y_{n,k(n)}) - x\| &\leq \|T_{\lambda^*} \cdot T_\lambda(y_{n,k(n)}) - T_{\lambda^*}(y_n)\| + \|T_{\lambda^*}(y_n) - x\| \\
&\leq \|T_{\lambda^*}\|\|T_\lambda(y_{n,k(n)}) - y_n\| + \|T_{\lambda^*}(y_n) - x\| \\
&\leq (\|T_{\lambda^*}\| + 1)\epsilon.
\end{aligned}$$

Hence we have a sequence $x_n = y_{n,k(n)}$ such that $\Delta_\lambda(T)(x_n) \xrightarrow{n} x$, thus $x \in \overline{\operatorname{Ran}(\Delta_\lambda(T))}$.

Finally, we will see that $\sigma_c(T) = \sigma_{S,c}(T)$. Assume $\lambda \in \sigma_c(T)$, then T_λ is not bounded below and there is a sequence of unitary vectors $x_n \in X$ such that $T_\lambda(x_n) \to 0$. By continuity of T_{λ^*}, we have $\Delta_\lambda(T)(x_n) = T_{\lambda^*} \cdot T_\lambda(x_n) \to 0$, i.e., $\lambda \in \sigma_{S,c}(T)$.

On the other hand, if $\lambda \in \sigma_{S,c}(T)$, there is a sequence of unitary vectors x_n such that $\Delta_\lambda(T)(x_n) = T_\lambda \cdot T_{\lambda^*}(x_n) \to 0$. Then either $\liminf \|T_{\lambda^*}(x_n)\| \to 0$, in which case lemma 5 implies that $\lambda \in \sigma_c(T)$, or $\ell := \liminf \|T_{\lambda^*}(x_n)\| > 0$. We can take a subsequence x_{n_k} of x_n where $\|T_{\lambda^*}(x_{n_k})\| \geq \ell$. For simplicity we denote such subsequence by x_n. Let $y_n = T_{\lambda^*}(x_n)/\|T_{\lambda^*}(x_n)\|$. Clearly,

$$\|T_\lambda y_n\| = \frac{\|T_\lambda \cdot T_{\lambda^*}(x_n)\|}{\|T_{\lambda^*}(x_n)\|} = \frac{\|\Delta_\lambda(T)(x_n)\|}{\|T_{\lambda^*}(x_n)\|} \leq \frac{\|\Delta_\lambda(T)(x_n)\|}{\ell} \to 0,$$

and so $\lambda \in \sigma_c(T)$. □

The main result of the paper is now a direct consequence of theorem 7.

Theorem 8 *Let $T \in \mathcal{B}(X)$, then $\sigma(T) = \sigma_S(T)$.*

Author Contributions All the authors wrote the manuscript.

Funding Open access funding provided by FCT|FCCN (b-on).

Declarations

Conflict of interest The authors declare no conflict of interest.

References

1. Alpay, D., Colombo, F., Kimsey, D.P.: The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum. J. Math. Phys. **57**, 023503 (2016)
2. Alpay, D., Colombo, F., Kimsey, D.P., Sabadini, I.: The spectral theorem for unitary operators based on the S-spectrum. Milan J. Math., pp. 41–61 (2016)
3. Colombo, F., Sabadini, I., Struppa, D.C.: Noncommutative functional calculus. Theory and applications of slice hyperholomorphic functions. Prog. Math. , vol. **289**. Birkhauser/Springer Basel AG, Basel (2011)
4. Colombo, F., Sabadini, I.: On some notions of spectra for quaternionic operators and for $n-$ tuples of operators. C. R. Acad. Sci. Ser. **I**(350), 399–402 (2012)
5. Colombo, F., Gantner, J., Kimsey, D.: Spectral Theory and the S-Spectrum for Quaternionic Operators. Birkhauser/Springer Basel AG, Basel (2018)
6. Ghiloni, R., Moretti, V., Perotti, A.: Continuous slice functional calculus in quaternionic Hilbert spaces. Rev. Math. Phys. **25**(4), 1350006 (2013)

Authors and Affiliations

Luís Carvalho[1,2] · Cristina Diogo[1,2] · Sérgio Mendes[1,3] · Helena Soares[1,4]

✉ Cristina Diogo
cristina.diogo@iscte-iul.pt

Luís Carvalho
luis.carvalho@iscte-iul.pt

Sérgio Mendes
sergio.mendes@iscte-iul.pt

Helena Soares
helena.soares@iscte-iul.pt

1 Instituto Universitário de Lisboa (ISCTE-IUL), Av. das Forças Armadas, 1649-026 Lisbon, Portugal

2 Center for Mathematical Analysis, Geometry, and Dynamical Systems Mathematics Department, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

3 Centro de Matemática e Aplicações, Universidade da Beira Interior, Rua Marquês d'Ávila e Bolama, 6201-001 Covilhã, Portugal

4 Centro de Investigação em Matemática e Aplicações, Universidade de Évora Colégio Luís António Verney, Rua Romão Ramalho, 59, 7000-671 Évora, Portugal