Convex Programming Based on Hahn-Banach Theorem

Manuel Alberto M. Ferreira[1*]

DOI:

ABSTRACT

The main objective of this chapter is to present the separation theorems, important consequences of Hahn-Theorem theorem. Therefore, we begin with an overview on convex sets and convex functionals. Then go on with the Hahn-Banach theorem and separation theorems. Follow these results specification: first for normed spaces and then for a subclass of these spaces, the Hilbert spaces. In this last case plays a key role the Riesz representation theorem. Separation theorems are key results in convex programming. Then the chapter ends with the outline of applications of these results in convex programming, Kuhn-Tucker theorem, and in minimax theorem, two important tools in operations research, management and economics, for instance.

Keywords: Hahn-Banach theorem; separation theorems; convex programming; Kuhn-Tucker theorem; minimax theorem.

MSC 2010:46B25

1 Introduction

After a general overview on convex sets and convex functionals, see [1 − 5] and [24] , we present Hahn-Banach theorem with great generality, together with important separation theorems, see [7] and [9] . Then we specify these results: first for normed spaces, see again [7] and then for a subclass of these spaces, the Hilbert spaces, see [8], [11] .

Then we emphasize the fruitfulness of the results presented, in the last sections where we show that they permit to obtain results very important in the applications:

- First, the Kuhn-Tucker theorem, see for instance [6],the main result of the convex programming so important in operations research,
- Second, the minimax theorem, an important result in game theory, observe [12] and [22,23]; which applications in management and economic models are becoming greater and greater.

This chapter is the conference paper [13] corrected and enlarged version.

2 Convex Sets and Fields

Be L a real vector space.

Definition 2.1

A set $K \subset L$ is convex if and only if

$$\underset{x,y \in K}{\forall} \ \underset{\theta \in [0,1]}{\forall} \theta x + (1-\theta) y \in K \quad (2.1)\blacksquare$$

[1]*ISTAR-IUL - Information Sciences, Technologies and Architecture Research Center (ISTA), Portugal.*
**Corresponding author: E-mail: manuel.ferreira@iscte.pt, manuel.ferreira@iscte-iul.pt;*

Definition 2.2

The nucleus of a set $E \subset L$, designated $J(E)$, is the set of points $x \in E$ such that, given any $y \in L$, it is possible to determine $\varepsilon = \varepsilon(y) > 0$ such that $x + ty \in E$ since$|t| < \varepsilon$. ■

Definition 2.3

A convex set with non-empty nucleus is a convex field.■

Theorem 2.1

The nucleus $J(K)$ of any convex set *K* is also a convex set.

Dem.:

Suppose that $x, y \in J(K)$. Be $z = \theta x + (1-\theta)y, 0 \leq \theta \leq 1$. So, given any $a \in L$, it is possible to determine $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that $|t_1| < \varepsilon_1, |t_2| < \varepsilon_2$, the points $x + t_1 a$ and $y + t_2 a$ belong both to *K*. So, the point

$$\theta(x + ta) + (1-\theta)(y + ta) = z + ta$$

belongs to *K* for $|t| < \varepsilon = min\{\varepsilon_1, \varepsilon_2\}$, that is $z \in J(K)$. ■

Theorem 2.2

The intersection of any family of convex sets is a convex set.

Dem.:

Be $K = \underset{\alpha}{\cap} K_\alpha$, being each K_α a convex set. Consider any two points *x* and *y* from *K*.

So $\theta x + (1-\theta)y, 0 \leq \theta \leq 1$, belongs to every K_α and, in consequence, to *K*. So *K*

is a convex set.■

Obs.:

-The intersection of convex fields, being a convex set; it is not necessarily a convex field.

Definition 2.4

Be *A* a part, anyone, of a vector space *L*. Among the convex sets that contain *A* there is a minimal set: the intersection of the whole convex sets that contain *A* -there is at least one convex set that contains *A*: the space *L*.

This minimal set is the convex hull of *A*.■

3 Homogeneous Convex Functional

Definition 3.1

A functional *p*, defined in *L* is convex if and only if

$$\underset{x,y \in L}{\forall} \ \underset{\theta \in [0,1]}{\forall} p(\theta x + (1-\theta)y) \leq \theta p(x) + (1-\theta)p(y) \qquad (3.1)$$ ■

Definition 3.2

A functional p is positively homogeneous if and only if

$$\underset{x \in L}{\forall} \; \underset{\alpha > 0}{\forall} \; p(\alpha x) = \alpha p(x) \qquad (3.2). \blacksquare$$

