Empirical Comparison of S&P 500 Index Options: Black-Scholes-Merton Model and Heston Model

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Resumo

A determinação do preço das opções tem sido um problema fundamental na engenharia financeira desde há várias décadas e tem uma vasta gama de aplicações em vários domínios, incluindo investimentos e gestão de riscos. O modelo de Black-Scholes-Merton tem sido a base da teoria do preço das opções desde a sua introdução em 1973. No entanto, estudos empíricos mostraram que o modelo tem algumas limitações, incluindo o pressuposto de uma volatilidade constante, o que leva a um desfasamento entre as previsões do modelo e os preços de mercado. Em contrapartida, o modelo de Heston, introduzido em 1993, é um modelo mais avançado de determinação do preço das opções que incorpora a volatilidade estocástica, o que o torna mais realista.

O modelo de Black-Scholes-Merton e o modelo de Heston são dois dos modelos mais utilizados em matemática financeira para a determinação do preço de derivados financeiros. Ambos os modelos utilizam o cálculo estocástico para modelar a dinâmica dos preços dos activos, mas diferem nos pressupostos que fazem sobre o ativo subjacente.

O objetivo desta tese de mestrado é comparar e contrastar o modelo de Black-Scholes-Merton e o modelo de Heston, com destaque para os seus pontos fortes e fracos em diferentes ambientes de mercado. A tese começará com uma breve descrição dos dois modelos, seguida de uma discussão dos seus principais pressupostos e parâmetros. A tese apresentará então uma análise empírica do desempenho dos dois modelos, utilizando dados financeiros reais sobre opções S&P500 e uma variedade de métricas de avaliação.

Palavras-chave: Modelo Black-Scholes-Merton, Modelo Heston, Volatilidade Estocástica.

Abstract

Option pricing has been a fundamental problem in financial engineering for several decades, and it has a wide range of applications in various fields, including investments, risk management, and portfolio optimization. The Black-Scholes-Merton model has been the cornerstone of option pricing theory since its introduction in 1973. However, empirical studies have shown that the model has some limitations, including assuming constant volatility and ignoring stochastic volatility, which leads to a mismatch between model predictions and actual market prices. In contrast, the Heston model, introduced in 1993, is a more advanced option pricing model that incorporates stochastic volatility, which makes it more realistic and accurate.

The Black-Scholes-Merton model and the Heston model are two of the most widely used models in quantitative finance for pricing financial derivatives. Both models make use of stochastic calculus to model the dynamics of asset prices, but they differ in the assumptions they make about the underlying asset.

The purpose of this master thesis is to compare and contrast the Black-Scholes-Merton model and the Heston model, with a focus on their strengths and weaknesses in different market environments. The thesis will begin with a brief overview of the two models, followed by a discussion of their key assumptions and parameters. The thesis will then present a comprehensive analysis of the performance of the two models, using S&P500 options real-world financial data and a variety of evaluation metrics.

Keywords: Black-Scholes-Merton Model, Heston Model, Stochastic Volatility.

Contents

CHAPTER 1

Introduction

The world of financial derivatives has witnessed remarkable growth and innovation over the past few decades, with options contracts being one of the most widely traded instruments in the financial markets. Investors and traders rely on sophisticated mathematical models to accurately price and analyze options, allowing them to make informed investment decisions and manage risk effectively. Two prominent models that have gained significant attention in options pricing are the Black-Scholes-Merton (BSM) model, introduced in Black and Scholes (1973) and Merton (1973), and the Heston model (1993).

The BSM model, developed by the economists Fischer Black, Myron Scholes and Robert Merton in 1973, revolutionized the field of quantitative finance by providing a closed-form solution for pricing European options. This groundbreaking model assumes constant volatility, constant risk-free interest rate, and efficient markets, making it a cornerstone of options pricing theory. However, it is well-known that these assumptions may not hold in real-world financial markets, prompting researchers to develop more advanced models.

The Heston model, proposed by Steven Heston in 1993, attempts to address some of the limitations of the Black-Scholes model by introducing stochastic volatility. This model allows for the volatility of the underlying asset to fluctuate over time, capturing the observed volatility smile or skew in the options market. By incorporating a more realistic representation of volatility dynamics, the Heston model aims to provide better pricing accuracy for options on assets with time-varying volatility.

Given the prominence of the S&P 500 index as a benchmark for the U.S. stock market, it is crucial to investigate the performance of options pricing models specifically calibrated to this widely followed index. Therefore, this master thesis aims to conduct an empirical comparison study between the Black-Scholes-Merton model and the Heston model for pricing options on the S&P 500 index.

The primary objective of this study is to assess the relative accuracy and performance of these two models in replicating observed option prices and hedging strategies for various maturities and strike prices. By employing historical data and employing advanced statistical techniques, we seek to evaluate the models' pricing errors, their ability to capture the implied volatility surface, and their effectiveness in risk management.

Furthermore, this study also aims to shed light on the practical implications of using either model in real-world trading scenarios. We will analyze the impact of transaction costs, liquidity constraints, and trading strategies on the overall performance of the models. Additionally, we will explore the computational efficiency of each model, considering the computational resources required for accurate pricing and risk management.

Ultimately, the findings of this empirical comparison study will provide valuable insights into the strengths and limitations of the Black-Scholes-Merton and Heston models for options on the S&P 500 index. The results will be of interest to financial practitioners, option traders, and researchers seeking to enhance their understanding of options pricing models and improve their decision-making process in the dynamic and complex world of financial derivatives.

CHAPTER 2

Literature Review

This chapter provides a comprehensive review of the literature on option pricing, the Black-Scholes-Merton model and the Heston model. It examines the assumptions, derivation, and limitations of both models, as well as their empirical performance in fitting actual market prices. By reviewing prior research, we aim to identify the key findings, methodologies, and gaps in knowledge that will inform and support our empirical comparison study.

The chapter begins by introducing the fundamental concepts of options pricing and the significance of accurate pricing models in financial markets. We explore the historical development of options pricing theory, highlighting the pioneering work of Fischer Black, Myron Scholes and Robert Merton in formulating the Black-Scholes-Merton model. We examine the assumptions underlying the model, including constant volatility, risk-free interest rate, and efficient markets and discuss the model's strengths and limitations in capturing the complexities of real-world options pricing.

Building upon the foundation of the BSM model, we then delve into the literature surrounding the Heston model. We trace the evolution of this model, introduced by Steven Heston as a means to address the limitations of the BSM model, specifically focusing on its incorporation of stochastic volatility. We discuss the motivations behind the development of the Heston model and its ability to capture the observed volatility smile or skew present in options markets. Moreover, we explore the various extensions and modifications proposed in subsequent studies to enhance the Heston model's performance and applicability.

Next, we review empirical studies that have compared the BSM and Heston models in various financial contexts. We analyze the methodologies employed, the data sets utilized and the key findings of these studies. We investigate the accuracy of the models in pricing options, their ability to capture market dynamics, and their performance in risk management and hedging strategies. Through this analysis, we aim to identify the existing empirical evidence and the factors that contribute to the models' performance disparities.

Furthermore, we examine the limitations and criticisms of both the Black-Scholes-Merton and Heston models that have emerged from the literature. We explore alternative models and approaches proposed by researchers to address the weaknesses identified in these traditional models. We discuss advanced techniques such as stochastic volatility models, local volatility models, and jump-diffusion models, which have been developed to capture additional complexities and anomalies observed in options pricing.

Finally, we outline the gaps and limitations in the existing literature that motivate our empirical comparison study. We emphasize the need for empirical research specifically focused on options on the S&P 500 index, considering its significance as a benchmark in the financial markets. By conducting a comprehensive literature review, we aim to synthesize the current knowledge landscape, identify research gaps, and establish a robust foundation for our empirical investigation.

In conclusion, this chapter serves as a comprehensive review of the literature related to options pricing models, with a specific focus on the Black-Scholes-Merton model and the Heston model for options on the S&P 500 index. By critically examining prior studies, identifying gaps in knowledge, and analyzing the strengths and weaknesses of existing models, we lay the groundwork for our empirical comparison study. The insights gained from this literature review will contribute to the development of our research framework, methodology, and the interpretation of our findings.

Several studies have compared the performance of the Black-Scholes-Merton model and the Heston model in option pricing (Park, Kim and Lee, 2014). Most of these studies have found that the Heston model outperforms the Black-Scholes model in terms of accuracy, particularly for options with longer maturities. However, the Heston model is also more computationally intensive than the Black-Scholes-Merton model, which can limit its practical use in certain applications.

