# Implementing and Solving Games with Best Payoff Method

Manuel Alberto M. Ferreira and
Maria Cristina Peixoto de Matos



May 3, 2022

# Implementing and Solving Games with Best Payoff Method

**Manuel Alberto M. Ferreira[1*] and Maria Cristina Peixoto Matos[1]**

## ABSTRACT

It is our intention, in this chapter, to propose and discuss the Best Payoff Method, a new method to resolve games. This is made exemplifying the application of the method to a pay raise voting game, that is a perfect information sequential game, without having yet formulated it, and then deploying the algorithm for its implementation. In the next examples we consider an imperfect information game and a game with random characteristics. We finish confronting the equilibrium concepts mentioned in this work: Subgame Perfect Nash Equilibrium, Nash Equilibrium, and Best Payoff Equilibrium through the formulation of some conjectures, and with a short conclusions section.

*Keywords: Best payoff method; perfect information game; imperfect information game; selection process game.*

## 1. INTRODUCTION

When specifying a game, it is imperative to begin by defining these essential elements:

-who are the players,
-what are the strategies available to each player, and
-what rewards can each player receive.

But, to know all the elements that make up a game, it is not enough to know its solution(s), that is: the equilibrium(s) of the game.

Depending on the game being analyzed, there are several processes to find your solution. In this work, we intend to present a process to solve games with sequential movements. Most of the games that characterize real situations are referenced by the simultaneous actions of the players. But there are many cases where the actions of the players are carried out in a sequential method, that is: the players act according to a previously determined order. These last games are of great applicability in economics, business, and political science; see [7, 9 and 10].

It is with the purpose of solving, in a simpler way, this type of game that the Best Payoff Method *(BPM)* is presented. It consists, considering the players' rationality, in determining how the game should be played and in finding its equilibrium. The *BPM* main idea lies in the fact that a player is talented of predicting that other players will determine the strategy that leads to their best return and evaluate the credibility of their opponents' strategies. Therefore, how to select the ideal return, in each of the respective sets of information of the player with whom it commits to decide, it is easier to find the equilibrium of the game.

This selection is carried out following the order opposite to the game's development. That is, the process begins with the player who has the last word in the selection of payoffs and ends with the player who

*[1]Instituto Universitário de Lisboa (ISCTE – IUL), ISTAR-IUL, BRU - IUL Portugal.*
*[2]Instituto Politécnico de Viseu (IPV), Escola Superior deTecnologia e Gestão de Viseu (ESTGV), CI&DETS Portugal.*
**Corresponding author: E-mail: manuel.ferreira@iscte.pt.*

initiates the selection of payoffs in the game. It is a kind of *backwards' induction*. This process will be illustrated in the next sections. This chapter is the enlarged and updated version of [8].

## 2. PAY RAISE VOTING-SEQUENTIAL MOVES

Consider a game where three members of a company board of directors must vote the approval of their pay raise. If two of the directors vote affirmatively the increase will be consummated, on the contrary the salary will not be increased. The decision achieved will be communicated to the company employees, and because of this each one of the directors wants simultaneously to negatively vote the proposal and that it is approved. This position will guarantee a political advantage since, from the employees' point of view, the director will be recognized as an entrepreneur not ambitious and conscientious.

So, each player has two possible strategies: to approve the pay raise and not to approve the pay raise.

The payoffs are established considering the preference of every one of the players: pay raise approved without its vote. So,

-Increase approved/Yes:$A_y$
-Increase approved/No:$A_n$
-Increase not approved/Yes:$NA_y$
-Increase not approved/No:$NA_n$

Obviously the preferences relation of the payoffs, see [5], is $A_n \succ A_y \succ NA_n \succ NA_y$. Suppose, since it is a sequential game, that player 1 is the first to vote, player 2 votes next and player 3 votes at last. The game is illustrated in Table 1.

