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EMPIRICAL ESSAYS ON PORTFOLIO MANAGEMENT

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PhD in Finance

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Resumo

Esta dissertação é constituída por três estudos que testam empiricamente três tópicos relacionados com a gestão de carteiras com a mesma base de informação (preços de todas as ações transacionadas nos índices S&P 500 e STOXX 600 entre 2002 e 2019).

O primeiro estuda o número mínimo de ações necessário para uma carteira obter grande parte dos benefícios da diversificação em termos de risco e rendibilidade. Verificamos que os maiores benefícios da diversificação podem ser obtidos com uma carteira de 50 ou 64 ações consoante a ponderação da carteira seja de pesos iguais ou ponderada pela capitalização bolsista, respetivamente.

O segundo examina a relação entre a rendibilidade do mês seguinte e variáveis de risco (desvio padrão, assimetria e curtose). Geralmente, não vemos uma relação estritamente crescente ou decrescente entre as variáveis de risco e a rendibilidade do mês seguinte nem diferenças estatisticamente significativas entre a rendibilidade média das carteiras dos quintis com valores mais baixos e valores mais altos das variáveis de risco.

O terceiro centra-se na implementação de três estratégias de investimento baseadas no modelo Treynor-Black. A estratégia do modelo Treynor-Black com maior rácio de Sharpe (índice de mercado usado como *benchmark* da carteira passiva) tem menor rendibilidade e maior risco face à carteira passiva. Apontamos três razões que levam a carteira passiva a obter uma rendibilidade média ajustada pelo risco superior às estratégias baseadas no modelo Treynor-Black.

Classificação JEL: G11, G12

Palavras-chave: Diversificação, Dimensão da carteira, Rendibilidades de ações cross-section, Risco idiossincrático, Gestão ativa, Modelo Treynor-Black.

Abstract

This dissertation consists in three studies that test empirically three topics related with portfolio management with the same dataset (prices of all stocks traded in the S&P 500 and STOXX 600 between 2002 and 2019).

The first studies the minimum number of stocks that a portfolio should have to achieve the major benefits of diversification in terms of risk and return. We find that major benefits of diversification can be achieved with an equal-weighted portfolio with 50 stocks and a value-weighted portfolio with 64 stocks.

The second examines the relationship between next month return and risk variables (standard deviation, skewness, and kurtosis). Generally, we see no clear increasing or decreasing monotonic relationships between risk variables and next month return neither statistically significant differences between the average return of quintile portfolios formed with stocks of lowest values and stocks with highest values of risk variables.

The third focuses on the implementation of three investment strategies based on Treynor-Black model. The Treynor-Black model strategy with highest Sharpe measure (when the market index is used as the market portfolio) has lower return and higher risk than the passive portfolio. We point out three reasons that lead passive portfolio to achieve superior risk-adjusted average return over the Treynor-Black model strategies.

JEL Classification: G11, G12

Keywords: Diversification, Number of stocks, Cross-section of stock returns, Idiosyncratic risk, Active management, Treynor-Black model.

Sumário Executivo

O tema desta dissertação consiste no teste empírico de três tópicos relacionados com a gestão de carteiras. A base de informação usada é comum aos três tópicos e consiste, essencialmente, nos preços de todas as ações transacionadas nos índices S&P 500 (mercado americano) e STOXX 600 (mercado europeu) entre janeiro de 2002 e dezembro de 2019.

O primeiro objetivo de investigação desta dissertação é o estudo do número mínimo de ações que uma carteira deve incorporar para maximizar os benefícios de diversificação em termos de risco e rendibilidade. A principal motivação para esta investigação é o debate na literatura sobre o número mínimo necessário de ações numa carteira para se obterem benefícios de diversificação satisfatórios. Diversos estudos demonstram que 10 ações são suficientes para alcançar benefícios de diversificação satisfatórios, mas, por outro lado, existem estudos, especialmente recentes, que desafiaram este facto, mostrando que 100 ações ou mais são necessárias para alcançar benefícios de diversificação satisfatórios. Através da geração aleatória de carteiras (sem reposição) com diferentes tamanhos, investigamos como o risco e a rendibilidade das carteiras mudam à medida que o número de ações aumenta em carteiras de pesos iguais e ponderadas pela capitalização bolsista.

As principais contribuições do primeiro estudo são as seguintes. Primeiro, comparamos os benefícios da diversificação nos mercados dos EUA e da Europa, e não em apenas um mercado como a maioria dos estudos sobre diversificação. Segundo, embora os estudos empíricos se concentrem, maioritariamente, nas carteiras com pesos iguais, analisamos também os benefícios da diversificação das carteiras ponderadas por capitalização bolsista. Terceiro, analisamos empiricamente o resultado em mais de sete milhões de carteiras para obter rendibilidade e risco de carteiras com diferentes configurações, assegurando a robustez dos resultados através do aumento e diminuição do número de simulações para cada uma das dimensões das carteiras. Em quarto lugar, utilizamos uma técnica de amostragem que inclui todas as ações que em algum momento fizeram parte do índice representativo de cada mercado ao longo do período, evitando assim o enviesamento dos resultados devido aos dados conterem apenas ações que estiveram cotadas durante a totalidade do período histórico. Finalmente, enquanto a maioria dos estudos apenas considera os resultados de estratégias para o período completo da amostra, a nossa análise apresenta resultados segregados por ano.

Os principais resultados do primeiro estudo podem ser sumarizados a três níveis. Primeiro, os principais benefícios da diversificação, nos mercados dos EUA e da Europa, podem ser obtidos com uma carteira de pesos iguais com 50 ações e uma carteira ponderada por capitalização bolsista com 64 ações. Estas carteiras reduzem, no mínimo, 95% do risco diversificável. Segundo, observamos que o aumento do número de ações em carteiras de pesos iguais não tem um impacto significativo na riqueza média do final do período, enquanto o referido aumento tem um efeito ligeiramente negativo nas carteiras ponderadas por capitalização bolsista. Finalmente, o desvio padrão da riqueza final diminui à medida que o número de ações de uma carteira aumenta em ambos os mercados e em ambas as abordagens de ponderação. As carteiras de pesos iguais com 50 ações e as carteiras ponderadas por capitalização bolsista com 64 ações têm um desvio padrão da riqueza do final do período inferior a 0,05 por cada $1 ou 1€ de investimento nos mercados dos EUA e da Europa, respetivamente.

O segundo objetivo de investigação proposto nesta dissertação é o estudo da relação entre a rendibilidade do mês seguinte e diferentes variáveis de risco (desvio padrão, assimetria e curtose). Os resultados divergentes apresentados na literatura sobre a relação entre estas variáveis de risco e a rendibilidade são a principal motivação para esta investigação. Tanto são reportadas relações negativas como positivas entre a rendibilidade e as mesmas variáveis de risco.

As principais contribuições do segundo estudo são as seguintes. Primeiro, comparamos a relação entre risco e rendibilidade do mês seguinte no mercado dos EUA e da Europa, em vez de utilizar apenas um mercado como a maioria dos estudos sobre este tópico. Segundo, analisamos também se a referida comparação mostra diferenças significativas entre carteiras de pesos iguais e carteiras ponderadas por capitalização bolsista. Terceiro, estudamos a relação da rendibilidade do mês seguinte com nove variáveis, incluindo variáveis realizadas e esperadas, para cobrir a maior parte das abordagens utilizadas noutros estudos sobre este tema. Quarto, analisamos o desempenho de uma carteira autofinanciada que consiste em comprar ou vender o quintil de ações com o valor mais baixo de cada variável de risco e vender ou comprar o quintil de ações com o valor mais alto de cada variável de risco. Em quinto lugar, avaliamos se uma estratégia de investimento baseada em carteiras dos quintis de ações com valores baixos ou altos de variáveis de risco consegue uma rendibilidade mais elevada do que uma carteira de referência. Finalmente, utilizamos uma técnica de amostragem que inclui todas as ações que em algum momento fizeram parte do índice representativo de cada mercado ao longo do período, evitando assim o enviesamento dos resultados devido aos dados conterem apenas ações que estiveram cotadas durante a totalidade do período histórico.

Os resultados obtidos no segundo estudo sugerem a existência de algumas relações estritamente crescentes ou decrescentes entre as variáveis de risco e a rendibilidade do mês seguinte no mercado dos EUA. Adicionalmente, os resultados indicam diferenças estatisticamente significativas, a um nível de 5%, entre a rendibilidade média das carteiras dos quintis com valores mais baixos e valores mais altos das variáveis de risco (carteira autofinanciada). No entanto, raramente essas diferenças estão presentes, simultaneamente, em ambas as abordagens de ponderação ou em ambos os mercados analisados. No que diz respeito à diferença entre a rendibilidade média dos quintis extremos e uma carteira composta por todas as ações do índice de mercado, observamos que, em geral, esta diferença é inferior à rendibilidade média da carteira autofinanciada. Dada a falta de similaridade entre os resultados dos mercados americano e europeu para as mesmas variáveis de risco, parece que as relações entre as variáveis de risco e a rendibilidade do mês seguinte são originadas de forma aleatória e não por significância económica. Os resultados para ambos os mercados e ambos os esquemas de ponderação mostram que, pelo menos uma relação negativa e uma relação positiva, podem ser encontradas para as estimativas de desvio padrão, assimetria e curtose.

A terceira linha de investigação desta dissertação visa a implementação empírica de três estratégias de investimento baseadas no modelo Treynor-Black usando como *benchmark* para carteira passiva: a carteira tangente, a carteira tangente com restrições de vendas a descoberto e o índice de mercado. A principal motivação é o facto do modelo Treynor-Black ter tido pouco impacto na comunidade financeira e, consequentemente, os estudos empíricos sobre este tema serem bastante raros. Investigamos como o risco e a rendibilidade das referidas estratégias de investimento se comparam com uma estratégia passiva.

As principais contribuições do terceiro estudo são as seguintes. Primeiro, temos um grande conjunto de dados com 500 ações do mercado americano e 600 ações do mercado europeu, que é de maior dimensão do que a maioria dos estudos sobre o tema de seleção de carteiras. Segundo, uma vez que a maioria dos estudos empíricos se concentra na carteira de média-variância, analisamos uma estratégia de seleção de carteiras baseada no modelo Treynor-Black, o qual assume que as ações não têm preços eficientes. Terceiro, destacamos os principais problemas de utilizar o modelo Treynor-Black empiricamente. Finalmente, utilizamos uma técnica de amostragem que inclui todas as ações que em algum momento fizeram parte do índice representativo de cada mercado ao longo do período, evitando assim o enviesamento dos resultados devido aos dados conterem apenas ações que estiveram cotadas durante a totalidade do período histórico.

Os resultados do terceiro estudo mostram que, em ambos os mercados, a estratégia baseada no modelo Treynor-Black com maior rácio de Sharpe (quando o índice de mercado é a carteira de mercado), tem menor rendibilidade e maior risco que a carteira passiva. Apontamos três razões que levam o modelo Treynor-Black a não ter uma rendibilidade média ajustada pelo risco consistentemente superior à carteira passiva. Erro de estimativa nos *alphas* (posições longas com menor rendibilidade realizada do que as expectativas e posições curtas com maior rendibilidade realizada do que as expectativas), peso reduzido do investimento na carteira ativa quando a rendibilidade da carteira ativa é maior do que a rendibilidade da carteira passiva, e peso elevado, quando o inverso ocorre, e pesos extremos da carteira ativa que originam elevados níveis de risco.

A dissertação está organizada da seguinte forma. O Capítulo 2 descreve em detalhe os dados usados, os quais são comuns para os três estudos. O Capítulo 3 investiga o número mínimo de ações a incorporar num portfolio com vista a maximizar os benefícios decorrentes do efeito de diversificação. O Capítulo 4 estuda a relação entre a rendibilidade do mês seguinte e variáveis de risco (desvio padrão, assimetria e curtose). O Capítulo 5 estuda a implementação empírica de três estratégias de investimento baseadas no modelo Treynor-Black. O Capítulo 6 resume as conclusões.

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Glossary of Acronyms

AMEX	American Stock Exchange
AP	Active Portfolio
AR	Autoregressive
ARCH	Autoregressive Conditional Heteroskedastic
CAPM	Capital Asset Pricing Model
CRSP	Center for Research in Security Prices
EE	Estimation Error
EGARCH	Exponential General Autoregressive Conditional Heteroskedastic
EPW	End-of-Period Wealth
EPWSD	End-of-Period Wealth Standard Deviation
EW	Equal-weighted
FF-3	Fama and French (1993) three-factor model
GARCH	General Autoregressive Conditional Heteroskedastic
NYSE	New York Stock Exchange
OLS	Ordinary Least Squares
PP	Passive Portfolio
p.p.	Percentage points
S&P 500	Standard and Poor's 500
SR	Sharpe Measure
SS	Short Selling
STOXX 600	STOXX Europe 600
TB	Treynor-Black
TB-MI	Treynor-Black using market index as benchmark portfolio
TP	Tangency Portfolio
TP-Rest	Treynor-Black using tangency portfolio with short selling restriction as benchmark portfolio
TP-Unrest	Treynor-Black using tangency portfolio as benchmark portfolio
VW	Value-weighted

Chapter 1. Introduction/Executive Summary

This dissertation's main theme is the empirical test of three topics related with portfolio management. The dataset used is common to all three topics and consists, essentially, in prices of all stocks traded in the S&P 500 (U.S. market) and in STOXX 600 (European market) between January 2002 and December 2019.

The first research goal of this dissertation is the study of the minimum number of stocks that a portfolio should have to achieve the major benefits of diversification in terms of risk and return. The main reason for this investigation is the debate in the literature regarding the required minimum number of stocks in a portfolio to achieve the satisfactory benefits of diversification. Numerous studies have shown that 10 stocks are sufficient to achieve satisfactory benefits of diversification, but on the other hand, numerous works, especially recent, have challenged this fact by showing that 100 stocks or more are required for satisfactory benefits of diversification. Through random generation of portfolios (without replacement) with different sizes, we investigate how risk and return of a portfolio change as the number of stocks in equal-weighted and value-weighted portfolios increases.

The main contributions of the first study are as follows. First, we compare the benefits of diversification in the U.S. and European markets instead of using only one market as the majority of studies about diversification. Second, although most empirical studies focus on equal-weighted portfolios, we also analyze the benefits of diversification of value-weighted portfolios. Third, we simulate more than seven million portfolios to obtain return and risk of portfolios with different sizes, assuring the robustness of the results by increasing and decreasing the number of simulations performed for each portfolio size. Fourth, we use a sampling technique that deals with delisted stocks over the period to avoid survivorship bias. Finally, whereas most studies are concerned solely on the full sample period, our analysis is performed year by year.

The first study major findings are as follows. First, major benefits of diversification, in the U.S. and European markets, can be achieved with an equal-weighted portfolio with 50 stocks and a value-weighted portfolio with 64 stocks. These portfolios reduce, at least, 95% of diversifiable risk. Second, we observe that the increase of the number of stocks in equal-weighted portfolios has no significant impact on average end-of-period wealth, while the mentioned increase has a slight negative effect in value-weighted portfolios. Finally, end-of-period wealth standard deviation decreases as the number of stocks in a portfolio increases in both markets and in both weighting schemes. Equal-weighted portfolios with 50 stocks and value-weighted portfolios with 64 stocks have an end-of-period wealth standard deviation lower than 0.05 per $1 or 1€ of investment in the U.S. and European markets, respectively.

The second research goal purposed in this dissertation is the examination of the relationship between next month return and risk variables (standard deviation, skewness, and kurtosis). The mixed results presented in the literature that examined the relationship between risk variables and next month return are the primary reason for this investigation. Negative, as well as positive relationships between next month return and the same risk variables are reported.

The main contributions of the second study are as follows. First, we compare the relationship between risk and next month return in the U.S. and European markets, instead of using only one market as most studies about this topic. Second, we analyze if the referred comparison shows significant differences in equal-weighted versus value-weighted portfolios. Third, we study the relation of next month return with nine risk variables, including realized and expected variables, to cover most of the approaches used in other studies of this subject. Fourth, we analyze the performance of a self-financing portfolio that consists in buying or selling the quintile portfolio of stocks with the lowest value of each risk variable and selling or buying the quintile portfolio of stocks with the highest value of each risk variable. Fifth, we evaluate if an investment strategy based on quintile portfolios of stocks with lowest or highest values of risk variables achieves higher average return than a benchmark portfolio. Finally, to avoid survivorship bias we use a sampling technique that deals with delisted stocks over the period.

We found some monotonic relations between next month return and risk variables in the U.S. market. Additionally, the results indicate statistically significant differences, at a 5% level, between the average return of quintile portfolios formed with stocks of lowest values and stocks with highest values of risk variables (self-financing portfolio). Nevertheless, rarely these differences are present, simultaneously, in both weighting schemes or in both markets. With respect to the difference of average return between extreme quintile portfolios and a portfolio composed by all stocks traded in the market index, we observe that, generally, this difference yields lower average return than the self-financing portfolio using the same risk variables. Given the lack of similarity between the results in the U.S. and European markets for the same risk variables, it appears that relations between risk variables and next month return are originated randomly rather than by economic significance. The results for both markets and both weighting schemes show that, at least one negative relation and one positive relation, can be found for standard deviation, skewness, and kurtosis estimates.

The third line of research of this dissertation is about an empirically implementation of three investment strategies based on the Treynor-Black (hereafter TB) model using as passive benchmarks: the tangency portfolio, the tangency portfolio with short selling restriction and the market index. The primary motivation is the fact that TB model had little impact on the financial community, and consequently, empirically studies on this topic are quite rare. We investigate how risk and return of the referred investment strategies compare with a passive strategy.

The principal contributions of the third study are as follows. First, we have a large dataset with 500 stocks from the U.S. market and 600 stocks from the European market, which is larger than the majority of the studies on the topic of portfolio selection. Second, since most of the empirical studies focus on the mean-variance framework, we analyze a portfolio allocation strategy based on TB model, assuming inefficiently priced stocks. Third, we highlight the principal drawbacks of using TB model empirically. Finally, we use a sampling technique that deals with delisted stocks over the period to avoid survivorship bias.

The results show that, in the U.S. and European markets, the TB model strategy with highest Sharpe measure (when the market index is used as the market portfolio) has lower return and higher risk than the passive portfolio. We point out three reasons that lead TB model to have not a consistently superior risk-adjusted return over the passive portfolio. Alpha estimation error (long positions with lower realized return than expectations and short positions with higher realized return than expectations), small weight of investment on the active portfolio when the active portfolio return is larger than the passive portfolio return, and large when the inverse occurs, and extreme weights of the active portfolio that lead to high levels of risk.

The remainder of this dissertation is organized as follows. Chapter 2 describes the dataset in detail, which is common for the three studies. Chapter 3 investigates the minimum number of stocks that a portfolio should have to achieve the major benefits of diversification in terms of risk and return. Chapter 4 studies the relationship between next month return and risk variables (standard deviation, skewness and kurtosis). Chapter 5 studies the empirically implementation of three investment strategies based on TB model. Chapter 6 summarizes the conclusions.

Chapter 2. Data Description and General Methodology

In this Chapter, we describe the dataset used in this study, as well as, the methodological aspects common to the three studies that form the empirical analysis of this dissertation. In section 2.1, we identify the stocks, indexes and rates of the dataset, the period to which their prices refer and the details regarding the adjustments applied. In section 2.2, we present the methodologies used throughout the dissertation. More methodological aspects are explained in each of the chapters.

2.1 Data Description

All data used in this dissertation was collected from Bloomberg. The dataset is based on last prices provided by the exchange recorded each trading day of all stocks ever traded on Standard & Poor's 500 (hereafter S&P 500) and STOXX Europe 600 (hereafter STOXX 600), and their respective market indexes, during the period between January 2002 and December 2019. This range of dates corresponds to the longest annual sample available at the start of the time of writing this dissertation after the introduction of the euro currency. The prices are adjusted to reflect cash dividends, spin-offs, stock splits/consolidations, stock dividend/bonus and rights offerings/entitlement.

S&P 500 is a value-weighted stock market index tracking the prices of 500 large U.S. companies listed on stock exchanges. It is one of the most followed equity indices. As of December 31, 2020, more than $5.4 trillion was invested in assets tied to the performance of the index. The S&P 500 was launched in 1957 (S&P Dow Jones Indices, 2021).

STOXX 600 is a value-weighted stock index of European stocks designed by STOXX Ltd. This index has a fixed number of 600 components representing large, mid, and small capitalization companies among 17 European countries, covering approximately 90% of the free-float market capitalization of the European stock market. The countries that make up the index are the United Kingdom, France, Germany, Switzerland, Austria, Belgium, Denmark, Finland, Ireland, Italy, Luxembourg, the Netherlands, Norway, Poland, Portugal, Spain, and Sweden. The STOXX 600 was introduced in 1998 (STOXX, 2021).

Gilbert and Strugnell (2010) comment that it is well established in financial research that ignoring delisted companies when conducting historical research leads to survivorship bias in results. They conclude that including data for delisted stocks is likely to have a significant effect on the results reached and researchers should attempt to include such data in any empirical analysis of this sort. In our study, the survivorship bias is removed since the calculations are based in all available stocks traded on S&P 500 or STOXX 600 at the beginning of each period. Thus, the return of the delisted stocks before the portfolio rebalancing are included in our dataset.

For companies that list stocks on two or more different exchanges (dual listing), the market value considered is the market value of the stock listed in the market index in analysis.

In the European market, the market value of stocks issued by companies from countries[1] that do not use Euro as their domestic currency is translated to Euro using the exchange rate of the respective date.

Tables 2.1 and 2.2 present descriptive statistics in the U.S. and European markets, respectively, of annual return and risk by year for stocks listed in the corresponding market index at the beginning of each period.

Regarding annual return, the negative means in 2002, 2008 and 2018 of U.S. and European stocks are related with dot.com bubble in 2002, the global financial crisis in 2008 and bitcoin crash in 2018. Additionally, we observe a negative mean in 2011 in European stocks associated with the Euro crisis that reached a peak in 2011. In both markets, the standard deviation and kurtosis estimates of annual stock returns do not have the highest values when the stock market crashes. This suggests that in periods of negative returns, most stocks have negative returns reducing the dispersion among stock returns. On the other hand, in periods of high returns, the dispersion among stock return is higher, indicating that stocks could have high or low returns. In periods of negative returns in the stock market, the maximum return of a single stock tends to be lower compared with periods of positive returns. On the other hand, the minimum return of a single stock has a lower sensibility to increases or decreases in stock market values. Often, annual maximum return shows a difference of 100 percentage points (hereafter p.p.) between periods of expansion and recession, while annual minimum returns never have a difference higher than 70 p.p. Thus, standard deviation and kurtosis estimates of annual stock returns are generally lower when the prices in stock markets fall. Finally, we also observe in U.S. and European markets that annual stock returns mean tends to be superior to the median. This fact is related with the positive skewness, which tends to be higher when the positive difference between the mean and the median is higher.

[1] Stocks from companies belonging to Czech Republic, Denmark, Iceland, Norway, Poland, Sweden, Switzerland and United Kingdom.

With respect to the average annual risk, the largest observed values are associated with crashes or bubbles in stock market prices. The year with lowest return (2008) has the highest annual risk mean in the U.S. market (66.2%) and in the European market (58.6%) and the year with highest return (2009) has the second highest annual risk mean in the U.S. market (51.2%) and in the European market (45.7%). In every year and in both markets, the annual risk mean is higher than the median, which is also reflected in the positive skewness. Standard deviation of annual risk has the maximum values in 2002, 2008 and 2009 in the U.S. and European markets. These years are related with large negative (2002 and 2008) and large positive (2009) variations in stock market prices.

Table 2.3 presents the annual return and risk of equal-weighted and value-weighted portfolios composed by all the stocks in the market index in the U.S. and European markets. The results indicate that equal-weighted portfolios have higher return, on average, than value-weighted portfolios in both markets. This result suggests that stocks of smaller firms tend to outperform stocks of larger companies. In fact, this effect is included in Fama and French (1993) three-factor model to explain returns.

The scope of this dataset does not replicate the market index return using single stocks. There are two main reasons for the differences. First, market indexes have quarterly rebalancing, and we use annual rebalancing in Chapter 3 and monthly rebalancing in Chapters 4 and 5. Second, market indexes managers may substitute stocks before the rebalancing date due to merger, acquisition or bankruptcy and we do not substitute any stock before the rebalancing date.