2.1. Black-Scholes-Merton Model: Assumptions, Mathematical Formulation, Limitations and Critiques

The Black-Scholes-Merton model, developed by Fischer Black, Myron Scholes and Robert Merton in 1973, has been a cornerstone of options pricing theory and has revolutionized the field of quantitative finance.

The Black-Scholes-Merton model was the first widely used mathematical method to calculate the theoretical value of an option contract, using current stock prices, expected dividends (Black and Scholes do not consider dividends, Merton considers a dividend yield), the option's strike price, expected interest rates, time to expiration, and expected volatility. The initial equation was introduced in Black and Scholes 1973 paper, "The Pricing of Options and Corporate Liabilities" published in the Journal of Political Economy. Robert C. Merton helped edit that paper. Later that year, he published his own article, "Theory of Rational Option Pricing" in The Bell Journal of Economics and Management Science, expanding the mathematical understanding and applications of the model, and coining the term "Black–Scholes theory of options pricing".

In 1997, Scholes and Merton were awarded the Nobel Memorial Prize in Economic Sciences for their work in finding "a new method to determine the value of derivatives." Black had passed away two years earlier, and so could not be a recipient, as Nobel Prizes are not given posthumously; however, the Nobel committee acknowledged his role in the Black-Scholes model.

This section provides a comprehensive review of the BSM model, examining its fundamental assumptions, mathematical formulation, the challenges and criticisms it has faced over the years.

2.1.1. Assumptions of the Black-Scholes Model

The Black-Scholes-Merton model is built upon several key assumptions that simplify the complex dynamics of options pricing. Let's delve into each assumption in more detail:

(1) **Constant Volatility and Risk-Free Rate**: The model assumes the risk-free rate and volatility of the underlying asset are known and constant. While constant volatility is a convenient assumption for mathematical tractability, it does not accurately capture the time-varying nature of volatility observed in real financial markets.

(2) **No dividends**: The model assumes that no dividends are paid out during the life of the option. While the original BSM model did not consider the effects of dividends paid during the life of the option, the model is frequently adapted to account for dividends by determining the ex-dividend date value of the underlying stock (Section 2.1.4). Merton (1973) considers the existence of a dividend yield.

(3) **Efficient Markets**: The BSM model assumes that financial markets are efficient, meaning that there are no arbitrage opportunities and all relevant information is fully and immediately reflected in the stock price. This assumption implies that the market is frictionless and that there are no restrictions on trading or borrowing. While market efficiency is a fundamental concept in finance, in practice, markets may not always be perfectly efficient, leading to deviations between model prices and observed market prices.

(4) **Log-Normal Distribution of Stock Returns**: The model assumes that the distribution of stock returns follows a log-normal distribution. This assumption allows for the calculation of the probability of the underlying asset reaching a certain price at the option's expiration. However, in reality, stock returns often exhibit skewness and kurtosis, deviating from the symmetric and bell-shaped assumptions of the log-normal distribution.

(5) **Continuous Trading and Borrowing**: The BSM model assumes that it is possible to buy or sell the underlying asset continuously and that borrowing and lending can be done at the risk-free rate of interest. This assumption implies unlimited liquidity and continuous access to the market, which may not be realistic

in all situations. Transaction costs, liquidity constraints, and market frictions can impact the practical application of the model.

(6) **No Transaction Costs or Taxes**: The BSM model assumes that there are no transaction costs or taxes associated with buying or selling the underlying asset or the option itself. In reality, transaction costs, such as brokerage fees and bid-ask spreads, and taxes on capital gains can significantly impact options trading strategies and the profitability of options positions.

2.1.2. Mathematical Formulation of the Black-Scholes Model

The BS model provides a closed-form solution for pricing European options on non-dividend-paying stocks. Black-Scholes employed the Capital Asset Pricing Model to figure out how options' required rate of return relates to the expected rate of return of the underlying stock. Merton set up a portfolio that's risk-free, consisting of options and the underlying asset, and argued that this portfolio's return should be guaranteed for a brief time. Consequently, the return on this risk-free portfolio represents the risk-free interest rate under the assumption of no-arbitrage. Black–Scholes Merton model assumed that the stock price follows a Wiener process with the constant expected mean rate and constant variance given by:

$$dS_t = \mu S_t dt + \sigma S_t dz_t. \tag{2.1}$$

where z_t is a Wiener process. Black, Scholes and Merton claim that the price of the stock is positively related with the value of the option and the option is almost surely to be exercised if the stock price is greater than the strike price. Thus, the current value of the option is approximated as the stock price minus the price of a pure discount bond having the same maturity date and face value equal to the option's strike price. Contrarily, if the stock price is less than the strike price the option will not be exercised, that is the option value will be zero. Further, they claim that keeping the stock price unchanged the value of an option decreases as its maturity date approaches. Under the BSM's assumptions mentioned in the introduction, the value of the option V is a function of the price of the stock, time and the variables considered to be known constants. Mathematically, it can be written as $V = V(S_t, T, r, \sigma, K)$ or simply $V = V(S_t, T)$ by ignoring the constant terms.

THEOREM 2.1. *The BSM partial differential equation (PDE) relates the rate of return in the money market account in an infinitesimal period with the change in the option price by eliminating the individual stock price expectation μ. The PDE is given by:*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial S} rS - rV = 0, \tag{2.2}$$

where V is the value of the option, S is the underlying stock price, σ is the volatility of the stock price and r is a risk free interest rate.

PROOF. The stock price dynamic is modeled by the SDE $dS = \mu Sdt + \sigma Sdz$, where S is the underlying stock price, μ is the expected value of S , σ is the volatility of S and z is a Wiener process, according to the continuous random walk assumption which is proven to be a Brownian motion. Using the Itô's lemma, dV can be written as:

$$\begin{aligned} dV &= \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \\ &= \frac{\partial V}{\partial S}(\mu Sdt + \sigma Sdz) + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt. \end{aligned}$$

Collecting the dt terms gives:

$$dV = \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt + \frac{\partial V}{\partial S}\sigma Sdz. \tag{2.3}$$

The coefficient of dt in (2.3) is called a drift and the coefficient of dz is a stochastic term. The BSM PDE derivation underlies on eliminating the stochastic term which is a source of uncertainty. For this purpose we use a strategy called delta hedging to counter balance the risk that would arise due to the stochastic behavior.

Suppose we take a short position on a call option and we want to hedge it by buying $\triangle$ amounts of stocks. Suppose also that we borrow m units of money-market M_t with a risk free interest rate r to finance the transaction (dM_t is defined by $dM_t = rM_t dt$). We drop the time index in the notations hereafter for simplicity. Then our portfolio will be:

$$\Pi = \triangle\ S + mM.$$

Taking the derivative and substituting the dS from $dS = \mu Sdt + \sigma Sdz$ and $dM = rMdt$, we get:

$$\begin{aligned} d\Pi &= \triangle\ dS + mdM \\ &= \triangle\ (\mu Sdt + \sigma Sdz) + mrMdt \\ &= (\triangle\ \mu S + mrM)dt + \triangle\ \sigma Sdz. \end{aligned} \tag{2.4}$$

Now we want to get the value of $\triangle$ that makes stochastic terms (i.e. coefficients of d_z) in the total portfolio (i.e. in $dV + d\Pi$) zero:

$$\sigma S\frac{\partial V}{\partial S} + \triangle\ \sigma S = 0$$

$$\triangle = -\frac{\partial V}{\partial S} \tag{2.5}$$

Adding together dV from (2.3) and $d\Pi$ from (2.4) and substituting the $\triangle$ from 2.5 gives:

$$
\begin{aligned}
dV + d\Pi &= \left[\left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)\right) dt + \frac{\partial V}{\partial S}\sigma S dz\right] \\
&+ \left[\left(-\frac{\partial V}{\partial S}\mu S + mrM\right) dt + \triangle\, \sigma S dz\right] \\
&= \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - \frac{\partial V}{\partial S}\mu S + mrM\right) dt + \left(\frac{\partial V}{\partial S}\sigma - \frac{\partial V}{\partial S}\sigma\right) S dz \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + mrM\right) dt.
\end{aligned}
\tag{2.6}
$$

In this last equation of the total portfolio (2.6), we have a deterministic coefficient only. Thus, the portfolio is riskless and must instantaneously earn the same rate of return as other short-term risk-free securities. Therefore, the left hand side (LHS) of the total portfolio $dV + d\Pi$, which can also be written as $d(V + \Pi)$ will be:

$$
\begin{aligned}
d(V + \Pi) &= (V + \Pi) r dt \\
&= (V - \frac{\partial V}{\partial S} S + mM) r dt.
\end{aligned}
\tag{2.7}
$$

By equating (2.6) and (2.7), we get:

$$
\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + mrM\right) dt = (V - \frac{\partial V}{\partial S} S + mM) r dt.
$$

Canceling the dt from both sides and rearranging the above expression gives us the Black-Scholes-Merton PDE:

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial S} r S - rV = 0.
$$

□

In this PDE the expected value of the stock price μ is eliminated, implying that the investors expectation or attitude towards risk do not have an impact in the BSM model.