**Table 1. Pay raise voting game**

| | | | | |
|---|---|---|---|---|
| 1 | $(1,\text{"Y"})$ | | | |
| 2 | | $(2,\text{"Y"})$ | | |
| 3 | | | $(3,\text{"Y"})$ | $(A_y, A_y, A_y)$ |
| | | | $(3,\text{"N"})$ | $(A_y, A_y, A_n)$ |
| 2 | | $(2,\text{"N"})$ | | |
| 3 | | | $(3,\text{"Y"})$ | $(A_y, A_n, A_y)$ |
| | | | $(3,\text{"N"})$ | $(NA_y, NA_n, NA_n)$ |
| 1 | $(1,\text{"N"})$ | | | |
| 2 | | $(2,\text{"Y"})$ | | |
| 3 | | | $(3,\text{"Y"})$ | $(A_n, A_y, A_y)$ |
| | | | $(3,\text{"N"})$ | $(NA_n, NA_y, NA_n)$ |
| 2 | | $(2,\text{"N"})$ | | |
| 3 | | | $(3,\text{"Y"})$ | $(NA_n, NA_n, NA_y)$ |
| | | | $(3,\text{"N"})$ | $(NA_n, NA_n, NA_n)$ |

## 3. BEST PAYOFF METHOD

To resolve the game is to make a good predicting on what each player will do identifying the payoffs resulting when the game is played. So, if a good predicting is got, none of the players will want to do anything different.

When using *BPM* to determine the strategy that each player will choose when it must decide, they believe that the next player will always choose the strategy that guarantees it optimal payoffs.
In the pay raise voting game, there are three players being the last to play player 3. For it there are four information sets, each one with two associated payoffs:

$\{(A_y, A_y, A_y), (A_y, A_y, A_n)\}$,
$\{(A_y, A_n, A_y), (NA_y, NA_n, NA_n)\}$,
$\{(A_n, A_y, A_y), (NA_n, NA_y, NA_n)\}$, and
$\{(NA_n, NA_n, NA_y), (NA_n, NA_n, NA_n)\}$.

Obviously, player 3 prefers the strategy that guarantees it a best payoff. Considering the first information set, player 3 prefers $A_n$, that is:

-the increase is approved without its vote,

since this action gives it a better payoff.

For the same reasoning in the second and third information sets it prefers $A_y$, and in the fourth information set it prefers $NA_n$.

As for player 2, what must it do? This player can predict the preferences of player 3 in the decision moments. So, player 2 analyses the preferences of player 3 being sure that those preferences will determine its payoff. Then player 2 will only consider the strategies that guarantee a greater payoff to player 3 and, among these ones, it will make its choice. Consequently, player 2 will analyze the payoffs selected by player 3 referring to its information sets. That is:

$\{(A_y, A_y, A_n), (A_y, A_n, A_y)\}$
and
$\{(A_n, A_y, A_y), (NA_n, NA_n, NA_n)\}$.

Facing the possible choices for player 2, it will choose $A_n$ in the first information set and the strategy $A_y$ in the second information set.

Player 1 will proceed analogously. Being the available possibilities:

$\{(A_y, A_y, A_n), (A_n, A_y, A_y)\}$.

Evidently it will prefer the salary non-increase.

Generally, *BPM* may be described as the following algorithm, see [8]:

**Algorithm 3.1**

**a)** Call *i* the player that takes the last decision in the game.
**b)** Consider the various player *i* information sets associated to its decision.
**c)** In each information set select the player *i* optimal payoff.
**d)** Consider the subgame obtained excluding the column referring to player *i*.
**e)** Call $i^{'}$ the player that takes the decision immediately before player *i*.
**f)** Make $i^{'} = i$. Repeat the whole process beginning in a) till finding the player that takes the first decision.
**g)** Select the strategies profile induced by the payoff vector obtained. ■

## 4. BEST PAYOFF EQUILIBRIUM

In pay raise voting game, *BPM* gives the following equilibrium strategy: Player 1 does not approve the pay raise and player 2 and player 3 approve both the pay raise. So, it is admissible to state:

## Definition 4.1

The Best Payoff Equilibrium (*BPE*) is a set of strategies, one for each player, such that given the strategies of the other players no one player can increase its payoff biasing for another decision point. ■

*BPE* for the pay raise game is represented in Table 2 and identifies the actions that each player takes to obtain the optimal payoff. It is indeed an equilibrium since, as it is possible to observe, each player in each decision point chooses the actions that guarantee to it the best payoffs as a function of its opponents' choices.