The dataset also includes the 1-month USD LIBOR, used as proxy for the risk-free rate in the U.S. market, in line with Fabozzi, Cheng and Chen (2007), and 1-month Euribor, used as proxy for the risk-free rate in the European market, in line with Breitenfellner and Wagner (2012) and Elyasiani, Gambarelli and Muzzioli (2020). The 1-month USD LIBOR is the average interest rate at which several banks in London are prepared to lend to one another in USD with a maturity of one month. The 1-month Euribor is the interest rate at which several European banks lend one another funds denominated in euros with a maturity of one month.

Table 2.4 presents risk-free rate annual return and risk by year in the U.S. and European markets. We measure the risk-free rate return as the daily logarithmic differences of the implicit prices in pure discount bonds having a yield that corresponds to our risk-free rates proxies. The results show a positive return in 2008 and 2009, in both markets, which is related with the decrease in interest rates originated by the global financial crisis in 2008. In 2019, the positive return in the U.S. market is due the Federal Reserve cuts on interest rates that reverse nearly all the 2018's rate increases.

2.2 General Methodology

For each stock *i*, the return (geometric return) is defined as:

$$r_i = \exp\left(\sum_{d=1}^{n} \ln\left(\frac{p_d}{p_{d-1}}\right)\right) - 1 \tag{2.1}$$

where p_d is the adjusted price of stock i at day d and p_{d-1} is the adjusted price of stock i on the day $d-1$, n is the number of trading days considered in each period. For each portfolio p, the rate of return is the weighted average return of all stocks included in the portfolio, and is defined as:

$$r_p = \sum_{i=1}^{m} w_i \times r_i \quad s.t. \quad \sum_{i=1}^{m} w_i = 1 \tag{2.2}$$

where m is the number of stocks in the portfolio p and w_i is the weight of stock i in the portfolio p. For each stock i, the risk is defined as the standard deviation of daily logarithmic returns[2]. For each portfolio p, portfolio risk is defined as:

$$s_p = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j Cov\left(r_i, r_j\right)} \tag{2.3}$$

In the following chapters, we use a market model defined as

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i\left(r_{M,t} - r_{f,t}\right) + \varepsilon_{i,t} \tag{2.4}$$

[2] Levy (1968) argues that since the geometric return is the correct measure of central tendency, it should be the reference point for the computation of dispersion. And since the geometric return is simply the antilog of the average logarithm of a set of numbers, the appropriate measure of dispersion is the standard deviation of the logarithm returns.

where $r_{f,t}$ is the risk-free rate, α_i and β_i are the estimation parameters, $r_{M,t}$ is the market portfolio return and $\varepsilon_{i,t}$ is the random error at period *t*. Equation (2.4) is based on a concept of market equilibrium in which the expected excess return on any single risky asset is proportional to the expected excess return on the market portfolio. The coefficient β_i represents the asset return's sensitivity to changes in the market portfolio return. So, β_i is a sensitivity risk measure relative to the market risk factor. For simplicity, we refer to this risk measure as the CAPM beta of the asset. The term inside the brackets on the right-hand side is the market risk premium: it is the additional return (above the risk-free rate) that investors can expect to be compensated for the risk of holding the market portfolio. In equilibrium no single asset may have an abnormal return where it earns a return above (or below) the risk-free rate without taking any market risk. Therefore, any asset with a positive alpha has an expected return that is more than its equilibrium return and should be bought, and any asset with a negative alpha has an expected return that is below its equilibrium return, and so, should be sold. The stock's specific return and the respective intercept term α_i should be zero if the equilibrium holds. We assume that (i) the error term is independent and identically distributed with mean equal to zero and some specific disturbance and (ii) the error terms for two different stocks are contemporaneously uncorrelated.

Frankfurter (1976) affirms that usually a stock market index is substituted as a proxy for the market portfolio in CAPM. Following Lim, Durand and Yang (2014), for the U.S. market, we use S&P 500 as market index and for the European market, we use STOXX 600 under the same rational.

Table 2.1: Descriptive statistics of annual return and risk in the U.S. market

Table 2.1 presents descriptive statistics of annual return (Panel A) and risk (Panel B) by year for the stocks traded in the U.S. market (S&P 500) at the beginning of each period.

Panel A: Annual return								
	Mean	Median	StdDev	Min	Max	Skew	Kurt	Count
2002	-18.1%	-15.0%	28.5%	-98.3%	82.8%	-0.4	3.2	500
2003	41.6%	31.8%	43.8%	-33.7%	428.2%	3.0	19.4	500
2004	16.6%	14.6%	27.8%	-54.0%	209.6%	1.8	12.9	500
2005	8.5%	5.4%	26.6%	-96.3%	128.5%	0.5	6.1	500
2006	15.5%	15.3%	22.5%	-89.4%	152.9%	0.5	6.9	500
2007	3.3%	2.8%	33.4%	-84.2%	140.1%	0.7	4.8	500
2008	-38.6%	-38.0%	25.8%	-98.0%	38.6%	0.2	2.9	500
2009	43.9%	33.3%	55.5%	-94.5%	419.4%	2.5	13.9	500
2010	20.9%	19.4%	24.4%	-51.0%	142.7%	0.6	4.1	500
2011	0.7%	0.9%	24.7%	-74.1%	101.0%	0.1	3.9	500
2012	16.7%	14.7%	25.5%	-68.6%	187.8%	1.2	9.5	500
2013	36.4%	34.3%	31.8%	-53.6%	296.8%	2.3	18.0	500
2014	14.1%	14.8%	22.7%	-71.7%	126.3%	0.2	5.6	500
2015	-2.0%	-0.2%	25.8%	-76.8%	134.4%	0.2	5.2	502
2016	13.8%	14.2%	24.8%	-73.1%	227.0%	1.7	16.0	504
2017	18.8%	18.0%	26.7%	-84.1%	133.7%	0.2	4.4	505
2018	-7.2%	-7.6%	22.5%	-73.3%	79.6%	0.3	3.3	505
2019	28.7%	29.1%	24.9%	-54.2%	148.4%	0.2	4.7	505
Panel B: Annual risk								
	Mean	Median	StdDev	Min	Max	Skew	Kurt	Count
2002	49.1%	40.3%	25.7%	9.1%	205.4%	2.3	10.2	500
2003	32.2%	29.1%	13.8%	11.6%	130.4%	2.0	9.7	500
2004	25.5%	22.5%	10.4%	7.1%	82.5%	1.6	6.3	500
2005	24.6%	22.4%	10.2%	9.7%	110.6%	3.3	23.5	500
2006	24.2%	22.3%	9.5%	5.4%	84.2%	1.4	6.6	500
2007	28.3%	26.8%	9.5%	3.2%	94.9%	1.7	9.9	500
2008	66.2%	57.8%	32.3%	7.4%	255.0%	2.4	11.5	500
2009	51.2%	44.7%	26.3%	17.7%	206.1%	1.8	7.6	500
2010	29.7%	28.7%	9.8%	10.8%	73.2%	0.8	3.9	500
2011	34.4%	33.7%	11.1%	4.1%	72.9%	0.5	3.5	500
2012	25.3%	23.3%	10.6%	4.4%	93.0%	1.6	8.1	500
2013	22.4%	20.7%	7.7%	5.9%	71.9%	2.0	10.0	500
2014	22.2%	20.6%	7.3%	11.6%	66.2%	1.5	6.9	500
2015	27.0%	24.3%	10.1%	4.8%	89.3%	2.2	10.6	502
2016	27.9%	25.1%	11.8%	3.0%	110.1%	2.2	11.2	504
2017	21.2%	19.3%	8.6%	7.7%	72.4%	1.8	7.7	505
2018	27.8%	26.3%	8.4%	4.2%	87.4%	1.8	9.9	505
2019	25.6%	23.7%	10.1%	3.4%	143.4%	3.8	39.8	505

Table 2.2: Descriptive statistics of annual return and risk in the European market

Table 2.2 presents descriptive statistics of annual return (Panel A) and risk (Panel B) by year for the stocks traded in the European market (STOXX 600) at the beginning of each period.

Panel A: Annual return								
	Mean	Median	StdDev	Min	Max	Skew	Kurt	Count
2002	-29.5%	-28.2%	27.3%	-98.5%	62.5%	0.0	3.1	602
2003	22.1%	16.9%	32.9%	-95.2%	223.5%	1.4	8.4	600
2004	15.3%	14.0%	25.2%	-69.0%	259.0%	1.4	17.1	600
2005	25.1%	23.3%	23.8%	-43.2%	123.3%	0.5	3.9	600
2006	25.5%	21.8%	29.1%	-87.2%	150.5%	0.8	6.4	600
2007	-3.5%	-5.0%	29.7%	-93.5%	186.5%	1.1	7.4	600
2008	-49.1%	-50.5%	22.8%	-99.9%	60.2%	0.6	4.1	600
2009	44.1%	32.5%	54.4%	-69.2%	520.1%	2.5	16.3	600
2010	17.1%	16.3%	33.9%	-75.0%	156.8%	0.5	4.1	600
2011	-15.9%	-16.2%	26.8%	-92.0%	158.3%	0.5	6.3	600
2012	19.6%	19.6%	29.2%	-89.1%	168.7%	0.3	5.1	600
2013	20.4%	18.2%	30.7%	-80.3%	242.3%	0.7	8.0	600
2014	4.5%	5.9%	22.8%	-87.2%	102.4%	-0.2	4.4	600
2015	10.5%	11.4%	28.5%	-99.7%	154.1%	-0.1	5.8	601
2016	-1.0%	-3.2%	26.2%	-87.8%	234.4%	1.9	16.9	600
2017	11.9%	9.6%	25.4%	-93.5%	180.5%	0.8	8.7	600
2018	-14.0%	-15.2%	22.8%	-89.3%	117.3%	0.9	6.6	600
2019	24.6%	25.1%	28.1%	-87.1%	176.7%	0.4	5.2	600

Panel B: Annual risk								
	Mean	Median	StdDev	Min	Max	Skew	Kurt	Count
2002	45.3%	40.7%	22.1%	5.0%	195.1%	2.4	12.4	602
2003	35.2%	32.6%	16.5%	4.4%	252.4%	5.0	56.4	600
2004	24.0%	22.2%	8.7%	4.9%	81.9%	2.0	10.1	600
2005	21.4%	20.5%	7.1%	4.0%	112.7%	4.6	51.6	600
2006	25.5%	24.1%	9.3%	4.5%	112.2%	3.5	27.4	600
2007	28.9%	27.7%	9.1%	4.8%	100.7%	1.9	13.7	600
2008	58.6%	55.0%	21.2%	3.4%	187.0%	1.7	9.2	600
2009	45.7%	42.6%	18.3%	7.8%	166.1%	2.1	10.9	600
2010	30.6%	29.2%	10.0%	7.9%	103.4%	2.3	14.3	600
2011	36.3%	33.8%	12.6%	3.5%	116.3%	1.6	8.4	600
2012	30.3%	27.8%	11.6%	10.9%	98.6%	1.6	7.4	600
2013	25.4%	23.3%	12.4%	7.7%	237.3%	9.4	147.6	600
2014	25.2%	23.3%	8.7%	7.1%	83.2%	2.5	13.9	600
2015	30.5%	27.9%	13.3%	3.0%	165.5%	5.8	50.0	601
2016	33.0%	30.1%	12.5%	7.4%	128.7%	2.2	12.7	600
2017	21.6%	19.8%	8.2%	6.2%	103.9%	3.4	26.3	600
2018	26.9%	25.1%	10.2%	3.5%	118.8%	2.8	19.4	600
2019	26.5%	25.2%	9.2%	8.7%	103.2%	2.5	16.9	600

Table 2.3: Annual return and risk of equal-weighted and value-weighted portfolios

Table 2.3 presents annual return and risk of equal-weighted (EW) and value-weighted (VW) portfolios composed by all the stocks in the market index by year in the U.S. and European markets. The market index is S&P 500 for the U.S. market and STOXX 600 for the European market.

	U.S. market				European market			
	Annual return		Annual risk		Annual return		Annual risk	
	EW	VW	EW	VW	EW	VW	EW	VW
2002	-18.1%	-21.9%	24.6%	25.8%	-29.5%	-31.6%	19.9%	27.0%
2003	41.6%	28.5%	18.4%	17.0%	22.1%	13.6%	16.1%	19.3%
2004	16.6%	10.0%	12.2%	11.0%	15.3%	10.0%	10.0%	11.1%
2005	8.5%	4.9%	11.2%	10.2%	25.1%	22.1%	8.5%	8.7%
2006	15.5%	15.8%	10.8%	9.9%	25.5%	17.4%	13.4%	12.9%
2007	3.3%	5.1%	16.0%	15.7%	-3.5%	0.8%	16.1%	15.5%
2008	-38.6%	-35.9%	41.0%	40.0%	-49.1%	-45.3%	33.8%	35.1%
2009	43.9%	26.6%	33.4%	26.8%	44.1%	26.1%	25.9%	23.3%
2010	20.9%	15.0%	20.2%	17.9%	17.1%	8.5%	18.9%	19.0%
2011	0.7%	2.4%	25.0%	23.1%	-15.9%	-12.7%	23.1%	23.3%
2012	16.7%	16.0%	14.0%	12.5%	19.6%	14.3%	17.4%	15.5%
2013	36.4%	31.9%	12.1%	11.0%	20.4%	17.0%	12.1%	12.4%
2014	14.1%	13.7%	11.7%	11.4%	4.5%	3.9%	12.5%	12.6%
2015	-2.0%	1.2%	15.2%	15.3%	10.5%	6.9%	16.8%	18.2%
2016	13.8%	11.6%	14.7%	13.0%	-1.0%	-1.4%	18.5%	17.9%
2017	18.8%	21.9%	7.2%	6.6%	11.9%	8.2%	7.8%	7.8%
2018	-7.2%	-4.2%	15.5%	16.9%	-14.0%	-12.9%	12.3%	12.3%
2019	28.7%	31.1%	12.7%	12.4%	24.6%	22.5%	12.1%	11.1%
Average	11.9%	9.7%	17.5%	16.5%	7.1%	3.7%	16.4%	16.8%

Table 2.4: Risk-free rate annual return and risk

Table 2.4 presents risk-free rate annual return and risk by year in the U.S. and European markets. We use 1-month USD LIBOR and 1-month Euribor as proxies for the risk-free rate in the U.S. and European markets, respectively.

	U.S. market		European market	
	Annual return	Annual risk	Annual return	Annual risk
2002	0.041%	0.025%	0.036%	0.022%
2003	0.022%	0.013%	0.066%	0.021%
2004	-0.106%	0.012%	-0.002%	0.007%
2005	-0.165%	0.012%	-0.023%	0.009%
2006	-0.077%	0.012%	-0.102%	0.012%
2007	0.060%	0.057%	-0.054%	0.063%
2008	0.347%	0.115%	0.140%	0.044%
2009	0.017%	0.008%	0.179%	0.021%
2010	-0.002%	0.005%	-0.027%	0.009%
2011	-0.003%	0.002%	-0.020%	0.016%
2012	0.007%	0.001%	0.076%	0.011%
2013	0.003%	0.001%	-0.009%	0.003%
2014	0.000%	0.002%	0.016%	0.006%
2015	-0.022%	0.006%	0.019%	0.002%
2016	-0.028%	0.006%	0.014%	0.002%
2017	-0.066%	0.009%	0.000%	0.001%
2018	-0.078%	0.010%	0.000%	0.001%
2019	0.062%	0.013%	0.006%	0.006%

Chapter 3. How Many Stocks Should be Included in a Portfolio to Cancel Out the Diversifiable Risk?

3.1 Introduction

The seminal work of Markovitz (1952) about portfolio theory is one of the most important theoretical developments in finance, implying that portfolios should be mean-variance optimized. In the traditional capital-asset pricing model proposed by Sharpe (1964), Lintner (1965) and Mossin (1966) diversification allows the investor to eliminate all but the risk resulting from swings in economic activity, even in efficient portfolios. The required number of stocks in a portfolio to achieve satisfactory benefits of diversification is still a debate in the literature. Numerous studies have shown that 10 stocks are sufficient to achieve satisfactory benefits of diversification, but on the other hand, various works, especially recently, have challenged this fact by showing that 100 stocks or more are required for satisfactory benefits of diversification.

Being the U.S. stock market the largest in the world, the great majority of empirical studies on diversification and portfolio size are concentrated in stocks of this market. Following a pioneer approach, Evans and Archer (1968) examine the rate at which the variance of returns for randomly selected portfolios is reduced as a function of the number of stocks included in the portfolio. Using semi-annual returns of 470 stocks traded on S&P 500 during the period between January 1958 and July 1967, the authors raise doubts about the economic justification for increasing the portfolio size beyond 10 stocks.

Latane and Young (1969) measure the effect of diversification with the expected geometric returns for portfolios with different dimensions. Using monthly returns of 224 stocks from the U.S. market during the period between January 1953 and December 1960, the authors' results indicate that 84% and 96% of the possible gains through diversification can be achieved with a portfolio size of 8 and 16 stocks, respectively.

Fisher and Lorie (1970) study the wealth ratios (ratio of the ending value of the investment portfolio to the initial amount invested in the portfolio) resulting from investment in portfolios of specified numbers of stocks, ranging from 1 through 128 and in all stocks traded in the market index. Using annual returns of stocks listed on the New York Stock Exchange (hereafter NYSE) during the 1926-1965 period, the results show that 95% of the achievable reduction in the dispersion of portfolio returns is obtained by holding 32 stocks and 99% by holding 128 stocks.

Jennings (1971) measures the effect of diversification by examining the trade-off between loss probability of the return being smaller than 75% of market return and gain probability of the return being higher than 115% of market return. Using annual returns of all stocks listed on the NYSE from January 1956 to December 1965, the author finds that a portfolio of approximately 15 stocks has an appropriate size in order to mitigate the diversifiable risk.

Wagner and Lau (1971) show how diversification can be used to offset the riskiness of individual stocks, so that portfolios with large numbers of high-risk stocks may be less risky than portfolios with small numbers of low-risk stocks, and earn a higher rate of return. Using monthly returns from the U.S. stock market over the period June 1960 through May 1970, they show that the rate of return on well-diversified low risk portfolios is significantly lower than the return on well-diversified higher risk portfolios.

Fielitz (1974) explores the issues of investing in an equal-weighted portfolio and reexamines the Evans and Archer (1968) regression. Using quarterly returns of 200 stocks listed on the NYSE during the period between January 1964 and September 1968, the author suggests that eight stocks should be enough to assure reasonable diversification.

Johnson and Shannon (1974) suggest an allocation method of stocks in a portfolio by resolving a quadratic programming problem. Using quarterly returns of 50 stocks listed on the NYSE during the period between April 1965 and December 1972, the results indicate that similar risk and superior returns can be obtained with less than 10 stocks in a portfolio.

Klemkosky and Martin (1975) test the relationship between market and residual risk and assess the significance of that association on the process of diversification. Using monthly returns of approximately 350 stocks listed on the NYSE in the period between July 1963 and June 1973, the results show that the levels of diversification achieved for high versus low beta portfolios for a given portfolio size were significantly different. High beta portfolios require a substantially larger number of stocks to achieve the same level of diversification as a low beta portfolio.

Bloomfield, Leftwich and Long (1977) assess the performance (before estimated transaction costs) of five different portfolio strategies that employ one or more cost reduction techniques. The authors used monthly returns of more than 800 stocks during the period between April 1953 and June 1970. The results were consistent with the well-documented relationship between portfolio size and portfolio efficiency. For all the strategies considered, the larger the portfolio size, the more efficient is the chosen portfolio.

Tole (1982) studies the adequate diversification levels by selecting stocks from recommendations of brokerage firms, research services, and other sources of investment information. Since these portfolios should contain stocks highly correlated with one another, the number of stocks should be higher than random selected portfolios for attain adequate diversification. The author concludes that an investor with a portfolio of stocks that have been selected in a biased manner should own substantially more than the 10 stocks suggested in previous studies, where portfolios were randomly constructed.

Newbould and Poon (1996) explore the return and risk of a portfolio, as a function of the number of stocks, considering two different investment styles: small and large companies from the U.S. market. Using monthly returns during the 1987-1993 period, the results show that an investor who wants to be within 5% of the average return and 5% of the average risk would need more than 100 stocks, either for a portfolio of small or large companies.

Beck, Perfect and Peterson (1996) develop alternative methods to reduce the effect that the number of repeated replications has in the results of some approaches used to indicate the number of stocks that constitute a well-diversified portfolio. Using monthly returns of 1,221 stocks listed on the NYSE or the American Stock Exchange during the period between January 1982 and December 1991, the authors find that the return variance of a sample portfolio with more than 19 stocks is not statistically different from the market return variance.

Domian, Louton and Racine (2003) examine returns and ending wealth over a 20-year holding period. Using returns of 100 large U.S. stocks from January 1979 through December 1998, the authors demonstrate that investors need more than 60 stocks to avoid a shortfall risk below 10%. Domian, Louton and Racine (2007) update their previous study by increasing the number of stocks of their dataset to 1,000 large U.S. stocks for the period between January 1985 and December 2004. The authors conclude that investors need at least 164 stocks to have a 99% chance of outperforming Treasury bonds in long-term.

Benjelloun (2010) uses two risk measures (standard deviation of returns and standard deviation of end-of-period wealth) and two weighting schemes (equal-weighted and value-weighted portfolios) to evaluate how large a well-diversified portfolio needed to be. Using monthly returns of all stocks listed in the Center for Research in Security prices tape during the 1980-2000 period, the author concludes that a portfolio of 40 to 50 stocks is straight enough to achieve a high level of diversification.

Empirical studies on diversification and portfolio size using stocks are rather limited outside U.S. market. Bird and Tippett (1986) derive an exact parametric relationship between portfolio standard deviation and size and emphasize the dangers of modeling the risk reduction advantages of naive diversification. Using monthly returns of 188 stocks from the Australian market during the period between January 1958 and December 1973, the authors argue that a portfolio of 22 stocks is required to obtain the same level of diversification calculated by Evans and Archer (1968) regression in a portfolio of 10 stocks.

Solnik (1974) studies risk reduction benefits from diversifying portfolios across foreign as well as domestic common stocks. Using more than 300 stocks from the U.S. and European markets for the period 1966-1971, the author concludes that adding 50 different stocks to a portfolio with 20 stocks would reduce total risk by a residual amount.

Copp and Cleary (1999) study the average standard deviation resulting from investing in portfolios with specified numbers of stocks, ranging from 1 through more than 200 stocks. Using monthly returns of stocks listed on the Toronto Stock Exchange during the period between January 1985 and December 1997, the results suggest that 30 to 50 stocks are required to capture most of the benefits associated with diversification. Nevertheless, substantial benefits occur with a portfolio of 10 stocks.

Kryzanowski and Singh (2010) use various metrics to measure the benefits of diversification to determine if a minimum portfolio size should be recommended to achieve a sufficiently well-diversified portfolio. Using monthly returns of all stocks listed on the Toronto Stock Exchange during the 1975-2003 period, the authors find that the minimum portfolio size required to achieve a sufficiently well-diversified portfolio is very sensitive to the performance metric used to measure such benefits. In order to achieve about 90% of the potential benefits from diversification, a portfolio with 20 to 25 stocks is required when diversification is measured with conventional metrics of risk or kurtosis; 45 stocks are required to achieve about 90% of the potential benefits from diversification when measured by average time-series measures; and only 2 stocks when measured by skewness.

Bradfield and Munro (2017) contrast the construction of equal-weighted portfolios with value-weighted portfolios. The authors use weekly returns of 167 stocks listed on Johannesburg Stock Exchange All Share Index during the period between June 2002 and December 2014, and show that, for levels of risk reduction between 90% and 95%, equal-weighted portfolios require between 15 and 29 stocks while value-weighted portfolios require between 33 and 60 stocks.

Analytical solutions have also been used to study diversification and portfolio size. Elton and Gruber (1977) study the effects on risk from introducing new stocks into a portfolio by developing an analytical relationship between the average variance of portfolio return and the size of the portfolio. They conclude that a portfolio with 15 stocks has 32% more risk than a portfolio with 100 stocks, with significant gains in decreasing risk from adding stocks beyond 15.

