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Pricing after the IBOR era

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Dedicated to my parents and my girlfriend, for the constant support and encouragement throughout the thesis. Thank you for believing in me.

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Resumo

O principal objetivo desta tese é explorar os fundamentos teóricos para a avaliação de swaps de taxas de juro. Para os swaps, existem dois tipos de taxas de juro subjacentes a analisar: taxas de juro subjacentes com maturidade fixa (por exemplo, a EURIBOR a 6 meses) e taxas de juros subjacentes num regime *overnight*, estas que são fixadas diariamente (por exemplo, a secured overnight financial rate). A avaliação destes swaps irá depender da medida de risco neutro, $\mathbb{Q}$, que assume a money-market account como numerário, e da *forward measure*, $\mathbb{Q}_t$, que será explicitada mais adiante. Após concluir a teoria para a avaliação de swaps de taxas de juro, testaremos a eficácia na avaliação das cotações mid para swaps EUSA com recurso a uma curva de obrigações de cupão zero extraída dos swaps EESWE.

Abstract

The main objective of this thesis is to explore the theoretical foundations for interest rate swaps pricing and valuation. For the swaps, there are two types of underlying interest rates on which we will look upon: underlying interest rates with some fixed maturity (for example, the 6-month EURIBOR), and underlying overnight interest rates, which are set on a daily basis (as, for instance, the secured overnight financial rate). The pricing of these swaps will rely heavily on both the risk-neutral measure, $\mathbb{Q}$, and on the forward measure, $\mathbb{Q}_t$, which will be defined later. After completing the theory of pricing interest rate swaps, we will test the pricing accuracy of EUSA mid swap quotes with resource to a bootstrapped zero-coupon bond curve derived from EESWE swaps.

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CHAPTER 1

Introduction

The London Interbank Offered Rate (LIBOR) has been the benchmark index for interest rates and interest rate derivatives until the end of last year. The LIBOR rates were published by the Intercontinental Exchange (ICE) on a daily basis and they were computed through the weighted average of quotes on borrowing rates of financial institutions with a good credit rating, such as major banks. Prior to the crisis of 2008, LIBOR was proven to be manipulated by various groups of traders in these financial institutions and, according to Jarrow and Li (2021) and Hou and Skeie (2014), this manipulation can be traced as early as 2003.

Due to how effortless this manipulation was, many regulations were implemented in both US and Europe. Hence, it was legislated that LIBOR was set to be extinct as an interest rate benchmark for interest rate derivatives by January 1, 2022. In 2017, in the US, the Alternative Reference Rates Committee (ARRC) created the Secured Overnight Financial Rate (SOFR), an overnight rate that is based on a volume weighted average of day-to-day transactions over the US Treasury bond market. The key factor is that the volume weighting makes it more transparent and reduces the possibility of manipulation. On the other hand, in Europe, the European Central Bank (ECB) created the euro short-term rate (ESTR) to represent the wholesale overnight euro borrowing cost of financial institutions, such as banks. It is calculated using daily confidential information of daily transactions in the money market.

Although the problem with manipulation is resolved with the emergence of these overnight rates based on the volume weightings of the money-market transactions, there is some concern in the financial community around the fact that these overnight rates are not the appropriate substitutes for the LIBOR as a benchmark for interest rates. According to Jarrow and Li (2021), "the issue underlying (...) is that a reference index rate, created for use in interest rate derivatives, should facilitate hedges for fixed and floating rate loans and be robust to manipulation". Jarrow and Li (2021) study the effectiveness of hedging on their paper, which is something that will not be covered during this thesis.

Of course, with a new type of interest rate swap also comes the need to know how to compute the fair value of these swaps. As it is known, swaps are key to understanding other market factors such as liquidity, supply and credit quality of the banks. Additionally, the swap curve is an important benchmark for credit rates as well, therefore it is important that the pricing is done correctly, as financial institutions also use them as a reference. Like said, this is what the thesis is mainly focus on: pricing and finding closed-form solutions for the swaps with each type of underlying interest rate mentioned above.

This work is structured as follows: the thesis starts in Chapter 2 by reminding some key concepts of financial mathematics that will help further on developing the pricing for the swaps, as well as some key assumptions over the market overall that are crucial to achieve our final results. In Chapter 3, we will price interest rate swaps under all possible conditions in terms of type of underlying and embedded credit risk, and we will also consider these two scenarios for valuating swaps that are still under some IBOR rate and that will make the switch to an overnight index rate. In Chapter 4, we will test the accuracy of pricing EUSA swaps with the aid of a bootstrapped zero-coupon bond curve from another swap, EESWE. Finally, on Chapter 5, we present final remarks and comments on the results and equations obtained throughout the thesis, and some key notes on work to follow this thesis.

CHAPTER 2

Stochastic calculus and the model

Before pricing swaps, it is crucial to remind some concepts that will help us further on. We begin by recalling definitions and concepts regarding stochastic processes. Furthermore, we will consider that we are working under some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ paired with a filtration $\mathcal{F}_t$.

DEFINITION 2.1. *The conditional expectation,*

$$\mathbb{E}^{\mathbb{P}}\left[X_T|\mathcal{F}_t\right], \tag{2.1}$$

is the expected value of the unknown future value that the random variable X will assume at time $T \geq t$, computed under some probability measure $\mathbb{P}$ and conditional to the information available until time t, $\mathcal{F}_t$.

It is also crucial to understand the notion of martingale as well:

DEFINITION 2.2. *Let $(X_t)_{t\geq 0}$ be a stochastic process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. A martingale, under the probability measure $\mathbb{P}$, is a continuous time and integrable stochastic process such that:*

$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s, \forall s < t \tag{2.2}$$

One important law that derives from conditional expectation is the law of iterated expectations, which will be crucial to derive the equations for swaps later on.

PROPOSITION 2.1. *(Law of iterated expectations) Let X_T be a $\mathcal{F}_T$-measurable random variable. Then,*

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X_T|\mathcal{F}_t\right]|\mathcal{F}_s\right] = \mathbb{E}^{\mathbb{P}}\left[X_T|\mathcal{F}_s\right], \forall s \leq t \leq T \tag{2.3}$$

Also important for the pricing of swaps (or any financial instrument at all) is the martingale probability measure $\mathbb{Q}$:

DEFINITION 2.3. *Consider the initial value of a money-market account, B_0. The money-market account is a deposit account that earns interest with continuous compounding at the risk-free rate of r_t. The time$-t$ value of the money-market account is equal to:*

$$B_t = B_0 e^{\int_0^t r_s ds} \tag{2.4}$$

DEFINITION 2.4. *The martingale probability measure $\mathbb{Q}$ is the probability measure that is equivalent to the physical measure $\mathbb{P}$, such that, for $t \leq T$ and considering X any financial asset:*

$$X_t = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{X_T}{B_T}\middle|\mathcal{F}_t\right] \tag{2.5}$$

Another stochastic calculus result that will be crucial is the following:

PROPOSITION 2.2. *(Change of numeraire) Let $\mathbb{Q}^N$ and $\mathbb{Q}^M$ be two equivalent martingale measures associated to the numeraire N_t and M_t, respectively. Then*

$$N_t \mathbb{E}^{\mathbb{Q}^N}\left[\frac{X_T}{N_T}\middle|\mathcal{F}_t\right] = M_t \mathbb{E}^{\mathbb{Q}^M}\left[\frac{X_T}{M_T}\middle|\mathcal{F}_t\right] \tag{2.6}$$

Going down the road, it is also important to characterize the model and assumptions we will be using throughout. Similarly to Jarrow and Li (2021), we will assume that the market trades: default-free zero coupon bonds of all maturities, a default-free money-market account just like the one defined, and risky bonds.

Having defined a money-market account, we can also define the time-t value of a (unit face value) default-free zero coupon bond expiring at time t_i, $P(t, t_i)$, where $0 \leq t \leq t_i$. On the other hand, risky zero coupon bonds, $D(t, t_i)$, are bonds where the issuer may default on one or more payments before the bond reaches its maturity. In case the bond defaults before the maturity, the bond will pay a stochastic recovery rate of $0 \leq R_t \leq 1$. Additionally, if we consider τ as the bond's default time, the risky bond payoff is:

$$D(t_i, t_i) = \mathbb{I}_{\tau > t_i} + \mathbb{I}_{\tau \leq t_i} R_\tau \frac{B_{t_i}}{B_\tau} \tag{2.7}$$

For the purpose of the pricing, we will also assume that both these markets are arbitrage free.

ASSUMPTION 2.1. *There exists an equivalent probability measure $\mathbb{Q}$ such that:*

$$\frac{P(t, t_i)}{B_t} \text{ and } \frac{D(t, t_i)}{B_t}, \forall t \leq t_i \tag{2.8}$$

are $\mathbb{Q} - martingales$.

Furthermore, we will also consider these are free of transaction costs and trading limitations, and we always act as price takers. Hence, our trades have no impact on the market price.

CHAPTER 3

Interest Rate Swaps

An interest rate swap (IRS) is an over-the-counter derivative that allows two parties to exchange a cash-flow at a fixed rate of interest, also known as the swap rate, for a cash-flow at a floating rate in the future. The settled floating rate is typically a LIBOR rate or an overnight rate. When one pays the fixed rate, the IRS is termed payer IRS, otherwise, it is termed receiver IRS. Going forward, we will solely consider the analysis of a payer IRS. Furthermore,

ASSUMPTION 3.1. *There is no counter-party risk while executing IRS contracts due to the collateral agreements in interest rate swaps.*

3.1. Interest rate swaps based on LIBOR rates

3.1.1. Floating rate loans based on LIBOR rates

Before evaluating interest rate swaps based on LIBOR rates, it is crucial to know how to evaluate a floating rate loan (FRL) under some variable rate (e.g, a LIBOR rate).

A floating rate loan is a financial security with notional 1, that expires at some maturity date t_N. The payments of a FRL are made at times $t_1, t_2, \dots, t_N$, and these are a coupon rate equal to some market index. Assuming that the FRL is default-free, the payment at time t_i for $i = 1, ..., N$ is

$$\delta L(t_{i-1}, t_i) = \frac{1}{P(t_{i-1}, t_i)} - 1 \tag{3.1}$$

where we denote $L(t_{i-1}, t_i)$ as the variable rate, L is the market index, and δ is the duration (in years) of the coupon period, i.e, $\delta = t_i - t_{i-1}$, for $i = 1, \dots, N$. Computing the arbitrage-free value of floating rate loans,

THEOREM 3.1. *(Default-free floating rate loans) The fair value of a default-free floating rate loan at time t ($t_0 \leq t < t_1$) is*

$$\frac{P(t, t_1)}{P(t_0, t_1)} \tag{3.2}$$

PROOF. Start by considering some arbitrary t such that $t_0 \leq t < t_1$. Then, the time$-t$ value of the FRL is equal to:

$$FRL_t = P(t, t_N) + P(t, t_1) \times \delta L(t_0, t_1) + \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}} \left[\frac{B_t}{B_{t_i}} \delta L(t_{i-1}, t_i) \middle| \mathcal{F}_t \right]$$

We need to know how to evaluate the present value of each coupon payment (besides the next one) in order to proceed with the proof. For such, consider the same time t as in the start of the proof, and two arbitrary coupon dates t_{i-1}, t_i, such that $t_1 < t_{i-1} < t_i < t_N$.

At time t, we will hedge the FRL cash-flow at time t_i, $\delta L(t_{i-1}, t_i)$. For such, consider the following portfolio with the following investment strategy: firstly, consider a loan starting at t and maturing at time t_i in the amount of $P(t, t_i)$, which will generate a cash-flow of -1 at t_i. Secondly, consider a deposit starting at t_i and maturing at t_{i-1} in the amount of $P(t, t_{i-1})$, which will generate a cash-flow of 1 at its maturity. The deposit will mature at t_{i-1}. Given this, at the deposit's maturity date we will renew the same deposit. This new deposit will start at t_{i-1} and will mature at t_i, while also generating a cash-flow of $1 + \delta L(t_{i-1}, t_i)$.

