Chapter 8

The Study of Maintenance Costs of Non-Autonomous Pension Funds through a Diffusion Process

Manuel Alberto M. Ferreira*

Instituto Universitário de Lisboa (ISCTE – IUL), ISTAR- IUL, Lisboa, Portugal

Abstract

The case of certain pensions funds that are not auto financed, and are systematically maintained with an outside financing effort, is considered in this work. As a representation of the unrestricted reserves value process of this kind of funds, a time homogeneous diffusion process with finite expected time to ruin is proposed. Then it is admitted a financial tool that regenerates the diffusion at some level with positive value, every time the diffusion hits a barrier at the origin. So, the financing effort can be modeled as a renewal-reward process if the regeneration level is kept constant. The evaluation of the perpetual maintenance cost expected values and of the finite time maintenance cost are studied. Also, we present an application of this approach when the unrestricted reserves value process behaves as a generalized Brownian motion process.

Keywords: pensions fund, diffusion process, first passage times, perpetuity, renewal equation

1. Introduction

* Corresponding Author's Email: manuel.ferreira@iscte-iul.pt.

Along this paper, we intend to deal with the protection cost present value expectation for a non-autonomous pensions fund. Two problems are considered in this context:

- One concerning the case of the above-mentioned expectation when the protection effort is perpetual,
- Other concerning the case of the protection effort for a finite time.

It is admitted that the unrestricted fund reserves behavior may be modeled as a time homogeneous diffusion process and use then a regeneration scheme of the diffusion to include the effect of an external financing effort.

A similar work is [1], where is considered a Brownian motion process conditioned by a particular reflection scheme. Less constrained, but in different conditions, exact solutions were then obtained for both problems.

The work presented in [2], on asset-liability management aspects, motivated the use of an application of the Brownian motion example in that domain.

Part of this work was presented at the Fifth International Congress on Insurance: Mathematics & Economics, [3]. Other works on this subject are [4, 5].

2. Pensions Fund Reserves Behaviour Representation

Be $X(t), t \geq 0$ the reserves value process of a pensions fund given by an initial reserve amount $a, a > 0$, added to the difference between the total amount of contributions received up to time *t* and the total amount of pensions paid up to time *t*. It is assumed that $X(t)$ is a time homogeneous diffusion process, with $X(0) = a$, defined by drift and diffusion coefficients:

$$\lim_{h \to 0} \frac{1}{h} E[X(t+h) - X(t) | X(t) = x] = \mu(x).$$
$$\lim_{h \to 0} \frac{1}{h} E\left[\left(X(t+h) - X(t)\right)^2 | X(t) = x\right] = \sigma^2(x).$$

Call S_a the first passage time of $X(t)$ by 0, coming from a. The funds to be considered in this work are non-autonomous funds. So

$$E[S_a] < \infty, \text{for any } a > 0 \qquad (2.1)$$

that is: funds where the pensions paid consume in finite expected time any initial positive reserve and the contributions received, so that other financing resources are needed in order that the fund survives.

The condition (2.1) may be fulfilled for a specific diffusion process using criteria based on the drift and diffusion coefficients. Here the work presented in [6], pg. 418-422, is followed in this context. Begin accepting that $P(S_a < \infty) = 1$ if the diffusion scale function

$$q(x) = \int_{x_0}^{x} e^{-\int_{x_0}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy} dz,$$

where x_0 is a diffusion state space fixed arbitrary point, fulfilling $q(\infty) = \infty$. Then the condition (2.1) is equivalent to $p(\infty) < \infty$, where

$$p(x) = \int_{x_0}^{x} \frac{2}{\sigma^2(z)} e^{\int_{x_0}^{z} \frac{2\mu(y)}{\sigma^2(y)} dy} dz,$$

is the diffusion speed function.

It is admitted, whenever the exhaustion of the reserves happens, that an external source places instantaneously an amount $\theta, \theta > 0$ of money in the fund, so that it may go on effective.

The reserves value process conditioned by this financing scheme is represented by the modification $\check{X}(t)$ of $X(t)$ that restarts at the level θ whenever it hits 0. Note that since $X(t)$ was defined as a time homogeneous diffusion, $\check{X}(t)$ is a regenerative process. Call $T_1, T_2, T_3, \ldots$ the sequence of random variables where T_n denotes the n^{th} $\check{X}(t)$ passage time by 0. It is obvious that the sequence of time intervals between these hitting times $D_1 = T_1, D_2 = T_2 - T_1, D_3 = T_3 - T_2, \ldots$ is a sequence of independent random variables where D_1 has the same probability distribution as S_a and $D_2, D_3, \ldots$ the same probability distribution as S_θ.