Statman (1987) analyses the trade-off between costs and benefits of increasing the number of stocks in a portfolio. The benefit is defined as the expected increase in return, calculated using the Capital Market Line equation, from investing in a portfolio considered well-diversified (Vanguard Index Trust that mimics the S&P 500) compared to a portfolio with less stocks. The cost is defined as the mean return differential between S&P 500 and the fund since the fund underperforms the S&P 500 due to transaction costs and administrative expenses. The results show that the optimal number of stocks in a portfolio is, at least, 30 stocks for a borrowing investor, and 40 stocks for a lending investor. Using the same procedure of his former study, Statman (2004) finds that the optimal number of stocks in a portfolio is, at least, 300 when using Vanguard Total Stock Market Index Fund with 3,444 stocks as the well-diversified portfolio instead of Vanguard Index Trust. However, the analytical method used in both studies is highly dependent on the number of stocks included in the well-diversified portfolio.

Tang (2004) shows through an analytical solution that a portfolio of 20 stocks is required to eliminate 95% of the diversifiable risk in a universe of 600 stocks. A portfolio of 86 stocks is required to eliminate 99% of diversifiable risk.

Haensly (2020) examines several methods for decomposing total portfolio risk into systematic and unsystematic components and then carries out simulations to compare cross-sectional distributions of estimated and true risk as number of stocks increases in equal-weighted portfolios. The author shows that risk and magnitude of shocks in return due to diversifiable risk are not negligible, even for a portfolio with 300 stocks.

Studies about portfolio diversification, generally, focus only on stocks from one market and on equal-weighted portfolios. In addition, datasets fail to include delisted stocks due to merger, acquisition or bankruptcy, for example. Finally, to the best of our knowledge, the results of similar studies only report the results for the full sample period studied without analyzing the results year by year. In our study, we try to incorporate these issues.

The main contributions of this Chapter are as follows. First, we compare the benefits of diversification in the U.S. and European markets instead of using only one market as the majority of studies about diversification. Second, although most empirical studies focus on the equal-weighted portfolios, we also analyze the benefits of diversification of value-weighted portfolios. Third, we simulate more than seven million portfolios to obtain return and risk of portfolios with different sizes, assuring the robustness of the results by increasing and decreasing the number of simulations performed for each portfolio size. Fourth, we use a sampling technique that deals with delisted stocks over the period to avoid survivorship bias.

Finally, whereas most studies are concerned solely on the full sample period, our analysis is performed year by year.

The present Chapter proceeds as follows. Section 3.2 describes the methodology of portfolio construction used to study the benefits of diversification and define the diversification measures used. Section 3.3 reports and discuss the results of the empirical analysis, and Section 3.4 concludes.

3.2 Methodology

In line with Jennings (1971), we assume long-only portfolios that contain only common stocks and where purchases are financed without borrowing. In addition, investment transactions do not influence the price or dividend of any stock and taxes and transaction costs are not considered. In the following subsections, we describe the portfolio construction method and the diversification metrics used in our study.

3.2.1 Portfolio Construction

This subsection describes the method of portfolio construction employed as a means of understanding the diversification benefits from adding stocks to a portfolio.

We randomly generate 10,000[3] portfolios (without replacement) with different sizes from stocks that were traded on S&P 500 or STOXX 600 at the beginning of each year, during the period between 2002 and 2019. Most of the studies on diversification and portfolio size uses random generation technique to select portfolios (see, for example, Evans and Archer, 1968; Latane and Young, 1969; Fisher and Lorie, 1970; Jennings, 1971; Wagner and Lau, 1971; Fielitz, 1974; Johnson and Shannon, 1974; Solnik, 1974; Klemkosky and Martin, 1975; Bloomfield et al., 1977; Bird and Tippett, 1986; Beck et al., 1996; Newbould and Poon, 1996; Copp and Cleary, 1999; Domian et al., 2003; Domian et al., 2007; Benjelloun, 2010; Kryzanowski and Singh, 2010; Bradfield and Munro, 2017). As a robustness check[4], we generate 5,000 and 20,000 portfolios for all the results presented and we find that the conclusions of this study would be the same. This suggests that the results reported are robust,

[3] To the best of our knowledge, 10,000 is the one of the highest number of simulations used in the literature and was the same number of simulations used by Benjelloun (2010).

[4] Results are available upon request.

in contrast with Beck et al. (1996) that provide evidence that results of diversification studies are sensitive to the number of replications.

Table 3.1 describes in tabular form the portfolio selection process for one computer run. The portfolio sizes are 2, 4, 6, 8, 10, 12, 16, 25, 32, 50, 64, 100 and 128 stocks. The number of simulations is 10,000 for each portfolio size specified and for each year. For a portfolio with one stock, the number of simulations are the number of stocks traded in the market index. For a portfolio with all stocks traded in the market index, the number of simulations is one. For every simulation, portfolios have 1-year holding period.

3.2.2 Portfolio End-of-Period Wealth

The end-of-period wealth standard deviation (hereafter EPWSD) is used to analyze changes in the dispersion of 1 EUR or 1 USD investment as the number of stocks in a portfolio increases. Other studies also analyze end-of-period wealth (see, for example, Domian et al., 2003; Domian et al., 2007; Benjelloun, 2010). The portfolio's end-of-period wealth (*EPW*) is defined as:

$$EPW = r_p + 1 \tag{3.1}$$

where r_p is the portfolio return defined in equation (2.2). The EPWSD is the standard deviation of the EPW.

3.2.3 Risk Distributions Test

The Kolmogorov-Smirnov test[5] is used to compare risk distributions of all simulations of different portfolio sizes. The risk obtained by each portfolio simulation described in subsection 3.2.1 for a specific portfolio size constitute the empirical distribution. Therefore, each portfolio size has one risk distribution that is compared with the corresponding distribution of the next portfolio size in ascending order.

The Kolmogorov-Smirnov test is a nonparametric test of the equality of continuous, not divided into bins, one-dimensional data samples. It assumes that a list of data points can be easily converted to a cumulative distribution function. The test uses the maximum absolute

[5] Bibliography on Kolmogorov-Smirnov test is given by Massey (1951).

difference between two cumulative distribution functions. The null distribution of this statistic is calculated under the null hypothesis that the samples are drawn from the same distribution.

3.2.4 Evans and Archer (1968) Regression Revisited

This subsection describes the procedure used to revisit the topic of adequate diversification under the assumptions of Evans and Archer (1968) regression.

Based on Evans and Archer (1968) regression, standard deviation decreases to an asymptote as the number of stocks increases. The regression for fitting this relationship, using least-squares method, is defined as:

$$\overline{ea}_p = A + B\left(\frac{1}{m}\right) \tag{3.2}$$

where $\overline{ea}_p$ is the average standard deviation of all simulated equal-weighted portfolios with m stocks, A is the intercept that stands for systematic risk component and B is the slope that stands for unsystematic risk component. The authors choose this method because there is a relatively stable and predictable relationship between the number of stocks included in a portfolio and the level of portfolio dispersion.

To compare the systematic risk under Evans and Archer (1968) regression, for equal-weighted portfolios, we express the squared portfolio risk equation defined in equation (2.3), into two separate components:

$$s_p^2 = \frac{1}{m}\sum_{i=1}^{m}\frac{1}{m}\sigma_i^2 + \sum_{\substack{i=1 \\ i\neq j}}^{m}\sum_{j=1}^{m}\frac{Cov\left(r_i, r_j\right)}{m^2} \tag{3.3}$$

where the portfolio p average variance is:

$$\bar{s}_p^2 = \sum_{i=1}^{m}\frac{1}{m}\sigma_i^2 \tag{3.4}$$

and the portfolio p average covariance is:

$$\overline{Cov_p} = \frac{1}{m(m-1)} \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} Cov(r_i, r_j) \tag{3.5}$$

expressing portfolio variance as:

$$s_p^2 = \frac{1}{m} \overline{s}_p^{\,2} + \frac{m-1}{m} \overline{Cov_p} \tag{3.6}$$

From equation (3.6), we observe that average variance reduces as the number of stocks in a portfolio increases, leaving only the average covariance. Consequently, the average covariance could be used as a proxy of systematic risk.

The comparison between equations (3.2) and (3.6) is important to assess if Evans and Archer (1968) regression provides biased estimates for the absolute level of systematic risk. Value-weighted portfolios are not studied in this subsection because equation (3.3) only holds for equal-weighted portfolios.

3.3 Empirical Results

Armed with the 10,000 simulations for each portfolio size following the approach described in subsection 3.2.1, we can now study the changes in risk, end-of-period wealth standard deviation and risk distributions as the number of stocks in a portfolio increases.

Figures 3.1 and 3.2 show the empirical return and risk density distributions, respectively, of equal-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the equal-weighted portfolio composed by all stocks traded in the S&P 500 (U.S. market). Figures 3.3 and 3.4 show the same information as Figures 3.1 and 3.2, respectively, for value-weighted portfolios instead of equal-weighted portfolios. Figures 3.5, 3.6, 3.7 and 3.8 show the same information as Figures 3.1, 3.2, 3.3 and 3.4, respectively, for the STOXX 600 (European market) instead of the U.S. market. In all figures mentioned, we present the results for the years 2003, 2008, 2013 and 2018 as examples. The pattern of the distributions for the years not reported are equivalent.

Regardless the number of stocks included in the portfolio, for equal-weighted and value-weighted portfolios and for both markets, the return distribution shows an average close to the return of the portfolio with all stocks traded in the market index. Concerning risk, the average risk tends to decrease as the number of stocks in a portfolio increases. In both distributions, the dispersion of return and risk reduces as the number of stocks in a portfolio increases.

3.3.1 Portfolio Risk

Tables 3.2 and 3.3 present the average risk for the U.S. and European markets, respectively. The results are presented by year, for portfolios with the specified number of stocks, ranging from one through all stocks traded in the market index, and for equal-weighted and value-weighted portfolios. This risk measure is widely used in studies on diversification and portfolio size (see, for example, Evans and Archer, 1968; Latane and Young, 1969; Fisher and Lorie, 1970; Wagner and Lau, 1971; Johnson and Shannon, 1974; Solnik, 1974; Bloomfield et al., 1977; Bird and Tippett, 1986; Beck et al., 1996; Newbould and Poon, 1996; Copp and Cleary, 1999; Benjelloun, 2010; Kryzanowski and Singh, 2010; Bradfield and Munro, 2017).

The results show that average risk always decreases when portfolio size increases, independently of the year, market, and weighting scheme. In the U.S. market, when the number of stocks in a portfolio increases from one to all stocks traded in the market index, the decrease in average risk is 14 p.p. and 15 p.p., on average, in equal-weighted and value-weighted portfolios, respectively. In the European market, when the same increase in the portfolio size is present, the decrease in average risk is 15 p.p. in both weighting schemes.

As expected, the decrease in risk that arises from increasing the number of stocks in a portfolio from one to all stocks traded in the market index, is higher when high levels of volatility are present. According to Table 2.3, the years with highest risk in the U.S. and European markets were 2008 and 2009. In these years, regardless the market and the weighting scheme, the risk difference between a portfolio with one stock and a portfolio with all stocks traded in the market index is superior compared to years with lower risk. In the U.S. market, the highest risk reduction occurs in 2008 (-26 p.p. in equal-weighted portfolios and -27 p.p. in value-weighted portfolios) while the lowest risk reduction occurs in 2010 in equal-weighted portfolios (-9 p.p.) and in 2014 in value-weighted portfolios (-11 p.p.). In the European market, the highest risk reduction occurs in 2002 in equal-weighted portfolios (-26 p.p.) and in 2008 in value-weighted portfolios (-24 p.p.) while the lowest risk reduction occurs in 2010 (-11 p.p. in both weighting schemes).

When comparing equal-weighted with value-weighted portfolios in the U.S. market, we observe that equal-weighted portfolios with 12 stocks or less have, on average, less risk than value-weighted portfolios of the same size (difference lower than 0.3 p.p.). For portfolios with 16 stocks or more, equal-weighted portfolios have, on average, more risk than value-weighted portfolios. However, the difference is also small (inferior to 1.1 p.p.). In the European market, equal-weighted portfolios have, on average, less risk than value-weighted portfolios, independently of the number of stocks in a portfolio. The difference of risk, on average, between the two weighting schemes, as in the U.S. market, is not substantial (inferior to 1.8 p.p.).

Tables 3.4 and 3.5 summarize the difference of average risk in the U.S. and European markets, respectively, presented by year and for equal-weighted and value-weighted portfolios. The mentioned difference is the difference of risk between a portfolio with 1 stock and a portfolio with a specified portfolio size, ranging from 1 to 128, as a percentage of the difference of average risk between a portfolio with 1 stock and a portfolio with all stocks traded in the market index. The results show that risk reduction is not a linear function of the number of stocks included in the portfolio

We observe that when the number of stocks in an equal-weighted portfolio of the U.S or the European markets increases from 1 to 16 stocks, on average, we obtain 91% of the difference between a portfolio with 1 stock and a portfolio with all available stocks in the respective market index (hereafter diversifiable risk reduction). With an equal-weighted portfolio of 32 stocks, we obtain, on average, at least 95% of the diversifiable risk reduction, in both markets. To achieve more than 90% of diversifiable risk reduction, on average, using value-weighted portfolios, in the U.S. and European markets, we need 25 and 32 stocks, respectively. To obtain 95% of diversifiable risk reduction, on average, we need to double the number of stocks in a portfolio: 50 and 64 stocks, in the U.S. and European markets, respectively.

Considering the diversifiable risk reduction by year, equal-weighed portfolios with 50 stocks in the U.S. and European markets achieve at least 95% of risk reduction every year. With respect to value-weighted portfolios, in the U.S. market a portfolio with 64 stocks achieves at least 95% of risk reduction every year except for 2013 (94% of diversifiable risk reduction). In the European market, only in 2008 (87% of diversifiable risk reduction) and in 2017 (94% of diversifiable risk reduction) a portfolio with 64 stocks does not have a risk reduction of at least 95%.

These results suggest that, in the U.S. and European markets most benefits of risk reduction in equal-weighted portfolios are obtained by holding a portfolio with 50 stocks. If we add stocks to an equal-weighted portfolio of 50 stocks until all stocks traded in the market index are included in the portfolio, we will not reduce the risk more than 1 p.p. in any of the years considered in our sample period. Additionally, equal-weighted portfolios seem to require fewer stocks than value-weighted portfolios to have approximately the same risk as a portfolio with all stocks traded in the market index using the same weighting scheme. This finding is in line with Bradfield and Munro (2017).

Regarding value-weighted portfolios, in the U.S. market most benefits of diversifiable risk reduction are obtained by holding a portfolio with 64 stocks. If we add stocks to a value-weighted portfolio of 64 stocks until all stocks traded in the market index are included in the portfolio, we will not reduce the risk more than 1 p.p. in any of the years considered in our sample period. In the European market, the referred reduction of 1 p.p. is only not possible in 2008 with a value-weighted portfolio with 64 stocks.

Our results contrast with studies that suggest a number of stocks lower than 25 (for example, Evans and Archer, 1968; Latane and Young, 1969; Jennings, 1971; Fielitz, 1974; Johnson and Shannon, 1974; Elton and Gruber, 1977; Bird and Tippett, 1986; Beck et al., 1996; Tang, 2004) and those who suggest the inclusion of more than 100 stocks (Newbould and Poon, 1996; Statman, 2004). Nevertheless, our findings are in line with Copp and Cleary (1999), Benjelloun (2010) and Bradfield and Munro (2017).

3.3.2 Portfolio End-of-Period Wealth

Tables 3.6 and 3.7 summarize, for the U.S. and European markets, respectively, the average end-of-period wealth. The results are presented by year, for all the portfolios sizes specified and the portfolio with all stocks traded in the market index, and for equal-weighted and value-weighted portfolios.

The results show that increasing the number of stocks in a portfolio has no significant impact on average end-of-period wealth for equal-weighted portfolios in the U.S. and European markets. For value-weighted portfolios, the number of stocks increase had a slight negative effect on average end-of-period wealth in both markets.

The change in end-of-period wealth from higher/lower than 1 to lower/higher than 1 as the number of stocks in a portfolio increases is very rare. The exceptions are value-weighted portfolios in the U.S. market in 2015 (the average end-of-period wealth from 0.98 to 1.01 when the portfolio size changes from 1 to 8 stocks) and value-weighted portfolios in the European market in 2007 (the average end-of-period wealth from 0.99 to 1.01 when the portfolio size changes from 1 to 2 stocks).

Tables 3.8 and 3.9 summarize, for the U.S. and European markets, respectively, the EPWSD. The results are presented by year, for all the portfolios sizes specified, and for equal-weighted and value-weighted portfolios.

The results show that EPWSD decreases as the number of stocks included in a portfolio increases, in both markets and weighting schemes. Our results are in line with those presented by Domian et al. (2007). We find no substantial differences on EPWSD between equal-weighted and value-weighted portfolios in both markets. Equal-weighted portfolios with 50 stocks have an EPWSD lower than 0.05 in the U.S. and European markets. About value-weighted portfolios, a portfolio with 64 stocks is required to achieve an EPWSD lower than 0.05 in the U.S. and European markets. Those portfolios sizes are required to obtain a dispersion lower than 0.05 per $1 or 1€ of investment in the U.S. and European markets, respectively.

3.3.3 Risk Distributions

The results presented in the previous subsections show that the benefits of adding stocks to a portfolio are quite superior when portfolios have few stocks rather than when portfolios have a high number of stocks. This conclusion applies to the U.S. and to the European markets, and for equal-weighted and value-weighted portfolios. Nevertheless, there are evidence of benefits (even residual) of adding stocks to portfolios with a large number of stocks.

When we compare the distribution of risk of all simulations with a specific portfolio size with the corresponding distribution of the next portfolio size in ascending order, we reject the null hypothesis of Kolmogorov-Smirnov test that samples are drawn from the same distribution. The *p*-values are inferior to 5% for every portfolio size, year, market and weighting scheme, thus the results are not reported.

From Figures 3.2, 3.4, 3.6 and 3.8, we observe that risk distributions average decreases as we increase the number of stocks in a portfolio. Additionally, the dispersion of risk also reduces as the number of stocks in a portfolio increases. Hence, these findings in conjunction with the rejection of the null hypothesis that the samples are drawn from a population with the same risk distribution shows evidence that the benefits of increasing the number of stocks in a portfolio exists, even in portfolios with a large number of stocks. Nevertheless, in subsections 3.3.1 and 3.3.2 we find that an equal-weighted portfolio with 50 stocks and a value-weighted portfolio with 64 stocks almost exhaust all the benefits of diversification. This suggest that the benefits of increasing the number of stocks beyond 50 in an equal-weighted portfolio and 64 in a value-weighted portfolio are negligible.

3.3.4 Evans and Archer (1968) Regression Revisited

Table 3.10 summarizes, in the U.S. and European markets, the coefficient of determination, the slope, and the intercept (systematic risk) of Evans and Archer (1968) regression procedure described in subsection 3.2.4. The squared intercept and the average covariance of a portfolio with all stocks traded in the market index are also reported.

The results show that the regression yields a good fit, as indicated by the coefficient of determination close to one. The slope, which stands for unsystematic risk component, exhibits higher values in years associated with largely negative returns (2002 and 2008). This shows evidence that increasing the number of stocks in a portfolio have more impact in risk reduction when the market plunge. The squared intercept, which stands for the systematic variance, is higher than the average covariance of equal-weighted portfolios of all stocks traded in the S&P 500 and STOXX 600 for every year considered. Accordingly to equation (3.6), as the number of stocks in a portfolio increases, the variance of a portfolio converges to the average covariance. In this context, average covariance is, in fact, a proxy of the systematic variance. This result suggests that Evans and Archer (1968) regression overestimates systematic risk since the portfolios with all stocks traded in the market index, on average, have lower average covariance than the systematic risk component of Evans and Archer regression, and confirms the findings of Bird and Tippett (1986).

Table 3.11 summarizes the average covariance of portfolios with a specified number of stocks, ranging from 2 through 128 and with all stocks traded in the S&P 500 and STOXX 600.

The average covariance of portfolios with 10 stocks is only 0.2 p.p. lower, on average, than the average covariance of portfolios with all stocks traded in the U.S. and European markets. However, analyzing the results year by year, we conclude that a portfolio with 16 stocks is required to have an average covariance with a maximum difference of 1.0 p.p. to the average covariance of a portfolios with all stocks traded in the market index. The referred difference is higher in periods of negative returns (especially 2008), where the covariance between stocks is higher. These results may be of sufficient importance to use average covariance of portfolios with at least 16 stocks as a proxy of systematic risk.

3.4 Conclusion

This Chapter analyzes the changes in risk, end-of-period wealth standard deviation and risk distributions as the number of stocks in a portfolio increases. We also revisit the topic of adequate diversification under the assumptions of Evans and Archer (1968) regression. We randomly generate equal-weighted and value-weighted portfolios with different sizes from stocks traded on S&P 500 or STOXX 600, in the beginning of each year, during the period between 2002 and 2019. The results of the simulations are averaged and used to study the benefits of diversifications as the portfolio size increases. We also perform a comparison between the U.S. and European markets, as well as, between equal-weighted and value-weighted portfolios.

Our first finding is that equal-weighted portfolios require fewer stocks than value-weighted portfolios to have the same benefits of diversification. Equal-weighted portfolios require fewer stocks than value-weighted portfolios to have approximately the same risk than a portfolio with all stocks traded in the market index in the respective weighting scheme, in line with Bradfield and Munro (2017). If we add stocks to an equal-weighted portfolio with 50 stocks or to a value-weighted portfolio with 64 stocks, until all stocks traded in the market index are in the portfolio, we will not reduce more than 1 p.p. of risk in any of the years considered in our sample period.

Secondly, the results show that, in the U.S. and European markets, the increase of the number of stocks in equal-weighted portfolios has no significant impact on average end-of-period wealth, while the mentioned increase has a slight negative effect in value-weighted portfolios. Regarding EPWSD, the results show that EPWSD decreases as the number of stocks in a portfolio increases in both markets and in both weighting schemes, in line with Domian et al. (2007). Equal-weighted portfolios with 50 stocks and value-weighted portfolios with 64 stocks have an EPWSD lower than 0.05. Thus, a dispersion lower than 0.05 per $1 or 1€ of investment can be obtained in the U.S. and European markets, respectively.

The results of this Chapter suggest that the major benefits of diversification, in the U.S. and European markets, can be achieved with an equal-weighted portfolio with 50 stocks and a value-weighted portfolio with 64 stocks. These portfolios reduce, at least, 95% of diversifiable risk. This finding contrasts with studies that suggest a lower number of stocks than 25 (for example, Evans and Archer, 1968; Latane and Young, 1969; Jennings, 1971; Fielitz, 1974; Johnson and Shannon, 1974; Elton and Gruber, 1977; Bird and Tippett, 1986; Beck et al., 1996; Tang, 2004) and those who suggest the inclusion of more than 100 stocks (Newbould and Poon, 1996; Statman, 2004). Nevertheless, our findings are in line with Copp and Cleary (1999), Benjelloun (2010) and Bradfield and Munro (2017).

Finally, we find that Evans and Archer (1968) regression overestimates systematic risk since portfolios with all stocks traded in the market index, on average, have lower average covariance than the systematic risk component of Evans and Archer regression, confirming the findings of Bird and Tippett (1986). Additionally, we find that portfolios with 16 stocks are required to obtain an average covariance that can be used as a proxy of systematic risk.

Table 3.1: Portfolio selection process

Table 3.1 describes in tabular form the portfolio selection process for one computer run. Stocks are ordered alphabetically and attributed numbers that identify their position in the sample. For a portfolio with two stocks, we randomly select two stocks without replacement (228 and 84). For the following portfolio (portfolio with four stocks), we randomly select two more stocks (16 and 497) which are different from the previous stocks. This process is repeated until we have one portfolio for each specified size.

	Position of the stocks selected for each portfolio size								
	2	4	6	8	10	12	16	.	128
1	228	228	228	228	228	228	228	.	228
2	84	84	84	84	84	84	84	.	84
3		16	16	16	16	16	16	.	16
4		497	497	497	497	497	497	.	497
5			144	144	144	144	144	.	144
6			313	313	313	313	313	.	313
7				5	5	5	5	.	5
8				358	358	358	358	.	358
9					316	316	316	.	316
10					472	472	472	.	472
11						172	172	.	172
12						155	155	.	155
13							26	.	26
14							417	.	417
15							233	.	233
16							364	.	364
.								.	.
128									291

Table 3.2: Average risk in the U.S. market

Table 3.2 summarizes the portfolio average risk for the U.S. market. The results are presented by year, for portfolios with a specified number of stocks, ranging from 1 through 128 and with all stocks traded in the S&P 500 (All). Panel A reports the results for equal-weighted portfolios and Panel B reports the results for value-weighted portfolios.