Summing up these operations, we can calculate each net cash-flow at times t, t_{i-1}, and t_i. By doing such, we will have a net cash-flow of $P(t, t_{i-1}) - P(t, t_i)$ at t, 0 at t_{i-1} and $\delta L(t_{i-1}, t_i)$ at t_i. In other words, our hedging strategy proves that: in order to receive $\delta L(t_{i-1}, t_i)$ at t_i, we will need to invest, at time t, an amount of $P(t, t_{i-1}) - P(t, t_i)$, i.e,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}\delta L(t_{i-1}, t_i)\middle|\mathcal{F}_t\right] = P(t, t_{i-1}) - P(t, t_i)$$

Therefore,

$$FRL_t = P(t, t_N) + P(t, t_1) \times \delta L(t_0, t_1) + \sum_{i=1}^{N}[P(t, t_{i-1}) - P(t, t_i)]$$

Simplifying the sum, we get that,

$$\sum_{i=1}^{N}[P(t, t_{i-1}) - P(t, t_i)] = P(t, t_1) - P(t, t_N),$$

which leads to,

$$\begin{aligned} FRL_t &= P(t, t_N) + P(t, t_1) \times \delta L(t_0, t_1) + \sum_{i=1}^{N}[P(t, t_{i-1}) - P(t, t_i)] \\ &= P(t, t_N) + P(t, t_1) \times \delta L(t_0, t_1) + P(t, t_1) - P(t, t_N) \\ &= P(t, t_1)[1 + \delta L(t_0, t_1)] \end{aligned}$$

Using equation (3.1), we can finally derive that,

$$FRL_t = \frac{P(t, t_1)}{P(t_0, t_1)}$$

which concludes the proof. □

In reality, in most of the cases, there is credit risk associated to a FRL. Therefore, we will need to reconsider our analysis to include such risk. Consider the FRL mentioned previously, but now with the assumption that it is under some credit-risky reference rate. The payment at time t_i, if default doesn't occur before time t_{i-1}, of this FRL is

$$\begin{aligned} &\mathbb{I}_{\tau > t_{i-1}}\left[(\delta L(t_{i-1}, t_i))\mathbb{I}_{\tau > t_i} + R_\tau(1 + \delta L(t_{i-1}, t_i))\mathbb{I}_{\tau \leq t_i}\right] \\ &= \mathbb{I}_{\tau > t_{i-1}}\left[\left(\frac{1}{D(t_{i-1}, t_i)} - 1\right)\mathbb{I}_{\tau > t_i} + R_\tau\frac{1}{D(t_{i-1}, t_i)}\mathbb{I}_{\tau \leq t_i}\right], \end{aligned} \tag{3.3}$$

for $i = 1, \ldots, N$, where R_τ is the stochastic recovery rate, τ is the default time, and $D(t_{i-1}, t_i)$ is a credit risky zero coupon bond according to its FRL rating. The fair value of this floating rate loan is given by the following theorem:

THEOREM 3.2. *(Defaultable floating rate loans) The fair value of a defaultable floating rate loan at time t ($t_0 \leq t < t_1$) is*

$$\frac{D(t, t_1)}{D(t_0, t_1)} \tag{3.4}$$

PROOF. Start by considering the process $X_t^{t_i}$, which represents the present value of a dollar paid at time t_i, in case the underlying firm hasn't defaulted before t_i:

$$X_t^{t_i} = \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}\mathbb{I}_{\tau \geq t_i}\middle|\mathcal{F}_t\right] = B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\mathbb{I}_{\tau \geq t_i}\middle|\mathcal{F}_t\right]$$

Considering a change of numeraire from the neutral-risk measure, $\mathbb{Q}$, based on the numeraire money-market account, to a forward measure, $\mathbb{Q}^{t_i}$, which takes the zero coupon bond, $P(t, t_i)$, as the numeraire. Then

$$\begin{aligned} X_t^{t_i} &= P(t, t_i)\mathbb{E}^{\mathbb{Q}_{t_i}}\left[\frac{1}{P(t_i, t_i)}\mathbb{I}_{\tau \geq t_i}\middle|\mathcal{F}_t\right] \\ &= P(t, t_i)\mathbb{E}^{\mathbb{Q}_{t_i}}[\mathbb{I}_{\tau \geq t_i}|\mathcal{F}_t] \\ &= P(t, t_i)\mathbb{Q}_{t_i}(\tau \geq t_i|\mathcal{F}_t) \end{aligned}$$

We will now divide the proof in three parts: the first one being the payment at the first resettlement date, t_1; the second one being the payments between t_1 and t_N; the last one being the payment at the FRL maturity date, t_N. For the first part, given the equation (1.3), we can determine the payment at time t_1,

$$\left(\frac{1}{D(t_0, t_1)} - 1\right)\mathbb{I}_{\tau > t_1} + R_\tau\left(\frac{1}{D(t_0, t_1)}\right)\mathbb{I}_{\tau \leq t_1},$$

whose time$-t$ value is:

$$\begin{aligned} Y_t^{t_0} &= B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{\left(\frac{1}{D(t_0,t_1)} - 1\right)\mathbb{I}_{\tau > t_1} + R_\tau\left(\frac{1}{D(t_0,t_1)}\right)\mathbb{I}_{\tau \leq t_1}}{B_{t_1}}\middle|\mathcal{F}_t\right] \\ &= B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{\frac{\mathbb{I}_{\tau > t_1} + R_\tau\mathbb{I}_{\tau \leq t_1}}{D(t_0,t_1)} - \mathbb{I}_{\tau > t_1}}{B_{t_1}}\middle|\mathcal{F}_t\right] \\ &= \frac{1}{D(t_0, t_1)}B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau > t_1} + R_\tau\mathbb{I}_{\tau \leq t_1}}{B_{t_1}}\right] - B_t\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_1}}\mathbb{I}_{\tau \geq t_1}\middle|\mathcal{F}_t\right] \end{aligned}$$

By the definition of a defaultable zero coupon bond, we can conclude that expected value in the first term is just $D(t, t_1)B_t$. Furthermore, we can also notice that the second term of the difference is just the process $X_t^{t_1}$. Therefore, the time$-t$ value of the payment at time t_i is

$$\frac{D(t,t_1)}{D(t_0,t_1)} - X_t^{t_1},$$

which concludes the first part.

For the second part, we will consider any i from $i = 2, \ldots, N$. The payment at time t_i is given by equation (1.3). The value of this payment is,

$$\begin{aligned}
Y_t^{t_i} &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_i} + R_\tau \mathbb{I}_{\tau\le t_i}}{D(t_{i-1},t_i)}\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_i}} - \frac{\mathbb{I}_{\tau>t_i}}{B_{t_i}}\middle|\mathcal{F}_t\right]B_t \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_i} + R_\tau \mathbb{I}_{\tau\le t_i}}{D(t_{i-1},t_i)}\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_i}}\middle|\mathcal{F}_t\right]B_t - \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_i}}{B_{t_i}}\middle|\mathcal{F}_t\right]B_t
\end{aligned}$$

Since $t < t_1 < t_{i-1}$, given Proposition (2.1), we can use the fact that $\mathcal{F}_t \subseteq \mathcal{F}_{t_{i-1}}$,

$$\begin{aligned}
Y_t^{t_i} &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{D(t_{i-1},t_i)}\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_i} + R_\tau \mathbb{I}_{\tau\le t_i}}{B_{t_i}}\middle|\mathcal{F}_{t_{i-1}}\right]\middle|\mathcal{F}_t\right]B_t - X_t^{t_i} \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{D(t_{i-1},t_i)}\frac{D(t_{i-1},t_i)}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t - X_t^{t_i} \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t - X_t^{t_i} \\
&= X_t^{t_{i-1}} - X_t^{t_i}
\end{aligned}$$

For the third and final part of this proof, we will evaluate the notional payment at t_N. At t_N, the notional pays $\mathbb{I}_{\tau>t_N}$, whose time$-t$ is:

$$Y_t^{t_N} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_N}}{B_{t_N}}\middle|\mathcal{F}_t\right]B_t = X_t^{t_N}$$

The value of the defaultable floating rate loan will be simply the sum of these parts all together,

$$\begin{aligned}
FRL_t &= \left[\frac{D(t,t_1)}{D(t_0,t_1)} - X_t^{t_1}\right] + \sum_{i=2}^{N}\left(X_t^{t_{i-1}} - X_t^{t_i}\right) + X_t^{t_N} \\
&= \frac{D(t,t_1)}{D(t_0,t_1)}
\end{aligned}$$

which concludes our proof. Also notice that, if we ignore credit risk, we get the result derived in the proof of Theorem 1.1, by taking $D(t_{i-1},t_i) = P(t_{i-1},t_i)$

3.1.2. Interest rate swaps on a default-free reference rate

Consider a set of resettlement dates $t_0, t_1, \ldots, t_N$, and consider $\delta = t_i - t_{i-1}$, for $i = 1, \ldots, N$, such that δ is constant (i.e, payments happen at the same frequency). Furthermore, for further simplification, consider a principal of 1.

DEFINITION 3.1. *The payments in a plain-vanilla IRS are as follows:*

(1) Payments will be made and received at times $t_i = t_0 + i\delta$, for $i = 1, \ldots, N$.

(2) For $i = 1, \ldots, N$, at every period $[t_{i-1}, t_i]$, the LIBOR rate $L(t_{i-1}, t_i)$ is set at t_{i-1} and the floating leg,

$$\delta L(t_{i-1}, t_i)$$

is paid at time t_i.

(3) For $i = 1, \ldots, N$, at every period $[t_{i-1}, t_i]$, the fixed leg payment

$$\delta x(t, t_N)$$

is paid at time t_i, where $x(t, t_N)$ is the swap rate and t is such that $t_0 \leq t < t_1$.

At time t, the value of a plain-vanilla IRS is simply equal to 0. Recalling that in an interest rate swap, a set of floating rate payments (floating leg) are exchanged for a set of fixed payment (fixed leg), this means that the time$-t$ of both legs is equal to one another.

Considering the fixed leg of the IRS as a fixed coupon rate bond with coupon rate equal to the fixed swap rate $x(t, t_N)$ (i.e, assuming an hypothetical exchange of capital), we can easily evaluate the present value at time t of a plain-vanilla IRS fixed leg.

THEOREM 3.3. *(Fixed leg of an IRS on a default-free reference rate) The time$-t$ value of the fixed leg of a plain-vanilla IRS with $t_0 \leq t < t_1$ on a default-free reference rate is*

$$\delta x(t, t_N) \sum_{i=1}^{N} P(t, t_i) \tag{3.5}$$

PROOF. Consider some time t such that $t_0 \leq t < t_1$. The payment of the fixed leg at any given time t_i, where $i = 1, \ldots, N$, is just

$$\delta x(t, t_N),$$

and its time$-t$ value is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\delta x(t, t_N)}{B_{t_i}}\middle|\mathcal{F}_t\right] B_t = \mathbb{E}^{\mathbb{Q}_{t_i}}\left[\frac{\delta x(t, t_N)}{P(t_i, t_i)}\middle|\mathcal{F}_t\right] P(t, t_i) = \delta x(t, t_N) P(t, t_i)$$

Therefore, the time$-t$ value of the fixed leg is the sum of the fair value of all these payments:

$$\sum_{i=1}^{N} \delta x(t, t_N) P(t, t_i) = \delta x(t, t_N) \sum_{i=1}^{N} P(t, t_i)$$

$\square$

Moving forward to the floating leg valuation, we can evaluate the present value of the floating leg of an IRS with aid of the results derived in Section 3.1.1.

