Journal of Applied Mathematics

volume III (2010), number III

M|G|∞ SYSTEM TRANSIENT BEHAVIOR WITH TIME ORIGIN AT THE BEGINNING OF A BUSY PERIOD MEAN AND VARIANCE

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Abstract. The $M|G|\infty$ queue system transient probabilities, with time origin at the beginning of a busy period, are determined. It is highlighted the obtained distribution mean and variance study as time functions. In this study it is determinant the hazard rate function service time. The results obtained are applied in modeling disease and unemployment situations.

Key words and phrases. $M|G|\infty$, hazard rate function, disease, unemployment.

Mathematics Subject Classification. 60K35.

1 Introduction

In the $M|G|\infty$ queue system the customers arrive according to a Poisson process at rate λ, receive a service which time is a positive random variable with distribution function $G(.)$ and mean α and, when they arrive, they find immediately an available server. Each customer service is independent from the other customers' services and from the arrivals process. The traffic intensity is $\rho = \lambda\alpha$.

$N(t)$ is the number of occupied servers (or the number of customers being served) at instant t, in a M $|$G $|\infty$ system.

From (Takács, 1962), as $p_{0n}(t) = P[N(t) = n\,|N(0) = 0]$, $n = 0, 1, 2, ...$,

$$p_{0n}(t) = \frac{\left(\lambda \int_0^t [1 - G(v)]\, dv\right)^n}{n!} e^{-\lambda \int_0^t [1-G(v)]dv}, \quad n = 0, 1, 2, ... \tag{1}$$

So, the transient distribution, when the system is initially empty, is Poisson with mean $\lambda \int_0^t [1 - G(v)]dv$.

The stationary distribution is the limit distribution:

$$\lim_{t\to\infty} p_{0n}(t) = \frac{\rho^n}{n!}\ e^{-\rho}, \quad n = 1, 2, \dots . \tag{2}$$

This queue system, as any other, has a sequence of busy periods and idle periods. A busy period begins when a customer arrives at the system finding it empty.

Be $p_{1'n} = P[N(t) = n \,|N(0) = 1']$, $n = 0, 1, 2, \dots$, meaning $N(0) = 1'$ that the time origin is an instant at which a customer arrives at the system jumping the number of customers from 0 to 1.

That is: a busy period begins.
At $t \geq 0$ possibly:

- The customer that arrived at the initial instant either abandoned the system, with probability $G(t)$, or goes on being served, with probability $1 - G(t)$;

- The other servers, that were unoccupied at the time origin, either go on unoccupied or occupied with 1, 2, ... customers, being the probabilities $p_{0n}(t)\,, \quad n = 0, 1, 2, \dots$.

Both subsystems, the one of the initial customer and the one of the servers initially unoccupied, are independent and so

$$\begin{aligned} p_{1'0}(t) &= p_{00}(t)\, G(t) \\ p_{1'n}(t) &= p_{0n}(t)\, G(t) + p_{0n-1}(t)\,(1 - G(t))\,, \quad n = 1, 2, \dots . \end{aligned} \tag{3}$$

It is easy to see that

$$\lim_{t\to\infty} p_{1'n}(t) = \frac{\rho^n}{n!}\ e^{-\rho}, \quad n = 0, 1, 2, \dots . \tag{4}$$

Denoting $\mu(1', t)$ and $\mu(0, t)$ the distributions given by (3) and (1) mean values, respectively,

$$\begin{aligned} \mu(1', t) &= \sum_{n=1}^{\infty} n p_{1'n}(t) = \sum_{n=1}^{\infty} nG(t)\, p_{00}(t) + \sum_{n=1}^{\infty} n p_{0n-1}(t)\,(1 - G(t)) = \\ &= G(t)\,\mu(0, t) + (1 - G(t)) \sum_{j=0}^{\infty} (j+1) p_{0j}(t) = \mu(0, t) + (1 - G(t))\,, \end{aligned}$$

that is

$$\mu(1', t) = 1 - G(t) + \lambda \int_0^t [1 - G(v)]dv \tag{5}$$

As

$$\begin{aligned}\sum_{n=0}^{\infty} n^2 p_{1'n}(t) &= G(t)\sum_{n=1}^{\infty} n^2 p_{0n}(t) + (1-G(t))\sum_{n=1}^{\infty} n^2 p_{0n-1}(t) = \\ &= G(t)\left(\mu^2(0,t)+\mu(0,t)\right) + (1-G(t))\left(\mu^2(0,t)+3\mu(0,t)+1\right) = \\ &= \mu^2(0,t) + (3-2G(t))\,\mu(0,t) + 1 - G(t)\,,\end{aligned}$$

denoting $V(1',t)$ the variance associated to the distribution defined by (3), it is obtained

$$V(1',t) = \mu(0,t) + G(t) - G^2(t)\ . \tag{6}$$

The main target is to study $\mu(1',t)$ and $V(1',t)$ as time functions. It will be seen that, in its behavior as time functions, plays an important role the hazard rate function service time given by, see for instance (Ross, 1983),

$$h(t) = \frac{g(t)}{1-G(t)} \tag{7}$$

and $g(.)$ is the density associated to $G(.)$.

