Department of Mathematics
Faculty of Mechanical Engineering
Slovak University of Technology in Bratislava

5th International Conference
APLIMAT 2006

RESOURCES DYNAMICS FROM THE POINT OF VIEW OF THE REGENERATED ORNSTEIN-UHLENBECK MODEL

FIGUEIRA João (PT), FERREIRA Manuel Alberto Martins (PT)

Abstract. We consider the problem of assessing resources dynamics in the context of an economic system subject to Gaussian consumption and deterministic productivity. A discrete time recursive equation is presented to justify the use of the Ornstein-Uhlenbeck diffusion process. Under assumptions about regeneration of the process, we observe the system equilibrium as to what concerns resources depreciation or accumulation.

1 Introduction

The main goal of this presentation is the study of a certain type of harvesting rules of an economic system where resources dynamics are given by an Ornstein-Uhlenbeck diffusion process. We mean the diffusion process satisfying the stochastic differential equation

$$dX(t) = (a + bX(t))dt + dB(t), \qquad X(0) = x, \tag{1}$$

where $B(t)$ denotes a standard Brownian motion process.

The problem we observe may be described briefly in the following way. If such a system is explored according to some management scheme or harvesting rules, what kind of conclusions may we take about the amount of resources made available. In particular, we observe the way we can equate some concept of equilibrium between the above mentioned economic system and the external exploration efforts with origin on the chosen management scheme.

Suppose that every time the resources process hits the set $\{\eta, \theta\}$, one collects an amount of size $\eta - x$ or $\theta - x$, with value $Q(\eta)$ or $Q(\theta)$, which takes instantaneously the process to the initial level x, and that the procedure is repeated continuously. In this case, the economic resources stock is represented clearly by a modification of $X(t)$ obtained by regeneration at the states η and θ. Write this modification as $\tilde{X}(t)$.

Let $N(t)$ denote the counting process of the regeneration epochs. Then, the cumulative process $R(t) = \sum_{n=0}^{N(t)} Q_n$ gives the accumulated resources harvested in such economic environment.

Note that $\{Q_n\}$ is a sequence of *i.i.d.* random variables, with values on the set $\{Q(\eta), Q(\theta)\}$, representing the flow of the harvesting procedure. We propose to take conclusions on the quality of the adopted management scheme with the help of the classical renewal theorem (see [1]) establishing

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}(Q)}{\mathbb{E}(T)} \ , \qquad a.s. \ . \tag{2}$$

The random variable T characterizes the sequence of recurrence times of the economic resources process $\tilde{X}(t)$ on the set $\{\eta, \theta\}$. It corresponds to the travel time of $X(t)$ from x to $\{\eta, \theta\}$.

In particular, we observe some special results on the sign of the function

$$\gamma(\theta) = \lim_{\delta\to\Delta} \frac{\mathbb{E}(Q)}{\mathbb{E}(T)} \ , \tag{3}$$

where δ and Δ are variables to define later, depending on η, x and θ.

About the agenda: we present in the next section more details on the process in (1). We postpone to the last section the presentation of the main results concerning (2) and (3). Recall, on what is related with the present theme, references [3], [4], [5] and [6].

2 More on the resources dynamics model

We consider here the economic system with resources dynamics suggested by the stochastic difference equation

$$Y_0 = x, \qquad Y_{nh} = e^{bh}(Y_{(n-1)h} + W_{nh}), \qquad n = 1, 2, \ldots \ . \tag{4}$$

In the above equation, $\{W_{nh}\}$ is a sequence of *i.i.d.* random variables with Gaussian distribution, mean ah and variance h. Assume $a < 0$ and $b, h > 0$.

Consequently, $\{Y_{nh}\}$ is also a Gaussian sequence with parameters (which may be easily evaluated)

$$\begin{aligned} &\mathbb{E}(Y_{nh}) = xe^{nbh} + ahe^{bh}\tfrac{1-e^{nbh}}{1-e^{bh}} \ , \\ &\mathbb{C}\mathrm{ov}(Y_{mh}, Y_{nh}) = he^{(2+n-m)bh}\tfrac{1-e^{2mbh}}{1-e^{2bh}} \ , \qquad 0 \le m \le n. \end{aligned} \tag{5}$$

The general idea behind (4) is that of an economic system with a deterministic productivity, multiplying in each period of time the stock available after Gaussian consumption.

The following identities suggest the approximation of $\{Y_{nh}\}$, as $h \to 0$, by a continuous time process satisfying the equation (1). Using a Taylor expansion around 0 of the left side, one obtains ($t = nh$)

$$\begin{aligned} &\mathbb{E}(Y_{t+h} - Y_t \mid Y_t = y) = (a + by)h + o(h), \\ &\mathbb{E}((Y_{t+h} - Y_t)^2 \mid Y_t = y) = h + o(h). \end{aligned} \tag{6}$$

The solution to (1), as is well known, is the Ornstein-Uhlenbeck process

$$X(t) = e^{bt}\left(x + a\int_0^t e^{-bs}ds + \int_0^t e^{-bs}dB(s)\right), \tag{7}$$

with parameters

$$\begin{aligned} &\mathbb{E}(X(t)) = xe^{bt} + \tfrac{a}{b}(e^{bt}-1)\ , \\ &\mathbb{C}\mathrm{ov}(X(s),X(t)) = e^{b(s+t)}\tfrac{1-e^{-2bs}}{2b}\ , \qquad 0 \le s \le t. \end{aligned} \tag{8}$$