3. First Passage Times Laplace Transforms

Call $f_a(s)$ the probability density function of $S_a(D_1)$. The corresponding probability distribution function is denoted by $F_a(s)$. The Laplace transform of S_a is

$$\varphi_a(\lambda) = E\left[e^{-\lambda S_a}\right] = \int_0^\infty e^{-\lambda s} f_a(s)ds, \lambda > 0.$$

Consequently, the density, distribution and transform of S_θ $(D_2, D_3, \ldots)$ will be denoted by $f_\theta(s), F_\theta(s)$ and $\varphi_\theta(\lambda)$, respectively.

The transform $\varphi_a(\lambda)$ satisfies the second order differential equation

$$\frac{1}{2}\sigma^2(a)u_\lambda''(a) + \mu(a)u_\lambda'(a) = \lambda u_\lambda(a), u_\lambda(a) = \varphi_a(\lambda), u_\lambda(0)=1,$$
$$u_\lambda(\infty) = 0 \quad (3.1)$$

see [7], pg. 478, [8] pg. 243 and [9], pg. 89.

4. Perpetual Maintenance Cost Present Value

Consider the perpetual maintenance cost present value of the pensions fund that is given by the random variable

$$V(r, a, \theta) = \sum_{n=1}^{\infty} \theta e^{-rT_n}, \text{ r} > 0,$$

where r represents the appropriate discount rate. Note that $V(r, a, \theta)$ is a random perpetuity. What matters is its expected value which is easy to get using Laplace transforms. Since the T_n Laplace transform is $E\left[e^{-\lambda T_n}\right] = \varphi_a(\lambda)\varphi_\theta^{n-1}(\lambda)$,

$$v_r(a, \theta) = E[V(r, a, \theta)] = \frac{\theta\varphi_a(r)}{1-\varphi_\theta(r)} \quad (4.1)$$

It is relevant to note that

$$\lim_{\theta \to 0} v_r(a, \theta) = \frac{u_r(a)}{-u_r'(0)} \quad (4.2)$$

5. Finite Time Period Maintenance Cost Present Value

Define the renewal process (t) , generated by the extended sequence $T_0 = 0, T_1, T_2, ...$, by $N(t) = \sup\{n: T_n \leq t\}$. The present value of the pensions fund maintenance cost up to time *t* is represented by the stochastic process

$$W(t; r, a, \theta) = \sum_{n=1}^{N(t)} \theta e^{-rT_n}, W(t; r, a, \theta) = 0 \; if \;\; N(t) = 0.$$

The important now is the expected value function of the process evaluation: $w_r(t; a, \theta) = E[W(t; r, a, \theta)]$. Begin noting that it may be expressed as a numerical series. In fact, evaluating the expected value function conditioned by $N(t) = n$, it is obtained

$$E[W(t; r, a, \theta)|N(t) = n] = \theta\varphi_a(r)\frac{1 - \varphi_\theta^n(r)}{1 - \varphi_\theta(r)}.$$

Repeating the expectation:

$$w_r(t; a, \theta) = E[E[W(t; r, a, \theta)]|N(t)] = \theta\varphi_a(r)\frac{1-\gamma\left(t, \varphi_\theta(r)\right)}{1-\varphi_\theta(r)} \qquad (5.1)$$

where $\gamma(t, \xi)$ is the probability generating function of $N(t)$.

Denote now the T_n probability distribution function by $G_n(s)$ and assume $G_0(s) = 1,$ for $s \geq 0$. Recalling that $P(N(t) = n) = G_n(t) - G_{n+1}(t)$, the above-mentioned probability generating function is

$$\gamma(t, \xi) = \sum_{n=0}^{\infty} \xi^n \, P(N(t) = n) = 1 - (1 - \xi) \sum_{n=1}^{\infty} \xi^{n-1} \, G_n(t) \qquad (5.2)$$

Substituting (5.2) in (5.1), $w_r(t; a, \theta)$ is expressed in the form of the series

$$w_r(t; a, \theta) = \theta\varphi_a(r) \sum_{n=1}^{\infty} \varphi_\theta^{n-1}(r) \, G_n(t) \qquad (5.3)$$

Then, using (5.3), we will show that $w_r(t; a, \theta)$ satisfies a renewal type integral equation.