Proposition 3.1

For any convex positively homogeneous functional:

$$i) p(x+y) \leq p(x) + p(y) \qquad (3.3)$$

$$ii) p(0) = 0 \qquad (3.4)$$

$$iii) p(x) + p(-x) \geq 0, \underset{x \in L}{\forall} \qquad (3.5)$$

$$iv)\ p(\alpha x) \geq \alpha p(x), \underset{\alpha \in \mathbb{R}}{\forall} \qquad (3.6)$$

Dem:

i) Indeed, $p(x+y) = 2p\left(\frac{x+y}{2}\right) \leq 2\left(p\left(\frac{x}{2}\right) + p\left(\frac{y}{2}\right)\right) = p(x) + p(y)$.

ii) $p(0) = p(\alpha 0) = \alpha p(0), \underset{\alpha > 0}{\forall}$. So $p(0) = 0$.

iii) $0 = p(0) = p\big(x + (-x)\big) \leq p(x) + p(-x), \underset{x \in L}{\forall}$.

iv) The result is evident for $\alpha \geq 0$. With $\alpha < 0$, $0 \leq p(\alpha x) + p(-\alpha x) = p(ax) + p(|\alpha| x) = p(\alpha x) + |\alpha| p(x)$. So, $p(\alpha x) \geq -|\alpha| p(x)$, that is $p(\alpha x) \geq \alpha p(x)$. ■

4 Minkowski Functional

Definition 4.1

A convex body in L is a compact convex set with non-empty interior. ■

Definition 4.2

Be L any vector space and A a convex body in L which nucleus contains 0. The functional

$$p_A(x) = inf\left\{r : \frac{x}{r} \in A\right\} \qquad (4.1)$$

is the Minkowski functional of the convex body A.■

Theorem 4.1

A Minkowski functional is convex positively homogeneous and assumes only positive values. Reciprocally, if *p(x)* is a positively homogeneous functional, assuming only positive values, and M a positive number, the set

$$A = \{x : p(x) \leq M\} \qquad (4.2)$$

is a convex body with nucleus $\{x: p(x) < M\}$, which contains the point 0. If in (4.2) $M = 1$, the initial functional *p(x)* will be the Minkowski functional of *A*.

Dem:

Given any element $x \in L, \frac{x}{r}$ belongs to A if r is great enough. Therefore, the number $p_A(x)$ defined by (4.1) is positive and finite. But, given $t > 0$ and $y = tx, p_A(y) = inf\left\{r > 0: \frac{y}{r} \in A\right\} = inf\left\{r > 0: \frac{tx}{r} \in A\right\} = inf\left\{tr' > 0: \frac{x}{r'} \in A\right\} = tinf\left\{r' > 0: \frac{x}{r'} \in A\right\} = tp_A(x)$. So,

$$p_A(tx) = tp_A(x), \underset{t > 0}{\forall} \quad (4.3),$$

consequently $p_A(x)$ is positively homogeneous.

Suppose now that $x_1, x_2 \in L$. Given any $\varepsilon > 0$, choose the numbers $r_i, i = 1,2$ in order that $p_A(x_i) < r_i < p_A(x_i) + \varepsilon$. So $\frac{x_i}{r_i} \in A$. Making $r = r_1 + r_2$, the point $\frac{x_1+x_2}{r} = \frac{r}{rr_1}x_1 + \frac{r_2}{rr_2}x_2$ will belong to the set of points $S = \left\{z: z = \theta\frac{x_1}{r_1} + (1-\theta)\frac{x_2}{r_2}, \underset{\theta \in [0,1]}{\forall}\right\}$. As A is a convex set, $S \subset A$ and , in particular, $\frac{x_1+x_2}{r} \in A$. So, $p_A(x_1 + x_2) \leq r = r_1 + r_2 < p_A(x_1) +, p_A(x_2) + 2\varepsilon$. As ε is arbitrary, $p_A(x_1 + x_2) \leq p_A(x_1) + p_A(x_2)$. So,

$$p_A(\theta x + (1-\theta)y) \leq p_A(\theta x) + p_A((1-\theta)y) = \theta p_A(x) + (1-\theta)p_A(y), \underset{x, y \in L}{\forall}, \underset{\theta \in [0,1]}{\forall},$$

since it was already shown that $p_A(x)$ is positively homogeneous.