THEOREM 2.2. *The pricing formula, known as the Black-Scholes equation, calculates the theoretical option price based on the following variables:*

$$
\textit{Call option price}: C_t = S_0 N(d_1) - K e^{-r\tau} N(d_2) \tag{2.8}
$$

and

$$
\textit{Put option price}: P_t = K e^{-r\tau} N(-d_2) - S_0 N(-d_1), \tag{2.9}
$$

with

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \quad and \quad d_2 = \frac{ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau},$$

where:

- *Stock Price (S_0): The current price of the underlying asset. The model assumes that the stock price follows a geometric Brownian motion.*
- *Strike Price (K): The agreed-upon price at which the option can be exercised. The strike price represents the level at which the option holder can buy (in the case of a call option) or sell (in the case of a put option) the underlying asset.*
- *Time to Expiration (τ): The remaining time until the option's expiration. The longer the time to expiration, the greater the potential for the option to be profitable.*
- *Risk-Free Interest Rate (r): The continuously compounded risk-free interest rate for the option's term. This rate is typically derived from government bond yields with a similar term to expiration as the option.*
- *Volatility (σ): The standard deviation of the underlying asset's returns, assumed to be constant. Volatility represents the level of uncertainty or fluctuation in the stock price and is a critical input in options pricing models.*
- *$N()$: Cumulative standard normal distribution function.*

The Black-Scholes equation combines these variables to determine the fair value of the option. The equation considers the relationship between the strike price, the stock price, the time to expiration, the risk-free rate, and the volatility to produce an option price estimation.

Despite its simplifications and assumptions, the Black-Scholes model has been influential in the field of options pricing and provided a framework for subsequent models that aim to address its limitations. In the next sections, we will explore the empirical comparison between the Black-Scholes-Merton model and the Heston model, which takes into account stochastic volatility, for pricing options on the S&P 500 index.

2.1.3. Limitations and Critiques of the Black-Scholes Model

Despite its significant contributions to options pricing theory, the Black-Scholes (BS) model has faced several limitations and criticisms, which have prompted researchers to develop more advanced models. The following are some of the key critiques:

(1) **Assumption of Constant Volatility**: The model's assumption of constant volatility may not hold in real-world financial markets, where volatility exhibits time-varying patterns. This limitation hinders the model's accuracy in pricing options, especially during periods of high market turbulence or sudden shifts in

volatility. Researchers and practitioners have recognized the need for models that can capture volatility dynamics more effectively.

(2) **Failure to Capture Skewness and Kurtosis**: The BS model assumes a log-normal distribution of stock returns, which implies that the model cannot adequately capture the observed skewness (asymmetric distribution) and kurtosis (fat-tailed distribution) present in options markets. This limitation leads to the mispricing of options with non-normal return distributions. In reality, options exhibit a volatility skew, where out-of-the-money puts tend to have higher implied volatility than out-of-the-money calls, indicating a preference for downside protection.

(3) **Lack of Consideration for Dividend Payments**: The BS model assumes that the underlying asset does not pay any dividends during the option's life. This assumption restricts the model's applicability to options on non-dividend-paying stocks. For options on dividend-paying stocks, the exclusion of dividends can lead to pricing errors, as dividends impact the stock price and should be factored into the option's value.

(4) **Inadequate Risk Management**: The BS model assumes that the market is frictionless, with no transaction costs or taxes. However, in reality, these costs and taxes significantly affect options trading strategies and can impact the model's performance in practical applications. Transaction costs, bid-ask spreads, and liquidity constraints can lead to deviations between the model's theoretical prices and actual market prices.

(5) **Sensitivity to Input Parameters**: The BS model is sensitive to the accuracy of its input parameters, especially the estimation of volatility and the risk-free interest rate. Small changes in these parameters can result in significant variations in the model's output, leading to potential pricing errors. Accurately estimating these parameters can be challenging, particularly for longer-dated options.

(6) **Limited Scope to Handle Complex Options**: The BS model is primarily designed for European options. It is not easily adaptable to price more complex options, such as American options, options on futures, or options on assets with multiple sources of cash flows. Expanding the model to accommodate these complexities requires additional modifications and assumptions.

(7) **Crisis and Market Anomalies**: The BS model struggled to accurately capture and predict extreme events, such as market crises and anomalies. The model

assumes that stock returns follow a continuous and predictable process, which is not always the case in turbulent market conditions. The model's failure to account for these events has highlighted the need for more robust models capable of incorporating jumps, stochastic volatility, and other market anomalies.

Over the years, researchers have developed various extensions and modifications to address these limitations and criticisms of the Black-Scholes model. These advancements have resulted in the emergence of more sophisticated models, such as the Heston model, which incorporates stochastic volatility. In the subsequent sections, we will delve into the literature surrounding the Heston model and explore its contributions, challenges, and empirical comparisons with the Black-Scholes-Merton model for pricing options on the S&P 500 index.

2.1.4. Modified Black-Scholes-Merton Model

The original BS model assumes that the underlying asset does not pay dividends during the life of the option. However, in practice, dividends are a significant factor in options pricing, particularly for options on dividend-paying stocks. To incorporate dividends into the BS framework, modifications are made to the model's formula.

The modified Black-Scholes-Merton model that accounts for dividends is commonly referred to as the Black-Scholes-Merton model with dividends. The inclusion of dividends requires adjusting the underlying stock price in the model's calculations to reflect the expected cash flows from dividend payments.

Theorem 2.3. *Black–Scholes–Merton formula with dividends is a closed form formula for the price of the European option given by:*

$$\textit{Call option price}\text{: } C_t = S_0 e^{-q\tau} N(d_1) - X e^{-r\tau} N(d_2) \tag{2.10}$$

and

$$\textit{Put option price}\text{: } P_t = K e^{-r\tau} N(-d_2) - S_0 e^{-q\tau} N(-d_1), \tag{2.11}$$

with

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} \qquad \textit{and} \qquad d_2 = d_1 - \sigma\sqrt{\tau},$$

where q is the dividend yield (annualized dividend rate expressed as a continuous yield).

In the modified formula, the term $e^{-q\tau}$ adjusts the stock price by discounting the expected dividends during the option's life. By incorporating the dividend yield (q) into the model, the modified Black-Scholes-Merton model considers the impact of dividends on the underlying stock price and, consequently, on the option's value.

It's important to note that accurately estimating the dividend yield is crucial for the reliable application of the modified Black-Scholes model. Dividend expectations and dividend payment dates should be carefully considered when using this modified formula.

By accounting for dividends, the modified Black-Scholes-Merton model enhances the accuracy of options pricing for dividend-paying stocks, providing a more realistic valuation framework that aligns with market dynamics.

2.2. Heston Model: Assumptions, Mathematical Formulation, Limitations and Critiques

In the twenty years since its introduction in 1993, the Heston model has become one of the most important models, if not the single most important model, in a then-revolutionary approach to pricing options known as stochastic volatility modeling. To understand why this model has become so important, we must revisit an event that shook financial markets around the world: the stock market crash of October 1987 and its subsequent impact on mathematical models to price options.

The exacerbation of smiles and skews in the implied volatility surface that resulted from the crash brought into question the ability of the Black-Scholes model to provide adequate prices in a new regime of volatility skews, and served to highlight the restrictive assumptions underlying the model. The most tenuous of these assumptions is that of continuously compounded stock returns being normally distributed with constant volatility. An abundance of empirical studies since the 1987 crash have shown that this assumption does not hold in equities markets. It is now a stylized fact in these markets that returns distributions are not normal. Returns exhibit skewness, and kurtosis - fat tails - that normality cannot account for. Volatility is not constant in time, but tends to be inversely related to price, with high stock prices usually showing lower volatility than low stock prices. A number of researchers have sought to eliminate this assumption in their models, by allowing volatility to be time-varying.

One popular approach for allowing time-varying volatility is to specify that volatility be driven by its own stochastic process. The models that use this approach, including the Heston (1993) model, are known as stochastic volatility models. The models of Hull and White (1987), Scott (1987), Wiggins (1987), Chensey and Scott (1989), and Stein and Stein (1991) are among the most significant stochastic volatility models that pre-date Steve Heston's model. The Heston model was not the first stochastic volatility model to be introduced to the problem of pricing options, but it has emerged as the most important and now serves as a benchmark against which many other stochastic volatility models are compared.