Panel A: Equal-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	49.6%	37.0%	31.1%	29.0%	27.9%	27.3%	26.8%	26.2%	25.6%	25.4%	25.1%	24.9%	24.8%	24.7%	24.6%
2003	32.1%	26.9%	23.4%	22.0%	21.2%	20.7%	20.3%	19.9%	19.4%	19.1%	18.9%	18.7%	18.6%	18.5%	18.4%
2004	25.6%	20.3%	16.8%	15.5%	14.7%	14.3%	13.9%	13.5%	13.0%	12.8%	12.6%	12.5%	12.4%	12.3%	12.2%
2005	24.8%	19.3%	15.8%	14.5%	13.7%	13.3%	13.0%	12.5%	12.1%	11.9%	11.6%	11.5%	11.4%	11.4%	11.2%
2006	24.3%	19.1%	15.6%	14.2%	13.5%	13.0%	12.6%	12.2%	11.7%	11.5%	11.3%	11.2%	11.0%	11.0%	10.8%
2007	28.4%	23.0%	19.8%	18.6%	18.0%	17.6%	17.4%	17.0%	16.6%	16.5%	16.3%	16.2%	16.1%	16.1%	16.0%
2008	66.6%	53.4%	47.3%	45.2%	44.2%	43.5%	43.1%	42.5%	41.9%	41.7%	41.4%	41.3%	41.1%	41.1%	41.0%
2009	51.0%	43.2%	38.7%	37.1%	36.3%	35.7%	35.3%	34.9%	34.3%	34.1%	33.9%	33.8%	33.6%	33.5%	33.4%
2010	29.6%	25.6%	23.2%	22.2%	21.7%	21.4%	21.2%	20.9%	20.7%	20.6%	20.4%	20.3%	20.3%	20.3%	20.2%
2011	34.5%	30.0%	27.7%	26.8%	26.4%	26.1%	25.9%	25.7%	25.4%	25.3%	25.2%	25.1%	25.1%	25.0%	25.0%
2012	25.2%	20.8%	17.8%	16.6%	16.0%	15.6%	15.3%	15.0%	14.6%	14.5%	14.3%	14.2%	14.1%	14.0%	14.0%
2013	22.4%	18.2%	15.6%	14.6%	14.0%	13.6%	13.4%	13.1%	12.7%	12.6%	12.4%	12.3%	12.2%	12.2%	12.1%
2014	22.2%	17.8%	15.1%	14.1%	13.5%	13.2%	13.0%	12.6%	12.3%	12.2%	12.0%	11.9%	11.8%	11.8%	11.7%
2015	26.9%	21.9%	18.8%	17.7%	17.1%	16.8%	16.5%	16.2%	15.8%	15.7%	15.5%	15.4%	15.3%	15.3%	15.2%
2016	28.0%	22.6%	19.1%	17.8%	17.1%	16.6%	16.4%	15.9%	15.5%	15.3%	15.1%	15.0%	14.9%	14.8%	14.7%
2017	21.3%	15.9%	12.4%	11.0%	10.1%	9.6%	9.3%	8.8%	8.2%	8.0%	7.7%	7.6%	7.4%	7.4%	7.2%
2018	28.1%	22.9%	19.6%	18.4%	17.7%	17.3%	17.0%	16.7%	16.2%	16.1%	15.9%	15.8%	15.7%	15.6%	15.5%
2019	25.7%	20.3%	17.0%	15.7%	15.0%	14.6%	14.3%	13.9%	13.5%	13.3%	13.1%	13.0%	12.9%	12.8%	12.7%
Average	31.5%	25.5%	21.9%	20.6%	19.9%	19.5%	19.1%	18.8%	18.3%	18.1%	17.9%	17.8%	17.7%	17.7%	17.5%

Panel B: Value-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	49.6%	39.7%	34.3%	32.3%	31.2%	30.4%	29.8%	29.0%	28.0%	27.6%	27.0%	26.7%	26.4%	26.2%	25.8%
2003	32.1%	25.9%	22.4%	21.1%	20.3%	19.9%	19.5%	19.0%	18.4%	18.1%	17.7%	17.5%	17.3%	17.2%	17.0%
2004	25.6%	20.5%	17.2%	15.8%	15.0%	14.4%	14.0%	13.4%	12.7%	12.4%	11.9%	11.7%	11.4%	11.3%	11.0%
2005	24.8%	19.4%	16.2%	14.8%	14.0%	13.4%	13.0%	12.5%	11.8%	11.4%	11.0%	10.8%	10.6%	10.5%	10.2%
2006	24.3%	19.4%	16.2%	14.8%	13.9%	13.3%	12.9%	12.3%	11.6%	11.2%	10.8%	10.6%	10.3%	10.2%	9.9%
2007	28.4%	23.5%	20.5%	19.3%	18.6%	18.1%	17.8%	17.4%	16.9%	16.6%	16.3%	16.2%	16.0%	15.9%	15.7%
2008	66.6%	54.9%	49.1%	46.7%	45.4%	44.6%	44.0%	43.1%	42.1%	41.6%	41.1%	40.8%	40.5%	40.3%	40.0%
2009	51.0%	41.5%	36.1%	33.9%	32.7%	31.8%	31.1%	30.3%	29.1%	28.7%	28.0%	27.7%	27.3%	27.2%	26.8%
2010	29.6%	25.3%	22.5%	21.4%	20.8%	20.4%	20.1%	19.6%	19.0%	18.8%	18.5%	18.3%	18.2%	18.1%	17.9%
2011	34.5%	30.2%	27.5%	26.5%	25.9%	25.4%	25.1%	24.7%	24.2%	24.0%	23.6%	23.5%	23.4%	23.3%	23.1%
2012	25.2%	20.2%	17.3%	16.1%	15.5%	15.1%	14.7%	14.3%	13.8%	13.6%	13.2%	13.1%	12.9%	12.8%	12.5%
2013	22.4%	18.2%	15.8%	14.8%	14.2%	13.8%	13.4%	13.0%	12.4%	12.2%	11.8%	11.6%	11.4%	11.3%	11.0%
2014	22.2%	18.3%	15.8%	14.8%	14.2%	13.8%	13.5%	13.1%	12.6%	12.3%	12.0%	11.9%	11.7%	11.6%	11.4%
2015	26.9%	22.1%	19.4%	18.4%	17.8%	17.4%	17.1%	16.7%	16.3%	16.1%	15.8%	15.7%	15.6%	15.5%	15.3%
2016	28.0%	22.2%	18.7%	17.3%	16.5%	15.9%	15.5%	15.0%	14.4%	14.1%	13.7%	13.6%	13.4%	13.3%	13.0%
2017	21.3%	16.0%	12.7%	11.4%	10.6%	10.0%	9.6%	9.1%	8.3%	8.0%	7.5%	7.3%	7.1%	6.9%	6.6%
2018	28.1%	23.4%	20.7%	19.8%	19.2%	18.9%	18.7%	18.3%	17.9%	17.7%	17.5%	17.3%	17.2%	17.1%	16.9%
2019	25.7%	20.6%	17.5%	16.3%	15.6%	15.1%	14.7%	14.3%	13.7%	13.4%	13.1%	13.0%	12.8%	12.7%	12.4%
Average	31.5%	25.6%	22.2%	20.9%	20.1%	19.5%	19.2%	18.6%	18.0%	17.7%	17.3%	17.1%	16.8%	16.7%	16.5%

Table 3.3: Average risk in the European market

Table 3.3 summarizes the portfolio average risk for the European market. The results are presented by year, for portfolios with a specified number of stocks, ranging from 1 through 128 and with all stocks traded in the STOXX 600 (All). Panel A reports the results for equal-weighted portfolios and Panel B reports the results for value-weighted portfolios.

Panel A: Equal-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	45.6%	33.2%	27.0%	24.8%	23.6%	22.9%	22.4%	21.8%	21.1%	20.9%	20.5%	20.4%	20.2%	20.1%	19.9%
2003	34.7%	27.2%	22.6%	20.7%	19.7%	19.0%	18.6%	18.0%	17.3%	17.1%	16.7%	16.6%	16.4%	16.3%	16.1%
2004	23.7%	18.3%	14.9%	13.5%	12.7%	12.3%	11.9%	11.5%	11.0%	10.8%	10.5%	10.4%	10.3%	10.2%	10.0%
2005	21.1%	16.3%	13.0%	11.7%	11.0%	10.6%	10.2%	9.8%	9.3%	9.1%	8.9%	8.8%	8.7%	8.6%	8.5%
2006	25.7%	20.6%	17.5%	16.2%	15.6%	15.2%	14.9%	14.5%	14.1%	14.0%	13.8%	13.7%	13.6%	13.5%	13.4%
2007	28.8%	23.3%	20.1%	18.9%	18.2%	17.8%	17.5%	17.2%	16.8%	16.7%	16.5%	16.4%	16.3%	16.2%	16.1%
2008	58.7%	47.0%	41.0%	39.0%	38.0%	37.3%	36.8%	36.2%	35.4%	35.0%	34.6%	34.4%	34.2%	34.1%	33.8%
2009	44.5%	36.7%	31.8%	30.0%	29.0%	28.5%	28.1%	27.5%	26.9%	26.7%	26.4%	26.2%	26.1%	26.0%	25.9%
2010	30.0%	25.1%	22.2%	21.2%	20.6%	20.3%	20.1%	19.8%	19.5%	19.3%	19.2%	19.1%	19.0%	19.0%	18.9%
2011	36.4%	30.2%	26.9%	25.7%	25.0%	24.7%	24.4%	24.1%	23.7%	23.6%	23.4%	23.4%	23.3%	23.2%	23.1%
2012	30.2%	24.8%	21.5%	20.3%	19.6%	19.2%	18.9%	18.5%	18.1%	18.0%	17.8%	17.7%	17.6%	17.5%	17.4%
2013	25.3%	19.9%	16.6%	15.3%	14.5%	14.1%	13.8%	13.4%	12.9%	12.7%	12.5%	12.4%	12.3%	12.2%	12.1%
2014	24.8%	19.7%	16.6%	15.4%	14.7%	14.3%	14.0%	13.7%	13.3%	13.1%	12.9%	12.8%	12.7%	12.7%	12.5%
2015	29.4%	23.8%	20.7%	19.5%	18.8%	18.4%	18.2%	17.9%	17.5%	17.3%	17.1%	17.1%	17.0%	16.9%	16.8%
2016	31.9%	26.1%	22.7%	21.4%	20.7%	20.3%	20.0%	19.7%	19.2%	19.1%	18.9%	18.8%	18.7%	18.6%	18.5%
2017	21.3%	16.2%	12.8%	11.4%	10.6%	10.1%	9.8%	9.3%	8.8%	8.6%	8.3%	8.2%	8.1%	8.0%	7.8%
2018	26.8%	20.9%	17.1%	15.7%	15.0%	14.4%	14.1%	13.7%	13.1%	13.0%	12.7%	12.6%	12.5%	12.4%	12.3%
2019	26.1%	20.4%	16.8%	15.4%	14.6%	14.1%	13.8%	13.4%	12.9%	12.7%	12.5%	12.4%	12.3%	12.2%	12.1%
Average	31.4%	25.0%	21.2%	19.8%	19.0%	18.5%	18.2%	17.8%	17.3%	17.1%	16.8%	16.7%	16.6%	16.6%	16.4%

Panel B: Value-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	45.6%	37.3%	33.6%	32.2%	31.5%	30.8%	30.4%	29.7%	29.0%	28.6%	28.1%	27.8%	27.5%	27.4%	27.0%
2003	34.7%	28.8%	25.4%	24.0%	23.2%	22.7%	22.3%	21.7%	21.0%	20.7%	20.2%	20.0%	19.7%	19.6%	19.3%
2004	23.7%	19.0%	16.2%	15.0%	14.4%	14.0%	13.6%	13.2%	12.6%	12.3%	11.9%	11.7%	11.4%	11.3%	11.1%
2005	21.1%	16.8%	14.0%	12.8%	12.1%	11.7%	11.3%	10.8%	10.2%	9.9%	9.5%	9.3%	9.1%	9.0%	8.7%
2006	25.7%	21.1%	18.2%	16.9%	16.2%	15.7%	15.3%	14.9%	14.2%	14.0%	13.6%	13.4%	13.2%	13.2%	12.9%
2007	28.8%	23.8%	20.7%	19.4%	18.7%	18.2%	17.8%	17.4%	16.8%	16.5%	16.1%	16.0%	15.8%	15.7%	15.5%
2008	58.7%	49.9%	45.7%	43.9%	42.9%	42.3%	41.7%	41.0%	40.0%	39.5%	38.6%	38.1%	37.1%	36.7%	35.1%
2009	44.5%	37.3%	32.8%	30.8%	29.7%	28.7%	28.1%	27.1%	25.9%	25.4%	24.7%	24.3%	24.0%	23.8%	23.3%
2010	30.0%	25.9%	23.6%	22.6%	22.0%	21.6%	21.3%	20.9%	20.3%	20.0%	19.7%	19.5%	19.3%	19.2%	19.0%
2011	36.4%	31.6%	28.8%	27.6%	26.8%	26.3%	25.9%	25.3%	24.7%	24.4%	24.0%	23.8%	23.6%	23.5%	23.3%
2012	30.2%	25.0%	21.8%	20.4%	19.5%	19.0%	18.6%	17.9%	17.2%	16.8%	16.4%	16.2%	15.9%	15.8%	15.5%
2013	25.3%	20.5%	17.6%	16.4%	15.7%	15.2%	14.8%	14.3%	13.7%	13.4%	13.1%	12.9%	12.7%	12.6%	12.4%
2014	24.8%	20.3%	17.6%	16.4%	15.8%	15.3%	15.0%	14.5%	13.9%	13.6%	13.3%	13.1%	12.9%	12.8%	12.6%
2015	29.4%	24.6%	22.2%	21.3%	20.7%	20.4%	20.1%	19.7%	19.2%	19.0%	18.7%	18.6%	18.5%	18.4%	18.2%
2016	31.9%	26.7%	23.7%	22.4%	21.6%	21.1%	20.7%	20.2%	19.4%	19.1%	18.7%	18.5%	18.3%	18.2%	17.9%
2017	21.3%	16.9%	13.9%	12.6%	11.9%	11.4%	11.0%	10.4%	9.7%	9.3%	8.9%	8.6%	8.3%	8.2%	7.8%
2018	26.8%	21.2%	17.9%	16.5%	15.8%	15.2%	14.9%	14.3%	13.7%	13.4%	13.0%	12.8%	12.6%	12.5%	12.3%
2019	26.1%	20.7%	17.2%	15.8%	14.9%	14.4%	14.0%	13.4%	12.7%	12.4%	11.9%	11.7%	11.5%	11.4%	11.1%
Average	31.4%	26.0%	22.8%	21.5%	20.7%	20.2%	19.8%	19.3%	18.6%	18.2%	17.8%	17.6%	17.3%	17.2%	16.8%

Table 3.4: Average risk reduction in the U.S. market

Table 3.4 summarizes the difference in average risk between a portfolio with 1 stock and a portfolio with a specified portfolio size, ranging from 2 to 128, as a percentage of the difference in average risk between a portfolio with 1 stock and a portfolio with all stocks traded in the S&P 500 (U.S. market). The results are presented by year, for equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B).

Panel A: Equal-weighted portfolios													
	Number of stocks												
	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	50%	74%	82%	86%	89%	91%	93%	96%	97%	98%	98%	99%	99%
2003	38%	63%	74%	80%	83%	86%	89%	93%	94%	97%	97%	98%	99%
2004	40%	65%	75%	81%	84%	87%	90%	93%	95%	97%	98%	99%	99%
2005	41%	66%	76%	81%	85%	87%	90%	94%	95%	97%	98%	99%	99%
2006	38%	64%	75%	80%	84%	87%	90%	93%	95%	97%	98%	99%	99%
2007	43%	69%	78%	84%	87%	89%	92%	95%	96%	97%	98%	99%	99%
2008	51%	75%	83%	87%	90%	92%	94%	96%	97%	98%	99%	99%	99%
2009	45%	70%	79%	84%	87%	89%	92%	95%	96%	97%	98%	99%	99%
2010	42%	69%	78%	84%	87%	89%	92%	95%	96%	98%	98%	99%	99%
2011	47%	71%	80%	85%	88%	90%	93%	96%	97%	98%	98%	99%	99%
2012	40%	66%	76%	82%	85%	88%	91%	94%	95%	97%	98%	99%	99%
2013	40%	66%	76%	82%	85%	87%	90%	94%	95%	97%	98%	99%	99%
2014	42%	67%	77%	82%	86%	88%	91%	94%	95%	97%	98%	99%	99%
2015	43%	69%	79%	83%	87%	89%	92%	95%	96%	97%	98%	99%	99%
2016	40%	67%	77%	82%	85%	87%	90%	94%	95%	97%	98%	99%	99%
2017	38%	63%	73%	79%	83%	85%	89%	93%	94%	96%	97%	98%	99%
2018	41%	67%	77%	83%	86%	88%	91%	94%	96%	97%	98%	99%	99%
2019	41%	67%	77%	82%	86%	88%	91%	94%	95%	97%	98%	99%	99%
Average	42%	68%	77%	83%	86%	88%	91%	94%	96%	97%	98%	99%	99%

Panel B: Value-weighted portfolios													
	Number of stocks												
	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	41%	64%	73%	77%	81%	83%	87%	91%	92%	95%	96%	98%	98%
2003	41%	64%	73%	78%	81%	83%	87%	91%	92%	95%	96%	98%	98%
2004	35%	58%	67%	73%	76%	79%	83%	88%	90%	94%	95%	97%	98%
2005	37%	59%	68%	74%	78%	81%	84%	89%	91%	94%	96%	97%	98%
2006	34%	56%	66%	72%	76%	79%	83%	88%	91%	94%	95%	97%	98%
2007	39%	62%	72%	77%	81%	84%	87%	91%	93%	96%	97%	98%	99%
2008	44%	66%	75%	80%	83%	85%	88%	92%	94%	96%	97%	98%	99%
2009	39%	62%	71%	76%	79%	82%	85%	90%	92%	95%	96%	98%	98%
2010	37%	60%	70%	75%	79%	81%	86%	90%	92%	95%	96%	97%	98%
2011	38%	61%	70%	76%	79%	82%	85%	90%	92%	95%	96%	98%	98%
2012	40%	62%	72%	77%	80%	83%	86%	90%	92%	94%	96%	97%	98%
2013	36%	58%	67%	72%	76%	78%	82%	87%	90%	93%	94%	97%	97%
2014	36%	59%	68%	74%	78%	80%	84%	89%	91%	94%	96%	97%	98%
2015	42%	65%	74%	79%	82%	85%	88%	92%	93%	96%	97%	98%	98%
2016	39%	62%	72%	77%	81%	83%	87%	91%	93%	95%	96%	98%	98%
2017	36%	58%	67%	73%	77%	80%	83%	88%	90%	94%	95%	97%	98%
2018	42%	66%	75%	79%	82%	84%	88%	91%	93%	95%	96%	98%	98%
2019	38%	61%	71%	76%	80%	83%	86%	90%	92%	95%	96%	97%	98%
Average	39%	61%	70%	76%	79%	82%	86%	90%	92%	95%	96%	97%	98%

Table 3.5: Average risk reduction in the European market

Table 3.5 summarizes the difference in average risk between a portfolio with 1 stock and a portfolio with a specified portfolio size, ranging from 2 to 128, as a percentage of the difference in average risk between a portfolio with 1 stock and a portfolio with all stocks traded in the STOXX 600 (European market). The results are presented by year, for equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B).

Panel A: Equal-weighted portfolios													
	Number of stocks												
	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	48%	72%	81%	86%	88%	90%	93%	95%	96%	98%	98%	99%	99%
2003	40%	65%	75%	81%	84%	87%	90%	94%	95%	97%	98%	99%	99%
2004	39%	65%	75%	80%	84%	86%	89%	93%	95%	97%	98%	98%	99%
2005	38%	64%	74%	80%	83%	86%	89%	93%	95%	97%	97%	98%	99%
2006	41%	67%	77%	82%	86%	88%	91%	94%	96%	97%	98%	99%	99%
2007	44%	69%	78%	84%	87%	89%	92%	95%	96%	97%	98%	99%	99%
2008	47%	71%	79%	83%	86%	88%	90%	93%	95%	97%	97%	98%	99%
2009	42%	68%	78%	83%	86%	88%	91%	95%	96%	97%	98%	99%	99%
2010	44%	70%	79%	84%	88%	90%	92%	95%	96%	97%	98%	99%	99%
2011	47%	72%	81%	86%	89%	91%	93%	96%	97%	98%	98%	99%	99%
2012	42%	68%	78%	83%	86%	88%	91%	94%	96%	97%	98%	99%	99%
2013	41%	66%	76%	81%	85%	87%	90%	94%	95%	97%	98%	99%	99%
2014	42%	67%	77%	82%	85%	88%	91%	94%	95%	97%	98%	99%	99%
2015	44%	69%	79%	84%	87%	89%	92%	95%	96%	97%	98%	99%	99%
2016	44%	69%	78%	83%	86%	89%	91%	95%	96%	97%	98%	99%	99%
2017	38%	63%	74%	79%	83%	86%	89%	93%	94%	96%	97%	98%	99%
2018	41%	66%	76%	82%	85%	87%	90%	94%	95%	97%	98%	99%	99%
2019	40%	66%	76%	82%	85%	88%	91%	94%	95%	97%	98%	99%	99%
Average	42%	68%	77%	83%	86%	88%	91%	94%	95%	97%	98%	99%	99%

Panel B: Value-weighted portfolios													
	Number of stocks												
	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	44%	65%	72%	76%	79%	82%	85%	90%	92%	94%	96%	97%	98%
2003	38%	60%	69%	74%	78%	80%	84%	89%	91%	94%	95%	97%	98%
2004	37%	59%	68%	73%	77%	79%	83%	88%	90%	94%	95%	97%	98%
2005	35%	57%	67%	73%	76%	79%	83%	88%	90%	93%	95%	97%	98%
2006	36%	59%	69%	74%	78%	81%	85%	90%	92%	95%	96%	97%	98%
2007	37%	61%	70%	76%	79%	82%	86%	90%	92%	95%	96%	97%	98%
2008	37%	55%	63%	67%	69%	72%	75%	79%	81%	85%	87%	91%	93%
2009	34%	55%	65%	70%	74%	78%	82%	88%	90%	94%	95%	97%	98%
2010	37%	58%	68%	73%	76%	79%	83%	88%	90%	94%	95%	97%	98%
2011	37%	58%	67%	73%	77%	80%	84%	89%	91%	94%	96%	97%	98%
2012	35%	57%	67%	73%	76%	79%	84%	89%	91%	94%	96%	97%	98%
2013	37%	59%	69%	74%	78%	81%	85%	90%	92%	95%	96%	97%	98%
2014	37%	59%	68%	73%	77%	80%	84%	89%	91%	94%	95%	97%	98%
2015	42%	64%	73%	77%	80%	83%	86%	90%	92%	95%	96%	98%	98%
2016	37%	59%	68%	74%	77%	80%	84%	89%	91%	94%	96%	97%	98%
2017	33%	55%	64%	70%	74%	77%	81%	86%	89%	93%	94%	96%	97%
2018	39%	61%	71%	76%	80%	82%	86%	90%	92%	95%	96%	98%	98%
2019	36%	59%	69%	74%	78%	81%	84%	89%	91%	94%	96%	97%	98%
Average	37%	59%	68%	73%	77%	80%	84%	88%	91%	94%	95%	97%	98%

Table 3.6: Average end-of-period wealth in the U.S. market

Table 3.6 summarizes the average end-of-period wealth in the U.S. market. The results are presented by year, for portfolios with a specified numbers of stocks, ranging from 1 through 128 and for portfolios with all stocks traded in the S&P 500 (All). Panel A reports the results for equal-weighted portfolios and Panel B reports the results for value-weighted portfolios.