Write for the $w_r(t; a, \theta)$ ordinary Laplace transform $\psi(\lambda) = \int_0^\infty e^{-\lambda s} w_r(s; a, \theta) ds$. Recalling that the probability distribution function $G_n(s)$ of T_n ordinary Laplace transform is given by

$\int_0^\infty e^{-\lambda s} G_n(s)ds = \varphi_a(\lambda)\frac{\varphi_\theta^{n-1}(\lambda)}{\lambda}$, and performing the Laplace transforms in both sides of (5.3), it is achieved

$$\psi(\lambda) = \frac{\theta\varphi_a(r)\varphi_a(\lambda)}{\lambda\left(1-\varphi_\theta(r)\varphi_\theta(\lambda)\right)}$$

or

$$\psi(\lambda) = \theta\varphi_a(r)\frac{\varphi_a(\lambda)}{\lambda} + \psi(\lambda)\varphi_\theta(r)\varphi_\theta(\lambda) \quad (5.4)$$

Inverting the transforms in both sides of (5.4) the following defective renewal equation

$$w_r(t;a,\theta) = \theta\varphi_a(r)F_a(t) + \int_0^t w_r(t-s;a,\theta)\varphi_\theta(r)f_\theta(s)ds \quad (5.5)$$

results.

Now an asymptotic approximation of $w_r(t;a,\theta)$ will be obtained through the key renewal theorem, following [7], pg. 376.

If in (5.5) $t \to \infty$

$$w_r(\infty;a,\theta) = \theta\varphi_a(r) + w_r(\infty;a,\theta)\varphi_\theta(r) \quad (5.6)$$

or

$$w_r(\infty;a,\theta) = \frac{\theta\varphi_a(r)}{1-\varphi_\theta(r)} = v_r(a,\theta).$$

That is: the expression (4.1) for $v_r(a,\theta)$is obtained again. Subtracting each side of (5.6) from each side of (5.5), and performing some elementary calculations the following, still defective, renewal equation

$$J(t) = j(t) + \int_0^t J(t-s)\varphi_\theta(r)f_\theta(s)ds \quad (5.7)$$

where $J(t) = w_r(\infty;a,\theta) - w_r(t;a,\theta)$ and $j(t) = \theta\varphi_a(r)\left(1-F_a(t)\right) + \frac{\theta\varphi_a(r)\varphi_\theta(r)}{1-\varphi_\theta(r)}\left(1-F_\theta(t)\right)$.

Now, to obtain a common renewal equation from (5.7), it must be admitted the existence of a value $k > 0$ such that

$$\int_0^\infty e^{ks}\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta(-k) = 1.$$

This imposes that the transform $\varphi_\theta(\lambda)$ is defined in a domain different from the one initially considered, that is a domain that includes a convenient subset of the negative real numbers.

Multiplying both sides of (5.7) by e^{kt} the common renewal equation desired is finally obtained:

$$e^{kt}J(t) = e^{kt}j(t) + \int_0^t e^{k(t-s)}J(t-s)e^{ks}\,\varphi_\theta(r)f_\theta(s)ds$$

from which, through the application of the key renewal theorem, it results

$$\lim_{t\to\infty} e^{kt}J(t) = \frac{1}{k_0}\int_0^\infty e^{ks}j(s)\,ds \qquad (5.8)$$

with $k_0 = \int_0^\infty se^{ks}\,\varphi_\theta(r)f_\theta(s)ds = \varphi_\theta(r)\varphi_\theta'(-k)$, provided that $e^{kt}j(t)$ is directly Riemann integrable. The integral in (5.8) may expressed in terms of transforms as

$$\int_0^\infty e^{ks}j(s)\,ds = \frac{\theta\varphi_a(r)\varphi_a(-k)}{k}.$$

Resuming this section:

- An asymptotic approximation, in the sense of (5.8) was obtained:

$$w_r(t;a,\theta) \approx v_r(a,\theta) - c_r(a,\theta)e^{-k_r(\theta)t} \qquad (5.9)$$

where $k_r(\theta)$ is the positive value of k that satisfies

$$\varphi_\theta(r)\varphi_\theta(-k) = 1 \qquad (5.10)$$

and

$$c_r(a,\theta) = \frac{\theta\varphi_a(r)\varphi_a\big(-k_r(\theta)\big)}{-k_r(\theta)\varphi_\theta(r)\varphi_\theta^{'}(-k_r(\theta))} \quad (5.11)$$

