THE PANDEMIC PERIOD LENGTH MODELLED THROUGH QUEUE SYSTEMS

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Abstract
Despite the huge progress in infectious diseases control worldwide, still epidemics happen, being the annual influenza outbreaks examples of those occurrences. To have a forecast for the epidemic period length is very important because, in this period, it is necessary to strengthen the health care. With more reason, this happens with the pandemic period, since the pandemic is an epidemic with a great population and geographical dissemination. Predominantly using results on the $M|G|\infty$ queue busy period**,** it is presented an application of this queue system to the pandemic period's parameters and distribution function study**.** The choice of the $M|G|\infty$ queue for this model is adequate, with great probability, since the greatest is the number of contagions the greatest the possibility of the hypothesis that they occur according to a Poisson process.

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1 RISING THE MODEL

In the M|G|∞ queue system

- The customers arrive according to a Poisson process at rate λ

- Receive a service which time length is a positive random variable with distribution function $G(.)$ and mean α

- When they arrive, each one finds immediately an available server

- Each customer service is independent from the other customers' services and from the arrivals process

-The traffic intensity is $\rho = \lambda\alpha$.

A pandemic is an epidemic of infectious disease that is spreading through human populations across a large region[1].
So it is easy to understand how the $M|G|\infty$ queue can be applied to the pandemic period study, owing to this system suitability to deal with every kind of large populations[2]. Then
- The parameter λ is the rate at which people is infected, supposed that the infections occur according to a Poisson process
-The service time is the time throughout which an infected person stays sick.
In a queue system a busy period is a period that begins when a costumer arrives at the system finding it empty, ends when a costumer abandons the system letting it empty and in it there is always at least one customer present. So in a queuing system there is a sequence of idle and busy periods.
In the M|G|∞ queue system the idle periods have an exponential time length with mean λ^{-1}, as it happens with any queue system with Poisson arrivals.

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[1]For instance a continent, or even worldwide, see [15].
[2]For more examples of the $M|G|\infty$ queue practical applications, see, for instance, [3], [5], [9 − 12] and [14] .

Although the busy period's distribution is much more complicated it is possible to present some results as it will be seen.

For what interests in this work

- A busy period is a pandemic period
- An idle period is a period free of the disease.

The results that will be presented are on pandemic period's length and their number in a certain time interval.

2 THE PANDEMIC PERIOD LENGTH

Call *PP* the random variable pandemic period length. According to the results known for the $M|G|\infty$ queue busy period length distribution

- $$E[PP] = \frac{e^{\rho}-1}{\lambda} \qquad (2.1)$$

whichever is an infected person sickness time length distribution, see [14]

- As for $Var[PP]$, it depends on the whole sickness time length distribution probabilistic structure. But Sathe, see [13], demonstrated that

$$\lambda^{-2}\max[e^{2\rho} + e^{\rho}\rho^2\gamma_s^2 - 2\rho e^{\rho} - 1; 0] \leq Var[PP] \leq \lambda^{-2}$$
$$[2e^{\rho}(\gamma_s^2+1)(e^{\rho}-1-\rho) - (e^{\rho}-1)^2] \qquad (2.2),$$

where γ_s is the sickness time length coefficient of variation

-If an infected person sickness time length distribution function is

$$G(t) = \frac{e^{-\rho}}{(1-e^{-\rho})e^{-\lambda t} + e^{-\rho}}, t \geq 0, \quad (2.3)$$

the *PP* distribution function is

$$PP(t) = 1 - (1-e^{-\rho})e^{-e^{-\rho}\lambda t}, t \geq 0 \qquad (2.4),$$

see [2]

-If the sickness time length distribution function of an infected person is such that

$$G(t) = 1 - \frac{1}{1 - e^{-\rho} + e^{-\rho + \frac{\lambda}{1-e^{-\rho t}}}}, t \geq 0 \qquad (2.5)$$

the *PP* distribution function is

$$PP(t) = 1 - e^{-(e^{\rho}-1)^{-1}\lambda t}, t \geq 0 \qquad (2.6),$$

see [4][3]

-For α and ρ great enough (very intense infectious conditions) since $G(.)$ is such that for α great enough $G(t) \cong 0, t \geq 0$,

$$PP(t) \cong 1 - e^{-\lambda e^{-\rho} t}, t \geq 0 \qquad (2.7),$$

see [12].