Panel A: Equal-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82	0.82
2003	1.41	1.42	1.42	1.42	1.42	1.42	1.42	1.42	1.42	1.42	1.42	1.41	1.42	1.42	1.42
2004	1.16	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17
2005	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08
2006	1.15	1.15	1.15	1.15	1.15	1.15	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16
2007	1.03	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
2008	0.62	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61	0.61
2009	1.43	1.44	1.44	1.43	1.44	1.44	1.44	1.44	1.44	1.44	1.44	1.44	1.44	1.44	1.44
2010	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21	1.21
2011	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01
2012	1.16	1.16	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17	1.17
2013	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36	1.36
2014	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14
2015	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98	0.98
2016	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14
2017	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19	1.19
2018	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93	0.93
2019	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29
Average	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12

Panel B: Value-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	0.82	0.81	0.80	0.79	0.79	0.79	0.79	0.79	0.79	0.79	0.78	0.78	0.78	0.78	0.78
2003	1.41	1.36	1.33	1.32	1.31	1.31	1.31	1.30	1.30	1.29	1.29	1.29	1.29	1.29	1.28
2004	1.16	1.16	1.14	1.13	1.12	1.12	1.12	1.11	1.11	1.11	1.10	1.10	1.10	1.10	1.10
2005	1.08	1.09	1.08	1.07	1.07	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.05	1.05	1.05
2006	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.16	1.16	1.16	1.16	1.16	1.16	1.16
2007	1.03	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05
2008	0.62	0.62	0.62	0.63	0.63	0.63	0.63	0.63	0.64	0.64	0.64	0.64	0.64	0.64	0.64
2009	1.43	1.37	1.33	1.32	1.31	1.30	1.30	1.29	1.29	1.28	1.28	1.27	1.27	1.27	1.27
2010	1.21	1.19	1.18	1.17	1.17	1.17	1.16	1.16	1.16	1.16	1.15	1.15	1.15	1.15	1.15
2011	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
2012	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16	1.16
2013	1.36	1.36	1.35	1.34	1.34	1.34	1.34	1.33	1.33	1.33	1.32	1.32	1.32	1.32	1.32
2014	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14
2015	0.98	0.99	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01
2016	1.14	1.13	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.12
2017	1.19	1.20	1.20	1.21	1.21	1.21	1.21	1.21	1.21	1.22	1.22	1.22	1.22	1.22	1.22
2018	0.93	0.94	0.94	0.95	0.95	0.95	0.95	0.95	0.95	0.95	0.96	0.96	0.96	0.96	0.96
2019	1.29	1.29	1.29	1.29	1.30	1.30	1.30	1.30	1.30	1.30	1.31	1.31	1.31	1.31	1.31
Average	1.12	1.11	1.11	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10	1.10

Table 3.7: Average end-of-period wealth in the European market

Table 3.7 summarizes the average end-of-period wealth in the European market. The results are presented by year, for portfolios with a specified numbers of stocks, ranging from 1 through 128 and for portfolios with all stocks traded in the STOXX 600 (All). Panel A reports the results for equal-weighted portfolios and Panel B reports the results for value-weighted portfolios.

Panel A: Equal-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72
2003	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26	1.26
2004	1.15	1.16	1.16	1.16	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15	1.15
2005	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24	1.24
2006	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25
2007	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99
2008	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55	0.55
2009	1.39	1.40	1.39	1.39	1.39	1.40	1.40	1.40	1.39	1.39	1.40	1.40	1.40	1.40	1.40
2010	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13	1.13
2011	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83	0.83
2012	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18
2013	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22
2014	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
2015	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08
2016	1.03	1.04	1.03	1.04	1.03	1.04	1.03	1.04	1.03	1.04	1.04	1.04	1.04	1.04	1.03
2017	1.14	1.14	1.15	1.15	1.15	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14	1.14
2018	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.86
2019	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.23	1.23	1.23	1.23	1.22
Average	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07	1.07

Panel B: Value-weighted portfolios															
	Number of stocks														
	1	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	0.72	0.71	0.71	0.70	0.70	0.70	0.70	0.70	0.70	0.70	0.69	0.69	0.69	0.69	0.69
2003	1.26	1.23	1.21	1.20	1.19	1.19	1.19	1.18	1.18	1.18	1.17	1.17	1.17	1.17	1.17
2004	1.15	1.14	1.13	1.13	1.12	1.12	1.11	1.11	1.11	1.11	1.10	1.10	1.10	1.10	1.10
2005	1.24	1.24	1.23	1.23	1.23	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22	1.22
2006	1.25	1.23	1.22	1.20	1.20	1.19	1.19	1.18	1.18	1.18	1.17	1.17	1.17	1.17	1.17
2007	0.99	1.01	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04
2008	0.55	0.56	0.57	0.57	0.57	0.58	0.58	0.58	0.58	0.58	0.59	0.59	0.59	0.59	0.59
2009	1.39	1.34	1.30	1.28	1.27	1.27	1.26	1.26	1.25	1.24	1.24	1.24	1.24	1.23	1.23
2010	1.13	1.11	1.10	1.08	1.08	1.07	1.07	1.07	1.06	1.06	1.06	1.06	1.06	1.06	1.06
2011	0.83	0.84	0.84	0.85	0.85	0.85	0.85	0.86	0.86	0.86	0.86	0.86	0.86	0.86	0.87
2012	1.18	1.17	1.16	1.15	1.15	1.15	1.14	1.14	1.14	1.14	1.14	1.13	1.13	1.13	1.13
2013	1.22	1.21	1.20	1.20	1.19	1.19	1.19	1.19	1.18	1.18	1.18	1.18	1.18	1.18	1.18
2014	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
2015	1.08	1.07	1.06	1.05	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.04	1.04	1.04	1.04
2016	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
2017	1.14	1.13	1.13	1.12	1.12	1.12	1.12	1.12	1.12	1.12	1.11	1.11	1.11	1.11	1.11
2018	0.86	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87	0.87
2019	1.22	1.21	1.21	1.20	1.20	1.20	1.20	1.20	1.20	1.20	1.20	1.20	1.20	1.20	1.20
Average	1.07	1.06	1.06	1.05	1.05	1.05	1.05	1.05	1.04	1.04	1.04	1.04	1.04	1.04	1.04

Table 3.8: End-of-period wealth standard deviation in the U.S. market

Table 3.8 summarizes the end-of-period wealth standard deviation in the U.S. market. The results are presented by year, for portfolios with a specified numbers of stocks, ranging from 1 through 128. Panel A reports the results for equal-weighted portfolios and Panel B reports the results for value-weighted portfolios.

Panel A: Equal-weighted portfolios

	Number of stocks													
	1	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	0.28	0.20	0.14	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02
2003	0.44	0.31	0.22	0.18	0.16	0.14	0.13	0.11	0.09	0.08	0.06	0.05	0.04	0.03
2004	0.28	0.20	0.14	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.02	0.02
2005	0.26	0.19	0.13	0.11	0.09	0.08	0.08	0.07	0.05	0.05	0.04	0.03	0.02	0.02
2006	0.22	0.16	0.11	0.09	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.03	0.02	0.02
2007	0.33	0.23	0.16	0.13	0.12	0.10	0.10	0.08	0.07	0.06	0.04	0.04	0.03	0.03
2008	0.26	0.18	0.13	0.11	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2009	0.55	0.39	0.28	0.23	0.20	0.18	0.16	0.14	0.11	0.10	0.07	0.07	0.05	0.04
2010	0.24	0.17	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2011	0.25	0.17	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2012	0.25	0.18	0.13	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2013	0.32	0.22	0.16	0.13	0.11	0.10	0.09	0.08	0.06	0.05	0.04	0.04	0.03	0.02
2014	0.23	0.16	0.11	0.09	0.08	0.07	0.06	0.06	0.04	0.04	0.03	0.03	0.02	0.02
2015	0.26	0.18	0.13	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2016	0.25	0.18	0.13	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2017	0.27	0.19	0.13	0.11	0.09	0.08	0.08	0.07	0.05	0.05	0.04	0.03	0.02	0.02
2018	0.22	0.16	0.11	0.09	0.08	0.07	0.06	0.06	0.04	0.04	0.03	0.03	0.02	0.02
2019	0.25	0.17	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
Average	0.29	0.20	0.14	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02

Panel B: Value-weighted portfolios

	Number of stocks													
	1	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	0.28	0.22	0.16	0.14	0.13	0.11	0.11	0.09	0.08	0.07	0.05	0.05	0.03	0.03
2003	0.44	0.26	0.19	0.16	0.14	0.13	0.12	0.11	0.09	0.08	0.06	0.06	0.04	0.04
2004	0.28	0.20	0.15	0.12	0.11	0.10	0.10	0.09	0.07	0.06	0.05	0.05	0.04	0.03
2005	0.26	0.19	0.14	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02
2006	0.22	0.16	0.13	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.02
2007	0.33	0.26	0.19	0.16	0.15	0.13	0.12	0.11	0.09	0.08	0.06	0.06	0.04	0.04
2008	0.26	0.20	0.16	0.14	0.13	0.12	0.11	0.10	0.08	0.07	0.06	0.05	0.04	0.03
2009	0.55	0.33	0.23	0.20	0.18	0.16	0.16	0.14	0.12	0.11	0.09	0.08	0.06	0.05
2010	0.24	0.18	0.14	0.12	0.10	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03
2011	0.25	0.19	0.15	0.13	0.12	0.11	0.10	0.09	0.07	0.07	0.05	0.05	0.04	0.03
2012	0.25	0.18	0.13	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02
2013	0.32	0.22	0.15	0.13	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03
2014	0.23	0.16	0.12	0.10	0.09	0.08	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.03
2015	0.26	0.19	0.15	0.13	0.12	0.11	0.10	0.09	0.07	0.06	0.05	0.04	0.03	0.03
2016	0.25	0.17	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2017	0.27	0.19	0.14	0.13	0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03
2018	0.22	0.17	0.13	0.11	0.10	0.09	0.09	0.08	0.06	0.06	0.05	0.04	0.03	0.03
2019	0.25	0.18	0.14	0.12	0.11	0.11	0.10	0.10	0.09	0.08	0.07	0.06	0.05	0.04
Average	0.29	0.20	0.15	0.13	0.12	0.11	0.10	0.09	0.07	0.06	0.05	0.05	0.04	0.03

Table 3.9: End-of-period wealth standard deviation in the European market

Table 3.9 summarizes the end-of-period wealth standard deviation in the European market. The results are presented by year, for portfolios with a specified numbers of stocks, ranging from 1 through 128. Panel A reports the results for equal-weighted portfolios and Panel B reports the results for value-weighted portfolios.

Panel A: Equal-weighted portfolios														
	Number of stocks													
	1	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	0.28	0.20	0.14	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
2003	0.33	0.24	0.17	0.14	0.12	0.10	0.10	0.08	0.07	0.06	0.04	0.04	0.03	0.03
2004	0.25	0.18	0.13	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2005	0.24	0.17	0.12	0.10	0.08	0.07	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2006	0.29	0.21	0.15	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02
2007	0.30	0.21	0.15	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.02
2008	0.25	0.17	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2009	0.51	0.36	0.25	0.20	0.18	0.16	0.14	0.13	0.10	0.09	0.07	0.06	0.05	0.04
2010	0.32	0.22	0.16	0.13	0.11	0.10	0.09	0.08	0.06	0.05	0.04	0.04	0.03	0.03
2011	0.26	0.19	0.13	0.11	0.09	0.08	0.08	0.06	0.05	0.05	0.04	0.03	0.02	0.02
2012	0.29	0.20	0.14	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02
2013	0.31	0.22	0.15	0.13	0.11	0.10	0.09	0.08	0.06	0.05	0.04	0.04	0.03	0.02
2014	0.22	0.16	0.11	0.09	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.03	0.02	0.02
2015	0.28	0.20	0.14	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
2016	0.28	0.20	0.14	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
2017	0.26	0.18	0.13	0.11	0.09	0.08	0.08	0.07	0.05	0.04	0.04	0.03	0.02	0.02
2018	0.23	0.16	0.11	0.09	0.08	0.07	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02
2019	0.28	0.19	0.14	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
Average	0.29	0.20	0.14	0.12	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02

Panel B: Value-weighted portfolios														
	Number of stocks													
	1	2	4	6	8	10	12	16	25	32	50	64	100	128
2002	0.28	0.20	0.16	0.13	0.12	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03
2003	0.33	0.23	0.16	0.13	0.12	0.11	0.10	0.09	0.07	0.06	0.05	0.04	0.03	0.03
2004	0.25	0.18	0.13	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
2005	0.24	0.18	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03
2006	0.29	0.21	0.15	0.12	0.11	0.10	0.09	0.08	0.06	0.06	0.05	0.04	0.03	0.03
2007	0.30	0.23	0.18	0.16	0.14	0.13	0.12	0.11	0.09	0.08	0.06	0.05	0.04	0.04
2008	0.25	0.19	0.16	0.14	0.13	0.12	0.11	0.10	0.08	0.07	0.06	0.05	0.04	0.03
2009	0.51	0.34	0.26	0.22	0.20	0.18	0.17	0.15	0.12	0.11	0.08	0.07	0.06	0.05
2010	0.32	0.23	0.18	0.15	0.14	0.13	0.12	0.10	0.08	0.07	0.06	0.05	0.04	0.03
2011	0.26	0.19	0.15	0.12	0.11	0.10	0.10	0.09	0.07	0.06	0.05	0.04	0.03	0.03
2012	0.29	0.21	0.16	0.14	0.13	0.11	0.11	0.09	0.08	0.07	0.05	0.05	0.04	0.03
2013	0.31	0.22	0.16	0.13	0.12	0.10	0.10	0.08	0.07	0.06	0.05	0.04	0.03	0.03
2014	0.22	0.16	0.12	0.10	0.09	0.08	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
2015	0.28	0.20	0.15	0.13	0.11	0.10	0.10	0.09	0.07	0.06	0.05	0.04	0.03	0.03
2016	0.28	0.20	0.15	0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03
2017	0.26	0.18	0.13	0.11	0.10	0.09	0.08	0.07	0.05	0.05	0.04	0.03	0.03	0.02
2018	0.23	0.16	0.12	0.10	0.09	0.08	0.08	0.07	0.06	0.05	0.04	0.03	0.03	0.02
2019	0.28	0.20	0.15	0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.03
Average	0.29	0.21	0.16	0.13	0.12	0.11	0.10	0.09	0.07	0.06	0.05	0.04	0.03	0.03

Table 3.10: Evans and Archer (1968) regression

Table 3.10 summarizes the coefficient of determination, the slope, and the intercept (systematic risk) of Evans and Archer (1968) regression procedure described in subsection 3.2.4, by year and in the U.S. and European markets. The squared intercept (Inter. Sq.) and the average covariance of a portfolio with all stocks traded in the market index (Av. Cov.) are also reported, as well as the difference between them (Dif.).

	U.S. market						European market					
	Coef. Det.	Slope	Intercept	Inter. Sq.	Av. Cov.	Dif.	Coef. Det.	Slope	Intercept	Inter. Sq.	Av. Cov.	Dif.
2002	100.0%	25.0%	24.6%	6.1%	6.0%	0.1%	99.9%	25.7%	20.2%	4.1%	3.9%	0.1%
2003	97.7%	14.1%	18.9%	3.6%	3.4%	0.2%	98.4%	18.9%	16.8%	2.8%	2.6%	0.2%
2004	98.3%	13.7%	12.6%	1.6%	1.5%	0.1%	98.1%	13.9%	10.6%	1.1%	1.0%	0.1%
2005	98.6%	13.8%	11.6%	1.4%	1.2%	0.1%	97.8%	13.0%	9.0%	0.8%	0.7%	0.1%
2006	98.0%	13.8%	11.3%	1.3%	1.2%	0.1%	98.8%	12.5%	13.7%	1.9%	1.8%	0.1%
2007	99.3%	12.7%	16.2%	2.6%	2.5%	0.1%	99.3%	12.8%	16.4%	2.7%	2.6%	0.1%
2008	100.0%	25.6%	40.9%	16.7%	16.7%	0.0%	99.6%	24.8%	34.4%	11.8%	11.4%	0.5%
2009	99.5%	17.8%	33.7%	11.4%	11.1%	0.3%	99.1%	19.0%	26.3%	6.9%	6.7%	0.3%
2010	99.2%	9.7%	20.3%	4.1%	4.1%	0.1%	99.5%	11.3%	19.1%	3.6%	3.6%	0.1%
2011	99.8%	9.6%	25.1%	6.3%	6.2%	0.1%	99.8%	13.4%	23.3%	5.4%	5.3%	0.1%
2012	98.5%	11.6%	14.3%	2.0%	1.9%	0.1%	99.1%	13.0%	17.7%	3.1%	3.0%	0.1%
2013	98.5%	10.5%	12.4%	1.5%	1.5%	0.1%	98.6%	13.5%	12.5%	1.6%	1.5%	0.1%
2014	98.9%	10.6%	12.0%	1.4%	1.4%	0.1%	98.8%	12.5%	12.9%	1.7%	1.6%	0.1%
2015	99.3%	11.9%	15.4%	2.4%	2.3%	0.1%	99.5%	12.8%	17.0%	2.9%	2.8%	0.1%
2016	98.7%	13.6%	15.1%	2.3%	2.1%	0.1%	99.3%	13.6%	18.8%	3.5%	3.4%	0.1%
2017	97.6%	14.4%	7.8%	0.6%	0.5%	0.1%	97.6%	13.8%	8.4%	0.7%	0.6%	0.1%
2018	98.9%	12.8%	15.8%	2.5%	2.4%	0.1%	98.6%	14.8%	12.7%	1.6%	1.5%	0.1%
2019	98.8%	13.2%	13.1%	1.7%	1.6%	0.1%	98.6%	14.3%	12.5%	1.6%	1.4%	0.1%

Table 3.11: Average covariance in the U.S. and European markets

Table 3.11 summarizes the average covariance of portfolios with a specified number of stocks, ranging from 2 through 128 and with all stocks (All) traded in the S&P 500 (U.S. market) and STOXX 600 (European market). The results are presented by year, and for the U.S. market (Panel A) and for the European market (Panel B).

Panel A: U.S. market														
	Number of stocks													
	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	-2.3%	2.6%	3.9%	4.5%	4.8%	5.0%	5.3%	5.5%	5.6%	5.8%	5.8%	5.9%	5.9%	6.0%
2003	3.7%	3.6%	3.6%	3.5%	3.5%	3.5%	3.5%	3.4%	3.4%	3.4%	3.4%	3.4%	3.4%	3.4%
2004	1.2%	1.4%	1.4%	1.4%	1.4%	1.4%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%
2005	0.5%	1.0%	1.1%	1.1%	1.1%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%
2006	0.9%	1.1%	1.1%	1.1%	1.1%	1.1%	1.1%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%	1.2%
2007	2.0%	2.3%	2.4%	2.5%	2.5%	2.5%	2.5%	2.5%	2.5%	2.5%	2.5%	2.5%	2.5%	2.5%
2008	5.4%	12.2%	13.9%	14.7%	15.1%	15.4%	15.7%	16.1%	16.2%	16.4%	16.5%	16.5%	16.6%	16.7%
2009	8.6%	10.2%	10.6%	10.8%	10.9%	10.9%	11.0%	11.0%	11.1%	11.1%	11.1%	11.1%	11.1%	11.1%
2010	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%	4.1%
2011	5.9%	6.1%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%	6.2%
2012	1.7%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%	1.9%
2013	1.4%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%
2014	1.2%	1.3%	1.3%	1.3%	1.3%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%
2015	1.7%	2.1%	2.2%	2.2%	2.2%	2.2%	2.3%	2.3%	2.3%	2.3%	2.3%	2.3%	2.3%	2.3%
2016	1.9%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%	2.1%
2017	0.1%	0.3%	0.4%	0.4%	0.5%	0.5%	0.5%	0.5%	0.5%	0.5%	0.5%	0.5%	0.5%	0.5%
2018	2.0%	2.3%	2.3%	2.4%	2.4%	2.4%	2.4%	2.4%	2.4%	2.4%	2.4%	2.4%	2.4%	2.4%
2019	1.2%	1.4%	1.5%	1.5%	1.5%	1.6%	1.6%	1.6%	1.6%	1.6%	1.6%	1.6%	1.6%	1.6%
Average	2.3%	3.2%	3.4%	3.5%	3.6%	3.6%	3.6%	3.7%	3.7%	3.7%	3.7%	3.7%	3.7%	3.8%

Panel B: European market														
	Number of stocks													
	2	4	6	8	10	12	16	25	32	50	64	100	128	All
2002	-2.4%	1.3%	2.3%	2.8%	3.0%	3.2%	3.4%	3.6%	3.7%	3.8%	3.8%	3.9%	3.9%	3.9%
2003	1.5%	2.2%	2.4%	2.5%	2.5%	2.5%	2.5%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%
2004	0.9%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%	1.0%
2005	0.4%	0.6%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%	0.7%
2006	1.3%	1.6%	1.7%	1.7%	1.7%	1.7%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%
2007	2.3%	2.5%	2.5%	2.5%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%	2.6%
2008	6.3%	9.8%	10.7%	11.1%	11.3%	11.3%	11.4%	11.4%	11.4%	11.4%	11.4%	11.4%	11.4%	11.4%
2009	6.1%	6.5%	6.6%	6.6%	6.6%	6.7%	6.6%	6.6%	6.7%	6.7%	6.7%	6.7%	6.7%	6.7%
2010	3.2%	3.4%	3.5%	3.5%	3.5%	3.5%	3.5%	3.5%	3.6%	3.6%	3.6%	3.6%	3.6%	3.6%
2011	4.3%	4.9%	5.1%	5.1%	5.2%	5.2%	5.2%	5.3%	5.3%	5.3%	5.3%	5.3%	5.3%	5.3%
2012	2.7%	2.9%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%	3.0%
2013	0.8%	1.2%	1.3%	1.3%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.5%
2014	1.3%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.6%	1.6%	1.6%	1.6%	1.6%	1.6%	1.6%
2015	1.2%	2.2%	2.5%	2.6%	2.6%	2.7%	2.7%	2.7%	2.8%	2.8%	2.8%	2.8%	2.8%	2.8%
2016	2.6%	3.1%	3.2%	3.3%	3.3%	3.3%	3.4%	3.4%	3.4%	3.4%	3.4%	3.4%	3.4%	3.4%
2017	0.0%	0.4%	0.5%	0.5%	0.5%	0.6%	0.6%	0.6%	0.6%	0.6%	0.6%	0.6%	0.6%	0.6%
2018	1.0%	1.3%	1.4%	1.4%	1.4%	1.4%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%	1.5%
2019	1.0%	1.3%	1.3%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%	1.4%
Average	1.9%	2.7%	2.8%	2.9%	3.0%	3.0%	3.0%	3.0%	3.0%	3.1%	3.1%	3.1%	3.1%	3.1%

Figure 3.1: Return distributions of equal-weighted portfolios in the U.S. market

Figure 3.1 shows the return density probability distributions of equal-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the equal-weighted portfolio with all stocks (EW) traded in the S&P 500 (U.S. market) in 2003, 2008, 2013 and 2019.