Look now to the set defined by (4.2). If $x, y \in A$ and $\theta \in [0,1]$, so $p(\theta x + (1-\theta)y) \leq \theta p(x) + (1-\theta)p(y) \leq M$. In consequence, A is a convex set. Suppose now that $p(x) < M, t > 0$ and $y \in L$. Under these conditions, $p(x \pm ty) \leq p(x) + tp(\pm y)$. If $p(-y) = p(y) = 0$, so $x \pm ty \in A$ for any t. If at least one of the numbers (positive) $p(y), p(-y)$ is not nul, so $x \pm ty \in A$ for

$$t < \frac{M - p(x)}{max\{p(y), p(-y)\}}.$$

From the definitions it results, p is the Minkowski functional of the set $\{x: p(x) \leq 1\}$. ■

5 Hahn-Banach Theorem

Definition 5.1

Consider a vector space L and its subspace L_0. Suppose that in L_0 it is defined a linear functional f_0. A linear functional *f* defined in the whole space *L* is an extension of the functional f_0 if and only if $f(x) = f_0(x), \underset{x \in L_0}{\forall}$. ■

Theorem 5.1 (Hahn-Banach**)**

Be p a positively homogeneous convex functional, defined in a real vector space L, and L_0 an L subspace. If f_0 is a linear functional defined in L_0, fulfilling the condition

$$f_0(x) \leq p(x), \underset{x \in L_0}{\forall} \quad (5.1),$$

there is an extension *f* of f_0 defined in *L*, linear, and such that $f(x) \leq p(x), \underset{x \in L}{\forall}$.

Dem.:

Begin showing that if $L_0 \neq L$, there is an extension of f_0, $f^{'}$ defined in a subspace $L^{'}$, such that $L \subset L^{'}$, in order to fulfill the condition (5.1).

Be $z \in L - L_0$. If $L^{'}$ is the subspace generated by L_0 and z, each point of $L^{'}$ is expressed in the form $tz + x$, being $x \in L_0$. If $f^{'}$ is an extension (linear) of the functional f_0 to $L^{'}$, it will happen that $f^{'}(tz + x) = tf^{'}(z) + f_0(x)$ or, making $f^{'}(z) = c$, $f^{'}(tz + x) = tc + f_0(x)$. Now choose c, fulfilling the condition (5.1) in $L^{'}$, that is: in order that the inequality $f_0(x) + tc \leq p(x + tz)$, for any $x \in L_0$ and any real number t, is accomplished. For $t > 0$ this inequality is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \leq p\left(\frac{x}{t} + z\right)$ or

$$c \leq p\left(\frac{x}{t} + z\right) - f_0\left(\frac{x}{t}\right) \qquad (5.2).$$

For $t < 0$ it is equivalent to the condition $f_0\left(\frac{x}{t}\right) + c \geq -p\left(-\frac{x}{t} - z\right)$, or

$$c \geq -p\left(-\frac{x}{t} - z\right) - f_0\left(\frac{x}{t}\right) \qquad (5.3).$$

Now it will be proved that there is always c satisfying simultaneously the conditions (5.2) and (5.3).

Given any $y^{'}$ and $y^{''}$ belonging to L_0,

$$-f_0(y^{''}) + p(y^{''} + z) \geq -f_0(y^{'}) - p(-y^{'} - z) \qquad (5.4)$$

This happens because

$$f_0(y^{''}) - f_0(y^{'}) \leq p(y^{''} - y^{'}) = p\big((y^{''} + z) - (y^{'} + z)\big) \leq p(y^{''} + z) + p(-y^{'} - z).$$

Be $c^{''} = \inf_{y^{''}}\big(-f_0(y^{''}) + p(y^{''} + z)\big)$ and $c^{'} = \sup_{y^{'}}\big(-f_0(y^{'}) - p(-y^{'} - z)\big)$.

As $y^{'}$ and $y^{''}$ are arbitrary, it results from (5.4) that $c^{''} \geq c^{'}$. Choosing c in order that $c^{''} \geq c \geq c^{'}$, it is defined the functional $f^{'}$ on $L^{'}$ through the formula $f^{'}(tz + x) = tc + f_0(x)$. This functional satisfies the condition (5.2). So any functional f_0 defined in a subspace $L_0 \subset L$ and subject in L_0 to the condition (5.1), may be extended to a subspace $L^{'}$. The extension $f^{'}$ satisfies the condition $f^{'}(x) \leq p(x)$, $\forall_{x \in L^{'}}$. If L has an algebraic numerable base $(x_1, x_2, \dots, x_n, \dots)$ the functional in L is built by finite induction, considering the increasing sequence of subspaces $L^{(1)} = (L_0, x_1), L^{(2)} = \big(L^{(1)}, x_2\big), \dots$ designating $\big(L^{(k)}, x_{k+1}\big)$ the L subspace generated by $L^{(k)}$ and x_{k+1}. In the general case, that is, when L has not an algebraic numerable base, it is mandatory to use a transfinite induction process, for instance the Haudsdorf maximal chain theorem.