Allowing for non-normality can be done by introducing skewness and kurtosis in the option price directly, as done, for example, by Jarrow and Rudd (1982), Corrado and Su (1997), and Backus, Foresi, and Wu (2004). In these models, skewness and kurtosis are specified in Edgeworth expansions or Gram-Charlier expansions. In stochastic volatility models, skewness can be induced by allowing correlation between the processes driving the stock price and the process driving its volatility. Alternatively, skewness can arise by introducing jumps into the stochastic process driving the underlying asset price.

The parameters of the Heston model are able to induce skewness and kurtosis, and produce a smile or skew in implied volatilities extracted from option prices generated by the model. The model easily allows for the inverse relationship between price level and volatility in a manner that is intuitive and easy to understand. Moreover, the call price in the Heston model is available in closed form, up to an integral that must be evaluated numerically. For these reasons, the Heston model has become the most popular stochastic volatility model for pricing equity options.

Another reason the Heston model is so important is that it is the first to exploit characteristic functions in option pricing, by recognizing that the terminal price density need not be known, only its characteristic function. This crucial line of reasoning was the genesis for a new approach for pricing options, known as pricing by characteristic functions.

2.2.1. Assumptions of the Heston Model

The Heston model is built upon several key assumptions that simplify the complex dynamics of options pricing. Let's delve into each assumption in more detail:

(1) **Geometric Brownian Motion**: The model assumes that the underlying asset follows a geometric Brownian motion, which means the logarithm of the asset price exhibits random fluctuations over time.

(2) **Stochastic Volatility**: The Heston model introduces volatility as a stochastic process, where the volatility itself follows a separate stochastic process. This assumption acknowledges that volatility is not constant but rather changes over time.

(3) **Mean Reversion**: The volatility process in the Heston model incorporates mean reversion, which implies that the volatility tends to move towards a long-term average level. This assumption ensures that extreme volatility values eventually return to a more typical level.

(4) **Correlation between Asset Returns and Volatility**: The model assumes a correlation between the asset returns and changes in volatility. Specifically, when the asset price experiences large movements, the volatility is expected to increase, and vice versa. This correlation captures the empirical observation that volatility tends to cluster during turbulent market periods.

(5) **No Arbitrage**: The model assumes the absence of arbitrage opportunities in the market. This assumption implies that it is not possible to make risk-free profits by trading options or the underlying asset.

(6) **Constant Interest Rate**: The Heston model assumes a constant risk-free interest rate throughout the option's lifetime. This assumption allows for consistent discounting of future cash flows.

(7) **Independent Processes**: The volatility process and the asset price process are assumed to be independent of each other. This assumption simplifies the mathematical formulation of the model but may not capture all real-world correlations between asset prices and volatilities.

(8) **Continuous Trading**: The Heston model assumes continuous trading in the underlying asset. It does not account for market discontinuities or jumps in the asset price.

(9) **Efficient Market**: The model assumes that the market is efficient, meaning that all relevant information is immediately reflected in the asset price. This assumption implies that the model does not consider market inefficiencies or investor behavioral biases.

It is important to note that these assumptions are simplifications made to facilitate the mathematical tractability of the Heston model. While the model has gained popularity, it is essential to be aware of its limitations and how these assumptions may affect its practical applicability. Practitioners and researchers often adapt or extend the model to better suit specific market conditions and empirical observations.

2.2.2. Mathematical Formulation of the Heston Model

Heston (1993) devised the stochastic volatility model for option pricing. Under the physical measure, the stochastic volatility model defined an underlying asset process as follows:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_{1,t}, \tag{2.12}$$

$$dv_t = k[\theta - v_t]dt + \xi \sqrt{v_t} dZ_{2,t} \tag{2.13}$$

and

$$dZ_{1,t} dZ_{2,t} = \rho dt, \tag{2.14}$$

where $Z_{1,t}$ and $Z_{2,t}$ are Wiener processes, respectively, μ is the drift of the process for the stock, $k > 0$ is the mean reversion speed for the variance, $\xi > 0$ is the volatility of the variance, ρ is the correlation between the log-returns and volatility, $\theta > 0$ is the mean reversion level for the variance and v_0 is the initial (time zero) level of variance. The volatility of an underlying asset, in the stochastic volatility model, behaves according to a mean-reverting diffusion process, as described in the equation above. Unlike the Black-Scholes model, which assumes constant volatility, this model treats the asset's volatility as a random process. This approach allows the model to account for the fact that stock returns often don't follow a normal distribution. Instead, it can explain characteristics like heavy tails and asymmetry in the distribution of stock returns.

THEOREM 2.4. *Denoting the log price variable $X_t = log(S_t)$, the characteristic function of X_t, following BañoRollin, FerreiroCastilla, and Utzet (2009), is defined by:*

$$\phi_T(u) = e^{A(u)+B(u)+C(u)}, \tag{2.15}$$

where

$$A(u) = iu(X_0 + rT), \tag{2.16}$$

$$B(u) = -\frac{(u^2+iu)(1-e^{\phi(u)T})v_0}{2\phi(u) - (\phi(u)-\tau(u))(1-e^{-\phi(u)T})}, \tag{2.17}$$

$$C(u) = -\frac{k\theta}{\xi^2}\left[2\log\left(\frac{2\phi(u) - (\phi(u)-\tau(u))(1-e^{-\phi(u)T})}{2\phi(u)}\right) + (\phi(u)-\tau(u))T\right], \tag{2.18}$$

with $\tau(u) = k - iu\rho\xi$ and $\phi(u) = \sqrt{\tau(u)^2 + \xi^2(u^2+iu)}$.

Given the characteristic function of such parametric model, we can implement the computations of option pricing by applying the Fourier transform methods.

THEOREM 2.5. *Following Carr and Madan (1999), when we know the characteristic function ϕ_T of the risk-neutral stock price process and the α-th moment of the stock price, the European call option price with strike price K and maturity T is given by:*

$$C(k,T) = \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{+\infty} e^{-iuk}\psi_T(u)du, \tag{2.19}$$

where k=ln(K) and

$$\psi_T(u) = \frac{e^{-rT}\phi_T(u-(\alpha+1)i)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u} \tag{2.20}$$

and the integration can be efficiently performed using the fast Fourier transform method.

2.2.3. Limitations and Critiques of the Heston Model

The Heston model, like any financial model, has certain limitations that should be considered when using it for option pricing or risk management purposes. Here are some of the key limitations of the Heston model:

(1) **Complex Calibration**: The model involves multiple parameters that need to be calibrated to market data, which can be a computationally intensive and challenging task. Moreover, due to the non-linear nature of the model, calibration

can be sensitive and prone to errors, especially when using limited or noisy data.

(2) **Mean Reversion Speed**: The Heston model assumes a constant mean reversion speed for the volatility process. However, in reality, the speed of mean reversion may vary over time and can be influenced by market conditions or events. Failing to capture this variability can impact the accuracy of the model's volatility dynamics.

(3) **Lack of Jumps and Fat Tails**: The Heston model assumes continuous asset price movements and a log-normal distribution for the underlying asset. It does not account for jumps or extreme events in prices, which are observed in real financial markets. Additionally, the model assumes normally distributed returns, whereas empirical evidence suggests that asset returns often exhibit fat tails.

(4) **No Stochastic Interest Rates**: The model assumes a constant risk-free interest rate, which may not accurately represent real-world interest rate dynamics, especially in environments with stochastic interest rates.

(5) **Arbitrage-Free but Unrealistic Dynamics**: Although the Heston model ensures no arbitrage opportunities by construction, its dynamics may not fully capture all the complexities and intricacies of real market behavior. The model's assumptions may oversimplify the dynamics of asset prices and volatilities, limiting its ability to accurately capture market phenomena.

(6) **Computational Intensity**: The Heston model can be computationally demanding, particularly when pricing complex options or performing risk management calculations. Simulating paths for both the asset price and the volatility process can require significant computational resources and time, making it challenging for real-time applications.

(7) **No Closed-Form Solutions**: Unlike the Black-Scholes-Merton model, which has closed-form solutions for option prices, the Heston model does not have analytical solutions. This means that option pricing and risk management typically require numerical methods, which can be computationally expensive.

(8) **Parameter Stability**: The parameters of the Heston model can be sensitive to changes in market conditions. Calibrating the model for one period may not be suitable for a different period, which can make it challenging to use for long-term forecasting.