THEOREM 3.4. *(Floating rate of an IRS on a default-free reference rate) The time−t value of the floating leg of a plain-vanilla IRS with $t_0 \leq t < t_1$ on a default-free reference rate is*

$$\frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N) \tag{3.6}$$

PROOF. We will divide the proof into two parts. Starting off by evaluating the first floating leg payment at t_1, we already know by equation (1.1) that this payment is

$$\frac{1}{P(t_0,t_1)} - 1,$$

whose time−t value is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\frac{1}{P(t_0,t_1)} - 1}{B_{t_1}} B_t \middle| \mathcal{F}_t\right] = \frac{1}{P(t_0,t_1)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_1}}\middle|\mathcal{F}_t\right] B_t - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_1}}\middle|\mathcal{F}_t\right] B_t$$

Switching to the forward measure $\mathbb{Q}_{t_1}$, based on the zero-coupon bond, $P(t,t_1)$:

$$\frac{1}{P(t_0,t_1)}\mathbb{E}^{\mathbb{Q}_{t_1}}\left[\frac{1}{P(t_1,t_1)}\middle|\mathcal{F}_t\right] P(t,t_1) - \mathbb{E}^{\mathbb{Q}_{t_1}}\left[\frac{1}{P(t_1,t_1)}\middle|\mathcal{F}_t\right] P(t,t_1) = \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_1),$$

which concludes the first part of the proof. For the second part, we already know that, by the same equation (1.1), the payment at time t_i, for $i = 2, \ldots, N$, is

$$\frac{1}{P(t_{i-1},t_i)} - 1,$$

whose time−t value is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\frac{1}{P(t_{i-1},t_i)} - 1}{B_{t_i}} B_t \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P(t_{i-1},t_i)}\frac{1}{B_{t_i}}\middle|\mathcal{F}_t\right] B_t - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\right] B_t$$

Since $\mathcal{F}_t \subseteq \mathcal{F}_{t_{i-1}}$, we can use Proposition (2.1), as well as the fact that $\frac{1}{P(t_{i-1},t_i)}$ is adapted to the filtration $\mathcal{F}_{t_{i-1}}$, to get:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P(t_{i-1},t_i)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\middle|\mathcal{F}_{t_{i-1}}\right]\middle|\mathcal{F}_t\right] B_t - P(t,t_i)\\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P(t_{i-1},t_i)}\frac{P(t_{i-1},t_i)}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right] B_t - P(t,t_i)\\
&= P(t,t_{i-1}) - P(t,t_i),
\end{aligned}$$

which concludes the second part of the proof. Note as well that this last result is equal to the one derived in the proof of Theorem 1.1. Finally, the arbitrage-free value of the floating leg will be the sum of all the payments,

$$\frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_1) + \sum_{i=2}^{N}[P(t,t_{i-1}) - P(t,t_i)] = \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N)$$

which concludes our proof. □

Since the time$-t$ value of a plain-vanilla IRS is the difference between the time$-t$ values of both floating and fixed legs, i.e, the difference between equations (1.5) and (1.6),

$$IRS_t = \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N) - \delta x(t,t_N)\sum_{i=1}^{N} P(t,t_i) \tag{3.7}$$

Setting such net value equal to 0 and solving for $x(t,t_N)$, we can derive the following equation for determining the fixed swap rate for a plain-vanilla IRS:

Corollary 3.1. *(Interest rate swap rate) The time$-t$ swap rate (with $t_0 \leq t < t_1$) of a plain-vanilla IRS on a default-free reference rate is*

$$x(t,t_N) = \frac{\frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N)}{\delta\sum_{i=1}^{N} P(t,t_i)} \tag{3.8}$$

One particular case of interest for this last result happens when we consider $t = t_0$, which gives the swap rate at the resettlement date t_0:

$$x(t_0,t_N) = \frac{\frac{P(t_0,t_1)}{P(t_0,t_1)} - P(t_0,t_N)}{\delta\sum\limits_{i=1}^{N} P(t_0,t_i)} = \frac{1 - P(t_0,t_N)}{\delta\sum\limits_{i=1}^{N} P(t_0,t_i)}$$

3.1.3. Interest rate swaps on a credit risky reference rate

As mentioned in section 3.1.1, there is usually credit risk attached to these derivatives. Therefore, once again, we will need to adapt our previous theorems and corollaries to include such risk.

Regarding the fixed leg, we can use Theorem 1.3 to cover this scenario, as the fixed leg is totally independent from the underlying FRL in the floating leg. Therefore, the focus will remain on the valuation of the floating leg under these conditions.

Theorem 3.5. *The time$-t$ value of the floating leg of a plain-vanilla IRS (with time $t_0 \leq t < t_1$) on a credit risky reference rate is*

$$\frac{P(t,t_1)}{D(t_0,t_1)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t - \sum_{i=1}^{N} P(t,t_i) \tag{3.9}$$

Proof. The proof for this theorem will be similar to the proof of Theorem 1.4, but now we will consider the payment being under some defaultable zero coupon bond, i.e,

$$\frac{1}{D(t_{i-1},t_i)} - 1$$

Indeed, for the payment at t_1, $\frac{1}{D(t_0,t_1)} - 1$, its time$-t$ value is

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\frac{1}{D(t_0,t_1)} - 1}{B_{t_1}} B_t \right| \mathcal{F}_t\right] \\
&= \frac{1}{D(t_0,t_1)} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{B_{t_1}}\right| \mathcal{F}_t\right] B_t - \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{B_{t_1}}\right| \mathcal{F}_t\right] B_t \\
&= \frac{P(t,t_1)}{D(t_0,t_1)} - P(t,t_1)
\end{aligned}
$$

On the other hand, for the payment at time t_i (for $i = 2, \ldots, N$), its present value is

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{\frac{1}{D(t_{i-1},t_i)} - 1}{B_{t_i}} B_t \right| \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{D(t_{i-1},t_i)} \frac{1}{B_{t_i}}\right| \mathcal{F}_t\right] B_t - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\right] B_t
\end{aligned}
$$

Again, since $\mathcal{F}_t \subseteq \mathcal{F}_{t_{i-1}}$, we can use Proposition (2.1), as well as the fact that $\frac{1}{D(t_{i-1},t_i)}$ is adapted to the filtration $\mathcal{F}_{t_{i-1}}$, to get that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{D(t_{i-1},t_i)} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{B_{t_i}}\right| \mathcal{F}_{t_{i-1}}\right] \right| \mathcal{F}_t\right] B_t - P(t,t_i) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{D(t_{i-1},t_i)} \frac{P(t_{i-1},t_i)}{B_{t_{i-1}}}\right| \mathcal{F}_t\right] B_t - P(t,t_i)
\end{aligned}
$$

which concludes the second part of this proof. Finally, the time$-t$ value of the defaultable floating leg will be the sum of the value of all these payments:

$$
\begin{aligned}
FL_t^{Default} &= \frac{P(t,t_1)}{D(t_0,t_1)} - P(t,t_1) + \sum_{i=2}^{N} \left(\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{D(t_{i-1},t_i)} \frac{P(t_{i-1},t_i)}{B_{t_{i-1}}}\right| \mathcal{F}_t\right] B_t - P(t,t_i) \right) \\
&= \frac{P(t,t_1)}{D(t_0,t_1)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{D(t_{i-1},t_i)} \frac{P(t_{i-1},t_i)}{B_{t_{i-1}}}\right| \mathcal{F}_t\right] B_t - P(t,t_1) - \sum_{i=2}^{N} P(t,t_i) \\
&= \frac{P(t,t_1)}{D(t_0,t_1)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)} \frac{1}{B_{t_{i-1}}}\right| \mathcal{F}_t\right] B_t - \sum_{i=1}^{N} P(t,t_i)
\end{aligned}
$$

$\square$

Notice that if we consider $D(t,t_i) \approx P(t,t_i)$, we can achieve an approximation for the last theorem,

$$
\frac{P(t,t_1)}{D(t_0,t_1)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)} \frac{1}{B_{t_{i-1}}}\right| \mathcal{F}_t\right] B_t - \sum_{i=1}^{N} P(t,t_i) \approx \frac{P(t,t_1)}{D(t_0,t_1)} - P(t,t_N)
$$

Having the equation for the floating rate, we can easily derive again the swap rate under these conditions, considering the net value at time $t_0 < t < t_i$, Corollary 3.2. *(Swap rate on a credit risky reference rate) The swap rate at time $t_0 < t \leq t_1$ of a plain-vanilla IRS on a credit risky reference rate is*

$$x(t, t_N) = \frac{\frac{P(t,t_N)}{D(t_0,t_N)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)} \frac{1}{B_{t_{i-1}}} \middle| \mathcal{F}_t\right] B_t - \sum_{i=1}^{N} P(t, t_i)}{\delta \sum_{i=1}^{N} P(t, t_i)} \tag{3.10}$$

Considering the same approximation used earlier, $D(t, t_i) \approx P(t, t_i)$, we can approximate the swap rate,

$$x(t, t_N) \approx \frac{\frac{P(t,t_1)}{D(t_0,t_1)} - P(t, t_N)}{\delta \sum_{i=1}^{N} P(t, t_i)}$$

3.2. Interest rate swaps based on overnight rates

3.2.1. Floating rate loans based on overnight rates

Like previously done for LIBOR rates, it is crucial to know how to price FRL based on overnight rates.

Once again, consider some maturity date t_N, as well as a set of times $t_1, t_2, \ldots, t_N$ where the FRL payments occur, and these payments include coupons that are equal to some market index. Assuming that the FRL is default-free, the payment at time t_i for $i = 1, \ldots, N$ is

$$\delta SOFR(t_{i-1}, t_i) = \prod_{k=1}^{M_i} \frac{1}{P\left(t_{i-1+\frac{k-1}{N}}, t_{i-1+\frac{k}{N}}\right)} - 1 \tag{3.11}$$

where $k = 1, \ldots, M_i$, M_i are the number of days in the $i-$th coupon period, $SOFR(t_{i-1}, t_i)$ is the overnight variable rate, SOFR is the market index, and δ is the coupon period duration (in years), $\delta = t_i - t_{i-1}$. We will also assume that Assumption 1.1 is valid for overnight rates as well. Calculating the arbitrage-free value of a floating rate loan, but now under an overnight rate,

Theorem 3.6. *(Default-free floating rate loans) The time$-t$ value of a default-free floating rate loan based on an overnight rate $\left(t_0 \leq t < t_{\frac{1}{M_1}}\right)$ is*

$$\frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \tag{3.12}$$

Proof. Consider some arbitrary t such that $t_0 \leq t < t_{\frac{1}{M_i}}$ and for the purpose of further simplification, consider that

$$Z_{t_i} = \prod_{k=1}^{M_i} W_k(t_i), \tag{3.13}$$

where $W_k(t_i) = \frac{1}{P\left(t_{i-1+\frac{k-1}{M_i}}, t_{i-1+\frac{k}{M_i}}\right)}$. Therefore, the time$-t$ value of the FRL is:

$$
\begin{aligned}
FRL_t &= \sum_{i=1}^{N-1} \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}} Z_{t_i} \middle| \mathcal{F}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_N}}\left(1 + Z_{t_N}\right) \middle| \mathcal{F}_t\right] \\
&= P(t, t_N) + \sum_{i=1}^{N} \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}} Z_{t_i} \middle| \mathcal{F}_t\right]
\end{aligned} \tag{3.14}
$$

From here, it will be useful to know how to evaluate Z_{t_i} for $i = 1, \ldots, N$. For evaluating Z_{t_i}, consider dividing the proof into two parts once again: first, evaluate the expression at t_1; second, consider some t_i such that $i = 2, \ldots, N$. Firstly,

$$
Z_{t_1} = \prod_{k=1}^{M_1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)} - 1,
$$

which means that the time$-t$ value of this payment is:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_1}} Z_{t_1} \middle| \mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_1}} \frac{\prod_{k=2}^{M_1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)}}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \middle| \mathcal{F}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_1}} \middle| \mathcal{F}_t\right] \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)} \frac{1}{B_{t_1}} \middle| \mathcal{F}_t\right] B_t - P(t, t_1) \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_1}} \middle| \mathcal{F}_{t_{\frac{M_1-1}{M_1}}}\right] \middle| \mathcal{F}_t\right] B_t - P(t, t_1) \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)} \frac{P\left(t_{\frac{M_1-1}{M_1}}, t_1\right)}{B_{t_{\frac{M_1-1}{M_1}}}} \middle| \mathcal{F}_t\right] B_t - P(t, t_1) \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P\left(t_{\frac{M_1-1}{M_1}}, t_1\right)} \prod_{k=2}^{M_1-1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)} \frac{P\left(t_{\frac{M_1-1}{M_1}}, t_1\right)}{B_{t_{\frac{M_1-1}{M_1}}}} \middle| \mathcal{F}_t\right] \times \\
&\quad \times B_t - P(t, t_1) \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)} \mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1} \frac{1}{P\left(t_{\frac{k-1}{M_1}}, t_{\frac{k}{M_1}}\right)} \frac{1}{B_{t_{\frac{M_1-1}{M_1}}}} \middle| \mathcal{F}_t\right] B_t - P(t, t_1)
\end{aligned}
$$

