$M|G|\infty$ QUEUE BUSY CYCLE RENEWAL FUNCTION FOR SOME PARTICULAR SERVICE TIME DISTRIBUTIONS

MANUEL ALBERTO M. FERREIRA

Abstract In this paper we present formulas to compute the busy cycle renewal function for the $M|G|\infty$ queue, considering service time distributions that arise when we study parameters related to the busy period. The busy cycle renewal function in t gives the mean number of busy periods that begin in $[0,t]$.

Keywords: $M|G|\infty$, busy cycle, renewal function.

1. INTRODUCTION

In the $M|G|\infty$ queueing system, λ is the customers Poisson process arrival rate, each customer gets a service that is a positive real value with distribution function $G(\cdot)$ and mean α, being $\alpha = \int_0^\infty [1-G(t)]\,dt$; there are infinite servers and the service time of each customer is independent of the others. The traffic intensity is $\rho = \lambda\alpha$.

In a queueing system we usually call busy period a period that begins when a customer arrives there, and it is empty; it ends when a customer leaves the system letting it empty, and there is always at least one customer present. So, in a queueing system, there is always a sequence of idle periods and busy periods.

Let's consider the $M|G|\infty$ system with time origin at the beginning of a busy period. The instant $0, t_1, t_2, \ldots$ at which a busy period begins, are the arrival epochs of a renewal process (Takács (1962)). We say that a cycle is complete when a renewal occurs, that is, a busy period begins. These cycles are busy cycles; and their length is a real value that we will call Z.

So

$$Z = B + I \tag{1.1}$$

where B is the busy period length and I the idle period one.

Takács (1962) proved that B and I are stochastically independent and, still, that the Z Laplace-Stieltjes transform, $\overline{Z}(s)$, is given by

$$\overline{Z}(s) = 1 - \frac{1}{(s+\lambda)P_{00}(s)} \tag{1.2},$$

where $P_{00}(s)$ is the $p_{00}(t) = e^{-\lambda\int_o^t[1-G(v)]dv}$ Laplace-Stieltjes transform, probability of the system emptiness at t having been initially empty.

That author showed also that

$$E[Z] = \frac{e^{\rho}}{\lambda} \tag{1.3}$$

and

$$E[Z^2] = 2\lambda^{-1}e^{2\rho}\int_0^{\infty}\left(e^{-\lambda\int_o^t[1-G(v)]dv} - e^{-\rho}\right)dt + 2\lambda^{-2}e^{\rho} \tag{1.4}.$$

Being I exponentially distributed with parameter λ its Laplace-Stieltjes transform is $\bar{I}(s) = \frac{\lambda}{\lambda + s}$ and the ratio $\frac{\bar{Z}(s)}{\bar{I}(s)}$ gives the expression $\bar{B}(s) = 1 + \frac{1}{\lambda}\left[s - \frac{1}{P_{00}(s)}\right]$ for the Laplace-Stieltjes transform of B (Stadje (1985)), whose inversion is a complex problem except for some service distributions (see Ferreira (1991), (1995), (1998)).

Our work in this paper will focus on the $M|G|\infty$ queue busy cycle, in particular in its renewal function study.

2. THE M|G|∞ QUEUE BUSY CYCLE RENEWAL FUNCTION

The renewal function, R of a renewal process is given for $R = 1 + F + F^{*2} + F^{*3} + \ldots$ where F^{*n} is the n-th interrenewal time convolution with itself distribution function (see, for instance, Çinlar (1975)). It gives the mean number of renewals in $[0, t]$. For instance, in the application of this model to unemployment situations, a busy period is a period of unemployment. And, in illness situations, a busy period is an epidemic period. See, about this kind of applications Ferreira (2003 and 2003a).

To compute the $M|G|\infty$ queue busy cycle renewal function we have, using the Laplace-Stieltjes transform, $\bar{R}(s) = \frac{1}{s} + \frac{1}{s}\bar{Z}(s) + \frac{1}{s}\bar{Z}^2(s) + \ldots + \frac{1}{s}\bar{Z}^n(s) = \frac{s^{-1}}{1-\bar{Z}(s)}$

$$= \frac{s^{-1}}{1 - 1 + \frac{1}{(s+\lambda)P_{00}(s)}} = \frac{(s+\lambda)P_{00}(s)}{s} = P_{00}(s) + \lambda\frac{1}{s}P_{00}(s).$$

So, $R(t) = p_{00}(t) + \lambda \int_0^t p_{00}(u)\,du$, and:

$$R(t) = e^{-\lambda \int_0^t [1-G(v)]dv} + \lambda \int_0^t e^{-\lambda \int_0^t [1-G(v)]dv} du \quad (2.1).$$

