BOUNDS FOR THE M|M|∞ QUEUEING SYSTEM BUSY PERIOD DISTRIBUTION FUNCTION

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1. INTRODUCTION

In a M|G|∞ queue system

- λ is the Poisson process arrival rate,
- α is the mean service time,
- $G(\cdot)$ is the service time d.f. and, so,

$$\alpha = \int_0^\infty [1 - G(t)]dt \qquad \textbf{(1.1)},$$

- $F(\cdot)$ is the service time equilibrium d.f. whose expression is

$$F(t) = \frac{1}{\alpha}\int_0^t [1 - G(x)]dx \qquad \textbf{(1.2)},$$

- $\rho = \lambda\alpha$ is the traffic intensity,
- B is the busy period length.

Note the importance of the busy period study, for this queueing system, because in it any customer when arrives finds a server available. So the problem is for how long the servers must be available that is how long is a busy period length.

The B d.f. has not a simple form and it can be written as (Stadje (1985))

$$P(B \le t) = 1 - \lambda^{-1}\sum_{n=1}^{\infty} c^{*n}(t) \qquad \textbf{(1.3)}$$

where c^{*n} is the *nth* convolution of *c* with itself being

$$c(t)=\lambda(1-G(t))e^{-\lambda\int_0^t[1-G(x)]dx} \quad \textbf{(1.4).}$$

Only for the service time d.f. collection given for (Ferreira (1998))

$$G(t)=1-\frac{(1-e^{-\rho})(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{(\lambda+\beta)t}-1\right)+\lambda},\ t\geq 0,\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \quad \textbf{(1.5)}$$

the expression **(1.3)** becomes simple:

$$P(B\leq t)=1-\frac{\lambda+\beta}{\lambda}(1-e^{-\rho})e^{-\rho(\lambda+\beta)t},\ t\geq 0,\ -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1} \quad \textbf{(1.6).}$$

This does not happen for the M|M|∞ queueing systems (service time exponential) and so we will give in this paper some simple bounds for $P(B\leq t)$ in that case.

Finally note that if the p.d.f. allows the study of the distribution structure only the d.f. allows the probabilities calculation.

2. BOUNDS FOR THE M|M|∞ QUEUEING SYSTEM $P(B\leq t)$

We can write $c(t)$ as

$$c(t)=\rho f(t)e^{-\rho F(t)} \quad \textbf{(2.1)}$$

where $f(t)=\frac{dF(t)}{dt}$. So,

$$c(t)\geq\rho f(t)e^{-\rho} \quad \textbf{(2.2)}$$

and

$$P(B \le t) \le 1 - \lambda^{-1} \sum_{n=1}^{\infty} f^{*n}(t) \rho^n e^{-n\rho} \qquad \textbf{(2.3)}$$

or

$$c(t) \le \rho f(t) \qquad \textbf{(2.4)}$$

and

$$P(B \le t) \ge 1 - \lambda^{-1} \sum_{n=1}^{\infty} f^{*n}(t) \rho^n \qquad \textbf{(2.5)}$$

f^{*n} is the p.d.f. of the sum of n i.i.d. r.v. whose d.f. is given by **(1.2)**. So the bounds given by **(2.3)** and **(2.5)** depend only on ρ, λ and $F(\cdot)$.

For the M|M|∞ queue, $G(t) = 1 - e^{-t/\alpha}$ and, so, $f(t) = \frac{1 - 1 + e^{-t/\alpha}}{\alpha} = \frac{1}{\alpha} e^{-t/\alpha}$. Then, $f^{*n}(t) = \left(\frac{e^{-t/\alpha}}{\alpha}\right)^{*n} = \frac{t^{n-1}}{\alpha^n (n-1)!} e^{-t/\alpha}$. And $\sum_{n=1}^{\infty} f^{*n}(t) \rho^n e^{-n\rho} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{\alpha^n (n-1)!} e^{-t/\alpha} \rho^n e^{-n\rho} =$ $= \frac{\rho}{\alpha} e^{-\rho} e^{-t/\alpha} \cdot \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{\rho}{\alpha} e^{-\rho} t\right)^{n-1} = \lambda e^{-\rho - \frac{t}{\alpha}} \cdot e^{\lambda e^{-\rho} t} = \lambda e^{-\rho + \left(\lambda e^{-\rho} - \frac{1}{\alpha}\right) t} = \lambda e^{-\rho + \frac{\rho e^{-\rho} - 1}{\alpha} t}$. So

$$P(B^M \le t) \le 1 - e^{-\rho - \frac{1 - \rho e^{-\rho}}{\alpha} t} \qquad \textbf{(2.6)}$$

after **(2.3)**. From **(2.5)**, as $\sum_{n=1}^{\infty} f^{*n}(t) \rho^n = \sum_{n=1}^{\infty} \frac{t^{n-1}}{\alpha^n (n-1)!} e^{-t/\alpha} \rho^n = \frac{\rho}{\alpha} e^{-t/\alpha} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{\rho}{\alpha} t\right)^{n-1} =$ $= \lambda e^{-t/\alpha} e^{-\lambda t}$ we conclude that

$$P(B^M \le t) \ge 1 - e^{-\frac{1 - \rho}{\alpha} t} \qquad \textbf{(2.7).}$$

The bound given by **(2.6)** is always less than **1.**. The one given by **(2.7)** is positive only for $\rho < 1$.

Otherwise $1 - e^{-\rho - \frac{1-\rho e^{-\rho}}{\alpha} t} \geq 1 - e^{-\frac{1-\rho}{\alpha} t} \Leftrightarrow -\rho - \frac{1-\rho e^{-\rho}}{\alpha} t \leq -\frac{1-\rho}{\alpha} t \Leftrightarrow \frac{1-\rho-1+\rho e^{-\rho}}{\alpha} t \leq \rho \Leftrightarrow$

$\Leftrightarrow \frac{\rho\left(e^{-\rho}-1\right)}{\alpha} t \leq \rho \Leftrightarrow t \geq \frac{\alpha}{e^{-\rho}-1} = -\frac{1}{1-e^{-\rho}} < 0,\ \rho > 0$ and the bound given by **(2.6)** is always greater than the one given by **(2.7)**.

In Ferreira e Ramalhoto (1994) we proved that

$$G(t)e^{-\rho} \leq P(B \leq t) \leq G(t) \qquad \textbf{(2.8)}$$

Consequentely

$$\left(1 - e^{-\frac{t}{\alpha}}\right) e^{-\rho} \leq P\left(B^M \leq t\right) \leq 1 - e^{-t/\alpha} \qquad \textbf{(2.9)}$$

The lower bound given in **(2.9)** is always positive, but for $\rho < 1$ the one given for **(2.7)** is better.

As $1 - e^{-\rho - \frac{1-\rho e^{-\rho}}{\alpha} t} \leq 1 - e^{-t/\alpha} \Leftrightarrow -\rho - \frac{1-\rho e^{-\rho}}{\alpha} t \geq -\frac{t}{\alpha} \Leftrightarrow \frac{1-1+\rho e^{-\rho}}{\alpha} t \geq \rho \Leftrightarrow t \geq \alpha e^{\rho}$ we conclude that for $t \geq \alpha e^{\rho}$ the bound given by **(2.6)** is better than the one given by **(2.9)**.

3. CONCLUSIONS

We presented bounds for $P(B \leq t)$ that can be used for any service time distribution. But they give simple expressions for exponential service times. It is even possible to compare them in order to make the better option in their use through very simple rules.

4. REFERENCES

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