With the aid of Proposition (2.1), we can apply the same reasoning repeatedly to get that,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_1}}Z_{t_1}\middle|\mathcal{F}_t\right] &= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P\left(t_{\frac{1}{M_1}}, t_{\frac{2}{M_1}}\right)}\frac{1}{B_{t_{\frac{2}{M_1}}}}\middle|\mathcal{F}_t\right]B_t - P(t,t_1) \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P\left(t_{\frac{1}{M_1}}, t_{\frac{2}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_{\frac{2}{M_1}}}}\middle|\mathcal{F}_{t_{\frac{1}{M_1}}}\right]\middle|\mathcal{F}_t\right]B_t - P(t,t_1) \\
&= \frac{1}{P\left(t_0, t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P\left(t_{\frac{1}{M_1}}, t_{\frac{2}{M_1}}\right)}\frac{P\left(t_{\frac{1}{M_1}}, t_{\frac{2}{M_1}}\right)}{B_{t_{\frac{1}{M_1}}}}\middle|\mathcal{F}_t\right]B_t - P(t,t_1) \\
&= \frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t,t_1)
\end{aligned} \tag{3.15}$$

For the second part of this proof, for $i > 1$, we know that the payment at time t_i is

$$Z_{t_i} = \prod_{k=1}^{M_i} W_k(t_i) - 1 = \prod_{k=1}^{M_i}\frac{1}{P\left(t_{i-1+\frac{k-1}{N}}, t_{i-1+\frac{k}{N}}\right)} - 1,$$

whose time$-t$ value is:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}Z_{t_i}\middle|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}\prod_{k=1}^{M_i}W_k(t_i)\middle|\mathcal{F}_t\right] - \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\prod_{k=1}^{M_i}W_k(t_i)\right)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\middle|\mathcal{F}_{t_{i-1+\frac{M_i-1}{M_i}}}\right]\middle|\mathcal{F}_t\right]B_t - P(t,t_i) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\prod_{k=1}^{M_i}W_k(t_i)\right)\frac{P\left(t_{i-1+\frac{M_i-1}{M_i}}, t_i\right)}{B_{t_{i-1+\frac{M_i-1}{M_i}}}}\middle|\mathcal{F}_t\right]B_t - P(t,t_i) \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\prod_{k=1}^{M_i-1}W_k(t_i)\right)\frac{1}{B_{t_{i-1+\frac{M_i-1}{M_i}}}}\middle|\mathcal{F}_t\right]B_t - P(t,t_i)
\end{aligned}$$

Following the same line of reasoning as we did for the payment at time t_1 and using Proposition (2.1), we can derive that:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}Z_{t_i}\middle|\mathcal{F}_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t - P(t,t_i) \\
&= P(t,t_{i-1}) - P(t,t_i)
\end{aligned} \tag{3.16}$$

which is exactly the same result derived in the proof of Theorem 1.4. All in all, joining all the payments given by the equations (1.14), (1.15) and (1.16) we have that:

$$
\begin{aligned}
FRL_t &= P(t,t_N) + \frac{P\left(t,t_{\frac{1}{M_1}}\right)}{P\left(t_0,t_{\frac{1}{M_1}}\right)} - P(t,t_1) + \sum_{i=2}^{N}[P(t,t_{i-1}) - P(t,t_i)] \\
&= \frac{P\left(t,t_{\frac{1}{M_1}}\right)}{P\left(t_0,t_{\frac{1}{M_1}}\right)}
\end{aligned}
$$

□

Again, like mentioned before, in most of the cases, there is credit risk associated to a FRL. The payment at time t_i, in case default does not occur before time t_{i-1}, is

$$
\begin{aligned}
&\mathbb{I}_{\tau>t_{i-1}}\left[(\delta SOFR(t_{i-1},t_i))\mathbb{I}_{\tau>t_i} + R_\tau(1+\delta SOFR(t_{i-1},t_i))\mathbb{I}_{\tau\le t_i}\right] \\
&= \mathbb{I}_{\tau>t_{i-1}}\left[\left(\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)} - 1\right)\mathbb{I}_{\tau>t_i}\right. \\
&\left.+ R_\tau\left(\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\mathbb{I}_{\tau\le t_i}\right]
\end{aligned}
\tag{3.17}
$$

for $i = 1,\ldots,N$, where R_τ is the recovery rate, and τ is the time when default occurs.

The fair value of this floating rate loan is given by the following theorem:

Theorem 3.7. *(Defaultable floating rate loan) The fair value of a defaultable floating rate loan at time t $\left(t_0 \le t < t_{\frac{1}{M_1}}\right)$ is*

$$
\frac{P\left(t,t_{\frac{1}{M_1}}\right)}{D\left(t_0,t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\beta_k(t_1)\middle|\mathcal{F}_t\right] + \sum_{i=2}^{N}\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_{i-1}}}\prod_{k=1}^{M_i-1}\beta_k(t_i)\middle|\mathcal{F}_t\right]B_t - X_t^{t_i}\right) + X_t^{t_N} - X_t^{t_1},
\tag{3.18}
$$

where $D(t_{i-1},t_i)$ is a credit risky zero coupon bond according to its FRL rating, and $\beta_k(t_i) = \frac{P\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}$.

Proof. Consider some arbitrary t such that $t_0 \le t < t_{\frac{1}{M_1}}$ and consider the same process Z_{t_i} consider in the proof for Theorem 1.6, but now for a credit risky zero coupon bond. Like before, we will divide the proof into three parts: 1) payment at t_1, 2) payment for $t_1 < t \le t_N$, and 3) the present value of the notional. For 1), at t_1, the FRL payment is

$$\left[\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}-1\right]\mathbb{I}_{\tau>t_1}+R_\tau\left[\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\right]\mathbb{I}_{\tau\leq t_1}$$

$$=\left[\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\right]\left(\mathbb{I}_{\tau>t_1}+R_\tau\mathbb{I}_{\tau\leq t_1}\right)-\mathbb{I}_{\tau>t_1}$$

whose time$-t$ value will be:

$$\begin{aligned}
Y_t^{t_1}&=\mathbb{E}^{\mathbb{Q}}\left[\left(\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\left(\mathbb{I}_{\tau>t_1}+R_\tau\mathbb{I}_{\tau\leq t_1}\right)-\mathbb{I}_{\tau>t_1}\right)\frac{B_t}{B_{t_1}}\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\frac{\mathbb{I}_{\tau>t_1}+R_\tau\mathbb{I}_{\tau\leq t_1}}{B_{t_1}}\middle|\mathcal{F}_t\right]B_t-\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_1}}{B_{t_1}}\middle|\mathcal{F}_t\right]B_t\\
&=\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_1}+R_\tau\mathbb{I}_{\tau\leq t_1}}{B_{t_1}}\middle|\mathcal{F}_{t_{\frac{M_1-1}{M_1}}}\right]\middle|\mathcal{F}_t\right]B_t-X_t^{t_1}\\
&=\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=1}^{M_1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\frac{D\left(t_{\frac{M_1-1}{M_1}},t_1\right)}{B_{t_{\frac{M_1-1}{M_1}}}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_1}\\
&=\frac{1}{D\left(t_0,t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\frac{1}{B_{t_{\frac{M_1-1}{M_1}}}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_1}\\
&=\frac{1}{D\left(t_0,t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_{\frac{M_1-1}{M_1}}}}\middle|\mathcal{F}_{t_{\frac{M_1-2}{M_1}}}\right]\middle|\mathcal{F}_t\right]B_t-X_t^{t_1}\\
&=\frac{1}{D\left(t_0,t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\frac{1}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\frac{P\left(t_{\frac{M_1-2}{M_1}},t_{\frac{M_1-1}{M_1}}\right)}{B_{t_{\frac{M_1-2}{M_1}}}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_1}
\end{aligned}$$

Following Proposition (2.1) and the same line of reasoning until t_0, we get that the time$-t$ value of the FRL payment at t_1 is:

$$Y_t^{t_1}=\frac{P\left(t,t_{\frac{1}{M_1}}\right)}{D\left(t_0,t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\frac{P\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}{D\left(t_{\frac{k-1}{M_1}},t_{\frac{k}{M_1}}\right)}\middle|\mathcal{F}_t\right]-X_t^{t_1}\tag{3.19}$$

which concludes the first part of the proof. For the second part, for the payments in t_i for $i=2,\dots,N$, we will use a simplified version of equation (1.15):

$$\mathbb{I}_{\tau>t_{i-1}}\left[\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right](\mathbb{I}_{\tau>t_i}+R_\tau\mathbb{I}_{\tau\le t_i})-\mathbb{I}_{\tau>t_i}$$

Calculating the time$-t$ value of this payment:

$$\begin{aligned}
Y_t^{t_i}&=\mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{I}_{\tau>t_{i-1}}\left[\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right](\mathbb{I}_{\tau>t_i}+R_\tau\mathbb{I}_{\tau\le t_i})-\mathbb{I}_{\tau>t_i}\right)\frac{B_t}{B_{t_i}}\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\tau>t_{i-1}}\left(\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\frac{\mathbb{I}_{\tau>t_i}+R_\tau\mathbb{I}_{\tau\le t_i}}{B_{t_i}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_i}\\
&=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\tau>t_{i-1}}\left(\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_i}+R_\tau\mathbb{I}_{\tau\le t_i}}{B_{t_i}}\middle|\mathcal{F}_{t_{i-1+\frac{M_i-1}{M_i}}}\right]\middle|\mathcal{F}_t\right]B_t\\
&-X_t^{t_i}\\
&=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\tau>t_{i-1}}\left(\prod_{k=1}^{M_i}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\frac{D\left(t_{i-1+\frac{M_i-1}{M_i}},t_i\right)}{B_{t_{i-1+\frac{M_i-1}{M_i}}}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_i}\\
&=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\tau>t_{i-1}}\left(\prod_{k=1}^{M_i-1}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\frac{1}{B_{t_{i-1+\frac{M_i-1}{M_i}}}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_i}\\
&=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\tau>t_{i-1}}\left(\prod_{k=1}^{M_i-1}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_{i-1+\frac{M_i-1}{M_i}}}}\middle|\mathcal{F}_{t_{i-1+\frac{M_i-2}{M_i}}}\right]\middle|\mathcal{F}_t\right]B_t\\
&-X_t^{t_i}\\
&=\mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\tau>t_{i-1}}\left(\prod_{k=1}^{M_i-1}\frac{1}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\right)\frac{P\left(t_{i-1+\frac{M_i-2}{M_i}},t_{i-1+\frac{M_i-1}{M_i}}\right)}{B_{t_{i-1+\frac{M_i-2}{M_i}}}}\middle|\mathcal{F}_t\right]B_t-X_t^{t_i}
\end{aligned}$$

Applying Proposition (2.1) and following the same reasoning, we get that:

$$Y_t^{t_i}=\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_{i-1}}}\prod_{k=1}^{M_i-1}\frac{P\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}{D\left(t_{i-1+\frac{k-1}{M_i}},t_{i-1+\frac{k}{M_i}}\right)}\middle|\mathcal{F}_t\right]B_t-X_t^{t_i}\tag{3.20}$$

which concludes the second part. For the third and final part of this proof, we will evaluate the notional on the payment at t_N, which we already know that it is, $\mathbb{I}_{\tau>t_N}$

$$Y_t^{t_N} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_N}}{B_{t_N}}\middle|\mathcal{F}_t\right]B_t = X_t^{t_N} \tag{3.21}$$

Joining all the three parts together, i.e, equations (1.19), (1.20) and (1.21), and defining $\beta_k(t_i) = \frac{P\left(t_{i-1+\frac{k-1}{N}}, t_{i-1+\frac{k}{N}}\right)}{D\left(t_{i-1+\frac{k-1}{N}}, t_{i-1+\frac{k}{N}}\right)}$, we can calculate the time$-t$ value of a defaultable FRL:

$$\begin{aligned}
FRL_t &= \frac{P\left(t, t_{\frac{1}{M_1}}\right)}{D\left(t_0, t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\beta_k(t_1)\middle|\mathcal{F}_t\right] \\
&- X_t^{t_1} + \sum_{i=2}^{N}\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_{i-1}}}\prod_{k=1}^{M_i-1}\beta_k(t_i)\middle|\mathcal{F}_t\right]B_t - X_t^{t_i}\right) + X_t^{t_N} \\
&= \frac{P\left(t, t_{\frac{1}{M_1}}\right)}{D\left(t_0, t_{\frac{1}{M_1}}\right)}\mathbb{E}^{\mathbb{Q}}\left[\prod_{k=2}^{M_1-1}\beta_k(t_1)\middle|\mathcal{F}_t\right] \\
&+ \sum_{i=2}^{N}\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{\mathbb{I}_{\tau>t_{i-1}}}{B_{t_{i-1}}}\prod_{k=1}^{M_i-1}\beta_k(t_i)\middle|\mathcal{F}_t\right]B_t - X_t^{t_i}\right) + X_t^{t_N} - X_t^{t_1}
\end{aligned}$$

$\square$

In practice, we can despise the embedded credit risk on the defaultable zero coupon bond, as the SOFR works almost as a risk-free rate, and the maturities on the defaultable zero coupon bonds are too small to reflect such credit risk. Furthermore, if we define the money-market account B_t with the same risk level as the overnight rate, the embedded zero coupon bonds will also be on the same level of risk. In fact, if we take consider $P(t_{i-1}, t_i) = D(t_{i-1}, t_i)$ in this last theorem, we get the result from Theorem 1.6. Therefore, from here on, credit risk will be ignored, and we will consider that the overnight rate is only based on a default-free zero coupon bond, as this will have little effect on our results.