So call $\mathcal{F}$ the set of the whole pairs$(L^{'}, f^{'})$, at which $L^{'}$ is a L subspace that contains L_0 and $f^{'}$ is an extension of f_0 to $L^{'}$ that fulfills (5.1). Order partially $\mathcal{F}$ so that $(L^{'}, f^{'}) \leq (L^{''}, f^{''})$ if and only if $L^{'} \subset L^{''}$ and $f^{''}_{|L^{'}} = f^{'}$. By the Haudsdorf maximal chain theorem, there is a chain, that is: a subset of $\mathcal{F}$ totally ordered, maximal, that is: not strictly contained in another chain. Call it Ω. Be Φ the family of the whole $L^{'}$ such that $(L^{'}, f^{'}) \in \Omega$. The family Φ is totally ordered by the sets inclusion; so, the union T of the whole elements of Φ is an L subspace. If $x \in$ T then $x \in L^{'}$ for some $L^{'} \in \Phi$. Define $\tilde{f}(x) = f^{'}(x)$, where $f^{'}$ is the extension of f_0 that is in the pair $(L^{'}, f^{'})$- the definition of $\tilde{f}$ is obviously coherent. It is easy to check that $\mathrm{T} = L$ and that $f = f^{'}$ satisfies the condition (5.1).■

Now it follows the Hahn-Banach theorem complex case. It is the Hahn contribution to the theorem.

Theorem 5.2 (Hahn-Banach**)**

Be p an homogeneous convex functional defined in a vector space L and f_0 a linear functional, defined in a subspace $L_0 \subset L$, fulfilling the condition$|f_0(x)| \leq p(x), x \in L_0$. Then, there is a linear functional f defined in L, satisfying the conditions

$|f(x)| \leq p(x), x \in L; f(x) = f_0(x), x \in L_0.$

Dem.:

Call L_R and L_{0R} the real vector spaces underlying, respectively, the spaces L and L_0 . As it is evident, p is an homogeneous convex functional in L_R and $f_{0R}(x) = Ref_0(x)$ a real linear functional in L_{0R} fulfilling the condition $|f_{0R}(x)| \leq p(x)$ and so, $f_{0R}(x) \leq p(x)$. Then, owing to Theorem 5.1, there is a real linear functional f_R , defined in the whole L_R space, that satisfies the conditions $f_R(x) \leq p(x), x \in L_R$; $f_R(x) = f_{0R}(x), x \in L_{0R}$. But, $-f_R(x) = f_R(-x) \leq p(-x) = p(x)$, and

$$|f_R(x)| \leq p(x), x \in L_R \qquad (5.5).$$

Define in L the functional f making $f(x) = f_R(x) - if_R(ix), i = \sqrt{-1}$. It is immediate to conclude that f is a complex linear functional in L such that $f(x) = f_0(x), x \in L_0; Ref(x) = f_R(x), x \in L.$

It is only missing to show that $|f(x)| \leq p(x), \underset{x \in L}{\forall}$.

Proceed by absurd: suppose that there is $x_0 \in L$ such that $|f(x_0)| > p(x_0)$. So, $f(x_0) = \rho e^{i\varphi}, \rho > 0$, and making $y_0 = e^{-i\varphi}x_0$, it would happen that $f_R(y_0) = Re[e^{-i\varphi}f(x_0)] = \rho > p(x_0) = p(y_0)$ that is contrary to (5.5).■

6 Vector Spaces Convex Parts Separation

In the next theorem, we present a very useful consequence of Hahn-Banach theorem, about vector space convex parts separation, remark [9]. Beginning with

Definition 6.1

Be M and N two subsets of a real vector space L. A linear functional f defined in L separates M and N if and only if there is a number c such that $f(x) \geq c$, for $x \in M$ and $f(x) \leq c$, for $x \in N$ that is, if $\inf_{x \in M} f(x) \geq \sup_{x \in N} f(x)$. A functional f separates strictly the sets M and N if and only if $\inf_{x \in M} f(x) > \sup_{x \in N} f(x)$. ■

Theorem 6.1 (Separation**)**

Suppose that M and N are two convex subsets of a vector space L such that the kernel of at least one of them, for instance M, is non-empty and does not intersect the other set. Therefore, there is a linear functional, non-null on L, which separates M and N.

Dem.:

Less than one translation, it is supposable that the point 0 belongs to the kernel of M, which we designate $\dot{M}$. So, given $y_0 \in N$, $-y_0$ belongs to the kernel of $M - N$ and 0 to the kernel of $M - N + y_0$. As $\dot{M} \cap N = \emptyset$, by hypothesis, 0 does not belong to the kernel of $M - N$ and y_0 does not belong to the one of $M - N + y_0$. Put $K = M - N + y_0$ and be p the Minkowski functional of $\dot{K}$. So $p(y_0) \geq 1$, since $y_0 \notin \dot{K}$.