**Table 2. Pay raise voting game *BPE***

| | | | | |
|---|---|---|---|---|
| 1 | $(1,"Y")$ | | | |
| 2 | | $(2,"Y")$ | | |
| 3 | | | $(3,"Y")$ | $(A_y, A_y, A_y)$ |
| | | | $(3,"N")$ | $(A_y, A_y, A_n)$ |
| 2 | | $(2,"N")$ | | |
| 3 | | | $(3,"Y")$ | $(A_y, A_n, A_y)$ |
| | | | $(3,"N")$ | $(NA_y, NA_n, NA_n)$ |
| 1 | $(1,"N")$ | | | |
| 2 | | $(2,"Y")$ | | |
| 3 | | | $(3,"Y")$ | $(A_n, A_y A_y)$ |
| | | | $(3,"N")$ | $(NA_n, NA_y, NA_n)$ |
| 2 | | $(2,"N")$ | | |
| 3 | | | $(3,"Y")$ | $(NA_n, NA_n, NA_y)$ |
| | | | $(3,"N")$ | $(NA_n, NA_n, NA_n)$ |

For this game, the *BPE* is

- (not to approve the increase, (not to approve the increase, to approve the increase), (to approve the increase, not to approve the increase, to approve the increase, not to approve the increase)),

-where (not to approve the increase) is the strategy of player 1, (not to approve the increase, to approve the increase) is the strategy of player 2 and (to approve the increase, not to approve the increase, to approve the increase, not to approve the increase) is the strategy of player 3.

## Proposition 4.1

Every perfect information finite sequential game has only one *BPE*. ■

**Notes:**

-Indeed, it is always possible to find the decisions to take such that, in face of the payoff values, any player may diverge from that decision to obtain a better payoff. If this happens any of the players was letting outside the decision that would allow it to obtain the optimal payoff.
-And this equilibrium is unique, because not to be so it means that the players have the possibility to choose, at least two optimal payoffs. Obviously, if there are at least two optimal payoffs they must be equal.

## 5. ENTRY IN THE MARKET

In the former example it was illustrated that *BPM* is a process applicable to perfect information sequential games-games at which the information sets have only one decision point.

Imagine now that we intended to analyze games that have simultaneously sequential moves and simultaneous moves. Is *BPM* useful in these circumstances?

**Consider the following example:** Suppose Company 1 is the monopolist in a certain market. Company 2 is considering entering in that market. If Company 2 decides to enter then both companies, simultaneously, decide to attack or to accommodate the situation. The strategies available for the companies are:

**Company 2**

It may decide between to enter-E- or to stay outside-O. If it decides to enter then it may choose between either to attack-T or to accommodate-A.

**Company 1**

If company 2 enters in the market, it may decide either to attack-T or to accommodate-A.

This situation leads to an imperfect information game. Table 3 illustrates the code form representation for the game, see [6] , where the first number of the payoff belongs to the company that wants to enter in the market and the second to the company already installed.

**Table 3. Entry in the market game representation**

| | | | | |
|---|---|---|---|---|
| 1 | $(2, "O")$ | | | (0,5) |
| | $(2, "E")$ | | | |
| 2 | | $(1, "T")$ | $(2, "T")$ | (-2, -1) |
| | | | $(2, "A")$ | (-3, 1) |
| | | $(1, "A")$ | $(2, "T")$ | (0, -3) |
| | | | $(2, "A")$ | (1, 2) |

As *BPM* is based in the circumstance that the players have the capacity to choose the actions optimally, given their opponents actions, it raises a problem with the application of this method:

-how to predict the moves once it is an imperfect information game?

It is out of question to use *BPM* in the whole game. But it is admissible to think that each player goes on trying to predict what the other players will do and react optimally, having in account that its opponents act in the same way. That is, each player chooses the strategy that is optimal in face of the other players' choices. So, what is intended is to find a *SPNE* - Subgame Perfect Nash Equilibrium, see [9] - in each nontrivial subgame associated to the initial game and then to apply *BPM.*

In this case there is only one nontrivial subgame which representation in normal form, see again [8], is in Table 4.