We shall need later some functionals concerning the Ornstein-Uhlenbeck process which we specify now. Let $0 \le \eta < x < \theta$ and recall the standard Gaussian distribution function: $\Phi(u) = \int_{-\infty}^{u} \phi(v)dv$ with $\phi(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$. Suppose $\rho(x)$ is the probability $X(t)$ reaches η before θ, coming from x. The evaluation of this functional in general diffusion processes is a well established result (see [2]). We have

$$\rho(x) = \frac{\Phi(\alpha+\beta\theta) - \Phi(\alpha+\beta x)}{\Phi(\alpha+\beta\theta) - \Phi(\alpha+\beta\eta)}\ . \tag{9}$$

Particular arguments of the expression above are $\alpha = a\sqrt{2}/\sqrt{b}$ and $\beta = b\sqrt{2}/\sqrt{b} = \sqrt{2b}$. Suppose also $\psi(x)$ is the expected value of the travel time of $X(t)$, from x to $\{\eta,\theta\}$. As to what respects the evaluation of this functional in general diffusion processes, see also [2]. In this case, we get

$$\begin{aligned} \psi(x) \ &= \tfrac{2}{\beta}\Big(\rho(x)\int_{\eta}^{x}\tfrac{\Phi(\alpha+\beta u)-\Phi(\alpha+\beta\eta)}{\phi(\alpha+\beta u)}\,du \\ &\quad +(1-\rho(x))\int_{x}^{\theta}\tfrac{\Phi(\alpha+\beta\theta)-\Phi(\alpha+\beta u)}{\phi(\alpha+\beta u)}\,du\Big)\ . \end{aligned} \tag{10}$$

3 Evaluation criteria of harvesting procedures

In the sequence of what has been mentioned earlier, considering equations (2) and (3), the main source of information on the managing procedures, observed here, is the ratio

$$\frac{\mathbb{E}(Q)}{\mathbb{E}(T)} = \frac{Q(\theta) + (Q(\eta) - Q(\theta))\rho(x)}{\psi(x)}\ . \tag{11}$$

Using as arguments the real variables δ and Δ and also the real function $Q(.)$, we intend to present conclusions about the sign of $\gamma(\theta) = \lim_{\delta\to\Delta}\mathbb{E}(Q)/\mathbb{E}(T)$ which enhance the way the described economic system is fragile as to what concerns the chosen harvesting technique.

(A) *Suppose $\delta = x$, $\Delta = \eta$ and $Q(y) = y - x$. Then the functions*

$$\gamma(\theta) \quad \text{and} \quad \Phi(\alpha+\beta\eta) + \beta\phi(\alpha+\beta\eta)(\theta-\eta) - \Phi(\alpha+\beta\theta) \tag{12}$$

share the same sign. In particular $\gamma(\theta)$ is positive whenever $-\alpha/\beta \le \eta < \theta$.

Arguments. Note that

$$\gamma(\theta) = \frac{-1-(\theta-\eta)\rho'(\eta)}{\psi'(\eta)}\ . \tag{13}$$

The conclusion follows from the fact that $\psi'(\eta)$ is always positive and the fact that the numerator of (13) has the sign of the function at the right in (12). As to what concerns the last conclusion notice that $\Phi(\alpha+\beta\eta) + \beta\phi(\alpha+\beta\eta)(\theta-\eta)$ is the tangent line to the graph of the function $\Phi(\alpha+\beta\theta)$ at η.

(B) *Suppose $\delta = x$, $\Delta = \theta$ and $Q(y) = y - x$. Then the functions*

$$\gamma(\theta) \quad \textit{and} \quad \Phi(\alpha + \beta\theta) + \beta\phi(\alpha + \beta\theta)(\eta - \theta) - \Phi(\alpha + \beta\eta) \tag{14}$$

share the same sign.

Arguments. Note that

$$\gamma(\theta) = \frac{1 + (\theta - \eta)\rho'(\theta)}{-\psi'(\theta)} \ . \tag{15}$$

The conclusion follows from the fact that $\psi'(\theta)$ is always negative and the fact that the numerator of (15) has the sign of the function at the right in (14). In the study of this result, notice that $\Phi(\alpha + \beta\theta) + \beta\phi(\alpha + \beta\theta)(\eta - \theta)$ is the tangent line to the graph of the function $\Phi(\alpha + \beta\eta)$ at θ.

Acknowledgements

Thanks are due to ISCTE and its research unit UNIDE for their support through the line L7 - Quantitative Methods in the Management Sciences.

References

[1] ASMUSSEN, S.: *Applied Probability and Queues*. Springer, New York, 2003.

[2] BHATTACHARYA, R.N.; WAYMIRE, E.C.: *Stochastic Processes with Applications*. John Wiley and Sons, New York, 1990.

[3] DASGUPTA, P.S.; HEAL, G.M.: *Economic Theory and Exaustible Resources*. Cambridge University Press, New York, 1979.

[4] LJUNGQVIST, L.; SARGENT, T.: *Recursive Macroeconomic Theory*. The MIT Press, Cambridge, 2000.

[5] PUU, T.: *Nonlinear Economic Dynamics*. Springer, Berlin, 1997.

[6] ROMER, D.: *Advanced Macroeconomics*. McGraw-Hill, Boston, 2001.

Current address

FIGUEIRA, João
ISCTE - Instituto Superior de Ciências do Trabalho e da Empresa
Av. Forças Armadas, 1649 026 Lisboa (Lisbon, Portugal), Tel. +351 217 903 000
e-mail: joao.figueira@iscte.pt

FERREIRA, Manuel Alberto Martins
ISCTE - Instituto Superior de Ciências do Trabalho e da Empresa
Av. Forças Armadas, 1649 026 Lisboa (Lisbon, Portugal), Tel. +351 217 903 000
e-mail: manuel.ferreira@iscte.pt