Define, now, the linear functional $f_0(\alpha y_0) = \alpha p(y_0)$. Note that f_0 is defined in a space with dimension 1, constituted by elements αy_0, and it is such that $f_0(\alpha y_0) \leq p(\alpha y_0)$. In fact, $p(\alpha y_0) = \alpha p(y_0)$, when $\alpha \geq 0$ and $f_0(\alpha y_0) = \alpha f_0(y_0) < 0 < p(\alpha y_0)$, when $\alpha > 0$. Under these conditions, after the Hahn-Banach theorem, it is possible to state the existence of linear functional f , defined in L, that extends f_0, and such that $f(y) \leq p(y), \underset{y \in L}{\forall}$. Then it results $f(y) \leq 1, \underset{y \in K}{\forall}$ and $f(y_0) \geq 1$. In consequence:

- f separates the sets K and $\{y_0\}$, that is

- f separates the sets *M*-*N* and $\{y_0\}$, that is
- f separates the sets *M* and *N*.■

7 Hahn-Banach Theorem for Normed Spaces

Definition 7.1

Consider a continuous linear functional *f* in a normed space *E*. The f norm designated $\|f\|$, $\|f\| = \sup_{\|x\|\leq 1} |f(x)|$ that is, the supreme of the values assumed by $|f(x)|$ in the *E* unitary ball.■

Obs.:

-The class of continuous linear functionals, with norm defined above, is a normed vector space, called the *E* dual space, designated E'.

Theorem 5.1 in normed spaces is, note [7]:

Theorem 7.1 (Hahn-Banach**)**

Call *L* a subspace of a real normed space *E*. Moreover, f_0 a bounded linear functional in *L*. So, there is a linear functional defined in *E*, extension of f_0, such that $\|f_0\|_{L'} = \|f\|_{E'}$.

Dem.:

It is enough to think in the functional g satisfying $g\|x\| = \|f_0\|_{L'}$. As it is convex and positively homogeneous, it is possible to put $p(x) = g\|x\|$ and apply Theorem 5.1.■

Obs.:

-To see an interesting geometric interpretation of this theorem, consider the equation $\|f_0(x)\| = 1$.It defines, in *L*, a hyperplane at distance $\frac{1}{\|f_0\|}$ of 0. Considering the $f_0$extension *f*, with norm conservation, it is obtained a hyperplane in *E*, which contains the hyperplane considered behind in *L*, at the same distance from the origin.

Theorem 5.2 in normed spaces is, see again [7] :

Theorem 7.2 (Hahn-Banach**)**

Be E a complex normed space and f_0 a bounded linear functional defined in a subspace $L \subset E$. So, there is a bounded linear functional *f*, defined in *E*, such that $f(x) = f_0(x), x \in L; \|f\|_{E'} = \|f_0\|_{L'}$.■

Two separation theorems, important consequences of the Hahn-Banach theorem, applied to the normed vector spaces, follow:

Theorem 7.3 (Separation**)**

Consider two convex sets *A* and *B* in a normed space *E*. If one of them, for instance *A*, has at least on interior point and $(intA) \cap B = \emptyset$, there is a continuous linear functional non-null that separates the sets *A* and *B*.■

Theorem 7.4 (Separation**)**

Consider a closed convex set *A*, in a normed space *E*, and a point $x_0 \in E$, not belonging to *A*. Therefore, there is a continuous linear functional, non-null, that separates strictly $\{x_0\}$ and *A*.■

8 Separation Theorems in Hilbert Spaces

Definition 8.1

A Hilbert space, designated *H* or *I*, is a complex vector space with inner product that, as a metric space, is complete.■

Definition 8.2

An inner product, in a complex vector space *H*, is a sesquilinear Hermitian functional, strictly positive on H.■

Obs.:

- Working with real vector spaces, **"**sesquilinear Hermitian" must be replaced by "bilinear symmetric**",**
- The inner product of two vectors *x* and *y* of *H*, by this order is designated $[x, y]$,
- The norm of a vector x will be $\|x\| = \sqrt{[x, x]}$.