3.2.2. Interest rate swaps on a default-free overnight reference rate

Like in Section 1.2.2, consider the same assumptions as done before. Additionally, considering N as the number of resettlement periods of the IRS contract, and M_i as the number of compounded days on the overnight rate during the i$-th$ coupon payment.

DEFINITION 3.2. *The payments in a plain-vanilla IRS are as follows:*

(1) Payments will be made and received at times $t_i = t_0 + i\delta$, for $i = 1, \dots, N$.

(2) For $i = 1, \dots, N$, at every period $[t_{i-1}, t_i]$, the floating leg,

$$\delta SOFR(t_{i-1}, t_i)$$

is paid at time t_i.

(3) For $i = 1, \dots, N$, at every period $[t_{i-1}, t_i]$, the fixed leg payment

$$\delta x(t, t_N)$$

is paid at time t_i, where $x(t, t_N)$ is the swap rate and t is such that $t_0 \le t < t_{\frac{1}{M_1}}$.

The same reasoning used in Section 1.2.2 for evaluating IRS under default-free reference rates will be used here as well. Furthermore, as the valuation of the fixed leg will not depend on the overnight rate, we can apply Theorem 1.3 to the IRS defined in Definition 1.2.

THEOREM 3.8. *(Fixed leg of an IRS on a default-free overnight reference rate) The time$-t$ value of the fixed leg of a plain-vanilla IRS with $t_0 \le t < t_{\frac{1}{M_1}}$ on a default-free overnight reference rate is*

$$\delta x(t, t_N) \sum_{i=1}^{N} P(t, t_i) \tag{3.22}$$

PROOF. Consider some time t such that $t_0 \le t < t_{\frac{1}{M_i}}$. The payment of the fixed leg at any given time t_i, where $i = 1, \dots, N$, is just

$$\delta x(t, t_N),$$

and its time$-t$ value is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{\delta x(t, t_N)}{B_{t_i}}\middle|\mathcal{F}_t\right] B_t = \mathbb{E}^{\mathbb{Q}_{t_i}}\left[\frac{\delta x(t, t_N)}{P(t_i, t_i)}\middle|\mathcal{F}_t\right] P(t, t_i) = \delta x(t, t_N) P(t, t_i)$$

Therefore, the time$-t$ value of the fixed leg is the sum of the fair value of all these payments:

$$\sum_{i=1}^{N} \delta x(t, t_N) P(t, t_i) = \delta x(t, t_N) \sum_{i=1}^{N} P(t, t_i)$$

□

Now considering the floating leg:

THEOREM 3.9. *(Floating leg of an IRS on a default-free overnight reference rate) The time$-t$ value of the floating leg of a plain-vanilla IRS with $t_0 \le t < t_{\frac{1}{M_1}}$ on a default-free overnight reference rate is*

$$\frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t, t_N) \tag{3.23}$$

PROOF. We will divide the proof into two parts: 1) the IRS floating payment at t_1, 2) the rest of the IRS floating payments for t_i, where $i = 2, \dots, N$. For both these proofs, we already know the time$-t$ values of these payments, as these were derived in the proof of Theorem 1.6. Therefore, we can calculate the time$-t$ value of this floating leg:

$$
\begin{aligned}
Float_t &= \frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t, t_1) + \sum_{i=2}^{N} [P(t, t_{i-1}) - P(t, t_i)] \\
&= \frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t, t_N)
\end{aligned}
$$

□

Therefore, like was done before, we can calculate the time$-t$ value of a plain-vanilla IRS:

$$
IRS_t = \frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t, t_N) - \delta x(t, t_N) \sum_{i=1}^{N} P(t, t_i) \tag{3.24}
$$

Having this, we can also derive the fixed swap rate for a plain-vanilla IRS, but now under an overnight rate.

Corollary 3.3. *(Interest rate swap rate) The time$-t$ swap rate, with $t_0 \leq t < t_{\frac{1}{M_1}}$, of a plain-vanilla IRS on a default-free overnight reference rate is*

$$
x(t, t_N) = \frac{\frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t, t_N)}{\delta \sum_{i=1}^{N} P(t, t_i)} \tag{3.25}
$$

3.3. Switching interest rate swaps based on LIBOR rates to overnight rates

3.3.1. Interest rate swaps based on a default-free reference rate

One final case of interest is to consider some current IRS whose floating leg payments are still under some LIBOR rate and, at some point in the future, will switch to an overnight rate. Again, consider a set of resettlement dates $t_0, t_1, \ldots, t_N$, a notional of 1, and consider $\delta = t_i - t_{i-1}$, for $i = 1, \ldots, N$, such that δ is constant. Additionally, consider a specific resettlement date where the IRS will make the switch from a LIBOR rate to an overnight rate. We will denote this resettlement date as t_S.

One particularity of these type of IRS is that after the switch to overnight rates is done, it is important to not alter the arbitrage dynamics of the IRS. To this end, according to the International Swaps and Derivatives Association (ISDA), we need to consider a spread adjustment in the floating leg's payments, s.

To calculate the time$-t$ value of this spread adjustment, s_t, we need to consider that it will have to be such that the time$-t$ value of an IRS under some LIBOR rate is the same as the time$-t$ value of an IRS under some overnight rate plus s_t. In practice, the most widely suggested approach for calculating s_t is, according to ISDA, the five year median for the LIBOR-SOFR spread. Utilizing this method has the advantage of avoiding market fluctuations. On the other hand, depending on the market conditions at the time (t) of determining the spread, the median could be lower or higher and also be subject to higher variance during times of economic stress. There is also the possibility of having a dynamic spread adjustment and that is one of the results aimed to be answered. This method is more economically fair as it will track market reality. However, it can be difficult to manage, while also requiring other decisions on the side such as frequency change (how much we want to keep track of the spread adjustment, i.e, second by second, minute by minute, or hourly) and calculation methodology (as we will see later on, the equation for the spread adjustment is dependent on the value of zero coupon bonds, which will make every spread adjustment value different from model to model).

Definition 3.3. *The payments in a plain-vanilla IRS switching to an overnight rate are as follows:*

(1) Payments will be made and received at times $t_i = t_0 + i\delta$, for $i = 1, \ldots, N$.

(2) For $i = 1, \ldots, S-1$, at every period $[t_{i-1}, t_i]$, the LIBOR rate $L(t_{i-1}, t_i)$ is set at t_{i-1} and the floating leg,

$$\delta L(t_{i-1}, t_i)$$

is paid at time t_i.

(3) For $j = S, \ldots, N$, at every period $[t_{j-1}, t_j]$, the floating leg,

$$\delta[SOFR(t_{j-1}, t_j) + s_t]$$

is paid at t_j, where s_t is the spread adjustment for the switch between rates.

(4) For $i = 1, \ldots, N$, at every period $[t_{i-1}, t_i]$, the fixed leg payment

$$\delta x(t, t_N)$$

is paid at time t_i, where $x(t, t_N)$ is the swap rate and t is such that $t_0 \leq t < t_{\frac{1}{M_1}} < t_1$.

Given the definition of an IRS switching from a LIBOR reference rate to a SOFR rate, it is possible to derive an equation for the spread adjustment, s_t. If we consider the IRS in the future at time t_S, we will just get an IRS that pays SOFR rate plus the spread adjusment until its maturity. We can compare the fair value of this IRS, IRS_1, with another IRS, IRS_2, that would pay LIBOR until the end of its maturity. Since the fixed leg of each IRS is equal, we will only need to compare their floating legs. Therefore, we want to have that:

$$Floating_t(IRS_1) = Floating_t(IRS_2) \tag{3.26}$$

For IRS_2, we already know the time$-t$ value of its floating leg:

$$\frac{P(t, t_1)}{P(t_0, t_1)} - P(t, t_N) \tag{3.27}$$

And we already know the time$-t$ value of IRS_1 floating leg if consider the time$-t$ floating of the same IRS but without the spread adjustment addition (Theorem 1.9) 3.27. If also we add the time$-t$ value of the spread adjustment which is, for $i = 1, \ldots, N$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\delta s_t \frac{B_t}{B_{t_i}}\middle|\mathcal{F}_t\right] &= \delta s_t \mathbb{E}^{\mathbb{Q}}\left[\frac{B_t}{B_{t_i}}\middle|\mathcal{F}_t\right] \\ &= \delta s_t P(t, t_i), \end{aligned} \tag{3.28}$$

then the time$-t$ value of IRS_1 floating leg is:

$$\frac{P\left(t, t_{\frac{1}{M_1}}\right)}{P\left(t_0, t_{\frac{1}{M_1}}\right)} - P(t, t_N) + \delta s_t \sum_{i=1}^{N} P(t, t_i) \tag{3.29}$$

Before using the floating legs to solve equation 3.26 for s_t, it is important to note that the zero coupon bonds, P, in each floating leg are different due to them being discounted in different ways. The discounting can be different because of factors such as, for example, liquidity and different underlying interest rates. Therefore, let us define $P^L(t_{i-1}, t_i)$ as a zero coupon bond linked to the discounting in a LIBOR reference rate, and $P^O(t_{i-1}, t_i)$ as a zero coupon bond linked to the discounting in an overnight rate. Therefore, using the floating legs 3.29 and 3.27 and equation 3.26 imply that:

$$\frac{P^O\left(t, t_{\frac{1}{M_1}}\right)}{P^O\left(t_0, t_{\frac{1}{M_1}}\right)} - P^O(t, t_N) + \delta s_t \sum_{i=1}^{N} P^O(t, t_i) = \frac{P^L(t, t_1)}{P^L(t_0, t_1)} - P^L(t, t_N)$$

which yields the following solution:

$$s_t = \frac{\frac{P^L(t,t_1)}{P^L(t_0,t_1)} - P^L(t, t_N) - \frac{P^O\left(t, t_{\frac{1}{M_1}}\right)}{P^O\left(t_0, t_{\frac{1}{M_1}}\right)} + P^O(t, t_N)}{\delta \sum\limits_{i=1}^{N} P^O(t, t_i)}, \tag{3.30}$$

where $t_0 \leq t < t_{\frac{1}{M_1}}$. Notice that for $t = t_0$, we get that

$$s_{t_0} = \frac{P^O(t_0, t_N) - P^L(t_0, t_N)}{\delta \sum\limits_{i=1}^{N} P^O(t_0, t_i)}$$

by a simple substitution.