6. Brownian Motion Example

Consider that the diffusion process $X(t)$, underlying the reserves value behavior of the pensions fund, is a generalized Brownian motion process, with drift $\mu(x) = \mu, \mu < 0$ and diffusion coefficient $\sigma^2(x) = \sigma^2, \sigma > 0$. Observe that the setting satisfies the conditions that were assumed before to the former work, namely $\mu < 0$ implies condition (2.1). Everything else remaining as previously stated, it will be proceeded to present the consequences of this particularization. In general, it will be added a $(*)$ to the notation used before because it is intended to use these specific results later.

To get the first passage time S_a Laplace transform it must be solved, remember (3.1),

$$\frac{1}{2}\sigma^2(a)u_\lambda^{*''}(a) + \mu(a)u_\lambda^{*'}(a) = \lambda u_\lambda^*(a), u_\lambda^*(a) = \varphi_a(\lambda), u_\lambda^*(0)=1,$$
$$u_\lambda^*(\infty) = 0 .$$

This is a homogeneous second order differential equation with constant coefficients, which general solution is

$$u_\lambda^*(a) = \beta_1 e^{\alpha_1 a} + \beta_2 e^{\alpha_2 a}, with\ \alpha_1, \alpha_2 = \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda\sigma^2}}{\sigma^2}.$$

Condition $u_\lambda^*(\infty) = 0$ implies $\beta_1 = 0$ and $u_\lambda^*(0)$=1 implies β_2=1 so that the solution is achieved:

$$u_\lambda^*(a) = e^{-K_\lambda a}\left(= \varphi_a^*(\lambda)\right), K_\lambda = \frac{\mu+\sqrt{\mu^2+2\lambda\sigma^2}}{\sigma^2} \quad (6.1)$$

In this case, the perpetual maintenance cost present value of the pensions fund is given by, following (4.1) and using (6.1),

$$v_r^*(a,\theta) = \frac{\theta e^{-K_r a}}{1-e^{-K_r \theta}} \tag{6.2}$$

Note that $v_r^*(a,\theta)$ is a decreasing function of a and an increasing function of θ. Proceeding as before, in particular

$$\lim_{\theta \to 0} v_r^*(a,\theta) = \frac{e^{-K_r a}}{K_r}\,. \tag{6.3}$$

This expression has been obtained in [1], expression number [7], in a different context and using different methods but, obviously, with identical meaning. In [1] the authors worked then with a generalized Brownian motion, with no constraints in what concerns the drift coefficient, conditioned by a reflection scheme at the origin.

To reach an expression for the finite time maintenance cost present value, start by the computation of $k_r^*(\theta)$, solving (5.10). This means finding a positive k satisfying

$e^{-K_r\theta} e^{-K_{-\lambda}\theta} = 1$ or $K_r + K_{-\lambda} = 0$. This identity is verified for the value of k

$$k_r^*(\theta) = \frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \tag{6.4}$$

Note that the solution is independent of θ in these circumstances. A simplified solution, independent of a and , for $c_r^*(a,\theta)$ was also obtained. Using (5.11) the result is

$$c_r^*(a,\theta) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2} \tag{6.5}$$

Combining these results as in (5.9) it is observable that the asymptotic approximation for this particularization reduces to $w_r^*(t;a,\theta) \approx v_r^*(a,\theta) - \pi_r(t)$, where the function $\pi_r(t)$ is, considering (6.4) and (6.5),

$$\pi_r(t) = \frac{2\sigma^2\left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)}{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}\, e^{-\frac{\mu^2 - \left(-2\mu - \sqrt{\mu^2 + 2r\sigma^2}\right)^2}{2\sigma^2} t}, \text{if } \mu < -\sqrt{\frac{2r\sigma^2}{3}} \tag{6.6}$$

7. Representation of the Assets and Liability Behaviour

It is proposed to consider now an application of the results obtained earlier to an asset-liability management scheme of a pensions fund. Assume that the assets value process of a pensions fund may be represented by the geometric Brownian motion process

$$A(t) = be^{a+(\rho+\mu)t+\sigma B(t)} \text{ with } \mu < 0 \text{ and } ab\rho + \mu\sigma > 0,$$

where $B(t)$ is a standard Brownian motion process. Suppose also that the liabilities value process of the fund performs as the deterministic process $L(t) = be^{\rho t}$.