It is crucial to be aware of these limitations and evaluate their impact in the specific context of your research or application. The choice of model should be made considering these limitations and the suitability of the Heston model for the particular objectives and market conditions under consideration.

CHAPTER 3

Methodology

3.1. Data

We used the S&P500 index call options prices to compare the performance of the Black-Scholes-Merton model and the Heston model. The real market data of the S&P500 index options, which is traded on various exchanges like Chicago Board Options Exchange and New York Stock Exchange Arca Options, was obtained from Refinitiv database for the period May 2022 to November 2022.

First, I've excluded the options with less than 7 days to expiration or more than 180 days to expiration because they are very sensitive to liquidity-related biases. Second, I have excluded very deep out-of-the-money and very deep in-the-money options, because they are not liquid options and their market prices may be quite different from their true values. An option is very deep in-the-money if its moneyness is greater than 12% and very deep out-of-the-money if its moneyness is less than -12%. The option moneyness is defined as the percentage difference between the current underlying price and the strike price: Moneyness(%) $= \frac{S}{K}$.

We define moneyness as OTM: $S/K < 0.95$, ATM: $0.95 \leq S/K < 1.05$ and ITM: $S/K \geq 1.05$.

Table 1 presents the sample properties of the S&P500 index used in this experiment. The average call prices, standard deviation of prices, total number of observations of each financial aspect, and time-to-maturity are described in this table.

Table 1
Sample properties of S&P500 index options: the notated numbers denote the average closing prices of call options, the standard deviation of the prices which are shown in parentheses and the total number of observations (in braces), for each moneyness maturity category. S is the spot S&P500 index level and K denotes the strike price. ITM, ATM, and OTM denote, respectively in-the-money, at-the-money, and out-of-the-money options.

		Days-to-Expiration		
	Moneyness (S/K)	$\tau < 1$ month	1 month $\leq \tau <$ 3 months	$\tau \geq 3$ months
OTM	<0.95	2.956	15.574	51.200
		{5.027}	{17.409}	{37.430}
		(440)	(392)	(420)
ATM	[0.95,1.05[	110.516	158.089	237.993
		{62.770}	{70.152}	{70.494}
		(931)	(445)	(419)
ITM	≥1.05	390.022	465.886	527.756
		{117.961}	{112.501}	{103.821}
		(863)	(484)	(467)

3.2. Procedures

We conducted a comprehensive analysis to compare pricing and forecasting methodologies. Our data collection involved the acquisition of N option data for multiple time periods. Subsequently, we computed the estimated call option prices denoted as $\hat{C}$, taking into consideration their moneyness and time-to-maturity, τ_i. To gauge the accuracy of our models, we formulated an error function for optimizing the parameter settings.

$$\epsilon_i\{C_{t,i}, \hat{C}(M_{i,t}, \tau_i, \phi_{t,i})\} = \hat{C}(M_{i,t}, \tau_i, \phi_{t,i}) - C_{t,i}, \tag{3.1}$$

where $C_{t,i}$ is the observed call prices. Then, we estimated the parameter sets $\phi_{t,i}$ by solving the following function:

$$\min SSE(t) = \min_{\phi_{t,i}} \sum_{i=1}^{N} |\epsilon_i\{C_{t,i}, \hat{C}(M_{i,t}, \tau_i, \phi_{t,i})\}|^2. \tag{3.2}$$

3.2.1. Modified Black-Scholes-Merton Model

We estimated the optimal value of volatility for pricing the Modified Black-Scholes-Merton model to describe the market prices. We used the Nelder-Mead simplex direct search method (Nocedal and Wright, 2006).

3.2.1.1. ***Nelder-Mead Simplex Direct Search Method***

The Nelder-Mead Simplex Direct Search Method is a versatile optimization algorithm designed to find the optimal solution for complex, nonlinear, and non-convex optimization problems. Unlike traditional gradient-based methods, it doesn't require the gradient information of the objective function, making it suitable for scenarios where obtaining such information might be challenging or impractical. The method operates by iteratively adjusting a simplex—a geometric figure resembling a simplex in the problem's parameter space—to explore and converge towards the optimal solution. The simplex reflects the objective function's behavior, allowing the algorithm to adaptively move in the search space. Relies on the construction of a simplex of N + 1 vertices, for an N-dimensional problem, then, it iteratively replaces its vertices for new ones with lower values of the cost function.

Key Features and Steps:

(1) **Initialization:** The algorithm begins by creating an initial simplex around a starting point in the parameter space.
(2) **Exploration and Reflection:** It evaluates the objective function at the vertices of the simplex. The worst vertex is reflected over the centroid of the other vertices to potentially reach a better solution.
(3) **Expansion:** If the reflected vertex improves the solution, the algorithm further expands along that direction to explore the search space more effectively.
(4) **Contraction:** If the reflected vertex's improvement is limited, the algorithm contracts the simplex towards the centroid, aiming to refine the solution.
(5) **Shrinkage:** In case none of the above steps lead to improvement, the algorithm contracts the entire simplex around its best vertex, which helps the algorithm converge towards the optimal solution.

Advantages:

(1) **Gradient-Free:** Nelder-Mead doesn't rely on gradient information, making it suitable for functions that lack analytical gradients or are noisy.
(2) **Versatility:** It works well for various problem types, including non-smooth, non-convex, and multimodal functions.
(3) **Ease of Implementation**: The algorithm's intuitive geometric approach makes it relatively easy to implement and understand.
(4) **Few Parameters:** It has minimal user-defined parameters, which simplifies the tuning process.
(5) **No Derivative Calculations:** The method doesn't require the often-complex calculations involved in gradient-based optimization.

Limitations:

(1) **Convergence Speed:** The Nelder-Mead method might converge slowly, particularly in high-dimensional spaces.
(2) **Sensitivity to Initialization:** Performance can be sensitive to the initial simplex configuration.

The Nelder-Mead Simplex Direct Search Method's ability to handle complex optimization problems without requiring gradient information has made it a valuable tool in various fields. However, its performance can vary based on problem characteristics and parameter settings, making it essential to understand its behavior and limitations for effective use.

3.2.2. Heston Model

To determine the parameter values for the Heston Model based on actual market option prices, we applied the Carr-Madan Fourier transform method. This method was originally introduced by Carr and Madan in 1999 and later used by Yang and Lee in 2012. To perform the necessary calculations within the Carr-Madan method, we employed Simpson's rule for numerical integration.

3.2.2.1. ***Carr-Madan Fourier Transform Method***

Carr and Madan used a Fourier transform to derive a call option price which is defined as a function of the natural logarithm of the strike price. They considered a modified version call price $C(k)$, where $k = ln(K)$. The expected discounted payoff $C = e^{-rT}E^{\mathbb{Q}}[\{S_T - K\}_+]$ is written in integral form and e^s is substituted in the place of S_T, where $s = ln(S_T)$, and e^k is substituted for K.

$$\begin{aligned} C &= e^{-rT}E^{\mathbb{Q}}[\{S_T - K\}_+] \\ &= \int_K^{\infty} e^{-rT}(S_T - K)\mathbb{Q}(ln(S_T))dlnS_T \\ C_T(k) &= \int_k^{\infty} e^{-rT}(e^s - e^k)\mathbb{Q}(s)ds. \end{aligned}$$

Definition 3.1. *A sufficient (but not necessary) condition for the existence of the Fourier transform of a function $f(x)$ and its inverse is:*

$$\int_{-\infty}^{\infty} |f(x)|dx < \infty, \tag{3.3}$$

which is called an integrability condition.

However, this $C_T(k)$ given above is not square-integrable as can be shown in the following lines. As $k \to -\infty$, $e^k \to 0$ and we get:

$$
\begin{aligned}
C_T(k) &= \int_{-\infty}^{\infty} e^{-rT} e^s \mathbb{Q}(s) ds \\
&= e^{-rT} E^{\mathbb{Q}}[e^s] \\
&= e^{-rT} E^{\mathbb{Q}}[S_T] \\
&= e^{-rT} S_0 e^{rT} = S_0.
\end{aligned}
$$

To tackle this problem, Carr and Madan modified $C(k)$ as follows:

$$
c_T(k) = e^{\alpha k} C_T(k),
$$

where $\alpha > 0$ is a damping parameter.
Now we check whether $c_t(k)$ is integrable:

$$
\begin{aligned}
\int_{-\infty}^{\infty} |c_T(k)| ds &= \int_{-\infty}^{\infty} |e^{\alpha k} C_T(k)| ds \\
&= e^{-rT} \int_{-\infty}^{\infty} |e^{\alpha k}(e^s - e^k)\mathbb{Q}(s)| ds \to 0 \;\; as \; k \to \infty.
\end{aligned}
$$

This last expression is integrable, thus the FT exists.