Note that:

$$\lim_{t\to\infty}\left[R(t) - \frac{\lambda}{e^{\rho}}t\right] = \lim_{t\to\infty}\left[e^{-\lambda\int_0^t [1-G(v)]dv} + \lambda e^{-\rho}\int_0^t e^{\rho-\lambda\int_0^u [1-G(v)]dv} du - \frac{\lambda}{e^{\rho}}t\right] =$$

$$= e^{-\rho} + \lambda e^{-\rho} \lim_{t\to\infty}\int_0^t \left(e^{\rho-\lambda\int_0^u [1-G(v)]dv} - 1\right) du = e^{-\rho} + \lambda e^{-\rho}\int_0^\infty \left(e^{\lambda\int_u^\infty [1-G(v)]dv} - 1\right) du$$

So, it is easy to see that:

$$\lim_{t\to\infty}\left[R(t) - \frac{t}{E[Z]}\right] = \frac{VAR[Z] + E^2[Z]}{2E^2[Z]} \quad (2.2)$$

as it has to be with a renewal function.

– As $e^{-\rho} \le p_{00}(t) \le 1$,

$$p_{00}(t) + \lambda e^{-\rho} t \le R(t) \le p_{00}(t) + \lambda t \quad (2.3)$$

and, still,

$$e^{-\rho}(1+\lambda t) \le R(t) \le 1 + \lambda t \quad (2.4),$$

we conclude that

$$\lim_{\alpha\to 0} R(t) = 1 + \lambda t \quad (2.5),$$

as it has to be because, when the service time is nul, when it arrives each customer begins a busy period.

And the arrival instants, in the $M|G|\infty$ system, occur according to a Poisson process at rate λ.

$$-\frac{d}{dt}R(t) = p_{00}(t)\{-\lambda[1-G(t)]\} + \lambda p_{00}(t) = \lambda G(t) p_{00}(t) \ge 0.$$

So $R(t)$ increases with t.

3. R(t) VALUES FOR SOME PARTICULAR SERVICE TIME DISTRIBUTIONS

A. The $M|G|\infty$ emptiness probability at time t, as time function, being the initial instant the one of the beginning of a busy period (at which a customer arrives at the system finding it empty) is determined by the sign of $\frac{g(t)}{1-G(t)}-\lambda G(t),\ t\geq 0$ (Ferreira (1996)) where $g(\cdot)$ and $G(\cdot)$ are, respectively the service time p.d.f. and d.f. .

Putting $\frac{g(t)}{1-G(t)}-\lambda G(t)=\beta(t)$ ($\beta(\cdot)$ is any time function) we get

$$\frac{dG(t)}{dt}=-\lambda G^2(t)-[\beta(t)-\lambda]\,G(t)+\beta(t) \tag{3.1}$$

that is a Ricatti equation about $G(\cdot)$.

Solving it, after noting that $G(t)=1,\ t\geq 0$ is a solution, we get

$$G(t)=1-\frac{1}{\lambda}\frac{\left(1-e^{-\rho}\right)e^{-\lambda t-\int_0^t\beta(u)du}}{\int_0^\infty e^{-\lambda w-\int_0^w\beta(u)du}dw-\left(1-e^{-\rho}\right)\int_0^t e^{-\lambda w-\int_0^w\beta(u)du}dw},$$

$$t\geq 0,\ -\lambda\leq\frac{\int_0^t\beta(u)\,du}{t}\leq\frac{\lambda}{e^{\rho}-1} \tag{3.2}.$$

Putting directly (3.2) in (2.1) we get the corresponding value for $R(t)$. If $\beta(t)=\beta$ (constant)

$$G(t)=1-\frac{\left(1-e^{-\rho}\right)(\lambda+\beta)}{\lambda e^{-\rho}\left(e^{-(\lambda+\beta)t}-1\right)+\lambda},\quad t\geq 0,\quad -\lambda\leq\beta\leq\frac{\lambda}{e^{\rho}-1}\ \text{(Ferreira (1998))} \tag{3.3}.$$

For this service time distributions collection the busy period length is exponentially distributed with an atom at the origin.

After (2.1) we get $R(t)=e^{-\rho}(1+\lambda t)+\left(1-e^{-\rho}\right)\frac{\beta}{\lambda+\beta}e^{-(\lambda+\beta)t}+\left(1-e^{-\rho}\right)\frac{\lambda}{\lambda+\beta}$, $-\lambda<\beta<\frac{\lambda}{e^{\rho}-1}$. It is easy to show that $R(t)-\lambda e^{-\rho}-\frac{VAR[Z]+E^2[Z]}{2E^2[Z]}=0,\ t\geq 0.$

For the collection given by (3.2) we can show that

$$R(t) \geq \frac{(1+\lambda t)+(1+e^{-\rho})\left(1+e^{\frac{\lambda}{1-e^{-\rho}}t}\right)}{e^{\rho}}, \quad t \geq 0, \quad -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \quad (3.4)$$

B. The $M|G|\infty$ populational process, with time origin at a busy period beginning instant, mean behaviour as time function determinated by the sign of $\frac{g(t)}{1-G(t)}-\lambda, \ t \geq 0$ (Ferreira (2003 and 2003a)).