2 Application in disease and unemployment situations

$M|G|\infty$ systems have great applicability in the modeling of real problems. See, for instance, the works of (Carrillo, 1991), (Ferreira, 1988), (Ferreira, Andrade and Filipe, 2009) (Hershey, Weiss and Morris, 1981) and (Kelly, 1979). The ones presented in this paper are very interesting and the results that will be shown in the sequence are particularly adequate to its study.

2.1 Disease

In this case the customers are the people that have a certain disease. They arrive at the system when they fall sick and their service time is the time during which they are sick. The time the first one falls sick, may be the beginning of an epidemic, is the beginning of a busy period. An idle period is a period of disease absence.

The service hazard rate function is the rate at which they get cured.

2.2 Unemployment

Now the customers are the unemployed in a certain activity. They arrive at the system when they loose their jobs and their service time is the time during which they are unoccupied.

An idle period is a full employment period. A busy period begins with the first worker loosing his job.

The hazard rate function is the rate at which the unemployed workers turn employees.

In both cases (3) is applicable. It must be checked if the people fall sick or loose their jobs according to a Poisson process. The failing of this hypothesis is more expectable in the unemployment situation. In some of this situations may be it is more adequate to consider a mechanism of batch arrivals.

The beginning of the epidemics or of the unemployment periods can be determined today with a great precision.

The results that will be presented can help to forecast the evolution of the situations.

Finally it is necessary to adjust the time distributions adequate to the disease and unemployment periods. In this last case the situation may not be the same for the various activities.

3 $\mu(1', t)$ study as time function

Lemma 3.1 *If $G(t) < 1$, $t > 0$ continuous and differentiable and*

$$h(t) \leq \lambda, t > 0 \tag{8}$$

$\mu(1', t)$ is non- decreasing.

Dem.:

It is enough to note, according to (5), that

$$\frac{d}{dt}\mu(1', t) = (1 - G(t))(\lambda - h(t)) .$$

Obs.:

- If the rate at which the services end is lesser or equal than the customers arrival rate $\mu(1', t)$ is non- decreasing.
- **Disease:** If the rate at which people get cured is lesser or equal than the rate at which they fall sick, the mean number of sick people is a non- decreasing time function.
- **Unemployment:** If the rate at which the workers loose their jobs is lesser than the rate at which they turn employees, the mean number of unemployed people is a non-decreasing time function.
- For the $M|M|\infty$ system (8) is equivalent to

$$\rho \geq 1 \tag{9}$$

- $\lim_{t\to\infty} \mu(1', t) = \rho.$
- **Disease:** If an epidemic lasts a vey long time, the mean number of sick people will be closer and closer from the traffic intensity.
- **Unemployment:** If an unemployment period lasts a very long time, the mean number of unemployed people will be closer and closer from the traffic intensity.

Making $h(t) - \lambda = \beta(t), \quad \beta(.)$ it is obtained

$$G(t) = 1 - (1 - G(0))\, e^{-\lambda t - \int_0^t \beta(u)du}, t \geq 0, \frac{\int_0^t \beta(u)\,du}{t} \geq -\lambda \tag{10}$$

So

Lemma 3.2 *If* $\beta = 0$

$$G(t) = 1 - (1 - G(0))\, e^{-\lambda t}, \quad t \geq 0 \tag{11}$$

and $\mu(1', t) = 1 - G(0) = \rho, \quad t \geq 0.$

Obs.:

- **Disease:** If the time that a patient is sick is a random variable, with a distribution function given by (11) the mean number of sick people is always equal to the traffic intensity.
- **Unemployment**: If the time of unemployment is a random variable, with a distribution function given by (11) the mean number of unemployed people is always equal to the traffic intensity.

For some particular service time distributions:

- Deterministic with value α

$$\mu(1', t) = \begin{cases} 1 + \lambda t, t < \alpha \\ \rho, t \geq \alpha \end{cases} \tag{12}$$

- Exponential

$$\mu(1', t) = \rho + (1 - \rho)\ e^{-\frac{t}{\alpha}}\ . \tag{13}$$

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$$\begin{aligned} G(t) &= 1 - \frac{(1 - e^{-\rho})(\lambda + \beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t} - 1\right) + \lambda}, t \geq 0, -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho} - 1} \\ \mu(1', t) &= \frac{(1 - e^{-\rho})(\lambda + \beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t} - 1\right) + \lambda} + \rho - \log\left(1 + (e^{\rho} - 1)\, e^{-(\lambda+\beta)\ t}\right) \end{aligned} \tag{14}$$