**Table 4. Entry in the market nontrivial subgame representation**

| | | | |
|---|---|---|---|
| 2 | $(1, "T")$ | $(2, "T")$ | (-2,-1) |
| | | $(2, "A")$ | (-3,1) |
| | $(1, "A")$ | $(2, "T")$ | (0,-3) |
| | | $(2, "A")$ | (1,2) |

This subgame represents a *SPNE*: (to accommodate, to accommodate). Now it is already possible to apply the *BPM*. So, Company 2 predicts that if it enters the market it will play the *SPNE* accommodating, receiving a payoff of *1* against to stay out of the market and receive a payoff of *0*. Consequently, the *BPE* is ((to enter, to accommodate), (to accommodate)).

Note that attending only to the Nash equilibrium, see [4 and 8], the game presents two credible equilibriums: ((to stay out, to attack),to attack) and ((to enter, to accommodate), to accommodate) and one non credible: ((to stay out, to accommodate), to attack).

## 6. A SELECTION PROCESS AS A GAME

Follows another example of *BPM* application in a context quite different from the former ones.

A presidential decree reduced the number of candidates to the vice-presidency to three people. Each of the three candidates are ranked on a scale from 1 (lowest) to 10 (highest).The presidential board attributed 10 points, 8 points and 5 points to the candidate classified in 1st place, 2nd place and 3rd place respectively. The probabilities of candidate *i* (*i*=1,2,3) accepting the *j*-th offer to run for the vice-presidency have been defined, considering that the first *j*-1 offers to the others have been declined, are denoted by $p_{ij}$ and the respective values are in Table 5.

**Table 5. Acceptance Probabilities**

| | | | |
|---|---|---|---|
| **Player 1** | $\boldsymbol{p_{11} = 0.5}$ | $\boldsymbol{p_{12} = 0.2}$ | $\boldsymbol{p_{13} = 0.0}$ |
| **Player 2** | $\boldsymbol{p_{21} = 0.9}$ | $\boldsymbol{p_{22} = 0.5}$ | $\boldsymbol{p_{23} = 0.2}$ |
| **Player 3** | $\boldsymbol{p_{31} = 1.0}$ | $\boldsymbol{p_{32} = 0.8}$ | $\boldsymbol{p_{33} = 0.4}$ |

**The question is:**

-What is the order in which the three potential candidates be offered the vice-presidential nomination if the presidential decree imposes the expected number of points maximization, supposing that no candidate is requested more than once and, each time a candidate rejects, another one is requested, until at least one has accepted or all have rejected.

This game which is made up of two parts – a selection process and an acceptance process – attests that the payoffs in the first part of the game (potential candidates) will be the intermediaries of the second (decision elements).Thus:

**Players:**

Presidential Board:4
Potential Candidates:
Player classified in 1st place –1;
Player classified in 2nd place – 2;
Player classified in 3rd place – 3.

**Strategies,** see [2] **:**

**Presidential Board:** The presidential board wants to establish the order in which the potential candidates will be invited to maximize the expected number of points. In this way the strategy for the presidential board will be the order in which the three potential candidates can be offered the vice-presidential nomination until at least one has accepted or all have rejected the offer - P.

**Potential Candidates:** The strategy of each potential candidate is to accept the offer – A - or to reject the offer - R.

**Payoffs:**
**Presidential Board:** The presidential board payoff is (a function of the attributed points in the daily preselection and of the potential candidates' probabilities of acceptance of the vice-presidency) the expected number of points[1] of each possibility in the order of the proposal presented to the potential

[1] *This expected number of points is the quantity to maximize through BPM application.*

candidates. Thus, for instance: Player 1 rejects the offer, 3 rejects the offer, 2 accepts the offer,presidential board payoff is: $\frac{1}{2}\times\frac{1}{5}\times\frac{1}{5}\times 8 = 0.16$, where the probability of player 1 rejecting the offer is $\frac{1}{2}$, $\frac{1}{5}$ is the probability of player 3 rejecting the offer and $\frac{1}{5}$ is the probability of player 2 accepting the offer.

**Potential Candidates:** For these players it is possible to define: if the player accepts the offer, it gets the "total prize", that is, it gets payoff 1. On the other hand, if it rejects the proposal, it does not get anything, so his/her payoff will be 0.