Again, the fixed leg will not change as it does not depend on the underlying rate. Therefore, we will refer to Theorem 1.8 to valuate the fixed leg. For the floating leg:

Theorem 3.10. *(Floating leg of an IRS on a default-free reference rate switching to an overnight rate) The time$-t$ value of the floating leg of a plain-vanilla IRS with $t_0 \leq t < t_{\frac{1}{M_1}} < t_1 < t_S < t_N$ on a default-free reference rate switching to an overnight* *rate is*

$$\frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N) + \delta s_t \sum_{i=S}^{N} P(t,t_i) \tag{3.31}$$

Proof. Start by considering some time$-t$ such that $t_0 \leq t < t_{\frac{1}{M_1}} < t_1$. We will divide the proof into three parts: 1) the first resettlement date, t_1; 2) the remaining LIBOR payments; 3) the overnight payments. For the payment at t_1, we have that,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_1}}\delta L(t_0,t_1)\middle|\mathcal{F}_t\right] B_t = \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_1), \tag{3.32}$$

which we already proved before in the proof of Theorem 1.4. Furthermore, from the proof of the same Theorem 1.4, we already know that, for $i > 1$:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\delta L(t_{i-1},t_i)\middle|\mathcal{F}_t\right] B_t = P(t,t_{i-1}) - P(t,t_i) \tag{3.33}$$

For the overnight payments and using equation (1.16),

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\delta[SOFR(t_{i-1},t_i)+s_t]\middle|\mathcal{F}_t\right] B_t \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\delta SOFR(t_{i-1},t_i)\middle|\mathcal{F}_t\right] B_t + \delta\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}s_t\middle|\mathcal{F}_t\right] B_t \\
&= P(t,t_{i-1}) - P(t,t_i) + \delta s_t P(t,t_i)
\end{aligned} \tag{3.34}$$

Therefore, we can join these last equations to derive the floating leg of this IRS:

$$\begin{aligned}
Float_t &= \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_1) + \sum_{i=2}^{S-1}[P(t,t_{i-1}) - P(t,t_i)] \\
&+ \sum_{i=S}^{N}[P(t,t_{i-1}) - P(t,t_i) + \delta s_t \cdot P(t,t_i)] \\
&= \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_1) + \sum_{i=2}^{S-1}[P(t,t_{i-1}) - P(t,t_i)] \\
&+ \sum_{i=S}^{N}[P(t,t_{i-1}) - P(t,t_i)] + \delta s_t \sum_{i=S}^{N} P(t,t_i) \\
&= \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_1) + \sum_{i=2}^{N}[P(t,t_{i-1}) - P(t,t_i)] + \delta s_t \sum_{i=S}^{N} P(t,t_i) \\
&= \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N) + \delta s_t \sum_{i=S}^{N} P(t,t_i)
\end{aligned} \tag{3.35}$$

□

Therefore, the time$-t$ value of this plain-vanilla IRS is

$$IRS_t = \frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N) + \delta s_t \sum_{i=S}^{N} P(t,t_i) - \delta x(t,t_N) \sum_{i=1}^{N} P(t,t_i), \tag{3.36}$$

and we can also calculate the spot swap rate of this IRS, by solving the equation $IRS_t = 0$.

Corollary 3.4. *(Interest rate swap rate) The time$-t$ swap rate, with $t_0 \le t < t_{\frac{1}{M_1}} < t_1 < t_S < t_N$, of a plain-vanilla IRS on a default-free reference rate switching to an overnight rate is*

$$x(t,t_N) = \frac{\frac{P(t,t_1)}{P(t_0,t_1)} - P(t,t_N) + \delta s_t \sum_{i=S}^{N} P(t,t_i)}{\delta \sum_{i=1}^{N} P(t,t_i)} \tag{3.37}$$

3.3.2. Interest rate swaps based on a credit risky reference rate

As done before for the other swaps, it is important to also include the case where the reference rate has some type of credit risk. Again, we can reuse Theorem 1.10 for an IRS under these conditions as the fixed leg, as said before, is not dependent on the underlying reference rate.

Following the same reasoning as in Section 3.4.1, and using Theorems 3.5 and 3.6, it is possible to calculate the solution for the spread adjustment under these conditions:

$$\begin{aligned}
&\frac{P^L(t,t_1)}{D^L(t_0,t_1)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}} \left[\frac{P^L(t_{i-1},t_i)}{D^L(t_{i-1},t_i)} \frac{1}{B_{t_{i-1}}} \middle| \mathcal{F}_t \right] B_t - \sum_{i=1}^{N} P^L(t,t_i) \\
&= \frac{P^O\left(t, t_{\frac{1}{M_1}}\right)}{P^O\left(t_0, t_{\frac{1}{M_1}}\right)} - P^O(t,t_N) + \delta s_t \sum_{i=1}^{N} P^O(t,t_i)
\end{aligned}$$

Solving for s_t:

$$s_t = \frac{\frac{P^L(t,t_1)}{D^L(t_0,t_1)} + \sum_{i=2}^{N} \mathbb{E}^{\mathbb{Q}} \left[\frac{P^L(t_{i-1},t_i)}{D^L(t_{i-1},t_i)} \frac{1}{B_{t_{i-1}}} \middle| \mathcal{F}_t \right] B_t - \sum_{i=1}^{N} P^L(t,t_i) - \frac{P^O\left(t,t_{\frac{1}{M_1}}\right)}{P^O\left(t_0,t_{\frac{1}{M_1}}\right)} + P^O(t,t_N)}{\delta \sum_{i=1}^{N} P^O(t,t_i)} \tag{3.38}$$

Concerning the time$-t$ value of the floating leg under these conditions:

Theorem 3.11. *(Floating leg of an IRS on a credit risky reference rate switching to an overnight rate) The time$-t$ value of the floating leg of a plain-vanilla IRS with $t_0 \le t < t_{\frac{1}{M_1}} < t_1 < t_S < t_N$ on a credit risky reference rate switching to an overnight rate is*

$$\frac{P(t,t_1)}{D(t_0,t_1)} + \sum_{i=2}^{S-1} \mathbb{E}^{\mathbb{Q}} \left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)} \frac{1}{B_{t_{i-1}}} \middle| \mathcal{F}_t \right] B_t - \sum_{i=1}^{S-2} P(t,t_i) + \delta s_t \sum_{j=S}^{N} P(t,t_j) - P(t,t_N) \tag{3.39}$$

PROOF. The time$-t$ value of floating leg, as seen before, will be the sum of all the time$-t$ values of each payment at each resettlement date. Therefore, using the equations derived in the proof of Theorem 3.5 for the time$-t$ value of each payment under some defaultable zero coupon bond:

$$
\begin{aligned}
Float_t &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\frac{1}{D(t_0,t_1)}-1}{B_{t_1}}B_t\middle|\mathcal{F}_t\right]+\sum_{i=2}^{S-1}\left(\mathbb{E}^{\mathbb{Q}}\left[\frac{\frac{1}{D(t_{i-1},t_i)}-1}{B_{t_i}}B_t\middle|\mathcal{F}_t\right]\right)\\
&+\sum_{j=S}^{N}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_{t_i}}\delta[SOFR(t_{i-1},t_i)+s_t]\middle|\mathcal{F}_t\right]B_t\\
&=\frac{P(t,t_1)}{D(t_0,t_1)}-P(t,t_1)+\sum_{i=2}^{S-1}\mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t-\sum_{i=2}^{S-1}P(t,t_i)\\
&+\sum_{i=S}^{N}[P(t,t_{i-1})-P(t,t_i)+\delta s_t\cdot P(t,t_i)]\\
&=\frac{P(t,t_1)}{D(t_0,t_1)}-P(t,t_1)+\sum_{i=2}^{S-1}\mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t\\
&-\left[\sum_{i=2}^{S-1}P(t,t_i)+\sum_{i=S}^{N}P(t,t_i)\right]+\sum_{i=S}^{N}P(t,t_{i-1})+\delta s_t\sum_{i=S}^{N}P(t,t_i)\\
&=\frac{P(t,t_1)}{D(t_0,t_1)}+\sum_{i=2}^{S-1}\mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t-\sum_{i=2}^{N}P(t,t_i)\\
&+\sum_{i=S}^{N}P(t,t_{i-1})+\delta s_t\sum_{i=S}^{N}P(t,t_i)\\
&=\frac{P(t,t_1)}{D(t_0,t_1)}+\sum_{i=2}^{S-1}\mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t-\sum_{i=1}^{S-2}P(t,t_i)+\delta s_t\sum_{j=S}^{N}P(t,t_j)-P(t,t_N)
\end{aligned}
$$

□

Joining this last equation with the fixed leg, we can calculate the time$-t$ value of this IRS:

$$
\begin{aligned}
IRS_t =&\frac{P(t,t_1)}{D(t_0,t_1)}+\sum_{i=2}^{S-1}\mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right]B_t-\sum_{i=1}^{S-2}P(t,t_i)+\delta s_t\sum_{j=S}^{N}P(t,t_j)-P(t,t_N)\\
&-\delta x(t,t_N)\sum_{i=1}^{N}P(t,t_i) \qquad (3.40)
\end{aligned}
$$

Solving the equation $IRS_t = 0$ for $x(t,t_N)$, we can calculate the spot swap rate of this IRS.

COROLLARY 3.5. *(Interest rate swap rate) The time$-t$ swap rate, with $t_0 \le t < t_{\frac{1}{M_1}} < t_1 < t_S < t_N$, of a plain-vanilla IRS on a credit risky overnight reference rate switching* *to an overnight rate is*

$$x(t,t_N) = \frac{\frac{P(t,t_1)}{D(t_0,t_1)} + \sum_{i=2}^{S-1} \mathbb{E}^{\mathbb{Q}}\left[\frac{P(t_{i-1},t_i)}{D(t_{i-1},t_i)}\frac{1}{B_{t_{i-1}}}\middle|\mathcal{F}_t\right] B_t - \sum_{i=1}^{S-2} P(t,t_i) + \delta s_t \sum_{j=S}^{N} P(t,t_j) - P(t,t_N)}{\delta \sum_{i=1}^{N} P(t,t_i)} \tag{3.41}$$

CHAPTER 4

Testing pricing accuracy on EUSA swaps through the zero coupon bond curve of EESWE overnight-based swaps

This section aims to test the pricing accuracy on EUSA swaps through bootstrapped zero coupon bonds on ESTR-based swaps, EESWE swaps. During this section it will be assumed that all swaps are based under default-free reference rates, the present moment of valuation is $t_0 = 0$, and that all dates are worked under the bond basis (30/360).

4.1. Valuating EUSA swaps

The Euro Swap Agreements (EUSA) are swaps that are linked to the EURIBOR 6M, on a semi-annual frequency. Until now, we have taken examples of swaps who assume that both fixed and floating legs have equal frequency regarding resettlement dates. However, the EUSA swaps, whose pricing accuracy will be tested, have different frequencies: like said above, the floating leg has semi-annual resettlement payments, whereas the fixed leg has annual resettlement payments. On a more formal definition:

Definition 4.1. *The payments in an EUSA swap are as follows:*

(1) The floating leg payments will be made and received at times $t_i = t_0 + i$, for $i = 1, \dots, N$.

(2) The fixed leg payments will be made and received at times $t_i = t_0 + 2i$, for $i = 1, \dots, \frac{N}{2}$.

(3) For $i = 1, \dots, N$, at every period $[t_{i-1}, t_i]$, the floating leg,

$$\frac{EUR_{6M}(t_{i-1}, t_i)}{2}$$

is paid at t_i, where EUR_{6M} is the six month EURIBOR index.

(4) For $i = 1, \dots, \frac{N}{2}$, at every period $[t_{2i-1}, t_{2i}]$, the fixed leg payment

$$EUSA(t, t_N)$$

is paid at t_{2i}, where $EUSA(t, t_N)$ is the EUSA swap rate and t is such that $t_0 \leq t < t_1$.