Definition 3.2. *Given a function $f(x)$, the FT is defined by:*

$$
\mathcal{F}[f(x)](v) = \psi(v) = \int_{-\infty}^{\infty} f(x) e^{-ivx} dx.
$$

Then, the Fourier transform $\psi(v)$ of $c_T(k)$ is derived as follows:

$$
\begin{aligned}
\mathcal{F}[c_T(k)](v) = \psi(v) &= \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk \\
&= \int_{-\infty}^{\infty} e^{ivk} \int_{k}^{\infty} e^{\alpha k} e^{-rT} (e^s - e^k) \mathbb{Q}(s) ds dk \\
&= \int_{-\infty}^{\infty} e^{-rT} \mathbb{Q}(s) \left(\int_{-\infty}^{s} e^{ivk} e^{\alpha k} (e^s - e^k) dk \right) ds \\
&= \int_{-\infty}^{\infty} e^{-rT} \mathbb{Q}(s) \left(e^s \int_{-\infty}^{s} e^{(iv+\alpha)k} dk - \int_{-\infty}^{s} e^{(iv+\alpha+1)k} dk \right) ds \\
&= \int_{-\infty}^{\infty} e^{-rT} \mathbb{Q}(s) \left(\left[\frac{e^s e^{(iv+\alpha)k}}{iv+\alpha} \right]_{-\infty}^{s} - \left[\frac{e^{(iv+\alpha+1)k}}{iv+\alpha+1} \right]_{-\infty}^{s} \right) ds \\
&= \int_{-\infty}^{\infty} e^{-rT} \mathbb{Q}(s) \left(\frac{e^{(1+iv+\alpha)s}}{iv+\alpha} - \frac{e^{(iv+\alpha+1)s}}{iv+\alpha+1} \right) ds \\
&= \int_{-\infty}^{\infty} e^{-rT} \mathbb{Q}(s) \left(\frac{e^{(1+iv+\alpha)s}[(iv+\alpha+1) - (iv+\alpha)]}{(iv+\alpha)(iv+\alpha+1)} \right) ds \\
&= e^{-rT} \int_{-\infty}^{\infty} \frac{e^{(1+iv+\alpha)s}}{(iv+\alpha)(iv+\alpha+1)} \mathbb{Q}(s) ds.
\end{aligned}
$$

Note that $(1 + iv + \alpha) = i(v - i(1 + \alpha))$ and, hence,

$$\begin{aligned}\psi(v) &= e^{-rT} \int_{-\infty}^{\infty} \frac{e^{i(v-i(1+\alpha))s}}{(iv+\alpha)(iv+\alpha+1)} \mathbb{Q}(s) ds \\ &= \frac{e^{-rT}}{(iv+\alpha)(iv+\alpha+1)} \int_{-\infty}^{\infty} e^{i(v-i(1+\alpha))s} \mathbb{Q}(s) ds.\end{aligned}$$

The expression in the integral will become $\phi(v - i(1 + \alpha))$ by the definition of a characteristic function. Then, we have:

$$\psi(v) = \frac{e^{-rT}\phi_T(v - i(1+\alpha))}{(iv+\alpha)(iv+\alpha+1)} = \frac{e^{-rT}\phi_T(v - i(1+\alpha))}{\alpha^2 + \alpha - v^2 + iv(2\alpha+1)}. \tag{3.4}$$

Inversion of Fourier Transform

We retrieve $C_T(k)$ by the method of inverse FT defined as follows.

DEFINITION 3.3. *The* ***inverse FT*** *is given by:*

$$\mathcal{F}^{-1}[\psi(v)](k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivk} \psi(v) dv.$$

In our case, since we transformed $c_T(k) = e^{\alpha k} C_T(k)$, we use $e^{-\alpha k}/2\pi$ instead of $1/2\pi$ to get the original function $C(k)$.

$$\begin{aligned}C_T(k) &= \frac{e^{-\alpha k}}{2\pi} \int_{\mathbb{R}} e^{-ivk} \psi(v) dv = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \psi(v) dv \\ &= \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \frac{e^{-rT}\phi_T(v - i(1+\alpha))}{\alpha^2 + \alpha - v^2 + iv(2\alpha+1)} dv.\end{aligned} \tag{3.5}$$

where ϕ_T is the characteristic function defined in section (2.2.2).

The modification $C_T(k)$ to $c_T(k)$ makes it integrable along the negative k axis and at the same time it affects the integrability along the positive k. For $c_T(k)$ to be integrable on both ends, the following sufficient condition has to be fulfilled:

$$\psi(0) < \infty.$$

We call this condition a sufficient condition for square-integrability,

$$\psi(0) = \frac{e^{-rT}\phi_T(-i(1+\alpha))}{\alpha^2 + \alpha}.$$

The expression $\frac{e^{-rT}}{\alpha^2+\alpha}$ is finite because both r and T are non–negative. Therefore, we need to check whether $\phi(-i(1+\alpha)) < \infty$ for the sufficient condition of square-integrability. To simplify, we use the definition of a characteristic function:

$$\begin{aligned}
\phi_{S_T}(u) &= E[e^{iuS_T}] \\
&= E[e^{iuln(S_T)}] \\
&= E[(e^{ln(S_T)})^{iu}] \\
&= E[S_T^{iu}].
\end{aligned}$$

By replacing u by $-i(1+\alpha)$, we proceed following the same manner,

$$\phi(-i(1+\alpha)) < \infty \Leftrightarrow E[S_T^{\{i(-i(1+\alpha))\}}] < \infty \Leftrightarrow E[S_T^{1+\alpha}] < \infty.$$

For equation (3.5) to be square integrable the $(1+\alpha)$th moment has to exist and be finite. The value of α has an impact on both the accuracy and efficiency of our derivation.

3.2.2.2. ***Trapezoidal Rule and Simpson's Rule***

The implementation of the FT requires discretization of the inverse FT using approximation methods for computing integrals numerically, for example trapezoidal and Simpson's rule.

Discretization using the Trapezoidal Rule

THEOREM 3.1. *Suppose that we have an integrand $f(x)$ on the interval $[a,b]$, and the interval is subdivided into N sub-intervals $[x_j, x_{j+1}]$ of width $h = (b-a)/N$ by using the equally spaced nodes $x_j = a + jh$ for $j = 0, 1, ..., N$. The* ***composite trapezoidal rule*** *for N subintervals can be expressed as:*

$$\int_a^b f(x)dx \approx T(f,h) = \frac{h}{2}\left(f(x_0) + 2f(x_1) + ... + 2f(x_{N-1}) + f(x_N)\right).$$

Truncating the upper limit of the integration for the inverse FT (3.5) to a finite number, say B, and applying the trapezoidal rule, gives the discrete Fourier transform (DFT).

$$\begin{aligned}
C_T(k) &= \frac{e^{-\alpha k}}{\pi}\int_0^{\infty} e^{-ivk}\psi(v)dv \\
&\approx \frac{e^{-\alpha k}}{\pi}\int_0^{B} e^{-ivk}\psi(v)dv \\
&\approx \frac{e^{-\alpha k}}{\pi}\sum_{j=1}^{N} e^{-iv_jk}\psi_T(v_j)\eta,
\end{aligned} \tag{3.6}$$

where $v_j = \eta(j-1)$, $\eta = B/N$.

Remark 3.1. *Note that η and B correspond to h and b respectively in Theorem 3.1. The count of summation starts in equation (3.6) from $j = 1$, as a result we would expect it to run to $N+1$. However, the last term is discarded because it's contribution will become insignificant as it becomes very small. Additionally, the lower limit for the integration of $C(k)$ is 0 thus $a = 0$. That is why we have $v_j = \eta(j-1)$, while $x_j = a + jh$ in the theorem.*

The weights are expressed in a slightly different way:

$$C_T(k) = \frac{e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) \frac{\eta}{2}(2 - \delta_{j-1}),$$

where $\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$.

Or, equivalently,

$$C_T(k) \approx \frac{\eta e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) w_j, \tag{3.7}$$

where $w_1 = w_N = \frac{1}{2}$ and $w_j = 1$ for $j = 2,...,N-1$.

While the DFT requires n multiplication, i.e. $O(n)$ for a single option price, the most efficient way would be to compute a number of option prices at once. The fast Fourier transform (FFT) algorithm, taking advantage of the complex roots of unity (see Definition 3.4 below), enables computation of a set of DFT with an n-point series that are power of two in $O(nlogn)$ multiplications that would have required $O(n^2)$ multiplications using the regular integration method.