The game is illustrated in Table 6. Reading from left to the right, the first column indicates the period number and the second column indicates the move, see $[1 \text{ and } 3]$ number. The following columns mention who moves when and in what circumstances and what action is played when somebody is called upon to move. Last column indicates the payoffs vector in accordance with the strategies chosen by the players. It is easy to check that the order in which the three potential candidates can be offered the vice-presidential nomination must be:

**-**To invite in the first place the candidate classified in 2nd place, 2; if it rejects the proposal, the candidate classified in third place, 3, should be invited and if it does not accept, the candidate classified in first place, 1 should be invited. The expected number of points is 7.6*.*

**Table 6. Selection process game**

| 1 | 1 | (4, P) | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | | (1, A,0.5) | | | (5,1,0,0) |
| | | | (1, R,0.5) | | | |
| | 3 | | | (2, A,0.5) | | (2,0,1,0) |
| | | | | (2, R,0.5) | | |
| | 4 | | | | (3, A,0.4) | (0.5,0,0,1) |
| | | | | | (3, R,0.6) | (0,0,0,0) |
| | 3 | | | (3, A,0.8) | | (2,0,0,1) |
| | | | | (3, R,0.2) | | |
| | 4 | | | | (2, A,0.2) | (0.16,0,1,0) |
| | | | | | (2, R,0.8) | (0,0,0,0) |
| | 2 | | **(2, A,0.9)** | | | (**7.2**,0,1,0) |
| | | | **(2, R,0.1)** | | | |
| | 3 | | | (1, A,0.2) | | (0.2,1,0,0) |
| | | | | (1, R,0.8) | | |
| | 4 | | | | (3, A,0.4) | (0.16,0,0,1) |
| | | | | | (3, R,0.6) | (0,0,0,0) |
| | 3 | | | **(3, A,0.8)** | | (**0.4**,0,0,1) |
| | | | | **(3, R,0.2)** | | |
| | 4 | | | | **(1, A,0)** | (**0**,1,0,0) |
| | | | | | **(1, R,1)** | (**0**,0,0,0) |
| | 2 | | (3, A,1) | | | (5,0,0,1) |
| | | | (3, R,0) | | | |
| | 3 | | | (1, A,0.2) | | (0,1,0,0) |
| | | | | (1, R,0.8) | | |
| | 4 | | | | (2, A,0.2) | (0,0,1,0) |
| | | | | | (2, R,0.8) | (0,0,0,0) |
| | 3 | | | (2, A,0.5) | | (0,0,1,0) |
| | | | | (2, R,0.5) | | |
| | 4 | | | | (1, A,0) | (0,1,0,0) |
| | | | | | (1, R,1) | (0,0,0,0) |

## 7. CONFRONTING EQUILIBRIUMS

Making a confrontation among the equilibrium concepts mentioned in this work: Subgame Perfect Nash Equilibrium, Nash Equilibrium and Best Payoff Equilibrium it seems plausible to conjecture that:

-Best Payoff Equilibrium is stronger than Nash Equilibrium,
-Not every Nash Equilibrium is a Best Payoff Equilibrium,
-Best Payoff Equilibrium is necessarily a Nash Equilibrium,
-Best Payoff Equilibrium prevents the players against non-credible threats,
-In games with perfect information, Best Payoff Equilibrium is equivalent to Subgame Perfect Nash Equilibrium.

## 8. CONCLUSIONS

As we have seen throughout this chapter, the key idea of *BPM* is to admit that one player is skilled at predicting that other players will determine the strategy that will lead to their best return, and will also assess the credibility of their opponents' strategies.

Therefore, how to select the ideal return, in each of the respective sets of information of the player with whom it undertakes to decide, it is easy to find the equilibrium of the game.

Thus, proper application of this method requires information in great quantity and quality.

Under these conditions, the algorithms are easily understood, although their application can be tedious and require great discipline. Obviously, these constraints are easily overcome using information technology.


## ACKNOWLEDGEMENTS

This work is financed by national funds through FCT -Fundação para a Ciência e Tecnologia, I.P., under the projects UIDB/04466/2020 and UIDB/MULTI/04016/ 2020. Furthermore, the authors thank the Instituto Universitário de Lisboa and ISTAR-IUL, and Instituto Politécnico de Viseu and CI&DETS for their support.


## COMPETING INTERESTS

Authors have declared that no competing interests exist.