An important theorem, about the representation of continuous linear functionals by elements of the space is the Riesz representation theorem, note $[1-3]$ and $[15,16]$:

Theorem 8.1 (Riesz representation**)**

Every continuous linear functional $f(\cdot)$ may be represented in the form $f(x) = [x, \tilde{q}]$ where $\tilde{q} = \frac{\overline{f(q)}}{[q,q]} q$. ■

Dem:

Begin noting that for every continuous linear functional $f(.)$, the *Nucleus* of $f(.)$ [1] is a closed vector subspace. If the functional under consideration is not the null functional, there is an element *y* such that $f(y) \neq 0$. Be *z* the projection of *y* over *Nuc*(*f*) and make $q = y - z$. So, q is orthogonal to *Nuc* (*f*), $f(q) = f(y)$ and, in consequence, $f(q) \neq 0$. Then, for every $x \in H, x - \frac{f(x)}{f(q)} q$ belongs, evidently to *Nuc*(*f*). So, $x - \frac{f(x)}{f(q)} q$ is orthogonal to q and, in consequence, $[x, q] - \frac{f(x)}{f(q)} [q, q] = 0 \Leftrightarrow [x, q] = \frac{f(x)}{f(q)} [q, q]\ that\ is: f(x) = \left[x, \frac{\overline{f(q)}}{[q,q]} q\right]$. ■

Obs.:

-From the theorem, it results $\|f\|_{H'} = \|\tilde{q}\|_H$, where the *H* dual space is H'.

From now on, we consider only real Hilbert spaces.

Note that the separation theorems, seen in the former section, are effective in Hilbert spaces. However, due to Riesz representation theorem, we can be formulate them in the subsequent way, follow [8] :

Theorem 8.2 (Separation**)**

Consider two convex sets *A* and *B* in a Hilbert space *H*. If one of them, for instance *A*, has at least one interior point and $(intA) \cap B = \emptyset$, there is a non-null vector *v* such that $\sup_{x \in A} [v, x] \leq \inf_{y \in B} [v, y]$. ■

Theorem 8.3 (Separation**)**

Consider a closed convex set *A*, in a Hilbert space *H*, and a point $x_0 \in H$, not belonging to *A*. So, there is a non-null vector *v*, such that $[v, x_0] < \inf_{x \in A} [v, x]$. ■
Another separation theorem:

Theorem 8.4 (Separation**)**

[1] *The Nucleus of $f(.)$ is designated Nuc(f) and Nuc(f)=$\{x: f(x) = 0\}$.*

Two closed convex subsets *A* and *B*, in a Hilbert space, at finite distance, that is: such that $\inf_{x\in A, y\in B}\|x-y\| = d > 0$ may be strictly separated: $\inf_{x\in A}[v,x] > \sup_{y\in B}[v,y]$. ■

It is also possible to establish:

Theorem 8.5 (Separation**)**

Being *H* a finite dimension Hilbert space, if *A* and *B* are disjoint and non-empty convex sets they always may be separated.■

9 Kuhn-Tucker Theorem

We outline now a class of convex programming problems, at which we intend to minimize[2] convex functionals subject to convex inequalities. Begin presenting a basic result that characterizes the minimum point of a convex functional subject to convex inequalities. Note that it is not necessary to impose any continuity conditions.

Theorem 9.1 (Kuhn-Tucker**)**

Be *f(x)*, $f_i(x), i = 1, \dots, n,$ convex functionals defined in a convex subset *C* of a Hilbert space. Consider the problem $\min_{x\in C} f(x)\ sub.\, f_i(x) \le 0, i = 1, \dots$. Be x_0 a minimizing point, supposed finite. Suppose also that for each vector *u* in E_n, Euclidean space with dimension *n*, non-null and such that $u_k \ge 0$, there is a point *x* in *C* such that $\sum_1 u_k f_k(x) < 0$,designating u_k the components of *u*. So,

i) There is a vector *v*, with non-negative components $\{v_k\}$, such that

$$\min_{x\in C}\left\{f(x) + \sum_1^n v_k f_k(x)\right\} = f(x_0) + \sum_1^n v_k f_k(x_0) = f(x_0) \qquad (9.1),$$

ii) For every vector u in E_n with non-negative components, that is: belonging to the positive cone of E_n ,

$$f(x) + \sum_1^n v_k f_k(x) \ge f(x_0) + \sum_1^n v_k f_k(x_0) \ge f(x_0) + \sum_1^n u_k f_k(x_0) \qquad (9.2).$$ ■

Having in mind Riesz representation theorem:

Corollary 9.1 (Lagrange duality**)**

In the conditions of Theorem 9.1 $f(x_0) = \sup_{u\ge 0}\inf_{x\in C} f(x) + \sum_1^n u_k f_k(x)$. ■

Obs.:

- This corollary is useful supplying a process to determine the problem optimal solution,
- If the whole v_k in expression (9.2) are positive, x_0 is a point that belonging to the border of the convex set defined by the inequalities,
- If the whole $v_k$are zero, the inequalities do not influence the problem, that is: the minimum is equal to the one of the restrictions free problem.