Definition 3.4. *A complex nth root of unity is a complex number ω such that $\omega^2 = 1$. There are exactly n complex nth roots of unity: $e^{2\pi ik/n}$ for $k = 0, 1, ..., n-1$. An exponential of a complex number can also be written using Euler's formula: $e^{iu} = \cos(u) + i\sin(u)$.*

Definition 3.5. *The FFT maps a vector $\boldsymbol{x} = (x_j)_{j=1}^N$ on to some vector $g(\boldsymbol{x_u})$ and the algorithm is given by:*

$$g(\boldsymbol{x_u}) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} x_j \quad \textit{for u=1,...,N.}$$

The conversion of the DFT call price to FFT is done in a few steps:

Step 1. Create a range, say $[-b, b]$, of the logarithm of the strike price k partitioned into N with equal spacing λ:

$$k_u = -b + \lambda(u-1) \quad \text{for u=1,...,N.}$$

Step 2. Substitute the v_j and k_u in the DFT equation (3.6):

$$
\begin{aligned}
C_T(k) &\approx \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-iv_j k_u}\psi_T(v_j)\eta \\
&= \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-iv_j(-b+\lambda(u-1))}\psi_T(v_j)\eta \\
&= \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{ibv_j - iv_j\lambda(u-1)}\psi_T(v_j)\eta \\
&= \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{iv_j\lambda(u-1)}e^{ibv_j}\psi_T(v_j)\eta \\
&= \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-i\eta(j-1)\lambda(u-1)}e^{ibv_j}\psi_T(v_j)\eta
\end{aligned}
$$

$$
C_T(k_u) \approx \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)}e^{ibv_j}\psi_T(v_j)\eta \quad u = 1,...,N. \tag{3.8}
$$

Step 3. Let $\lambda\eta = 2\pi/N$ and $e^{ibv_j}\psi_T(v_j)\eta = x_j$ to get the FFT that corresponds to Definition 3.5 for the European call price $C_T(k_u)$ at N number of strike points.

Discretization using Simpson's Rule

The fact that $\eta = \frac{B}{N}$ and $\lambda\eta = \frac{2\pi}{N}$ induces a trade off between η (which determines the accuracy of the integration) and λ (which is the spacing of the logarithmic strike price and determines the chance of getting relevant strike prices) for a fixed computational budget N. To give more weights to the more relevant points the Simpson rule as in Theorem 3.2 below can be applied to the inverse FT in equation (3.5).

THEOREM 3.2. *Suppose that $[a, b]$ is subdivided into $2N$ sub-intervals $[x_j, x_{j+1}]$ of equal width $h = (b-a)/2N$ by using $x_j = a + jh$ for j =0,1,...,2N. The composite Simpson rule for $2N$ sub-intervals can be expressed as:*

$$
\begin{aligned}
&\int_a^b f(x)dx \approx S(f, h) \\
&\qquad = \frac{h}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ... + 2f(x_{2N-2}) + 4f(x_{2N-1}) + f(x_{2N})).
\end{aligned}
$$

Applying Theorem 3.2, expression (3.8) becomes:

$$
C(k_u) = \frac{e^{-\alpha k_u}}{\pi}\sum_{j=1}^{N} e^{\frac{-i2\pi}{N}(j-1)(u-1)}e^{ibv_j}\psi_T(v_j)\frac{\eta}{3}(3 + (-1)^j - \delta_{j-1}), \tag{3.9}
$$

where $v_j = \eta(j-1)$, $\eta = B/N$ and $\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$.

Or, equivalently,

$$C_T(k) \approx \frac{\eta e^{-\alpha k}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) w_j, \tag{3.10}$$

where $w_1 = w_N = \frac{1}{3}$, $w_j = \frac{4}{3}$ when j is even and $w_j = \frac{2}{3}$ when j is odd.

In this computation, Carr and Madan are particularly interested at the option price of at-the-money (ATM) condition.

The use of FFT for calculating OTM option prices is similiar to (3.10). The only differences are that we replace the multiplication by $e^{-\alpha k}$ with a division by $\sinh(\alpha k)$ and the function call to $\psi(v)$ is replaced by a function call to $\gamma(v)$ defined in (3.11).

$$\begin{aligned} \gamma(v) &= \int_{-\infty}^{\infty} e^{ivk} \sinh(\alpha k) z_T(k) dk \\ &= \int_{-\infty}^{\infty} e^{ivk} \frac{e^{\alpha k} - e^{-\alpha k}}{2} z_T(k) dk \\ &= \frac{\zeta_T(v - i\alpha) - \zeta_T(v + i\alpha)}{2}, \end{aligned} \tag{3.11}$$

where $\zeta_T(v) = e^{-rT}\left(\frac{1}{1+iv} - \frac{e^{rT}}{iv} - \frac{\phi_T(v-i)}{v^2 - iv}\right)$ and $\zeta_T(v)$ is the Fourier transform of $z_T(k)$.

Proof. Let $\zeta_T(v)$ denote the Fourier transform of $z_T(k)$:

$$\zeta_T(v) = \int_{-\infty}^{\infty} e^{ivk} z_T(k) dk.$$

The prices of out-of-the-money options are obtained by inverting this transform:

$$z_T(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \zeta_T(v) dv.$$

For ease of notation, we will derive $\zeta_T(v)$ assuming $S_0 = 1$. We may then define $z_T(k)$ by

$$z_T(k) = e^{-rT} \int_{-\infty}^{\infty} \left[(e^k - e^s)\mathbf{1}_{s<k,k<0} + (e^s - e^k)\mathbf{1}_{s>k,k>0}\right] q_T(s) ds.$$

The expression for $\zeta_T(v)$ follows on noting that:

$$\begin{aligned} \zeta_T(v) = &\int_{-\infty}^{0} e^{ivk} e^{-rT} \int_{-\infty}^{k} (e^k - e^s) q_T(s) ds dk \\ &+ \int_{0}^{\infty} e^{ivk} e^{-rT} \int_{k}^{\infty} (e^s - e^k) q_T(s) ds dk. \end{aligned}$$

Reversing the order of integration yields

$$\begin{aligned} \zeta_T(v) = &\int_{-\infty}^{0} q_T(s) e^{-rT} \int_{s}^{\infty} \left(e^{(1+iv)k} - e^s e^{ivk}\right) dk ds \\ &+ \int_{0}^{\infty} q_T(s) e^{-rT} \int_{0}^{s} \left(e^s e^{ivk} - e^{(1+iv)k}\right) dk ds. \end{aligned}$$

Performing the inner integrations, simplifying, and writing the outer integration in terms of characteristic function, we get

$$\zeta_T(v) = e^{-rT}\left(\frac{1}{1+iv} - \frac{e^{rT}}{iv} - \frac{\phi_T(v-i)}{v^2 - iv}\right).$$

Although there is no issue regarding the behavior of $z_T(k)$ as k tends to positive or negative infinity, the time value at $k = 0$ can get quite steep as $T \to 0$, and this can cause difficulties in the inversion. The function $z_T(k)$ approximates the shape of a Dirac delta function near $k = 0$ when maturity is small, and thus the transform is wide and oscillatory. It is useful in this case to consider the transform of $sinh(\alpha k)z_T(k)$ as this function vanishes at $k = 0$. Define

$$\begin{aligned}\gamma_T(v) &= \int_{-\infty}^{\infty} e^{ivk}\sinh(\alpha k) z_T(k)dk \\ &= \int_{-\infty}^{\infty} e^{ivk}\frac{e^{\alpha k} - e^{-\alpha k}}{2} z_T(k)dk \\ &= \frac{\zeta_T(v - i\alpha) - \zeta_T(v + i\alpha)}{2}.\end{aligned}$$

Thus, the time value is given by

$$z_T(k) = \frac{1}{sinh(\alpha k)}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\gamma_T(v)dv.$$

□

3.3. Test of Forecasting Accuracy

We conducted tests to see if there were meaningful differences in the forecasting results of the methods we compared. These tests checked whether the mean squared errors of two sets of forecasts were significantly different. To do this, we used well-established statistical tests, namely the Diebold Mariano (DM) test and the Wilcoxon (WS) signed rank test (Diebold and Mariano, 1995).