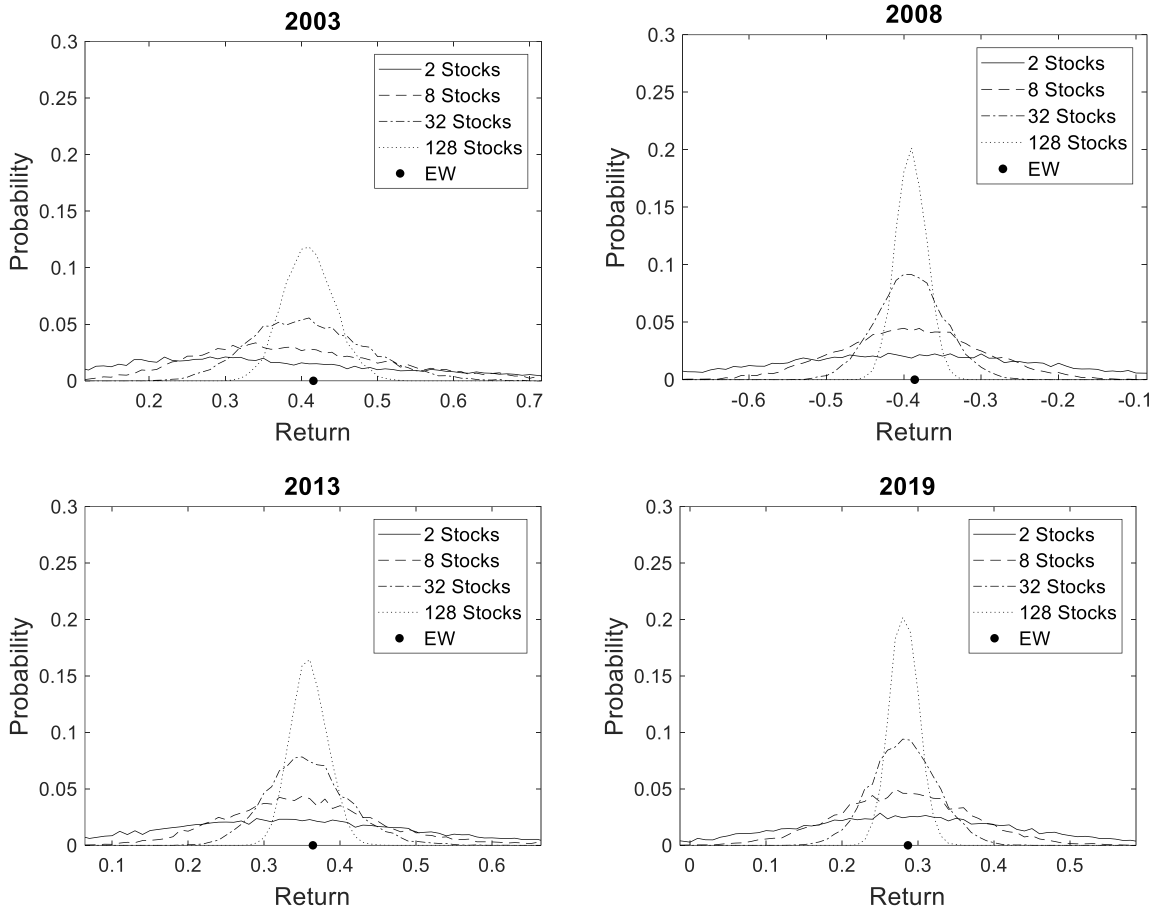


Figure 3.2: Risk distributions of equal-weighted portfolios in the U.S. market

Figure 3.2 shows the risk density probability distributions of equal-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the equal-weighted portfolio with all stocks (EW) traded in the S&P 500 (U.S. market) in 2003, 2008, 2013 and 2019.

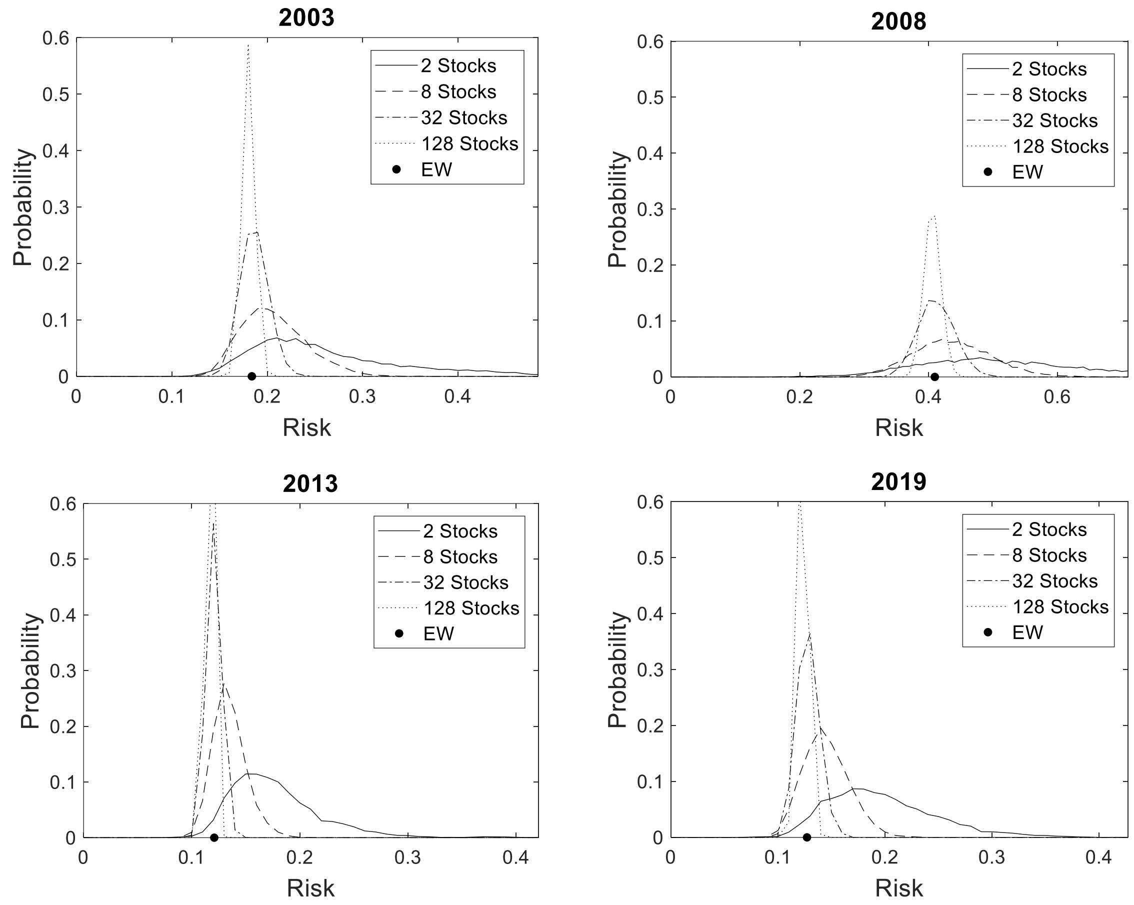


Figure 3.3: Return distributions of value-weighted portfolios in the U.S. market

Figure 3.3 shows the return density probability distributions of value-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the value-weighted portfolio with all stocks (VW) traded in the S&P 500 (U.S. market) in 2003, 2008, 2013 and 2019.

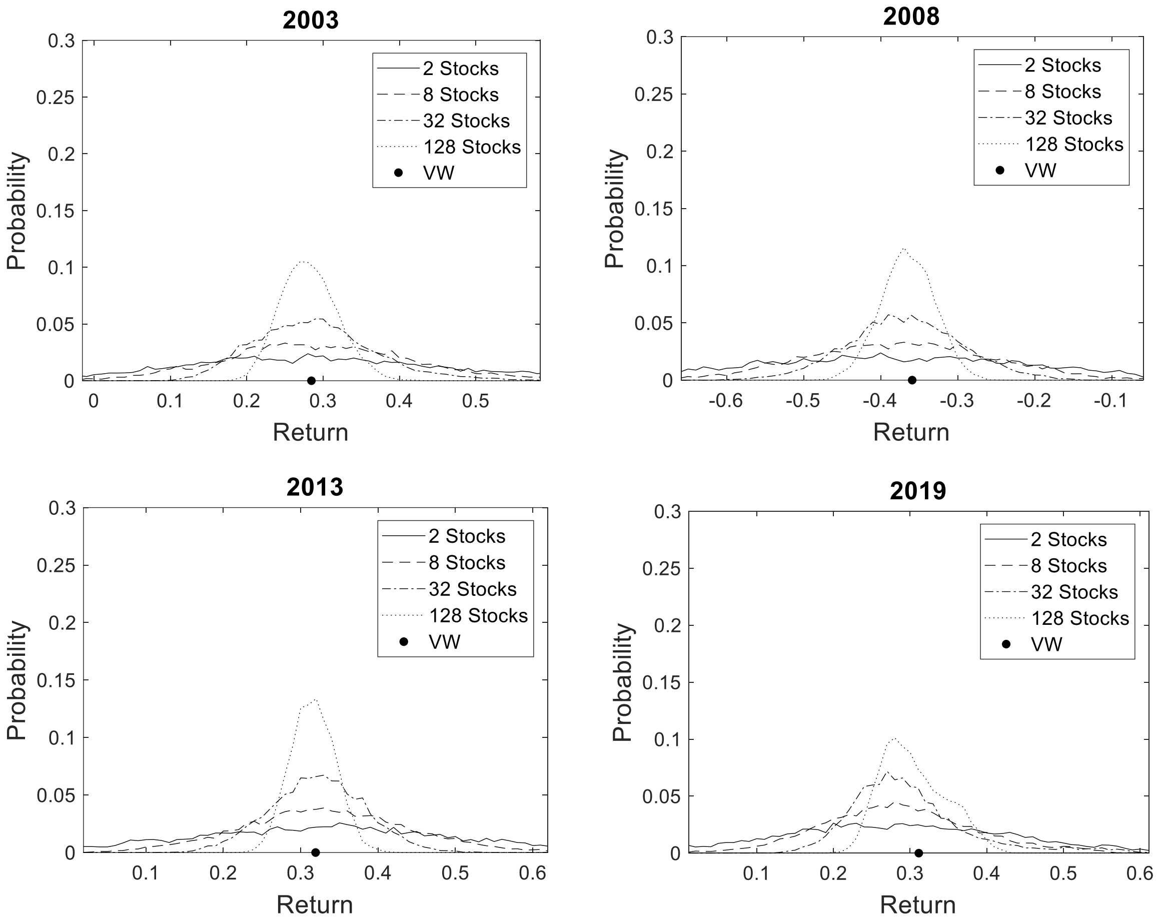


Figure 3.4: Risk distributions of value-weighted portfolios in the U.S. market

Figure 3.4 shows the risk density probability distributions of value-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the value-weighted portfolio with all stocks (VW) traded in the S&P 500 (U.S. market) in 2003, 2008, 2013 and 2019.

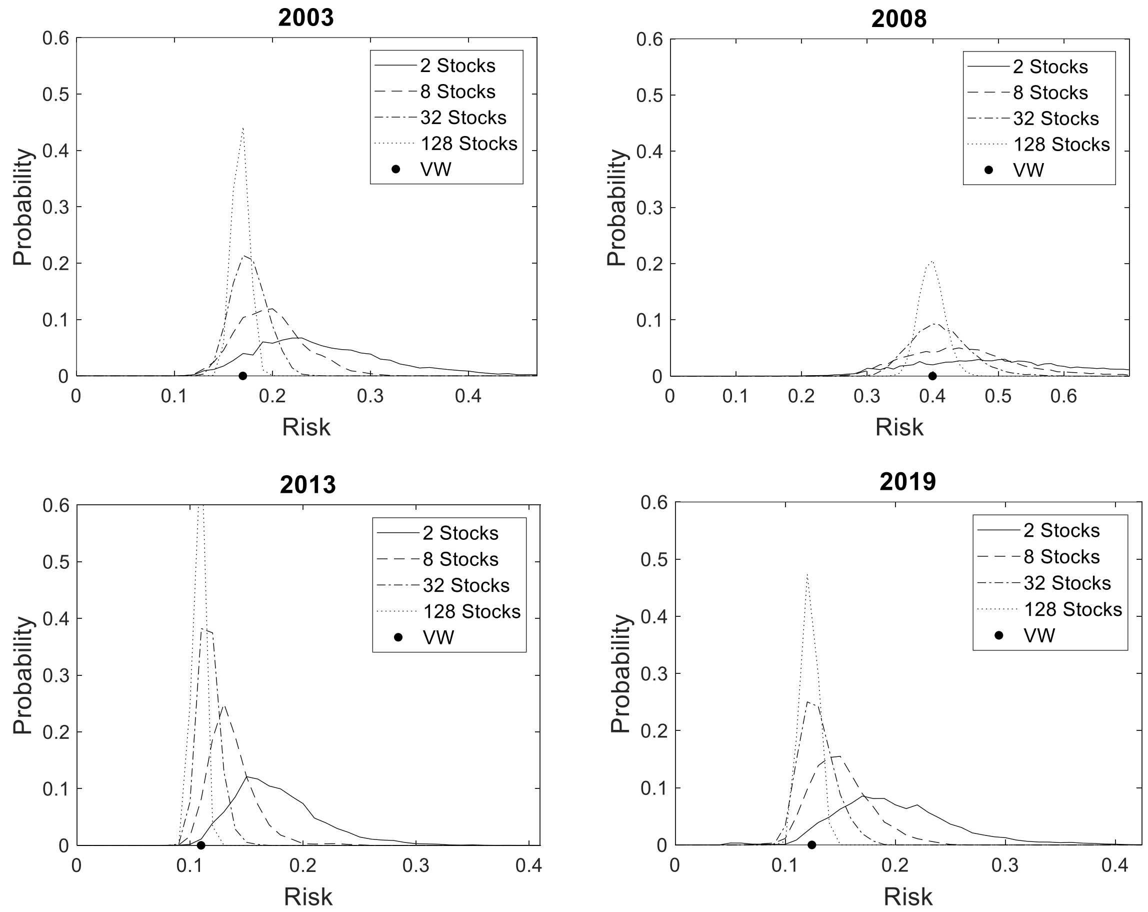


Figure 3.5: Return distributions of equal-weighted portfolios in the European market

Figure 3.5 shows the return density probability distributions of equal-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the equal-weighted portfolio with all stocks (EW) traded in the STOXX 600 (European market) in 2003, 2008, 2013 and 2019.

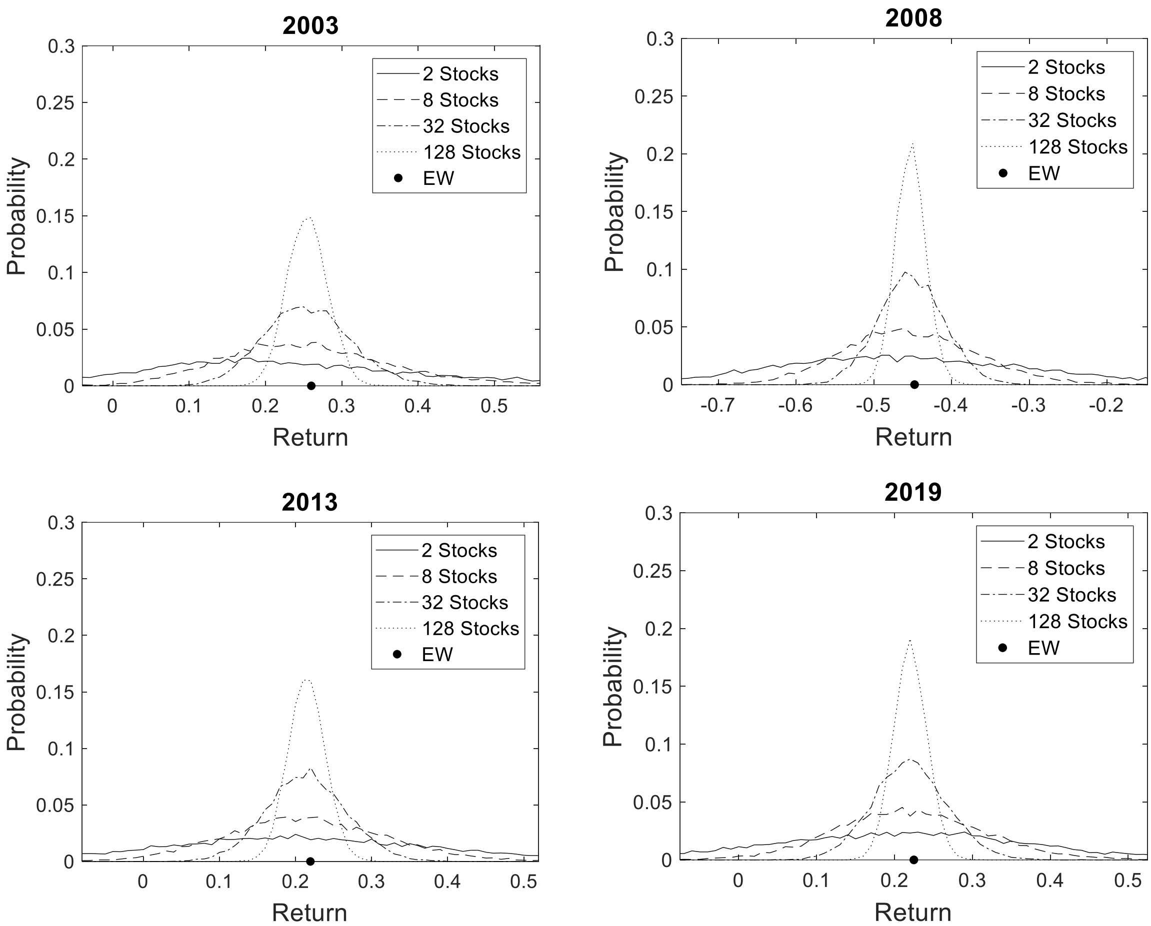


Figure 3.6: Risk distributions of equal-weighted portfolios in the European market

Figure 3.6 shows the risk density probability distributions of equal-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the equal-weighted portfolio with all stocks (EW) traded in the STOXX 600 (European market) in 2003, 2008, 2013 and 2019.

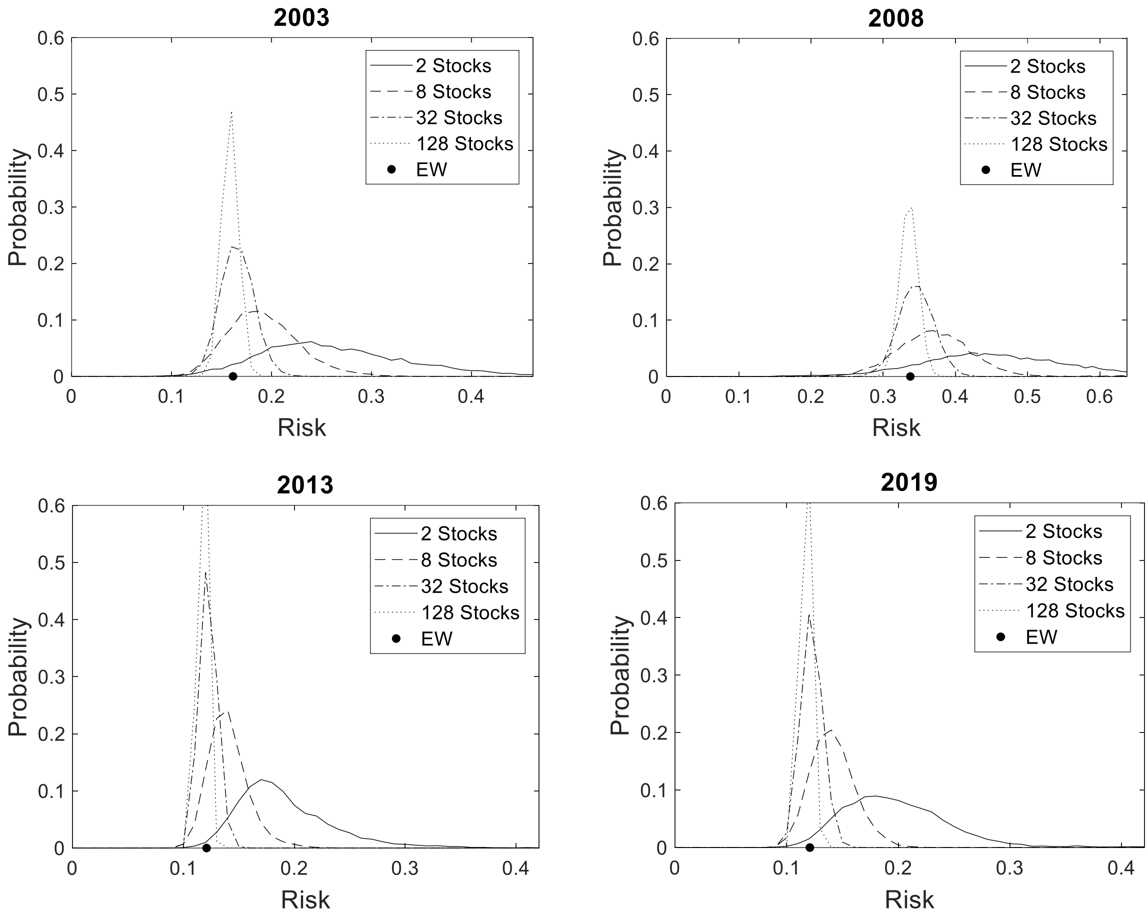


Figure 3.7: Return distributions of value-weighted portfolios in the European market

Figure 3.7 shows the return density probability distributions of value-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the value-weighted portfolio with all stocks (VW) traded in the STOXX 600 (European market) in 2003, 2008, 2013 and 2019.

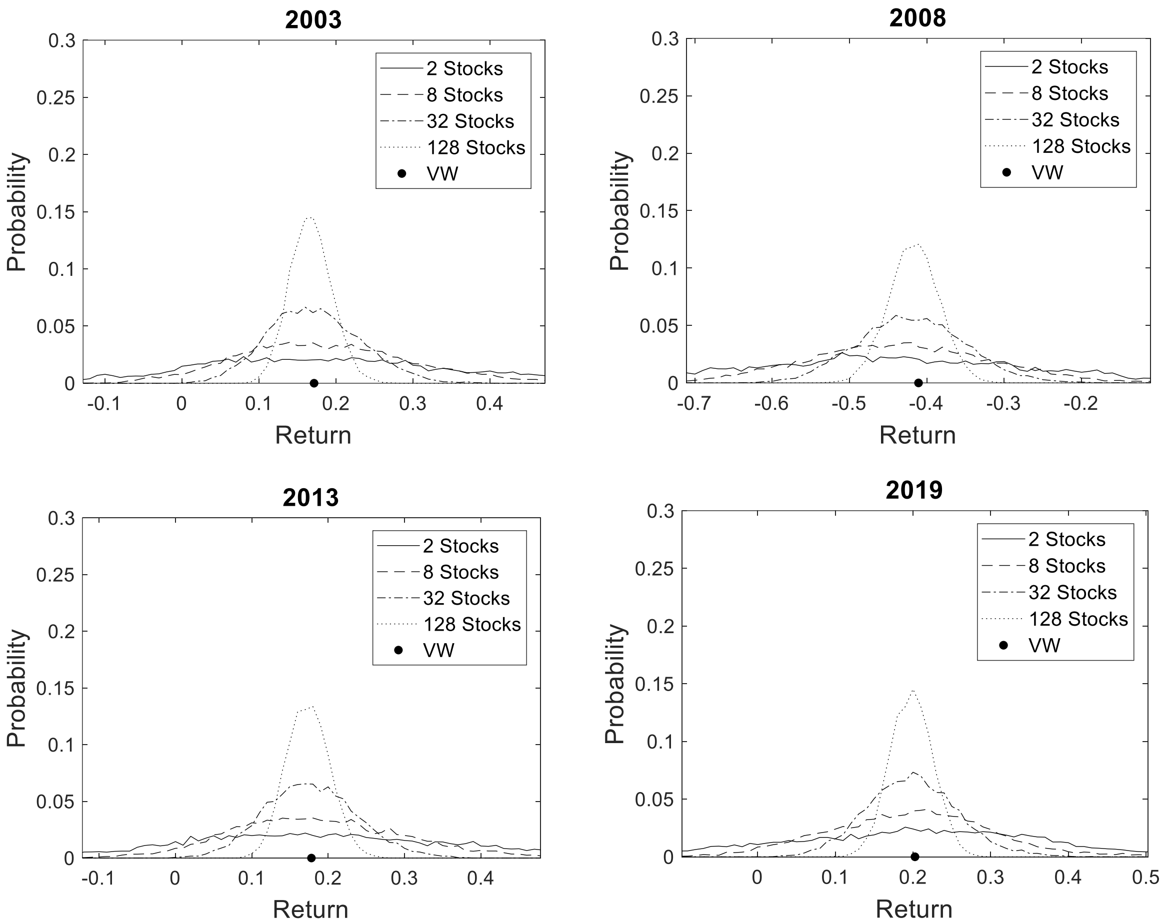
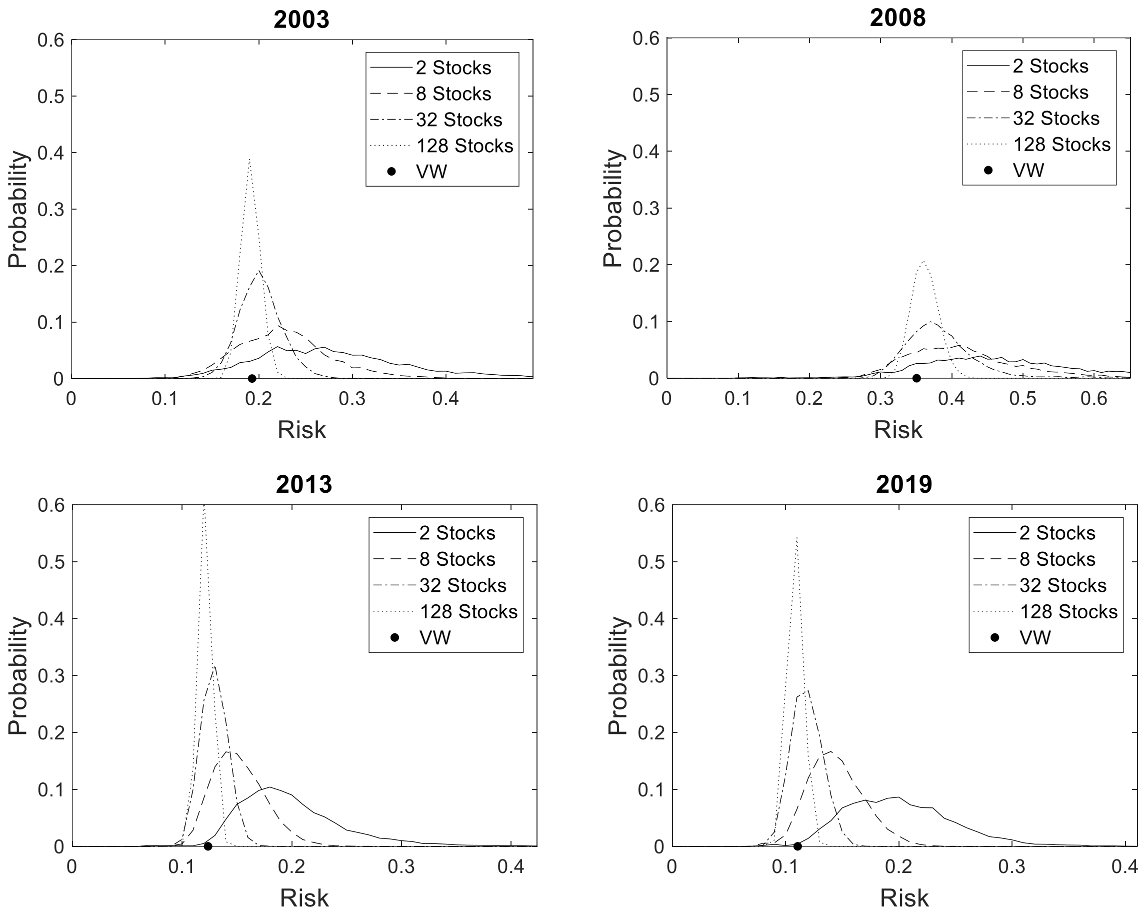


Figure 3.8: Risk distributions of value-weighted portfolios in the European market

Figure 3.8 shows the risk density probability distributions of value-weighted portfolios with 2, 8, 32 and 128 stocks, as well as the value-weighted portfolio with all stocks (VW) traded in the STOXX 600 (European market) in 2003, 2008, 2013 and 2019.