Considering non-finite inequalities and following [10]:
Theorem 9.2 (Kuhn-Tucker in infinite dimension**)**

[2] *To consider maximization, note that* $Max\ f = min -f$.

Be C a convex subset of a Hilbert space H and $f(x)$ a real convex functional defined in C. Be I a Hilbert space with a closed convex cone $\wp$, with non-empty interior, and $F(x)$ a convex transformation from H to I (convex in relation to the order introduced by cone $\wp$: if $x, y \in \wp, x \geq y \; if \; x - y \in \wp$). Be x_0 an $f(x)$ minimizing in C, subjected to the inequality $F(x) \leq 0$.Consider $\wp^* = \left\{x: [x, p] \geq 0, \underset{x \in \wp}{\forall}\right\}$ (dual cone). Admit that given any $u \in \wp^*$ it is possible to determine x in C such that $[u, F(x)] < 0$. So, there is an element v in the dual cone $\wp^*$, such that for x in C $f(x) + [v, F(x)] \geq f(x_0) + [v, F(x_0)] \geq f(x_0) + [u, F(x_0)]$, being u any element of $\wp^*$. ■

Having in mind Riesz representation theorem:

Corollary 9.2 (Lagrange duality in infinite dimension**)**

$f(x_0) = \sup_{v \in \wp^*} \inf_{x \in C} (f(x) + [v, F(x)])$ in the conditions of Theorem 6.2.■

10 Minimax Theorem

In a two players game with null sum be $\Phi(x, y)$ a real function of two variables $x, y \in H$ and A, B convex sets in H. One of the players chooses strategies (points) in A in order to maximize $\Phi(x, y)$ (or minimize $-\Phi(x, y)$): it is the maximizing player. The other player chooses strategies (points) in B in order to minimize $\Phi(x, y)$ (or maximize $-\Phi(x, y)$): it is the minimizing player. The function $\Phi(x, y)$, is the payoff function, see [18,19] . The $\Phi(x_0, y_0)$ value represents, simultaneously, the gain of the maximizing player and the loss of the minimizing player in a move at which they chose, respectively the strategies x_0 and y_0. Therefore, the gain of one of the players is equal to the other's loss. That is why the game is a null sum game. A game in these conditions value is c if

$$\sup_{x \in A} \inf_{y \in B} \Phi(x, y) = c = \inf_{y \in B} \sup_{x \in A} \Phi(x, y) \qquad (10.1).$$

If, for any (x_0, y_0), $\Phi(x_0, y_0) = c$, (x_0, y_0) is a pair of optimal strategies. There will be a saddle point if also

$$\Phi(x, y_0) \leq \Phi(x_0, y_0) \leq \Phi(x_0, y), x \in A, y \in B \qquad (10.2).$$

Theorem 10.1

Consider A and B closed convex sets in H, being A bounded. Be $\Phi(x, y)$ a real functional defined for x in A and y in B fulfilling:

- $\Phi(x, (1-\theta)y_1 + \theta y_2) \leq (1-\theta)\Phi(x, y_1) + \theta\Phi(x, y_2)$ for x in A and y_1, y_2 in B, $0 \leq \theta \leq 1$ (that is: $\Phi(x, y)$ is convex in y for each x),
- $\Phi\big((1-\theta)x_1 + \theta x_2, y\big) \geq (1-\theta)\Phi\big(x_1, y\big) + \theta\Phi(x_2, y)$ for y in B and x_1, x_2 in A, $0 \leq \theta \leq 1$ (that is: $\Phi(x, y)$ is concave in x for each y),
- $\Phi(x, y)$ is continuous in x for each y,

so (10.1) holds, that is: the game has a value. ■

The next corollary follows from the Theorem 10.1 hypothesis strengthen:

Corollary 10.1(Minimax**)**

Suppose that the functional $\Phi(x, y)$ in Theorem 10.1 is continuous in both variables, separately, and that B is bounded. Then, there is an optimal pair of strategies, with the property of being a saddle point.■

One last reference to Nash theorem, see [17] and [20, 21], which in a certain way generalizes minimax theorem:
Theorem 10.2 (Nash)

The mixed extension of every finite game has, at least, one strategic equilibrium. ■

Obs.:

- Its demonstration demands, among other results, an important contribution of Kakutani's theorem, see [14] and [17].

11 Conclusions

We began this chapter with an overview on convex sets and convex functionals notions and results, fundamental as support to the sequence of the text.