For this collection of service time distributions the busy period (and so also the time that an **epidemic** or an **unemployment period** lasts) is exponentially distributed with an atom at the origin

$$B^{\beta}(t) = 1 - \frac{\lambda + \beta}{\lambda}\left(1 - e^{-\rho}\right) e^{-e^{-\rho}(\lambda+\beta)t}, \qquad t \geq 0, \qquad -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho} - 1}. \tag{15}$$

4 $V(1', t)$ study as time function

Lemma 4.1 *If* $G(t) < 1, \quad t > 0$, *continuous and diffentiable and*

$$h(t) \geq -\frac{\lambda}{1 - 2G(t)} \tag{16}$$

$V(1', t)$ *is non-decreasing.*

Dem.:

It is enough to note, according to (6), that

$$\begin{aligned}\frac{d}{dt}V(1', t) &= \lambda(1 - G(t)) + g(t) - 2G(t)g(t) = \lambda(1 - G(t)) + g(t)(1 - 2G(t)) = \\ &= (1 - G(t))(h(t)(1 - 2G(t)) + \lambda).\end{aligned}$$

Obs.:

- Obviously $1 - 2G(t) < 0 \Leftrightarrow G(t) > \frac{1}{2}, \quad t > 0.$
- **Disease:** If the rate at which people get cured, the rate at which they fall sick and the sickness duration distribution function hold (16) the variance of the number of sick people is a non- decreasing time function.
- **Unemployment:** If the rate at which the workers loose their jobs, the rate at which they turn employees and the unemployment duration distribution function hold (16) the variance of the number of sick people is a non- decreasing time function.
- $\lim_{t\to\infty} V(1', t) = \rho.$
- **Disease:** If an epidemic lasts a vey long time, the variance of the number of sick people will be closer and closer from the traffic intensity.
- **Unemployment:** If an unemployment period lasts a very long time, the variance of the number of unemployed people will be closer and closer from the traffic intensity.

- **Disease:** If an epidemic lasts a very long time the number of sic people is distributed according to a Poisson distribution with mean ρ, see (4).
- **Unemployment:** If an unemployment period lasts a very long time the mean number of unemployed people is Poisson distributed with mean ρ, see (4).

Making $h(t) + \frac{\lambda}{1-2G(t)} = 0$ the following proposition holds:

Lemma 4.2 *If $G(.)$ is implicitly defined as*

$$\frac{1-G(t)}{1-G(0)} e^{2(G(t)-G(0))} = e^{-\lambda t}, t \geq 0 \tag{17}$$

$$V(1', t) = \rho, t \geq 0.$$

Obs.:

- The density associated to (17) is

$$g(t) = -\frac{\lambda e^{-\lambda t}(1-G(0))}{(1-2G(t))\, e^{2(G(t)-G(0))}} \tag{18}$$

- After (18), denoting S the associated random variable, it is easy to see that, with $G(0) > \frac{1}{2}$,

$$\frac{(1-G(0))\, n! e^{-2(1-G(0))}}{\lambda^n} \leq \mathrm{E}[S^n] \leq \frac{(1-G(0))\, n!}{(2G(0)-1)\, \lambda^n}, n = 1, 2, \ldots. \tag{19}$$

For some particular service time distributions:

- Deterministic with value α

$$V(1', t) = \begin{cases} \lambda t, t < \alpha \\ \rho, t \geq \alpha \end{cases}. \tag{20}$$

- Exponential

$$V(1', t) = \rho\left(1 - e^{-t/\alpha}\right) + e^{-\frac{t}{\lambda}} + e^{-\frac{2t}{\alpha}}. \tag{21}$$

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$$\begin{aligned} G(t) &= 1 - \frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda}, \qquad t \geq 0, \qquad -\lambda \leq \beta \leq \frac{\lambda}{e^{\rho}-1} \\ V(1', t) &= \rho - \log\left(1 + (e^{\rho}-1)\, e^{-(\lambda+\beta)t}\right) + \frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda} + \\ &+ \left(\frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda}\right)^2. \end{aligned} \tag{22}$$

5 Conclusions

With very simple probalistic reasoning, the M |G |∞ transient probabilities, being the time origin the beginning of a busy period instant, were determined. It is enough to condition to the service lasting of the first costumer.

It was possible to study $\mu(1', t)$ and $V(1', t)$, as time functions, playing here an important role the service time hazard rate function.

This model may be applied in modeling real situations being the difficulties the usual ones when theoretical models are applied to real situations.

Finally note that, in the disease application, the model is note applicable to contagious epidemics. In this situation it would be more realistic to consider arrival rates not constant. But, with this kind of rates, it is not possible to have results as interesting and useful as those presented in this work.

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