Furthermore, in this case, the floating leg has the same properties as the swaps defined in Section 3.2, so we can use Theorem 3.4 for the EUSA swaps floating leg. However, for the fixed leg:

THEOREM 4.1. *(Fixed leg of an EUSA IRS) The time−t value of the fixed leg of a plain-vanilla IRS with* $t_0 \leq t < t_1$ *on a default-free reference rate is*

$$EUSA(t, t_N) \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} P(t, t_{2i}), \tag{4.1}$$

where $\lfloor \cdot \rfloor$ *is the floor function.*

PROOF. Consider some time t such that $t_0 \leq t < t_1$. The payment of the fixed leg at any given time t_{2i}, where $i = 1, \ldots, \frac{N}{2}$, is just

$$EUSA(t, t_N),$$

and its time−t value is

$$\mathbb{E}^{\mathbb{Q}} \left[\frac{EUSA(t, t_N)}{B_{t_{2i}}} \middle| \mathcal{F}_t \right] B_t = \mathbb{E}^{\mathbb{Q}_{t_{2i}}} \left[\frac{EUSA(t, t_N)}{P(t_{2i}, t_{2i})} \middle| \mathcal{F}_t \right] P(t, t_{2i}) = EUSA(t, t_N) P(t, t_{2i})$$

Therefore, while also accounting the cases where N can be odd, the time−t value of the fixed leg is the sum of the fair value of all these payments, yielding equation (4.1). □

Therefore,

$$IRS_t = \frac{P(t, t_1)}{P(t_0, t_1)} - P(t, t_N) - EUSA(t, t_N) \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} P(t, t_{2i}), \tag{4.2}$$

and we can formulate the equation for an EUSA swap:

COROLLARY 4.1. *(EUSA swap rate) The time−t swap rate, with* $t_0 \leq t < t_1$ *and* $N \geq 2$*, of a plain-vanilla EUSA swap is*

$$EUSA(t, t_N) = \frac{\frac{P(t,t_1)}{P(t_0,t_1)} - P(t, t_N)}{\sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} P(t, t_{2i})} \tag{4.3}$$

4.2. Valuating EESWE swaps

The Euro Overnight Index Swaps (EESWE) are swaps that are linked to the ESTR index published on a daily basis by the European Central Bank (ECB), on an annual frequency. This is will the basis for our pricing. Therefore, it is also crucial to know how to value these swaps.

DEFINITION 4.2. *The payments in an EESWE swap are as follows:*

(1) The floating leg payments will be made and received at times $t_i = t_0 + i$*, for* $i = 1, \ldots, N$.

(2) The fixed leg payments will be made and received at times $t_i = t_0 + i$*, for* $i = 1, \ldots, N$.

(3) For $i = 1, \dots, N$, at every period $[t_{i-1}, t_i]$, the floating leg,

$$ESTR(t_{i-1}, t_i)$$

is paid at time t_i, where $ESTR$ is the european short-term rate index.

(4) For $i = 1, \dots, N$, at every period $[t_{i-1}, t_i]$, the fixed leg payment

$$EESWE(t, t_N)$$

is paid at time t_i, where $EESWE(t, t_N)$ is the EESWE swap rate and t is such that $t_0 \leq t < t_1$.

On contrary to EUSA swaps, EESWE swaps have the same frequency on both legs. Furthermore, it is a swap of the same category as the ones evaluated on section 1.3.2. Notice that if we consider $\delta = 1$ and $t = t_0$ for the equations on this same section, we can easily obtain the EESWE swap rate equation:

$$EESWE(t_0, t_N) = \frac{1 - P(t_0, t_N)}{\sum_{i=1}^{N} P(t_0, t_i)} \tag{4.4}$$

4.3. Testing the pricing

4.3.1. Preparations for bootstrapping the mid swap curves

Given that we will be bootstrapping the EESWE swaps, it is important to have a closed form equation for $P(t_0, t_N)$, where t_N is the EESWE swaps maturity. For a EESWE swap maturing at time t_1, we know that,

$$EESWE(t_0, t_1) = \frac{1 - P(t_0, t_1)}{P(t_0, t_1)}, \tag{4.5}$$

whose equation we can easily solve for $P(t_0, t_1)$:

$$P(t_0, t_1) = \frac{1}{1 + EESWE(t_0, t_1)} \tag{4.6}$$

For EESWE swaps maturing at t_N, for $N \geq 2$, we can start by rewriting equation (4.4):

$$EESWE(t_0, t_N) = \frac{1 - P(t_0, t_N)}{\sum_{i=1}^{N} P(t_0, t_i)} = \frac{1 - P(t_0, t_N)}{P(t_0, t_N) + \sum_{i=1}^{N-1} P(t_0, t_i)}$$

Solving for $P(t_0, t_N)$, we get that:

$$P(t_0, t_N) = \frac{1 - EESWE(t_0, t_N) \sum_{i=1}^{N-1} P(t_0, t_i)}{1 + EESWE(t_0, t_N)} \tag{4.7}$$

Moving to EUSA swaps, we can assume an EUSA swap maturing at t_1. In this swap, there will be a single floating leg payment done and no fixed leg payments. Therefore, we will not consider any swap here. For an EUSA swap maturing at t_2, now there will be two floating leg payments and one fixed leg payment. Therefore,

$$EUSA(t_0, t_2) = \frac{1 - P(t_0, t_2)}{P(t_0, t_2)} \iff P(t_0, t_2) = \frac{1}{1 + EUSA(t_0, t_2)} \tag{4.8}$$

For $N > 2$, it will be possible to consider some generic equation,

$$P(t_0, t_N) = \begin{cases} \dfrac{1 - EUSA(t_0, t_N) \sum\limits_{i=1}^{\frac{N}{2}-1} P(t_0, t_{2i})}{1 + EUSA(t_0, t_N)}, \text{ if } N \text{ is even} \\ \\ 1 - EUSA(t_0, t_N) \sum\limits_{i=1}^{\lfloor \frac{N}{2} \rfloor} P(t_0, t_{2i}), \text{ if } N \text{ is odd} \end{cases} \tag{4.9}$$

4.3.2. MATLAB implementation

There are two ways we can test the pricing accuracy: 1) we obtain the zero coupon bond curves from both EUSA and EESWE swaps; 2) we compare the EUSA original mid quotes with the ones calculated through the EESWE zero coupon bond curve.

Step 0 - Importing swap data: using Bloomberg's ICVS function, the raw money-market data in Tables 1 and 2 was exported on the day 04/11/2022.

Maturity (y)	Bloomberg Ticker	Bid	Ask	Mid
0.5	EUR006M	2.2780%	2.2780%	2.2780%
1	EUSA1	2.7860%	2.8336%	2.8098%
1.5	EUSA1F	2.9555%	3.0011%	2.9783%
2	EUSA2	3.0406%	3.0754%	3.0580%
3	EUSA3	3.0522%	3.0818%	3.0670%
4	EUSA4	3.0561%	3.0719%	3.0640%
5	EUSA5	3.0617%	3.0803%	3.0710%
6	EUSA6	3.0706%	3.0894%	3.0800%
7	EUSA7	3.0843%	3.0999%	3.0921%
8	EUSA8	3.0964%	3.1136%	3.1050%
9	EUSA9	3.1177%	3.1333%	3.1255%
10	EUSA10	3.1394%	3.1554%	3.1474%

Table 1. EUSA Swaps - 04/11/2022

Maturity (y)	Bloomberg Ticker	Bid	Ask	Mid
1	EESWE1	2.5779%	2.5966%	2.5872%
2	EESWE2	2.7683%	2.7937%	2.7810%
3	EESWE3	2.7608%	2.7852%	2.7730%
4	EESWE4	2.7410%	2.7910%	2.7660%
5	EESWE5	2.7461%	2.7959%	2.7710%
6	EESWE6	2.7537%	2.8003%	2.7770%
7	EESWE7	2.7677%	2.8063%	2.7870%
8	EESWE8	2.7874%	2.8286%	2.8080%
9	EESWE9	2.8057%	2.8563%	2.8310%
10	EESWE10	2.8334%	2.8846%	2.8590%

Table 2. EESWE Swaps - 04/11/2022

Step 1 - Find the zero coupon bond curves for both swaps: consider the equations for each zero coupon bond: (4.9), (4.8), (4.6) and (4.7). For EESWE swaps, we can extract directly the zero coupon curve as we have every mid quote for each year looking at Table 2, and we have annual frequency on both legs. Applying equations (4.6) and (4.7) we get the discount factors in Table 3.

Maturity (y)	ZCB - EESWE
1	0.9748
2	0.9466
3	0.9212
4	0.8966
5	0.8722
6	0.8484
7	0.8248
8	0.8010
9	0.7774
10	0.7537

Table 3. Zero coupon bond - EESWE swaps

For the EUSA swaps, since some maturities are missing from Table 1 (as they are not available on Bloomberg), we will have to use linear interpolation as an alternative for computing the missing mid quotes. These are presented in Table 4.

Maturity (y)	Mid
1	2.8098%
1.5	2.9783%
2	3.0580%
2.5	3.0625%
3	3.0670%
3.5	3.0655%
4	3.0640%
4.5	3.0675%
5	3.0710%
5.5	3.0755%
6	3.0800%
6.5	3.0861%
7	3.0921%
7.5	3.0986%
8	3.1050%
8.5	3.1152%
9	3.1255%
9.5	3.1364%
10	3.1474%

Table 4. EUSA swaps after interpolation for mid quotes

Having the interpolated the mid quotes, we can then calculate (in Table 5) the zero coupon bond curve for the EUSA swaps using equations (4.9) and (4.8).

Maturity (y)	ZCB - EUSA
1	0.9727
1.5	0.9710
2	0.9415
2.5	0.9414
3	0.9133
3.5	0.9133
4	0.8862
4.5	0.8861
5	0.8596
5,5	0.8594
6	0.8335
6.5	0.8331
7	0.8078
7,5	0.8074
8	0.7827
8.5	0.7820
9	0.7576
9.5	0.7568
10	0.7329

Table 5. Zero coupon bond - EUSA swaps

Step 3 - Comparing both zero coupon bond curves: the last step is to graph both zero coupon bond curves and calculate the difference between each curve. Before that, we need to interpolate our EESWE zero coupon bond curve, in Table 6.

Maturity (y)	ZCB - EESWE (interpolated)
1	0.9748
1.5	0.9607
2	0.9466
2.5	0.9339
3	0.9212
3.5	0.9089
4	0.8966
4.5	0.8844
5	0.8722
5.5	0.8603
6	0.8484
6.5	0.8366
7	0.8248
7.5	0.8129
8	0.8010
8.5	0.7892
9	0.7774
9.5	0.7655
10	0.7537

TABLE 6. Interpolated zero coupon bond - EESWE swaps

Both curves are plotted in Figure 1 and the difference is presented in Table 7.

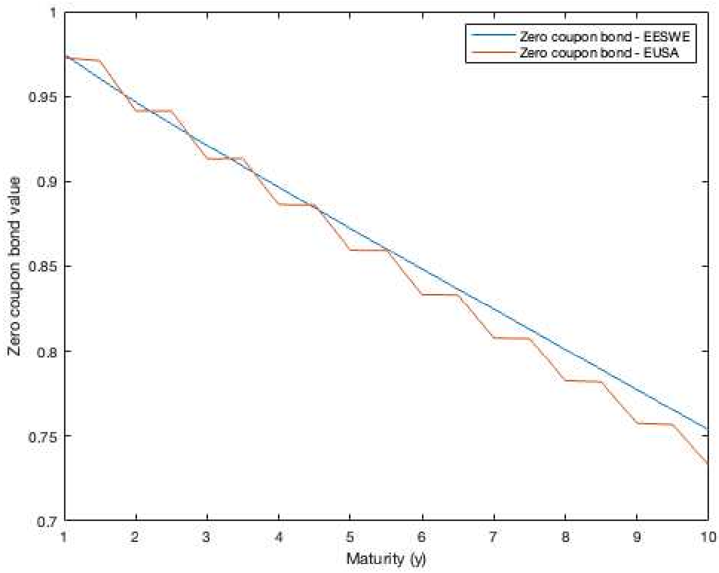

FIGURE 1. Graph of both zero coupon bond curves for each swap

Maturity (y)	Difference in ZCB — EUSA-EESWE
1	0.21%
1.5	-1.04%
2	0.51%
2.5	-0.75%
3	0.79%
3.5	-0.44%
4	1.04%
4.5	-0.17%
5	1.27%
5.5	0.10%
6	1.49%
6.5	0.35%
7	1.70%
7.5	0.55%
8	1.83%
8.5	0.72%
9	1.98%
9.5	0.88%
10	2.08%

TABLE 7. EUSA ZCB curve minus EESWE ZCB curve

Step 4 - Calculating EUSA mid swap quotes using EESWE zero coupon bond curve: the final step is to use data from Table 6 to price EUSA mid swap quotes for each maturity. Using equation (4.1) we get the results in Table 8 and we also calculate the difference in absolute value between the mid curves in Table 9. We also plot both mid curves in Figure 2 for comparison.