Under these assumptions, consider now the stochastic process $Y(t)$ obtained through the elementary transformation of $A(t)$

$$Y(t) = ln\frac{A(t)}{L(t)} = a + \mu t + \sigma B(t).$$

This is a generalized Brownian motion process exactly as the one studied before, starting at a and with drift μ and diffusion coefficient σ^2. Note also that the firs passage time of the assets process $A(t)$ by the mobile barrier T_n , the liabilities process, is the first passage time of $Y(t)$ by 0, with finite expected time under the condition, stated before, $\mu < 0$.

Also consider the pensions fund management scheme that raises the assets value by some positive constant θ_n, when the assets value falls equal to the liabilities process by the n^{th} time. This corresponds to consider the modification $\bar{A}(t)$ of the process $A(t)$ that restarts at times T_n when $A(t)$ hits the barrier $L(t)$ by the n^{th} time at the level $L(T_n) + \theta_n$. For purposes of later computations, it is a convenient choice the management policy where

$$\theta_n = L(T_n)\left(e^{\theta} - 1\right), \text{for some } \theta > 0 \qquad (7.1)$$

The corresponding modification $\tilde{Y}(t)$ of $Y(t)$ will behave as a generalized Brownian motion process that restarts at the level $ln\frac{L(T_n)+\theta_n}{L(T_n)} = \theta$ when it hits 0 (at times T_n).

Proceeding this way, it is reproduced via $\tilde{Y}(t)$ the situation observed before when the Brownian motion example was treated. The Laplace transform in (6.1) is still valid.

Similarly, to former proceedings, the results for the present case will be distinguished with the symbol $(\#)$. It is considered the pensions fund perpetual maintenance cost present value, because of the proposed asset-liability management scheme, given by the random variable:

$$V^{\#}(r,a,\theta)=\sum_{n=1}^{\infty}\theta_n e^{-rT_n}=\sum_{n=1}^{\infty}b\left(e^{\theta}-1\right)e^{-(r-\rho)T_n}, r>\rho$$

where *r* represents the appropriate discount interest rate. To obtain the above expression it was only made use of the $L(t)$ definition and (7.1). It is possible to express the expected value of the above random variable with the help of (6.2) as

$$v_r^{\#}(a,\theta)=\frac{b\left(e^{\theta}-1\right)}{\theta}v_{r-\rho}^{*}(a,\theta)=\frac{b\left(e^{\theta}-1\right)e^{-K_{r-\rho}a}}{1-e^{-K_{r-\rho}\theta}} \quad (7.2)$$

as $\theta\to 0$

$$\lim_{\theta\to 0}v_r^{\#}(a,\theta)=\frac{be^{-K_{r-\rho}a}}{K_{r-\rho}} \quad (7.3)$$

another expression explicit in [1].

In a similar way, the maintenance cost up to time *t* in the above-mentioned management scheme, is the stochastic process

$$W^{\#}(t;r,a,\theta)=\sum_{n=1}^{N(t)}b\left(e^{\theta}-1\right)e^{-(r-\rho)T_n}, \qquad W^{\#}(t;r,a,\theta)=0 \text{ if } N(t)=0,$$

with expected value function

$$w_r^{\#}(t;a,\theta)=\frac{b\left(e^{\theta}-1\right)}{\theta}w_{r-\rho}^{*}(t;a,\theta) \quad (7.4)$$

The results of section 6 with r replaced by $r-\rho$ may be combined as in (7.4) to obtain an asymptotic approximation.

Concluding Remarks

In the general diffusion setting, the main results are formulae (4.1) and (5.9). The whole work depends on the possibility of solving the equation (3.1) to obtain the Laplace transforms of the first passage times. Unfortunately, the solutions are known only for rare cases. An obvious case for which the solution of the equation is available is the one of the Brownian motion diffusion processes. The main results concerning this particularization are formulae (6.2) and (6.6). Certain transformations of the Brownian motion process that allowed us to make use of the available Laplace transform may be explored as it was done in section 7. Formulae (7.2) and (7.4) are this application most relevant results.

Other approaches, on the use of stochastic processes in the study of the issue of pension funds sustainability, can be seen in [10, 11, and 12].

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