Putting $\frac{g(t)}{1-G(t)}-\lambda=\beta(t)$ ($\beta(\cdot)$ is any time function) we get

$$G(t)=\{1-[1-G(0)]\}e^{-\lambda t-\int_0^t \beta(u)\,du}, \quad t \geq 0, \quad \frac{\int_0^t \beta(u)\,du}{t} \geq -\lambda \quad (3.5).$$

Putting directly (3.5) in (2.1) we get the corresponding value for $R(t)$.

For $\beta=0$, (3.5) becomes

$$G(t)=1-[1-G(0)]e^{-\lambda t}, \quad t \geq 0 \quad (3.6)$$

with $1-G(0)=\rho$. In this situation the $M|G|\infty$ populational process, with time origin at a busy period beginning instant, mean is constant with value rho (Ferreira (2003 and 2003a)). As for $R(t)$ we have

$$R(t)=e^{-\rho(1-e^{-\lambda t})}+\lambda\int_0^t e^{-\rho(1-e^{-\lambda u})}du \quad (3.7)$$

and also

$$e^{-\rho(1-e^{-\lambda t})}+\lambda e^{-\rho}t \leq R(t) \leq e^{-\rho(1-e^{-\lambda t})}+\lambda t \quad (3.8)$$

CONCLUSIONS

After a short study about $R(t)$ and its properties, we present the formulas for it in the situation of particular distribution functions related to important busy period parameters behaviour study as time functions.

We show also bounds that help, in this case, when the formulas are note so friendly.

REFERENCES

[1] BROWN, M.: *Further monotonicity properties for specialized renewal processes*. Ann. Prob. 9 (1981), 891-895.

[2] ÇINLAR, E.: *Introduction to Stochastic Processes*. New Jersey: Prentice-Hall, Inc.. 1975.

[3] COX, D. R.: *Renewal Theory*. Methen London. 1962.

[4] FERREIRA, M.A.M.: *Um sistema M|G|∞ com período de ocupação exponencial.* Actas das XV Jornadas Luso-espanholas de Matemática, Vol. IV, Universiade de Évora. Évora, 1991.

[5] FERREIRA, M.A.M.: *Cauda do período de ocupação de fila de espera M|G|∞.* Comunicação apresentada no III Congresso Anual da Sociedade Portuguesa de Estatística. Guimarães. 1995.

[6] FERREIRA, M.A.M.: *Comportamento Transeunte do Sistema M|G|∞, Aplicação em Problemas de Doença e de Desemprego*. In: Revista de Estatística, INE, Vol. 3, 3.ºQuadrimestre. Lisboa. 1996

[7] FERREIRA, M.A.M.: *Aplicação da Equação de Ricatti ao estudo do Período de Ocupação do Sistema M|G|∞.* Revista de Estatística, Vol. 1, INE. 1998.

[8] FERREIRA, M.A.M.(2003): *Comportamento Transeunte do Sistema M|G|∞ com Origem dos Tempos no Início de um Período de Ocupação – Média e Variância*. In: VI Congresso Galego de Estatística e Investigación de Operacións, Actas, Vigo. Espanha. 5-7 de Novembro, 2003.

[9] FERREIRA, M.A.M.(2003): *M|G|∞ System Transient Behaviour with Time Origin at a Busy Period Beginning Instant – Mean and Variance.* 9th International Scientific Conference, quantitative Methods in Economy – Compatibility of Methodologys and Practice with the EU Conditions,.Bratislava, Slovak Republic. November 13-14, 2003.

[10] ROSS, S.: *Stochastic Processes*. Wiley. New York. 1983.

[11] STADJE, W.: *The busy period of queueing systems M|G|∞.* J.A.P.. 22 (1985), 697-704.

[12] TAKÁCS; L.: *An introduction to Queueing Theory.* Oxford University Press. New York, 1962.

MANUEL ALBERTO M. FERREIRA
Associate Professor
ISCTE-Instituto Superior de Ciȩncias do Trabalho e da Empresa
Departamento de Métodos Quantitativos
Av. Das Forças Armadas, 1649-026 Lisboa, PORTUGAL
e-mail:vice.presidente@iscte.pt