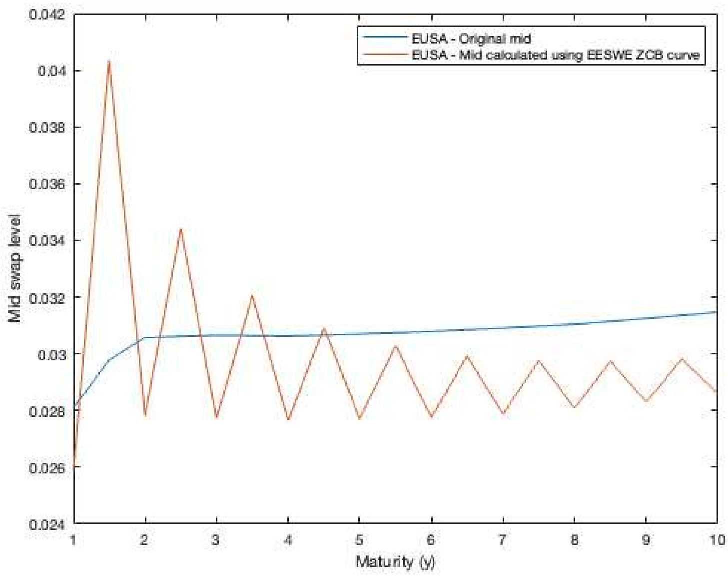

FIGURE 2. Graph of both EUSA mid swap quotes curves

Maturity (y)	EUSA mid using EESWE curve
1	2.5872%
1.5	4.0344%
2	2.7810%
2.5	3.4417%
3	2.7730%
3.5	3.2057%
4	2.7660%
4.5	3.0917%
5	2.7710%
5.5	3.0295%
6	2.7770%
6.5	2.9925%
7	2.7870%
7.5	2.9765%
8	2.8080%
8.5	2.9748%
9	2.8310%
9.5	2.9820%
10	2.8590%

TABLE 8. EUSA mid quotes calculated using EESWE curve

Maturity (y)	Difference in absolute value
1	0.2226%
1.5	1.0561%
2	0.2770%
2.5	0.3792%
3	0.2940%
3.5	0.1402%
4	0.2980%
4.5	0.0242%
5	0.3000%
5.5	0.0460%
6	0.3030%
6.5	0.0935%
7	0.3051%
7.5	0.1221%
8	0.2970%
8.5	0.1404%
9	0.2945%
9.5	0.1544%
10	0.2884%

TABLE 9. Difference in absolute value of EUSA mids minus EUSA mids using EESWE swaps

4.4. Pricing results assessement

Commenting on the accuracy of the pricing itself, it is safe to say that, in the majority of the cases, the zero coupon bond curve from EESWE swaps cannot replicate totally the original EUSA mid swap quotes, at least for most of the prices. For the record, we consider that a mid is replicated with success only if the difference in both mids is less than 10 basis points (0.10%).

Indeed, by looking at Table 9, we can see that the best results occur on the 4.5, 5.5, and 6.5 year maturities. However, for the remaining 16 resettlement dates, the results are appalling. One could argue that the pricing is not accurate due to various reasons. One possibility is that the current money-market conditions faces an unstable period (at the time of writing) due to some uncertainty caused by the rate hikes imposed by the central banks, causing the EUSA minus EESWE zero coupon bond curve to widen. Additionally, and above all, the fact that these swaps both have different frequencies on the floating legs is also not good for our pricing. As mentioned before, for $n = 1, \dots, 10$, not having the zero coupon bond value for the $\left(n + \frac{1}{2}\right)$-th year on the EESWE zero coupon bond curve forces the use of linear interpolation to compute these and there is not always a linear relationship between the year before (n) and the year after $(n+1)$, which can cause some disturbance. Furthermore, if we explore the floating leg frequency argument a bit further, we are also able to argue that the bootstrapped EESWE ZCB curve does not take into account any gaps on the fixed leg (even after being interpolated), since we have both fixed and floating payments occurring every resettlement date. But what if we did not? A suggestion to work around this problem in EESWE swaps would be to assume that in the $\left(n + \frac{1}{2}\right)$-th years only one payment occurs, the floating one. Since we are assuming these mid-term resettlement dates only for the purpose of EUSA swap pricing and since these also do not affect the pricing on the EESWE swaps at the original resettlement dates, we can assume that our zero coupon bond level at the $\left(n + \frac{1}{2}\right)$-th year will be the zero coupon bond level from year n plus some adjustment to take into account the discount factor for one more semester (0.5 years).

CHAPTER 5

Conclusions

5.1. Final remarks

We began the thesis by reminding and exploring some key swap concepts under IBOR rates and adapting these results to the new overnight rates, while also considering the possibility of credit risk, to reach closed-form solutions.

Looking back at the derived closed-form solutions for the new underlying overnight rates and using the already known ones for fixed maturity rates as a benchmark, we can say that the results are line with the market practice, at least for the most simple scenario where we assume default-risk. In this scenario, the equations obtained were almost similar in both underlying rates, with them only differing on the type of zero coupon bond (just like seen in Section 3.3 of Chapter 3) and the maturity of the zero coupon bonds embedded on the first floating rate loan.

Considering the solutions derived under credit risky reference rates, most of the equations assume some expected value that cannot be further simplified, under the measure $\mathbb{Q}$. In practice, for these equations, we can adopt some assumptions that can simplify the equation overall: we can consider some short-term interest rate model, like Vasiček (1977), or try to Monte-Carlo the expected value while also assuming some interest rate model.

In addition to the theoretical component of this thesis, we also present a numerical exercise where we test the pricing accuracy on EUSA swaps through the zero coupon bond curve of a EESWE swap. As we have previously noted in Section 4.4, the pricing overall is not that appealing mainly due to the frequency of the fixed leg of EUSA swaps not being equal to the frequency of the floating leg of the same swaps. Our proposed solution is to consider some adjustment to mid-term resettlement dates using the previous resettlement date, in order to adjust our EESWE zero coupon bond curve to match the shape of our EUSA zero coupon curve, as per Figure 1 shows.

As prospects of future research, finding closed-form model independent solutions for generalizing the pricing of swaptions and other OTC interest rate options (caps and floors) under some underlying overnight rate is the next step. When pricing swaptions and other OTC interest rate derivatives under some LIBOR underlying rate, for instance, we previously demonstrate that the forward swap rate is a martingale under the measure $\mathbb{Q}_S$ that assumes the annuity as a numeraire. However, this does not happen if our underlying is an overnight rate, which makes the pricing harder. Xu (2021) already approaches this pricing by modelling the dynamics of SOFR through a 1-factor Hull-White model for European interest rate options, that is easily extended to multi-factor affine Gaussian models. On the other hand, American interest rate options can not be computed through closed-form solutions, but numerical methods and Monte-Carlo are used to price these. Furthermore, Rutkowski and Bickersteth (2021) already cover some alternative pricing solutions for swaptions, as well as caps and floors under the SOFR, by reaching equations based on the conditional expectation of their respective payoffs.

APPENDIX A

MATLAB code

```
clear;
clc;

% Data reading

EUSA = readtable('EUSA.xlsx');
EESWE = readtable('EESWE.xlsx');

% Useful variables

freq_EESWE = 1;
freq_EUSA_fixed = 1;

t0 = 0;
max_maturity = max(EUSA.Tenor_y_);

% Creating vector for EUSA resettlement payment dates (all semesters ...
    within
% 10 years)

EUSA_resettlement_dates = zeros(2 * max_maturity,1);

for i = 1:length(EUSA_resettlement_dates)
    EUSA_resettlement_dates(i) = t0 + i * 0.5;
end

% EESWE ZCB Bootstrapping

EESWE.ZCB(1) = 1 / (1 + freq_EESWE * EESWE.Mid(1));

for i = 2:height(EESWE)
     EESWE.ZCB(i) = (1 - freq_EESWE * EESWE.Mid(i) * ...
         sum(EESWE.ZCB(1:i-1))) / (1 + freq_EESWE * EESWE.Mid(i));
end

EESWE_interpolated = table(EUSA_resettlement_dates);
EESWE_interpolated.Properties.VariableNames = {'Tenor_y_'};

```

```
EESWE_interpolated = outerjoin(EESWE_interpolated, EESWE);
EESWE_interpolated.Tenor_y__EESWE = [];
EESWE_interpolated.Ticker = [];

EESWE_interpolated = fillmissing(EESWE_interpolated, 'linear', ...
    'SamplePoints', EESWE_interpolated.Tenor_y__EESWE_interpolated);

% Getting EUSA data ready to merge with EESWE data + Computation

dt_swaps = table(EUSA_resettlement_dates);
dt_swaps.Properties.VariableNames = {'Tenor_y_'};

dt_swaps = outerjoin(dt_swaps, EUSA);
dt_swaps.Tenor_y__EUSA = []; %remove extra tenor column
dt_swaps.Ticker = [];

dt_swaps = fillmissing(dt_swaps, 'linear', 'SamplePoints', ...
    dt_swaps.Tenor_y__dt_swaps);
dt_swaps.N = dt_swaps.Tenor_y__dt_swaps * 2;

dt_swaps.EESWE_ZCB = EESWE_interpolated.ZCB;

dt_swaps.Mid_using_EESWE(1) = 1 - dt_swaps.EESWE_ZCB(1);

for i = 2:height(dt_swaps)
    dt_swaps.Mid_using_EESWE(i) = (1 - dt_swaps.EESWE_ZCB(i)) / ...
        (freq_EUSA_fixed * sum(dt_swaps.EESWE_ZCB(2:2:(floor(i/2))*2)));
end

dt_swaps.MidDifference_EUSA_minus_EUSA_using_EESWE_ZCB = dt_swaps.Mid ...
    - dt_swaps.Mid_using_EESWE;

% EUSA ZCB Bootstrapping

dt_swaps.EUSA_ZCB(1) = 1 - dt_swaps.Mid(1);
dt_swaps.EUSA_ZCB(2) = 1 / (1 + dt_swaps.Mid(2));

for i = 3:height(dt_swaps)
    if ¬mod(i,2) == 1
        dt_swaps.EUSA_ZCB(i) = (1 - ...
            dt_swaps.Mid(i)*sum(dt_swaps.EUSA_ZCB(2:2:((i/2)-1)*2)))  ...
            / (1 + dt_swaps.Mid(i));
    else
        dt_swaps.EUSA_ZCB(i) = 1 - dt_swaps.Mid(i) * ...
            sum(dt_swaps.EUSA_ZCB(2:2:(floor(i/2))*2));
    end
end
```

```
dt_swaps.ZCBDifference_EUSA_minus_EESWE = (dt_swaps.EUSA_ZCB - ...
    dt_swaps.EESWE_ZCB);

% Finalizing the table by removing first 0.5 year (first semester) ...
    from the
% final results

dt_swaps(1,:) = [];
dt_swaps

% Plotting figures and results

figure(1)
plot(dt_swaps.Tenor_y__dt_swaps, dt_swaps.Mid)
hold on
plot(dt_swaps.Tenor_y__dt_swaps, dt_swaps.Mid_using_EESWE)
legend('EUSA - Original mid', 'EUSA - Mid calculated using EESWE ZCB ...
    curve')
xlabel("Maturity (y)");
ylabel("Mid swap level")

figure(2)
plot(dt_swaps.Tenor_y__dt_swaps, dt_swaps.EESWE_ZCB)
hold on
plot(dt_swaps.Tenor_y__dt_swaps, dt_swaps.EUSA_ZCB)
legend('Zero coupon bond - EESWE','Zero coupon bond - EUSA')
xlabel("Maturity (y)")
ylabel("Zero coupon bond level")

best_results_condition = ...
    abs(dt_swaps.MidDifference_EUSA_minus_EUSA_using_EESWE_ZCB) < ...
    0.1/100;

best_results = dt_swaps(best_results_condition,:)
```

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