Chapter 4. Can Volatility, Skewness and Kurtosis Predict Stocks Returns?

4.1 Introduction

The capital asset pricing model (hereafter CAPM) proposed by Sharpe (1964), Lintner (1965) and Mossin (1966) implies a positive linear relationship between risk and return of a stock. However, there is no agreement in the literature whether the relationship between stock returns and standard deviation is negative or positive, or even if there is a statistically significant relation. Due to this uncertainty, empirical and theoretical studies began to consider moments of higher order than the second moment to explain stock returns.

Some authors contradict CAPM, arguing that the relationship between stock returns and volatility is negative. Campbell (1987) examines the relationship between conditional means and conditional variances of returns on stocks. Using stock returns, obtained from the Center for Research in Security Prices (hereafter CRSP) and measured by the value-weighted return on the New York Stock Exchange (hereafter NYSE), for the period between February 1959 and November 1983, the author finds that stock returns have a negative relationship with their conditional variance.

Glosten, Jagannathan and Runkle (1993) use a modified GARCH-M model for determining the estimated relationship between risk and return. Using monthly excess returns on the CRSP value-weighted stock index portfolio from April 1951 to December 1989, they find support for a negative relationship between conditional expected monthly return and conditional variance of monthly return. They also show that their conclusions do not change when they use Nelson (1991) EGARCH model modified to include the risk-free rate and/or seasonality.

Ang, Hodrick, Xing and Zhang (2006) examine the cross-sectional relationship between idiosyncratic volatility and expected returns, where idiosyncratic volatility is defined as the variance of the residuals from Fama and French (1993) three-factor model (hereafter FF-3). They form value-weighted portfolios every month, in the period between July 1963 and December 2000, using stocks from American Stock Exchange (hereafter AMEX), Nasdaq and NYSE. The results show a statistically significant difference of −1.06% per month between average return of the quintile portfolio with the highest idiosyncratic volatility stocks and the quintile portfolio with the lowest idiosyncratic volatility stocks. Ang, Hodrick, Xing and Zhang (2009) study if the anomalous relationship between idiosyncratic volatility and next month return in the U.S. market exists in other markets. They conclude that the difference in average return between the extreme quintile portfolios sorted by idiosyncratic volatility is negative, even after controlling for world market, size and value factors across 23 developed markets.

Guo and Savickas (2006) study the relationship between stock market returns and idiosyncratic stock volatility and aggregate stock market volatility through an ordinary least squares regression. Using the 500 stocks with the biggest market capitalization from CRSP during the period between July 1962 and December 2002, they find that value-weighted idiosyncratic stock volatility and aggregate stock market volatility jointly exhibit strong predictive power for excess stock market returns. The stock market risk–return relation is found to be positive, as stipulated by the CAPM. However, idiosyncratic volatility is negatively related with future stock market returns.

Despite the negative relationship between return and idiosyncratic volatility referred in the over mentioned studies, some research shows evidence of a positive relationship between return and standard deviation. Malkiel and Xu (2002) provide a theory of idiosyncratic risk and test some of the implications with constructed portfolio return, individual stock return and equity mutual fund return. Using monthly returns from stocks listed on NYSE, AMEX, Nasdaq and Tokyo Stock Exchange in the period between January 1935 and June 2000, they demonstrate that idiosyncratic volatility is more powerful than either beta or size measures in explaining the cross section of returns. Additionally, they also find a significantly positive relationship between idiosyncratic risk and the cross section of expected returns.

Goyal and Santa-Clara (2003) consider the average stock risk in addition to market risk in order to explain relationship between risk and return. Using CRSP data from July of 1962 to December of 1999, they find that the variance of the market (adjusted for autocorrelation) has no forecasting power for the excess market return. However, they identify a positive relationship between average stock variance (adjusted for autocorrelation) and excess market return.

Jiang and Lee (2006) examine the dynamic relationship between idiosyncratic volatility and return by addressing the problems associated with persistent volatility. Using CRSP daily data for the sample period from July 1962 to December 2002 to construct monthly market volatility, average stock volatility, and idiosyncratic volatility, they find significant positive effects of idiosyncratic volatility on stock returns, although the effects tend to be delayed. The positive effect is robust for small and large firm portfolios, equal-weighted and value-weighted volatilities, average volatility and idiosyncratic volatility, and for different sample periods.

Fu (2009) examines whether under-diversified investors are compensated for bearing idiosyncratic risk and explore the findings of Ang et al. (2006) study. Using stocks listed on

NYSE, the AMEX, and the Nasdaq during the period from July 1963 to December 2006, he shows that idiosyncratic volatilities are time varying and thus the findings of Ang et al. (2006) study should not be used to imply the relationship between idiosyncratic risk and expected return. Additionally, using an EGARCH model to estimate expected idiosyncratic volatility, he finds a significantly positive relationship between the estimated conditional idiosyncratic volatility and expected return.

Huang, Liu, Rhee and Zhang (2010) investigate the relationship between idiosyncratic risk and expected return with a particular interest in understanding the contrasting results between idiosyncratic risk estimated by daily data and monthly data. Using daily and monthly returns of NYSE, AMEX and Nasdaq common stocks from July 1963 to December 2004, they demonstrate the existence of a negative relationship between idiosyncratic risk and expected monthly returns in cross-sectional regressions. This negative relationship is present when the estimates of conditional idiosyncratic volatility are based on the time series of realized monthly idiosyncratic volatilities from daily returns. However, after controlling for return reversals, the negative relation is no longer significant. In addition, they estimate conditional idiosyncratic volatility with an EGARCH model, using monthly returns, and confirm the significantly positive relationship between this proxy for idiosyncratic risk and expected return. This relation is still significantly positive after return reversals control.

Besides the positive and negative relationship between risk and return, Bali and Cakici (2008) find no robustly significant relationship between idiosyncratic volatility and expected return. They use two different measures of idiosyncratic volatility (estimated using daily and monthly data), three weighting schemes (value-weighted, equal-weighted, inverse volatility-weighted), three breakpoints (CRSP, NYSE, equal market share) and two different samples (NYSE/AMEX/Nasdaq and NYSE).

Qadan (2019) tries to provide an explanation for the changes in the relationship between idiosyncratic volatility and a cross-section of expected return, using daily returns of all firms recorded by CRSP, covering stocks listed on NYSE, AMEX and Nasdaq during the 1980–2016 period and employing Fama-French's 5-factor model to extract the idiosyncratic volatility. The author establishes that the relationship between idiosyncratic volatility and expected return is not constant over time. That relationship is positive when considering pro-cyclical investor sentiment, but negative when using contrarian investor sentiment. These findings hold true for both value-weighted and equal-weighted portfolios and for different econometrics specifications.

The lack of agreement in the relationship between standard deviation and return is also present in the relationship between realized return of next periods and skewness or kurtosis. Nevertheless, most studies find a negative relationship between realized return of next periods and skewness. Regarding the relationship between next period return and kurtosis, studies tend to show a positive relation. Kraus and Litzenberger (1976) extend the CAPM to incorporate the effect of skewness on valuation. Using monthly excess rates of return of stocks that were listed on NYSE from January 1926 through June 1970, they predict a significant negative relationship between skewness and next period return.

Harvey and Siddique (2000) introduce skewness in asset pricing since they consider that there is considerable evidence that unconditional distributions of returns cannot be adequately characterized by mean and variance alone. Using monthly U.S. stock returns from CRSP, NYSE/AMEX and Nasdaq during the period between July 1963 and December 1993, they find that conditional skewness (the component of an asset's skewness related to the market portfolio's skewness) has a negative relation with the cross-asset variation of stock returns.

Fang and Lai (1997) examine the impact of conditional kurtosis (the component of an asset's kurtosis related to the market portfolio's kurtosis) on asset pricing using a four-moment capital asset pricing model. Armed with monthly returns of all the stocks that were continuously exchanged on NYSE over the period January 1969 through December 1988, they find that higher systematic variance and higher systematic kurtosis are related with higher expected return, while higher systematic skewness tends to be associated with lower expected return.

Boyer, Mitton and Vorkink (2010) investigate the pricing implications of idiosyncratic standard deviation and skewness. Based on daily returns of stocks listed on NYSE and AMEX, from January 1925 through December 2005, and of stocks listed on Nasdaq from January 1973 through December 2005, they sort stocks based on their level of expected standard deviation and skewness. Their results show that average return of low expected skewness quintile exceed the average return of high expected skewness quintile by 0.67% per month. The average return of low expected standard deviation quintile exceed the average return of the high expected standard deviation quintile by 1.09% per month.

Chang, Christoffersen and Jacobs (2013) extend the investigation of Ang et al. (2006) and examine if market skewness and kurtosis affect the cross-sectional returns. Using risk measures implied by S&P 500 index options, they find that stocks with high exposure to innovations in implied market skewness exhibit low return, on average, and stocks with high exposure to volatility and kurtosis exhibit, on average, higher return.

Conrad, Dittmar and Ghysels (2013) examine the importance of higher moments under the hypothesis that, if option and stock prices reflect the same information, then it is possible to use options market data to extract estimates of higher moments of the stocks' risk-neutral probability density function. Using option prices and stock returns from 1996 to 2005 for all individual stocks covered by CRSP with common shares outstanding, they find that risk-neutral volatility and skewness have a negative relation with subsequent return, while risk-neutral kurtosis has a significant positive relation with subsequent return.

Bali, Hu and Murray (2019) develop a forward-looking measure of a stock's expected return derived from analyst price targets. Using monthly returns from all U.S. based common stocks in CRSP with no missing data from March 1999 through December 2012, they show that ex-ante measures of volatility, skewness and kurtosis implied from stock option prices are positively related to the cross section of ex-ante expected stock return.

Ayadi, Cao, Lazrak and Wang (2019) examine the relationship between idiosyncratic volatility, skewness, and kurtosis and future realized stock return. Using daily and monthly returns of all stocks listed on NYSE, AMEX and Nasdaq during the period from January 1960 to December 2016, they show that portfolios sorted by the level of expected idiosyncratic volatility, skewness and kurtosis do not observe monotonic changes in portfolio deciles return.

Elyasiani, Gambarelli and Muzzioli (2020) investigate whether volatility, skewness, and kurtosis are priced in the European stock market and assess the signs and the magnitudes of the corresponding risk premium. Using stocks listed on STOXX Europe 600 Index from 21 January 2005 to 29 December 2017, based on availability, they find a negative volatility risk premium and a positive skewness risk premium. These findings are robust to different estimation methods.

Studies about the relation between risk and return, generally, focus only on stocks from one market and on one weighting scheme. In addition, the number of risk variables studied are generally inferior to four and datasets fail to include delisted stocks due to merger, acquisition, or bankruptcy, for example. To the best of our knowledge, studies on this subject do not analyze if the average return of extreme quintile portfolios is superior to a passive portfolio. In our study, we try to incorporate these issues.

The main contributions of this Chapter are as follows. First, we compare the relationship between risk and next month return in the U.S. and European markets, instead of using only one market as the majority of studies about this topic. Second, we analyze if the referred comparison shows significant differences in equal-weighted versus value-weighted portfolios. Third, we study the relation of next month return with nine variables, including realized and expected variables, to cover most of the approaches used in other studies of this subject. Fourth, we analyze the performance of a self-financing portfolio that consists in buying or selling the quintile portfolio of stocks with the lowest value of each risk variable and selling or buying the quintile portfolio of stocks with the highest value of each risk variable. Fifth, we evaluate if an investment strategy based on quintile portfolios of stocks with lowest or highest values of risk variables achieve higher return than a benchmark portfolio. Finally, to avoid survivorship bias we use a sampling technique that deals with delisted stocks over the period.

This Chapter proceeds as follows. Section 4.2 describes the methodology used to study the relationship between risk variables and next month return. Section 4.3 reports the results of the empirical analysis and section 4.4 concludes.

4.2 Methodology

From the stocks traded in the S&P 500 (U.S. market) and STOXX 600 (European market) in the beginning of each month, we form value-weighted and equal-weighted quintile portfolios for each market sorted by nine different variables: standard deviation, skewness and kurtosis of daily returns (total risk variables), standard deviation, skewness and kurtosis of the residuals of market model (idiosyncratic risk variables measured by market model) and standard deviation, skewness and kurtosis of the residuals of Fama and French (1993) three-factor model (hereafter FF-3) (idiosyncratic risk variables measured by FF-3). The sorting procedure is widely used in the literature (see for example Harvey and Siddique, 2000; Malkiel and Xu, 2002; Ang et al. 2006; Bali and Cakici, 2008; Ang et. al., 2009; Fu, 2009; Boyer et al., 2010; Huang et al., 2010; Chang et al., 2013; Conrad et al., 2013; Ayadi et al., 2019; Qadan, 2019). This procedure yields five portfolios for each risk variable corresponding to quintiles, with Portfolio 1 containing stocks with the lowest values of the chosen variable and Portfolio 5 containing stocks with the highest values. The next step is to compute the return of each portfolio in the next month. These 45 portfolios are rebalanced every month. Following Fu (2009), we only include stocks that have a minimum of 15 trading days in each month of the estimation period.

Forming portfolios sorted by risk variables allows us to examine the relationship between risk variables and next month return. In addition, this procedure can also be used to assess the performance of a self-financing strategy that buys or sells the Portfolio 1 and sells or buys the Portfolio 5. Armed with this self-financing strategy, we can analyze if it is possible to obtain a profit using the negative or positive relationship between risk variables and next month return. This type of strategy is considered extensively in the literature (see for example Ang et al. 2006;

Bali and Cakici, 2008; Ang et. al., 2009; Boyer et al., 2010; Huang et al., 2010; Chang et al., 2013; Conrad et al., 2013; Ayadi et al., 2019; Qadan, 2019).

Besides the self-financing portfolio, we compare the return of the extreme quintile portfolios with the return of a highly diversified portfolio that contains all stocks traded in the respective index at the respective date. We use a portfolio composed by all stocks traded in the market index instead of the market index itself to keep the analysis comparable. Our rebalancing procedure is twofold different from the rebalancing procedure in the market index. First, we rebalance the portfolio monthly while the market index is rebalanced quarterly; and second, we do not substitute stocks before the rebalancing date in case of merger, acquisition, or bankruptcy opposed to the market index case. With this comparison, we can analyze if a quintile portfolio composed by stocks with the highest or lowest risk variables can achieve higher return than an equal-weighted or value-weighted portfolio composed by all stocks traded in the market index. Investment transactions do not influence the price or dividend of any stock and taxes and transaction costs are not considered.

Total standard deviation, skewness and kurtosis are estimated from the daily returns of each stock. They are obtained by substituting the regression residuals in equations (4.2), (4.3) and (4.4) by the daily returns.

We compute idiosyncratic risk variables relative to the market model and FF-3 in the spirit of other studies related with the relationship between return and risk (see for example Malkiel and Xu, 2002; Ang et al. 2006; Bali and Cakici, 2008; Ang et. al., 2009; Fu, 2009; Boyer et al., 2010; Huang et al., 2010; Ayadi et al., 2019). Market model is defined in equation (2.4). FF-3 is defined as:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i \left(r_{M,t} - r_{f,t} \right) + \chi_i SMB_t + \delta_i HML_t + \varepsilon_{i,t} \tag{4.1}$$

where SMB_t is the return from a strategy involving a long position on a portfolio of small stocks and a short position on a portfolio of large stocks, HML_t is the return from a strategy involving a long position on a portfolio of high book-to-market (value) stocks and a short position on a portfolio of low book-to-market (growth) stocks. Also $r_{i,t}$ is the monthly geometric return of stock i, $r_{f,t}$ is the risk-free rate, α_i, β_i, χ_i and δ_i are the estimation parameters, $r_{M,t}$ is the market portfolio return and $\varepsilon_{i,t}$ is the random error at period t which follows the same assumptions identified in equation (2.4). Following Lim, Durand and Yang (2014), for the U.S. market, we use S&P 500 as market portfolio, the difference between S&P Small Cap 600 and

S&P 100 returns as *SMB* factor and the difference between S&P Value and S&P Growth returns as *HML* factor. For the European market, we follow the same rational used in the U.S. market. We use STOXX 600 as market portfolio, the difference between STOXX Europe Small 200 and STOXX Europe Large 200 returns as *SMB* factor and the difference between STOXX Europe TMI Value and STOXX Europe TMI Growth returns as *HML* factor.

We define idiosyncratic standard deviation (IV_i), idiosyncratic skewness (IS_i), and idiosyncratic kurtosis (IK_i) of stock *i* as:

$$IV_i = \left(\frac{1}{n} \sum_{d=1}^{n} e_{i,d}^2 \right)^{\frac{1}{2}} \tag{4.2}$$

$$IS_i = \frac{1}{n} \frac{\sum_{d=1}^{n} e_{i,d}^3}{IV_i^3} \tag{4.3}$$

$$IK_i = \frac{1}{n} \frac{\sum_{d=1}^{n} e_{i,d}^4}{IV_i^4} \tag{4.4}$$

where n is the number of trading days of the estimation period and $e_{i,d}$ is the regression residual obtained in equations (2.4) or (4.1) of day *d* for stock *i*.

For each risk variable, we use three estimation methods: two based on realized values and one based on estimated values. The methods based on realized values are distinguish on the estimation period. For one method, we obtain the realized values using the returns of the previous month, while for the other method we use the previous 12 months. In both methods, we sort portfolios into quintiles based on the realized values of each stock and then compute the return of each quintile portfolio in the following month. Ang et al. (2006, 2009) and Conrad et al. (2013) also use an estimation period of 1 month and 12 months. The third method relies on estimated values modeled with an EGARCH(1,1) for standard deviation and a first-order autoregressive (AR(1)) model for skewness and kurtosis. In line with the dominant literature (see for example Malkiel and Xu, 2002; Goyal and Santa-Clara, 2003; Bali and Cakici, 2008; Boyer et al., 2010; Ayadi et al., 2019), we use a rolling window period of 60 months to obtain the model used to predict risk variables of each stock in the following month. We sort portfolios into quintiles based on the estimated risk variables given by the model for each stock and then compute the return of each quintile portfolio in the following month.

Bali and Cakici (2008), Fu (2009) and Huang et al. (2010) also use an EGARCH developed by Nelson (1991). The explicit functional forms are as follows:

$$y_{i,t} \sim N\left(0, \sigma_{i,t}^2\right) \tag{4.5}$$

$$\ln \sigma_{i,t}^2 = a_i + b_i \ln \sigma_{i,t-1}^2 + c_i \left\{ \left| \frac{y_{i,t-1}}{\sigma_{i,t-1}} \right| - \left(\frac{2}{\pi} \right)^{\frac{1}{2}} \right\} \tag{4.6}$$

where $y_{i,t}$ is the standard deviation measure of stock i at time t and a_i, b_i and c_i are the estimation parameters. Bollerslev, Chou and Kroner (1992) show that the linear GARCH model is not able to capture the negative relationship between the future volatility and the current return on the stock since the conditional variance is only linked to past conditional variances. Thus, the sign of returns plays no role in affecting volatilities. This limitation of the standard ARCH formulation is one of the primary motivations for the EGARCH model developed by Nelson (1991). Engle and Mustafa (1992) use the option prices to compute the implied variances and find that simple GARCH and EGARCH are the best models. Engle and Ng (1993) test the specifications of various volatility models using Lagrange Multiplier tests and conclude that EGARCH does a good job in capturing the asymmetry of conditional volatilities.

Skewness and kurtosis display a mean reversion pattern since they tend to oscillate up and down around the mean without diverging from it indefinitely. Figures 4.1 and 4.2 show the mean reverting pattern of skewness in the U.S. and European markets, respectively. Figures 4.3 and 4.4 show the mean reverting pattern of kurtosis in the U.S. and European markets, respectively. In this context, an AR(1) is used to model and forecast skewness and kurtosis since it is one of the most popular model for mean reversion among practitioners. We define the AR(1) process as:

$$z_{i,t} = \alpha_i + \beta_i z_{i,t-1} + \varepsilon_{i,t} \tag{4.7}$$

where $z_{i,t}$ is the skewness or kurtosis of stock i at time t, α_i and β_i are the estimation parameters and $\varepsilon_{i,t}$ is the random error.

4.3 Empirical Results

Before analyzing the relationship between next month return and the risk variables mentioned in the previous section, we investigate the relationship between return and the referred risk variables in the same month.

Tables 4.1 and 4.2 report the monthly average return, total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3 of portfolio quintiles sorted by monthly return in the U.S. and European markets, respectively.

For each risk variable, we observe the same behavior in the U.S. and European markets. Standard deviation and kurtosis have their minimum value in the third quintile and their maximum value in the first quintile when portfolios are sorted by return. This suggests that stocks with low return are associated with high standard deviation and high kurtosis. On the other hand, skewness shows a monotonically increasing relationship as the return increases. This result shows evidence that stocks with high return are associated with high skewness and stocks with low return are associated with low skewness.

Since stocks with high absolute values of skewness have higher return than stocks with low absolute values of skewness, investing in stocks with high absolute values of skewness should be a profitable strategy. Stocks with high absolute values of standard deviation and of kurtosis are present in the first and fifth quintile when portfolios are sorted by return. This turns difficult to differentiate between stocks that have high or low return based only on standard deviation or kurtosis without considering skewness. Thus, a stock with higher absolute values of standard deviation and kurtosis, in conjunction with high absolute values of skewness, should be associated with high return.

