MEAN SOJOURN TIME IN STATE *k*, *k*=0,1,…, FOR THE M|G|∞ QUEUEING SYSTEM

(Exact and approximated expressions)

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1. Introduction

In a M|G|∞ queue system λ is the Poisson process arrival rate, α is the mean service time, $G(\cdot)$ is the service time distribution function and there are infinite servers.

When we consider practical situations to apply this model we do not want necessarily the physical presence of infinite servers. But we only guarantee that when a customer arrives at the system it always finds immediately a server available. Other situations occur when there is no distinction between a customer and its server.

So, often, it is very important to manage a group of servers, in order to guarantee that the system works as an infinite server queueing system as it was designed. For this purpose it is important to know the mean sojourn time in state k, $k = 0,1,\ldots$. Here, by state k, we mean that there are k customers in the system, or, that is the same, k servers occupied.

Unhappily, only for M|M|∞ (exponential service time) queueing systems we know that mean. But as it was proposed in (Ramalhoto and Girmes, 1977) we will consider that M|G|∞ systems are well approximated by a Markov Renewal Process. And so we will consider the mean sojourn time in state k, $k = 0,1,\ldots$ for that process as a good approximation to the ones of the M|G|∞ queueing systems.

Then we are going to show some results to the mean sojourn time in state k, $k = 0,1,\ldots$ distribution function for the Markov Renewal Process considered.

2. Mean Sojourn Time in State *k, k*=0,1,2,… for the Markov Renewal Process

Calling m_k the mean sojourn time in state k for the Markov Renewal Process we have

$$m_k = \int_0^\infty e^{-\lambda t}\left[\frac{\int_t^\infty [1-G(x)]dx}{\alpha}\right]^k dt,\ k = 0,1,\ldots \qquad (1).$$

We have

$$m_0 = \frac{1}{\lambda} \tag{2}$$

as it happens with any queueing system with Poisson process arrival, and

$$m_k \le \frac{1}{\lambda} \tag{3}$$

because $\alpha^{-1}\int_t^{\infty}[1-G(x)]dx \le 1$. So, the mean sojourn time in any state does not exceed the one of the state "0".

By Schwartz's inequality $m_k^2 \le \int_0^{\infty} e^{-2\lambda t}dt \int_0^{\infty}\left[\frac{\int_t^{\infty}[1-G(x)]dx}{\alpha}\right]^{2k} dt =$

$= \frac{1}{2\lambda\alpha^{2k}}\int_0^{\infty}\left[\int_t^{\infty}[1-G(x)]dx\right]^{2k}dt = \frac{1}{2\lambda\alpha^{2k}}\frac{2k\alpha^2}{2}\left(\gamma_s^2+1\right)\frac{\alpha^{2k-1}b_{2k-1}}{2k(2k+1)} \le \alpha\frac{\gamma_s^2+1}{2\lambda(2k+1)}$, where γ_s is the service time variation coefficient, because according to (Sathe, 1985) $\int_0^{\infty}\left[\int_t^{\infty}[1-G(x)dx]\right]^n dt = \frac{n\alpha^2}{2}\left(\gamma_s^2+1\right)\frac{\alpha^{n-1}b_{n-1}}{n(n-1)}$ with $b_n \le 2,\ n = 0,1,\ldots$. So

$$m_k \le \alpha\sqrt{\frac{\gamma_s^2+1}{2\rho(2k+1)}},\ k = 1,2,\ldots \tag{4}$$