3.3.1. Diebold Mariano Test

Park, Kim, and Lee (2014) defined the series of forecasting errors as follows

$$e_{t,i} = \hat{y}_{t,i} - y_t \text{ for i=1,2,} \tag{3.12}$$

where y_t is the actual option prices of time-series and $\hat{y}_{t,i}$ is the series of forecasting. To compare the forecasting performance of two methods, we calculate the difference between them using a loss function $h(.)$, which gives us $d_t = h(e_{t,1}) - h(e_{t,2})$. The test to verify if the forecasting accuracies are equal is based on the null hypothesis,

$$H_0 : E(d_t) = 0,$$

and the DM test statistics is given by

$$\mathrm{DM} = \frac{d}{\sqrt{2\pi \hat{f}_d(0)/T}}, \tag{3.13}$$

where $d = \sum_{t=1}^{T}(h(e_{t,1}) - h(e_{t,2}))/T$ and $f_d(.)$ is the spectral density of d_t.

3.3.2. Wilcoxon Signed Rank Test

If we have a finite set of actual observations, we can test the difference in sampled loss by considering the null hypothesis,

$$H_0 : \mathrm{Median}(d_t) = 0,$$

based on the ranks of the forecasting errors. When the differences in the sampled data are distributed symmetrically, this null hypothesis aligns with the null hypothesis used in the DM test. The WS test statistics here is given by

$$\mathrm{WS} = \sum_{t=1}^{T} I_{+}(d_t)Rank(|d_t|), \tag{3.14}$$

where

$$I_{+}(d_t) = \begin{cases} 1 & \text{if } \ d_t > 0 \\ 0 & \text{otherwise} \end{cases}.$$

CHAPTER 4

Results

This chapter presents the results of the comparative analysis of the Black-Scholes-Merton model and the Heston model. It includes a discussion of the model parameters, goodness-of-fit measures, and statistical tests used to evaluate the performance of both models.

4.1. In-sample Pricing Performance

The in-sample performance was measured by training a variety of option pricing methods. All models used S&P500 Index call option data for each period by minimizing the sum-of-squared error between the market and estimated prices. Table 2 reports the root mean-squared errors (RMSEs) defined by:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(C_{t,i} - \hat{C}_{t,i})^2},$$

respectively for each method.

Table 2

In-sample pricing errors: each model is optimized by minimizing the sum of squared pricing errors. The reported root-mean-squared error is the differences between the model price and the market price in a given moneyness-maturity category. The columns are divided into 'days-to-expiration'. Moneyness denotes S/K. BSM and Heston stands for Black–Scholes-Merton model and Heston model, respectively. The best performance results for each category are shown in bold.

		Pricing Errors (**RMSE**)		
Moneyness (S/K)	Model	$\tau < 1$ month	1 month $\leq \tau <$ 3 months	$\tau \geq 3$ months
<0.95	BSM	**0.657**	**3.278**	**8.158**
	Heston	1.605	5.626	14.044
[0.95,1.05[	BSM	5.818	10.584	12.957
	Heston	**4.010**	**4.007**	**9.066**
≥1.05	BSM	2.390	7.586	10.583
	Heston	**1.763**	**2.611**	**2.011**

For ITM options, the Heston model consistently demonstrates superior accuracy with significantly lower RMSE values, particularly for shorter maturities. This highlights the Heston model's ability to capture the dynamics of ITM options with precision, making it a preferred choice for risk assessment, investment strategies, and valuation in these scenarios.

On the other hand, for ATM options, the Heston model shines as well, boasting consistently lower RMSE values across all maturities. This finding underscores the Heston model's strength in predicting ATM option prices and suggests that it can be a robust tool for market practitioners seeking accurate pricing for these options.

However, it's essential to note that for ATM options with maturities extending beyond three months, the Heston model's RMSE values show a noticeable increase, indicating potential limitations in capturing long-term dynamics. In such cases, the choice between the BSM and Heston models may require careful consideration of the specific analysis context and time horizon.

For OTM options, the BSM model consistently outperforms the Heston model, as indicated by significantly lower RMSE values across various time horizons. This suggests that the BSM model provides more accurate pricing for OTM options, especially for longer maturities. The BSM model's ability to maintain better accuracy in OTM option pricing underscores its suitability for risk assessment, investment strategies, and valuation in these specific OTM scenarios, particularly when considering longer-term positions.

In conclusion, the data underscores that the Heston model is a strong choice for both ITM and ATM options, especially for shorter maturities. This underscores its applicability in various financial applications. Yet, for ATM options with extended maturities, it is important to be mindful of potential limitations and evaluate the trade-off between model accuracy and computational complexity. However, when addressing OTM options, the BSM model consistently outperforms the Heston model, especially for extended maturities. This outcome underscores the BSM model's superior accuracy in OTM option pricing, particularly in longer-term scenarios. Researchers, analysts, and financial professionals should weight these factors when selecting an option pricing model that aligns with their specific objectives and time horizons.

4.2. Results of the Statistical Tests

Table 3

Comparison statistics for pricing errors. Provides the 95% significance level of the difference between BSM model and Heston model. DM-test and WS-test denotes each p-value from the results of each statistical test. The significant results $\alpha = 0.5$ are shown in bold.

Moneyness	Test	$\tau < 1$ month	1 month $\leq \tau < 3$ months	$\tau \geq 3$ months
OTM	DM-test	**(0.0000)**	**(0.0000)**	**(0.0000)**
	WS-test	**(0.0000)**	(0.9200)	**(0.0000)**
ATM	DM-test	**(0.0000)**	**(0.0000)**	**(0.0000)**
	WS-test	**(0.0000)**	**(0.0000)**	**(0.0000)**
ITM	DM-test	**(0.0000)**	**(0.0000)**	**(0.0000)**
	WS-test	**(0.0000)**	**(0.0000)**	**(0.0000)**

The combined results of the Diebold-Mariano (DM) test and the Wilcoxon signed-rank test provide robust evidence of substantial differences in forecasting accuracy between the Heston model and the BSM model across a range of maturities and moneyness levels.

For ITM, ATM and OTM options, both the DM and Wilcoxon tests consistently yield highly significant results, indicating a marked divergence in forecasting performance. While the direction of these differences may vary, the agreement of both tests underscores the significant disparities between the models for ITM, ATM and OTM options. These disparities necessitate careful consideration in selecting the appropriate model, taking into account the specific context and the intended use of forecasts.

The convergence of findings from both tests underscores the significance of model selection and its implications for financial analysis and decision-making. Researchers and financial professionals must weight not only the statistical significance of the differences but also the practical implications, potential sources of bias, and model assumptions. Ultimately, the choice of model should align with the specific objectives and constraints of the analysis, recognizing the distinct strengths and limitations of each model in capturing market dynamics.

For OTM options with maturities between 1 month and 3 months, the Wilcoxon signed-rank test suggests that there is no statistically significant difference in performance between the Heston and BSM models. The data does not provide enough evidence to conclude that one model outperforms the other for this specific scenario.

In conclusion, the combination of the Diebold-Mariano and Wilcoxon tests strengthens the evidence of the forecasting accuracy differences between the Heston and BSM models, providing valuable insights for financial modeling, risk assessment, and investment decision-making.

CHAPTER 5

Conclusions

In conclusion, the empirical comparison of the BSM model and the Heston model in the context of S&P 500 index options has provided valuable insights into the pricing and risk assessment of these complex financial instruments. Through an extensive analysis of historical data and option pricing methodologies, this master thesis has shed light on the strengths and weaknesses of each model.

Our research has shown that the BSM model, while simplistic and easy to implement, falls short in accurately capturing the dynamics of S&P 500 index ITM and ATM options. Its assumptions of constant volatility and deterministic paths do not fully reflect the inherent stochastic nature of financial markets, leading to significant pricing errors, especially during periods of high volatility and extreme market events.

On the other hand, the Heston model, with its incorporation of stochastic volatility, has demonstrated superior performance in capturing the dynamic nature of S&P 500 index ITM and ATM options. It better accounts for changes in implied volatility over time, making it more suitable for pricing options in both calm and turbulent market conditions. However, it is essential to note that the Heston model's complexity and computational intensity may present challenges in real-time trading environments.

It's worth noting that, in our analysis, the BSM model has demonstrated superior performance for OTM options when compared to the Heston model. This suggests that for OTM options, the BSM model may offer a more suitable pricing approach, considering its relative simplicity and reasonable accuracy.

Furthermore, this research has highlighted the importance of model calibration and parameter estimation when employing the Heston model, as inaccurate parameter values can lead to suboptimal results. Additionally, the choice of numerical methods for solving the Heston model equations can significantly impact computational efficiency.

In summary, the choice between the BSM and Heston models for pricing S&P 500 index options depends on the specific requirements of market participants. While the BSM model may suffice for some simplistic applications, the Heston model offers a more accurate representation of option pricing dynamics, especially for sophisticated investors and risk managers seeking to navigate the complexities of modern financial markets.

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