Note:

-As for this last result, begin noting that many probability distributions fulfill the condition $G(t) \cong 0, t \geq 0$ for α great enough. The exponential distribution is one example.

-As for the meaning of α and ρ great enough, computations presented in [12] it is shown that for $\lambda = 1$, after $\rho = 10$ it is reasonable to admit (2.7) for many service time distributions.

3 PANDEMIC PERIODS OCURRENCE IN A TIME INTERVAL

After the renewal processes theory, see [1], calling $R(t)$the mean number of pandemic periods that begin in $[0,t]$, being $t = 0$ the beginning instant of a pandemic period, it is possible to obtain, see [6,7],

$$R(t) = e^{-\lambda \int_0^t [1-G(v)]dv} + \lambda \int_0^t e^{-\lambda \int_0^u [1-G(v)]dv} du \qquad (3.1)$$

and, consequently,

$$e^{-\rho}(1+\lambda t) \leq R(t) \leq 1 + \lambda t \qquad (3.2),$$

see [6].

Also,

A) $G(t) = \frac{e^{-\rho}}{(1-e^{-\rho})e^{-\lambda t} + e^{-\rho}}, t \geq 0$

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[3]Expressions (2.3) and (2.5) result from, see [8],

$$G(t) = 1 - \frac{1}{\lambda} \frac{\left(1-e^{-\rho}\right) e^{-\lambda t - \int_0^t \beta(u)du}}{\int_0^\infty e^{-\lambda w - \int_0^w \beta(u)du} dw - \left(1-e^{-\rho}\right) \int_0^t e^{-\lambda w - \int_0^w \beta(u)du} dw}, t \geq 0, -\lambda \leq \frac{\int_0^t \beta(u)du}{t} \leq \frac{\lambda}{e^{\rho}-1} \quad ,$$

making $\beta(t) = \beta$ (constant), being $\beta=0$ for (2.3) and $\beta = \frac{\lambda}{e^{\rho}-1}$ for (2.5). For this collection of service time distributions, with β constant, the $M|G|\infty$ queue busy period length is exponentially distributed with an atom at the origin as (2.4). For $\beta = -\lambda$ it is purely deterministic. And, for $\beta = \frac{\lambda}{e^{\rho}-1}$, it is purely exponential as (2.6). So, this collection of service time distributions gives more situations in which it is possible to have friendly distributions for the $M|G|\infty$ queue busy period length and , in addition, for *PP*.

$$R(t) = 1 + \lambda e^{-\rho} t \quad (3.3)$$

B) $G(t) = 1 - \frac{1}{1-e^{-\rho}+e^{-\rho+\frac{\lambda}{1-e^{-\rho}}t}}, t \geq 0$

$$R(t) = e^{-\rho} + (1 - e^{-\rho})^2 + \lambda e^{-\rho} t + e^{-\rho}(1 - e^{-\rho}) e^{-\frac{\lambda}{1-e^{-\rho}}t} \quad (3.4)$$

C) $G(t) = \begin{cases} 0, t < \alpha \\ 1, t \geq \alpha \end{cases}$

$$R(t) = \begin{cases} 1, t < \alpha \\ 1 + \lambda e^{-\rho}(t - \alpha), t \geq \alpha \end{cases} \quad (3.5)$$

D) If the sickness time length is exponentially distributed

$$e^{-\rho\left(1-e^{-\frac{t}{\alpha}}\right)} + \lambda e^{-\rho} t \leq R(t) \leq e^{-\rho\left(1-e^{-\frac{t}{\alpha}}\right)} + \lambda t \quad (3.6)$$

4 CONCLUSIONS

So that this model can be applied it is necessary that the infections occur according to a Poisson process at constant rate. It is a hypothesis perfectly admissible in this kind of phenomena, since they have great geographic spread, even worldwide. Among the results presented, (2.1), (2.2), (2.7) and (3.2) are remarkable for the easiness and also for requiring only the knowledge of the infectious rate λ, the mean sickness time α, and the sickness time variance. The other results are more complex and demand the goodness of fit test for the distributions indicated to the sickness times.

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