Then presented the Hahn-Banach theorem with great generality, real and complex version, followed by important separation theorems, its consequence.

Separation theorems are fundamental in optimization and of course are key results in convex programming.

We specified these results for normed spaces and then for a subclass of these spaces: The Hilbert spaces. Better saying, we reformulated them for Hilbert spaces using the Riesz representation theorem.

Examples of the fruitfulness of the results presented are patent in the last two sections, where we show they permit to obtain important results, for the applications, as Kuhn-Tucker and minimax theorems. Now the mathematical structures considered were the real Hilbert spaces. The problems studied are placed in the class of convex optimization problems in which, never hurts to emphasize, the separation theorems, dealt with in this chapter, are a key tool.

Kuhn-Tucker theorem is the convex programming main result so important in operations research. Minimax theorem is an important result in game theory, which consideration in management and economic problems resolution is greater and greater.

Acknowledgements

This work is financed by national funds through FCT - Fundação para a Ciência e Tecnologia, I.P., under the project UID/Multi/04466/2019. Furthermore, I would like to thank the Instituto Universitário de Lisboa and ISTAR-IUL for their support.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Aubin JP. Applied functional analysis. John Wiley & Sons Inc. New York; 1979.

[2] Balakrishnan AV. Applied functional analysis. Springer-Verlag New York Inc., New York; 1981.

[3] Brézis H. Analyse fonctionelle (Théorie et Applications). Masson, Paris; 1983.

[4] Ferreira MAM. Aplicação dos teoremas de separação na programação convexa em espaços de Hilbert. Revista de Gestão. 1986;I(2):41-44.

[5] Ferreira MAM, Andrade M, Matos MC. Separation theorems in Hilbert spaces convex programming. Journal of Mathematics and Technology. 2010;1(5):20-27.

[6] Ferreira MAM, Andrade M. Management optimization problems. International Journal of Academic Research. 2011;3(2, Part III):647-654.

[7] Ferreira MAM, Andrade M. Hahn-Banach theorem for normed spaces. International Journal of Academic Research. 2011a;3(4, Part I):13-16.

[8] Ferreira MAM, Andrade M. Riesz representation theorem in Hilbert spaces separation theorems. International Journal of Academic Research. 2011b;3(6, II Part):302-304.

[9] Ferreira MAM, Andrade M. Separation of a vector space convex parts. International Journal of Academic Research. 2012;4 (2):5-8.

[10] Ferreira MAM, Andrade M, Filipe JA. Kuhn-Tucker's theorem for inequalities in infinite dimension. Journal of Mathematics and Technology. 2012;3(1):57-60.

[11] Ferreira MAM, Andrade M, Filipe JA. Weak convergence in Hilbert spaces. International Journal of Academic Research. 2012;4(4):34-36.

[12] Ferreira MAM, Andrade M, Matos MC, Filipe JA, Coelho M. Minimax theorem and Nash equilibrium. International Journal of Latest Trends in Finance & Economic Sciences. 2012;2(1):36-40.

[13] Ferreira MAM. The hahn-banach theorem in convex programming. APLIMAT 2015 - 14th Conference on Applied Mathematics, Proceedings. 2015;283-291.

[14] Kakutani S. A generalization of Brouwer's fixed point theorem. Duke Mathematics Journal. 1941;8.

[15] Kantorovich LV, Akilov GP. Functional analysis. Pergamon Press, Oxford; 1982.

[16] Kolmogorov AN, Fomin SV. Elementos da teoria das funções e de análise funcional. Editora Mir; 1982.

[17] Matos MC, Ferreira MAM. Game representation-code form. Lecture Notes in Economics and Mathematical Systems. 2006;567:321-334.

[18] Matos MC, Ferreira MAM, Andrade M. Code form game. International Journal of Academic Research. 2010;2(1):135-141.

[19] Matos MC, Ferreira MAM, Filipe JA, Coelho M. Prisonner`s dilemma: Cooperation or treason? PJQM-Portuguese Journal of Quantitative Methods. 2010;1(1):43-52.

[20] Matos MC, Ferreira MAM, Filipe JA. Let the games begin and go on. International Journal of Business and Systems Research. 2018;12(1):43-52.

[21] Nash J. Non-cooperative games. Annals of Mathematics. 1951;54.

[22] Neumann J. Von, Morgenstern O. Theory of games and economic behavior. Princeton University Press, Princeton, New Jersey; 1947.

[23] Neumann J. Von, Morgenstern O. Theory of games and economic behavior. John Wiley & Sons Inc., New York; 1967.

[24] Royden HL. Real analysis. Mac Milan Publishing Co. Inc, New York; 1968.