We have also $m_k \le \int_0^{\infty}\left[\frac{\int_t^{\infty}[1-G(x)dx]}{\alpha}\right]^k dt \le \frac{1}{\alpha^k}\frac{k\alpha^2}{2}\left(\gamma_s^2+1\right)\frac{2\alpha^{k-1}}{k(k+1)} = \alpha\frac{\gamma_s^2+1}{k+1}$, according again to the result of (Sathe, 1985) cited above. So

$$m_k \le \alpha\frac{\gamma_s^2+1}{k+1},\ k = 1,2,\ldots \tag{5}$$

Simple, although rather fastidious, computations allow the following rules to choose, the best upper bound to m_k:

a) $\rho\left(\gamma_s^2+1\right) > \frac{2}{3}$

a₁) $k < \frac{1}{4}\rho\left(\gamma_s^2+1\right) - \frac{1}{2}$

$$m_k \le \frac{1}{\lambda}$$

a₂) $\frac{1}{4}\rho\left(\gamma_s^2+1\right)-\frac{1}{2}\le k\le 2\rho\left(\gamma_s^2+1\right)-1$

$$m_k\le\alpha\sqrt{\frac{\gamma_s^2+1}{2\rho(2k+1)}}$$

a₃) $2\rho\left(\gamma_s^2+1\right)-1<k<4\rho\left(\gamma_s^2+1\right)-1$

$$m_k\le\min\left\{\alpha\sqrt{\frac{\gamma_s^2+1}{2\rho(2k+1)}};\alpha\frac{\gamma_s^2+1}{k+1}\right\}$$

a₄) $k\ge 4\rho\left(\gamma_s^2+1\right)-1$

$$m_k\le\alpha\frac{\gamma_s^2+1}{k+1}$$

b) $\frac{1}{2}<\rho\left(\gamma_s^2+1\right)\le\frac{2}{3}$

b₁) $k=1$

$$m_1\le\min\left\{\alpha\sqrt{\frac{\gamma_s^2+1}{6\rho}};\alpha\frac{\gamma_s^2+1}{2}\right\}$$

b₂) $k=2,3,\ldots$

$$m_k\le\alpha\frac{\gamma_s^2+1}{k+1}$$

c) $\rho\left(\gamma_s^2+1\right)\le\frac{1}{2}$

$$m_k\le\alpha\frac{\gamma_s^2+1}{k+1},\ k=1,2,\ldots$$

If the service time is exponential we have

$$m_k=\frac{\alpha}{k+\rho},\ k=1,2,\ldots \qquad (6),$$

coincident to the one known to the M|M|∞ system.

If the service time is NBUE (New better than used in expectation), see for instance (Ross, 1983),

$$m_k\le\frac{\alpha}{k+\rho},\ k=1,2,\ldots \qquad (7).$$

If the service time is NWUE (New worse than used in expectation), see again (Ross, 1983),

$$m_k \geq \frac{\alpha}{k+\rho}, \; k = 1,2,\ldots \quad (8).$$

If the service time is DFR (Decreasing failure rate), see (Ross, 1983),

$$m_k \geq e^{k\left(\frac{1-\gamma_s^2}{2}\right)} \frac{\alpha}{k+\rho}, \; k = 1,2,\ldots \quad (9).$$

If the service time is IMRL (Increasing mean residual life), see (Brown, 1981) and (Cox, 1962),

$$m_k \geq e^{k\left(1-\frac{2}{3}\frac{\alpha}{\mu_2^2}\mu_3\right)} \frac{\mu_2}{\mu_2\lambda + 2k\alpha}, \; k = 1,2,\ldots \quad (10)$$

where μ_r is the r^{th} moment centered at the origin of $G(\cdot)$.

3. Concluding remarks

For, $k = 0$, whatever is the service time distribution, and for the M|M|∞ queueing system, whatever is k, the Markov Renewal Process mean sojourn time in state k is coincident with the ones known for the M|G|∞ queueing system. This allows us to hope that in the other cases it gives good approximations. We show results appliable to any service time distribution and for service time distributions important in reliability theory, NBUE, NWUE, DFR and IMRL.

4. Summary

Although, in generally, approximated we present some practical formulas for the mean sojourn time in state k, $k = 0,1,...$ for the M|G|∞ queue system. Its knowledge is of key importance for managing a group of servers in order to guarantee that the queueing systems act as an infinite server one.

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