## REFERENCES


1. Benoit J P, Krishna V.Finitely repeated games. Econometrica.1985;53:*890–904*.
2. Bicchieri C, Jeffrey R, Skyrms B. The Logic of Strategy. Oxford University;1999.
3. Eberwein C. The sensitivity of repeated bargaining outcomes to the choice of first proposer. June. 2000.
4. Ferreira MAM, Andrade M, Matos M C. Separation theorems in Hilbert spaces convex programming. Journal of Mathematics and Technology. 2010; 1(5): 20-27.
5. Ferreira MAM, Filipe JA. Preferences relations and consumer theory. 16th Conference on Applied Mathematics APLIMAT 2017 Proceedings. 2017. Bratislava; Slovakia: 545-552.
6. Ferreira MAM., Matos MCP. A code form game in a selection process.In:*Quantitative Methods in Economics (Multiple Criteria Decision Making XIX).*[online] Trenčianske Teplice,Slovakia: Letra Edu, s. r. o., pp. 96-103.
   Availableat: http://fhi.sk/files/katedry/kove/ssov/proceedings/Zbornik2018.pdf
   [Accessed 4 February 2020].

7. Ferreira, MAM. A look on mathematical fundamentals for minimax theorem and Nash equilibrium existence. 19th Conference on Applied Mathematics APLIMAT 2020 Proceedings. 2020. Bratislava; Slovakia: 418-424.
8. Matos MC, Ferreira MAM. O método do best payoff. In: Ferreira MAM, Menezes R, Catanas F (Edits.). Temas em métodos quantitativos 4. Edições Sílabo. Lisboa; 2004: 193- 201.
9. Matos MC, Ferreira MAM. Game representation - code form. In: Namatame A, Kaizouji T, Aruga Y (Edits). The complex network of economic interactions (essays in agent –based economics and econophysics). Lecture Notes in Economics and Mathematical Systems. 2006; 567:321-334.
DOI: 10.1007/3-540-28727-2_22
10. Matos MC, Ferreira MAM, Filipe JA. Let the games begin and go on. International Journal of Business and Systems Research. 2018;12(1):43-52.
DOI: 10.1504/ijbsr.2018.088463

**Biography of author(s)**

Manuel Alberto M. Ferreira is Electrical Engineer and has a master's degree in Applied Mathematics by Lisbon University; achieved his PhD in Management-Quantitative Methods and Habilitation in Quantitative Methods by ISCTE-Lisbon University Institute. Former Chairman of Board of Directors and Vice-President of ISCTE-Lisbon University Institute. Full Professor (retired) of ISCTE-Lisbon University Institute. Emeritus Professor of ISCTE-Lisbon University Institute. Former Director of Department of Mathematics in ISTA-School of Technology and Architecture. Member of ISTAR-IUL (Integrated Researcher) and BRU-IUL (Associate Researcher) research centres. Former member of UNIDE (Integrated Researcher) research centre. Research interests: Mathematics; Statistics; Stochastic Processes-Queues and Applied Probabilities; Game Theory; Chaos Theory; Bayesian Statistics: Application to Forensic Identification; Applications to Economics, Management, Business, Marketing, Finance, and Social Problems in general; Published 446 papers, in Scientific Journals and Conference Proceedings, 43 book chapters, and 40 books. Presented 180 communications in International Scientific Conferences. Has 53 papers indexed in Scopus, 23 in Web of Science, 10 in zbMATH, and 65 in Crossref. He is editor of several Scientific Journals and Conference Proceedings. Reviewed papers for many Scientific Journals and Conferences.

Maria Cristina Peixoto Matos received her PhD and master's degree in Quantitative Methods at ISCTE –Instituto Universitário de Lisboa, Portugal. She got her Licentiate degree in mathematics from University of Coimbra, Portugal. She is currently professor at ESTGV -Instituto Politécnico de Viseu, Portugal. Her work has focused on investigating Game Theory and its applications to different Sciences.

She published academics works in different international scientific journals, such as Springer, Applied Mathematical Sciences, International Journal of Academic Research, International Journal of Business and Systems Research, International Journal of Latest Trends in Finance and Economics Sciences, Journal of Economics and Engineering, The Complex Networks of Economic Interactions: Essays in Agent-Based Economics and Econophysics. She was a member of international congress committees as well as chairman of sessions of different congresses