Tables 4.3 and 4.4 report the average monthly return, in the U.S and European markets, respectively, of equal-weighted and value-weighted portfolios sorted by realized values of risk variables with an estimation period of one month.

Regarding the standard deviation, the results show a negative difference but not statistically significant between the next month return of the fifth and the first quintile of stocks in the U.S. and European markets for both weighting schemes. Ang et al. (2006), Guo and Savickas (2006) and Boyer et al. (2010) using stocks from the U.S. market and Ang et al. (2009) using stocks from France, Germany, Italy, U.K. and U.S. markets also find a negative relationship between idiosyncratic volatility and next month return. Bali and Cakici (2008) in their study attribute the negative relation to the value weighting scheme that yields an anomalously low return in quintile 5, which contains very small, illiquid, and low-priced stocks. However, this explanation is not applicable to our dataset since we only include liquid stocks, and both weighted schemes have a negative relation. Boyer et al. (2010) suggest that a starting point for understanding the negative relationship between idiosyncratic volatility and expected return lies in the preferences of investors since they find that forecasted skewness explains the mentioned negative relation. Huang et al. (2010) suggest that short-term return reversals are the primary reason for the negative relationship between realized idiosyncratic volatility and stock return. We highlight the monotonically decreasing relation in average return of value-weighted portfolios sorted by total standard deviation and by idiosyncratic standard deviation (market model) in the U.S. market.

However, this does not occur with equal-weighted portfolios, suggesting that the referred relation is more evident in stocks with large market capitalization and/or less in stocks with small market capitalization. The results of the European market confirm the findings of Fu (2009), who find that return of equal-weighted and value-weighted portfolios is not monotonically increasing or decreasing across portfolios sorted by idiosyncratic standard deviation. Finally, we do not find any statistically significant superiority of the average return of extreme quintile portfolios over the average return of all stocks traded in the market index when sorting portfolios by standard deviation.

Regarding skewness, in the U.S. market, we observe a negative relationship between skewness and next month return, for equal-weighted and value-weighted portfolios. This result is in line with Boyer et al. (2010), who find that, on average, stocks with low absolute values of skewness or idiosyncratic skewness in the previous month have higher return than stocks with high absolute values of skewness or idiosyncratic skewness. Nevertheless, the mentioned relation is not monotonically decreasing and the difference between the average return of quintile portfolios of stocks with high and low absolute values of skewness is only statistically significant, at a 5% level, for idiosyncratic skewness measured by market model in value-weighted portfolios. The value-weighted self-financing portfolio based on idiosyncratic skewness measured by market model yields 0.27% per month, which is statistically significant, at a 5% level[6]. In the European market, in both weighting schemes, the relationship between next month return and total skewness is negative while the relationship between next month return and idiosyncratic skewness is positive. The difference in average return between the

[6] A self-financing portfolio based on portfolios 2 and 4 does not have a statistically significant difference in average returns.

extreme quintile portfolios sorted by total skewness in equal-weighted portfolios is statistically significant, at a 5% level, with a self-financing portfolio yielding 0.24% per month.

The average return of lowest quintiles value-weighted portfolios sorted by skewness have a statistically significant superiority, at a 5% level, over the average return of all stocks traded in the market index, in the U.S. market. This superiority of value-weighted portfolios, in the European market, is present in the highest quintile portfolios of idiosyncratic skewness, with a statistically significant, at a 10% level, superiority over the average return of all stocks traded in the market index.

In terms of kurtosis, the results show, on average, a positive difference between the next month return of the fifth and the first quintile of stocks sorted by any of the kurtosis variables in the U.S. market in both weighting schemes. In the European market, the mentioned difference is mostly negative. The only positive relation in the European market occurs in value-weighted portfolios sorted by idiosyncratic kurtosis measured by FF-3. Despite the opposite sign between U.S. and European markets, there is not a monotonically increasing or decreasing relation in any market. In addition, the difference in average return between the first and fifth quintile portfolios and the difference in average return between the lowest quintile portfolio and the portfolio with all stocks traded in the market index are only statistically significant for total kurtosis in the European market in both weighting schemes. The statistical significance is 5% for equal-weighted portfolios and 10% for value-weighted portfolios.

Tables 4.5 and 4.6 report the average monthly return, in the U.S and European markets, respectively, of equal-weighted and value-weighted portfolios. The average return is sorted into quintiles based on each variable computed from daily data over the previous 12 months.

In the U.S. market, the results of value-weighted portfolios of stocks sorted by total standard deviation and idiosyncratic standard deviation (FF-3) show, on average, a positive relation with next month return, in contrast with Ang et al. (2006, 2009) and Boyer et al. (2010). The same risk variables have a negative relation in equal-weighted portfolios. Nevertheless, none of the differences in average return between the quintile portfolio of stocks with lowest standard deviation and the quintile portfolio of stocks with highest standard deviation are statistically significant. In addition, the average return of extreme quintile portfolios is not statistically different from the average return of a portfolio composed by all stocks traded in the market index.

In the European market, the relationship between next month return and each of the standard deviation variables is, on average, negative in both weighting schemes. Additionally, as in the U.S. market, the difference in average return between the quintile portfolio of stocks with lowest standard deviation and the quintile portfolio of stocks with highest standard deviation is not statistically significant. The difference between average return of extreme quintile portfolios and the average return of a portfolio composed by all stocks traded in the market index is also not statistically significant.

Concerning skewness, in the U.S. market the findings are similar for equal-weighted and value-weighted portfolios. There is not a monotonically decreasing or increasing relationship, on average, between skewness and the next month return. Additionally, the difference in average return between the quintile portfolios of stocks with lowest skewness and the quintile portfolios of stocks with highest skewness is not statistically significant.

In the European market, we highlight the idiosyncratic skewness measured by market model and FF-3 of equal-weighted portfolios, where the self-financing portfolio yields 0.33% and 0.34% per month, respectively, with a statistical significance at a 1% level[7]. In addition, the differences in average return between the fifth quintile of equal-weighted portfolios sorted by idiosyncratic skewness measured by market model and FF-3 and the portfolio composed by all stocks traded in the market index are significant at a 1% level.

Regarding kurtosis, in the U.S. market we observe that the relationship between kurtosis and next month return is, on average, mostly positive. The only negative relation occurs in value-weighted portfolios sorted by total kurtosis. We highlight that value-weighted portfolios sorted by idiosyncratic kurtosis (market model) average return has a monotonically increasing relationship with the risk variable, but the average return difference between the portfolios of stocks with lowest values and highest values of idiosyncratic kurtosis (market model) is only statistically significant, at a 10% level. On the other hand, equal-weighted portfolios sorted by idiosyncratic kurtosis average return has not a monotonically increasing relationship with the risk variables but the average return difference between the portfolios of stocks with lowest values and highest values of idiosyncratic kurtosis is statistically significant, at a 5% level. The average return difference between the equal-weighted quintile portfolios of stocks with highest values of idiosyncratic kurtosis and the portfolio composed by all stocks traded in the market index is also statistically significant, at a 5% level.

In the European market, the results show, on average, that none of the variables considered has a monotonically increasing or decreasing relation with next month return in both weighting schemes. In addition, for equal-weighted and value-weighted portfolios, the self-financing

[7] A self-financing portfolio based on portfolios 2 and 4 does not have a statistically significant difference in average returns.

portfolio average return is not statistically significant and the extreme quintile portfolios average return is not statistically different from the average return of a portfolio composed by all stocks traded in the market index.

So far, in the U.S. and European markets, the relationship between the majorities of realized values of standard deviation and next month return, on average, do not show evidence of being monotonically increasing or decreasing. In addition, the self-financing portfolio average return is not statistically different from zero in any case. These findings confirm those presented by Fu (2009) but are not in line with those of Ang et al. (2006), who find a statistically significant difference in the average return between the quintile portfolios of stocks with lowest and highest standard deviation. Fu (2009) argues that using this month's idiosyncratic volatility to approximate the value in the next month could introduce severe measurement errors. As a result, Ang et al. (2006) findings should not be used to draw inference on the relationship between idiosyncratic risk and next month return. Huang et al. (2010) find that value-weighted and equal-weighted portfolios with the highest idiosyncratic volatility stocks have lower average returns than portfolios composed by low idiosyncratic volatility stocks. Although only value-weighted portfolios have statistically significant difference in average returns between the quintile portfolios of stocks with highest and lowest idiosyncratic volatility. Huang et al. (2010) study contradict our results for value-weighted portfolios but are in line with respect to equal-weighted portfolios.

Tables 4.7 and 4.8 report the average monthly return, in the U.S and European markets, respectively, of equal-weighted and value-weighted portfolios. The average monthly return is sorted into quintiles based on the estimated value modeled by a EGARCH(1,1) for standard deviation and by an AR(1) for skewness and kurtosis, using monthly data over the previous 60 months.

With respect to standard deviation, the results for the U.S. and European markets in both weighting schemes indicate that none of the variables considered have a monotonically increasing or decreasing relation with next month return, on average. These results are not in line with Fu (2009). In addition, the self-financing portfolio average return is not statistically different from zero and the extreme quintile portfolios average return is not statistically different from the average return of a portfolio composed by all stocks traded in the market index.

Regarding skewness, in the U.S. and European markets and in both weighting schemes, the relationship between skewness and next month return is, on average, positive. This result is not in line with Boyer et al. (2010) that find a monotonically decreasing relation. In the U.S. market, the average return of equal-weighted portfolios sorted by idiosyncratic skewness (market model) and value-weighted portfolios sorted by idiosyncratic skewness (FF-3) has a monotonically increasing relation with the risk variables.

In the European market, on average, we do not observe any monotonically increasing relation between skewness and next month return, opposite to the U.S. market. About self-financing portfolio, the results indicate that, in the U.S. market, the value-weighted self-financing portfolio sorted by total skewness and idiosyncratic skewness (FF-3) has a statistically significant average return, at a 5% level[8]. In the European market, the equal-weighted self-financing portfolio sorted by any skewness variable has a statistically significant average return, at a 5% level. With respect to the average return of extreme quintile portfolios that is higher than the average return of a portfolio composed by all stocks traded in the market index, we observe mixed results between markets. In the U.S. market, there is a statistically significant difference, at a 1% level, of value-weighted quintile portfolios of stocks with high total skewness. In the European, there is a statistically significant difference, at a 5% level, for both weighting schemes, of quintile portfolios of stocks with high idiosyncratic skewness.

As for kurtosis, in the U.S. market we observe a monotonically increasing relationship between most of the kurtosis variables and next month return, on average, in both weighting schemes, as well as a statistically significant difference, at a 5% level, between the average return of quintile portfolios of stocks with lowest and highest values of kurtosis. The average return difference between the fifth quintile portfolio and a portfolio with all stocks traded in the market index is statistically significant, at a 5% level and at a 1% level, for equal-weighted and value-weighted portfolios, respectively, for every kurtosis variable.

In the European market, the relationship between kurtosis is positive, as in the U.S. market, although, on average, none of the kurtosis variables have a monotonically increasing relation with next month return. Despite the absence of any monotonically increasing relation, the average return of a value-weighted self-financing portfolio sorted by idiosyncratic kurtosis (FF-3) is statistically significant, at a 5% level, yielding 0.27% per month. The average return difference between the fifth quintile portfolio and a portfolio with all stocks traded in the market index is statistically significant, at a 5% level, when value-weighted portfolios are sorted by total kurtosis or idiosyncratic kurtosis (FF-3) and when equal-weighted portfolios are sorted by idiosyncratic kurtosis (FF-3).

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[8] A self-financing portfolio based on portfolios 2 and 4 does not have a statistically significant difference in average returns.

In the U.S. and European markets for both weighting schemes, the results of realized and estimated variables are similar to Bali and Cakici (2008) and Ayadi et al. (2019). As the mentioned studies, generally, we see no monotonic relationship between risk variables and next month return. In the European market, none of the nine risk variables obtained by three different methods used to sort portfolios by two weighting schemes observe a monotonically relation with next month return. In the U.S. market, we have nine situations where the relationship between the risk variable and next month return is monotonically increasing or decreasing. However, only idiosyncratic kurtosis measured by market model modeled by an AR(1) has a monotonically relation with next month return, simultaneously, in equal-weighted and value-weighted portfolios.

Regarding the self-financing portfolio average return (average return difference between the quintile portfolios of stocks with lowest and highest values of risk variables), we observe some statistically significant differences, at a 5% level. In the U.S. market, the results show 12 situations where a self-financing portfolio yields a statistically significant positive average return, at a 5% level. The most profitable strategies respect to estimated risk variables, where value-weighted portfolios yield an average return per month superior to 0.50% using kurtosis variables. In the European market, we observe eight situations where a self-financing portfolio yields an average return that is statistically significant. Nevertheless, the results do not indicate any situation where the self-financing portfolio, for the same risk variable, yields an average return statistically different from zero, in the U.S. and European markets, simultaneously.

Finally, we find that when the difference in average return between extreme quintile portfolios and a portfolio composed by all stocks traded in the market index is statistically significant, generally, the self-financing portfolio has an average return statistically different from zero. Nevertheless, the self-financing portfolio tends to exhibit higher average return than the difference in average return between extreme quintile portfolios and a portfolio composed by all stocks traded in the market index. From all the statistically significant differences, at a 5% level, between extreme quintile portfolios and a portfolio composed by all stocks traded in the market index, in the U.S. market, the self-financing portfolio has always higher average return for the respective risk variable. In the European market, the self-financing portfolio has lower average return than the difference between extreme quintile portfolios and a portfolio composed by all stocks traded in the market index, when we consider equal-weighted portfolios sorted by realized idiosyncratic skewness (market model) with an estimation period of one month and by estimated values of idiosyncratic kurtosis (FF-3) and value-weighted portfolios sorted by estimated values of idiosyncratic skewness (FF-3).

4.4 Conclusion

This Chapter analyzes the relationship between next month return and risk variables (standard deviation, skewness, and kurtosis). We analyze the mentioned relationship by forming value-weighted and equal-weighted quintile portfolios sorted by risk variables. We present a large set of results to understand the conflicting evidence presented in the literature about the relationship between risk variables and next month return. Negative, as well as positive, relationships between next month return and the same risk variables are reported. In addition, we analyze if it is possible to obtain a positive average return using the negative or positive relationship between risk variables and next month return by buying or selling the quintile portfolio formed with stocks with lowest risk variables and by selling or buying the quintile portfolio formed with stocks with highest risk variables (self-financing portfolio). Besides the self-financing portfolio, we also compare the average return of the extreme quintile portfolios with the average return of a portfolio composed by all stocks traded in the market index.

The first finding confirms the results of Bali and Cakici (2008) and Ayadi et al. (2019). Generally, we see no clear increasing or decreasing monotonic relationships between risk variables and next month return. These relations are only present in the U.S. market.

Secondly, although we observe some statistically significant differences, at a 5% level, between the average return of quintile portfolios formed with stocks of lowest values and stocks with highest values of risk variables, these differences are rarely statistically different from zero, simultaneously, in both weighting schemes or in both markets. These cases are important if a self-financing portfolio consisting in buy or sell a quintile portfolio of stocks with the lowest risk variables values and sell or buy a quintile portfolio of stocks with the highest risk variables values is implemented.

Thirdly, with respect to the difference of average return between extreme quintile portfolios and a portfolio composed by all stocks traded in the market index, we observe that, generally, this difference yields lower average return than the self-financing portfolio using the same risk variables.

Given the lack of similarity between the results in the U.S. and European markets for the same risk variables, it appears that relations between risk variables and next month return are originated randomly rather than by economic significance. The results for both markets and both weighting schemes show that, at least one negative relation and one positive relation, can be found for standard deviation, skewness, and kurtosis estimates.

Table 4.1: Risk variables for portfolios sorted by monthly return in the U.S. market

Table 4.1 reports the monthly average return, total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3 in the U.S. market. Portfolios are sorted by monthly return into quintiles each month between January 2007 and December 2019 (156 months).

Variable	Quintile				
	Low (1)	2	3	4	High (5)
Return	-8.8%	-2.3%	0.8%	3.9%	10.6%
Total standard deviation	10.3%	7.7%	7.2%	7.4%	9.3%
Total skewness	-0.55	-0.17	-0.02	0.12	0.48
Total kurtosis	4.84	3.79	3.69	3.76	4.66
Idiosyncratic standard deviation (market model)	8.1%	5.5%	5.1%	5.3%	7.2%
Idiosyncratic skewness (market model)	-0.63	-0.17	0.05	0.24	0.68
Idiosyncratic kurtosis (market model)	5.30	4.15	4.01	4.12	5.16
Idiosyncratic standard deviation (FF-3)	7.3%	4.9%	4.6%	4.8%	6.6%
Idiosyncratic skewness (FF-3)	-0.52	-0.14	0.04	0.19	0.56
Idiosyncratic kurtosis (FF-3)	4.79	3.91	3.78	3.86	4.69

Table 4.2: Risk variables for portfolios sorted by monthly return in the European market

Table 4.2 reports the monthly average return, total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3 in the European market. Portfolios are sorted by monthly return into quintiles each month between January 2007 and December 2019 (156 months).

Variable	Quintile				
	Low (1)	2	3	4	High (5)
Return	-9.5%	-3.0%	0.3%	3.6%	10.3%
Total standard deviation	10.3%	7.9%	7.5%	7.6%	9.2%
Total skewness	-0.47	-0.11	0.03	0.16	0.43
Total kurtosis	4.78	3.89	3.78	3.81	4.36
Idiosyncratic standard deviation (market model)	8.3%	5.9%	5.6%	5.8%	7.4%
Idiosyncratic skewness (market model)	-0.56	-0.15	0.04	0.22	0.55
Idiosyncratic kurtosis (market model)	5.10	4.14	4.00	4.04	4.68
Idiosyncratic standard deviation (FF-3)	7.5%	5.4%	5.1%	5.2%	6.7%
Idiosyncratic skewness (FF-3)	-0.47	-0.13	0.03	0.19	0.46
Idiosyncratic kurtosis (FF-3)	4.68	3.91	3.78	3.83	4.31

Table 4.3: Average monthly return of portfolios sorted by realized values of risk variables in the U.S. market (estimation period of one month)

Table 4.3 reports the average monthly return of equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B) in the U.S. market. Quintile portfolios are formed every month from January 2007 to December 2019 (156 months) by sorting stocks based on the risk variable computed over the previous month. The risk variables are total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3. We also report the average return difference between quintile 5 and quintile 1 (5-1), between quintile 1 and full sample portfolio (1-FS) and between quintile 5 and full sample portfolio (5-FS). The full sample portfolio is composed by all stocks traded in the market index. t-statistics of the null hypothesis that the mentioned differences are zero are also reported. ∗∗∗, ∗∗, ∗ indicate significance at the 1%, 5%, and 10% levels, respectively.

Panel A: Equal-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.88%	0.91%	0.88%	0.78%	0.85%	-0.03%	-0.07	0.02%	0.08	-0.01%	-0.05
Total skewness	0.98%	0.84%	0.73%	0.90%	0.84%	-0.14%	-0.93	0.12%	1.26	-0.02%	-0.24
Total kurtosis	0.75%	0.79%	0.81%	1.00%	0.96%	0.21%	1.57*	-0.11%	-1.51	0.10%	1.22
Idiosyncratic standard deviation (market model)	0.92%	0.83%	0.94%	0.73%	0.88%	-0.04%	-0.12	0.06%	0.37	0.02%	0.07
Idiosyncratic skewness (market model)	0.97%	0.93%	0.82%	0.76%	0.82%	-0.15%	-1.23	0.11%	1.40*	-0.04%	-0.69
Idiosyncratic kurtosis (market model)	0.84%	0.80%	0.79%	0.98%	0.89%	0.05%	0.45	-0.02%	-0.30	0.03%	0.47
Idiosyncratic standard deviation (FF-3)	0.92%	0.89%	0.84%	0.75%	0.90%	-0.01%	-0.04	0.06%	0.33	0.04%	0.18
Idiosyncratic skewness (FF-3)	0.95%	0.93%	0.76%	0.79%	0.87%	-0.08%	-0.60	0.09%	1.06	0.01%	0.06
Idiosyncratic kurtosis (FF-3)	0.79%	0.87%	0.88%	0.91%	0.86%	0.07%	0.69	-0.07%	-1.12	0.00%	-0.07
Panel B: Value-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.94%	0.90%	0.75%	0.74%	0.61%	-0.33%	-0.77	0.14%	0.91	-0.19%	-0.64
Total skewness	1.01%	0.79%	0.53%	0.84%	0.78%	-0.23%	-1.29*	0.21%	1.83**	-0.02%	-0.20
Total kurtosis	0.83%	0.65%	0.86%	0.77%	0.86%	0.03%	0.18	0.03%	0.39	0.06%	0.70
Idiosyncratic standard deviation (market model)	0.85%	0.83%	0.82%	0.69%	0.65%	-0.21%	-0.58	0.05%	0.47	-0.15%	-0.59
Idiosyncratic skewness (market model)	0.99%	0.87%	0.76%	0.58%	0.72%	-0.27%	-2.06**	0.19%	2.37***	-0.08%	-0.87
Idiosyncratic kurtosis (market model)	0.62%	0.85%	0.72%	0.85%	0.83%	0.22%	1.58*	-0.18%	-2.06	0.03%	0.43
Idiosyncratic standard deviation (FF-3)	0.85%	0.88%	0.68%	0.79%	0.70%	-0.16%	-0.45	0.05%	0.51	-0.10%	-0.39
Idiosyncratic skewness (FF-3)	0.94%	0.82%	0.69%	0.71%	0.74%	-0.20%	-1.40*	0.14%	1.71**	-0.06%	-0.57
Idiosyncratic kurtosis (FF-3)	0.75%	0.72%	0.84%	0.77%	0.80%	0.04%	0.30	-0.05%	-0.50	0.00%	-0.01

Table 4.4: Average monthly return of portfolios sorted by realized values of risk variables in the European market (estimation period of one month)

Table 4.4 reports the average monthly return of equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B) in the European market. Quintile portfolios are formed every month from January 2007 to December 2019 (156 months) by sorting stocks based on the risk variable computed over the previous month. The risk variables are total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3. We also report the average return difference between quintile 5 and quintile 1 (5-1), between quintile 1 and full sample portfolio (1-FS) and between quintile 5 and full sample portfolio (5-FS). The full sample portfolio is composed by all stocks traded in the market index. t-statistics of the null hypothesis that the mentioned differences are zero are also reported. ∗∗∗, ∗∗, ∗ indicate significance at the 1%, 5%, and 10% levels, respectively.

Panel A: Equal-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.33%	0.26%	0.31%	0.42%	0.16%	-0.17%	-0.42	0.05%	0.29	-0.12%	-0.52
Total skewness	0.32%	0.37%	0.34%	0.37%	0.09%	-0.24%	-2.09**	0.04%	0.62	-0.19%	-2.80
Total kurtosis	0.39%	0.28%	0.32%	0.32%	0.18%	-0.21%	-1.70**	0.11%	1.75**	-0.10%	-1.27
Idiosyncratic standard deviation (market model)	0.34%	0.31%	0.42%	0.29%	0.13%	-0.20%	-0.64	0.06%	0.41	-0.15%	-0.77
Idiosyncratic skewness (market model)	0.32%	0.29%	0.19%	0.30%	0.40%	0.08%	0.76	0.04%	0.56	0.12%	1.95**
Idiosyncratic kurtosis (market model)	0.30%	0.24%	0.35%	0.38%	0.23%	-0.06%	-0.60	0.02%	0.28	-0.05%	-0.71
Idiosyncratic standard deviation (FF-3)	0.31%	0.34%	0.42%	0.29%	0.14%	-0.17%	-0.56	0.03%	0.21	-0.14%	-0.80
Idiosyncratic skewness (FF-3)	0.28%	0.33%	0.25%	0.28%	0.35%	0.07%	0.71	0.00%	-0.01	0.07%	1.11
Idiosyncratic kurtosis (FF-3)	0.33%	0.24%	0.39%	0.26%	0.27%	-0.07%	-0.69	0.05%	1.05	-0.01%	-0.17
Panel B: Value-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.20%	0.15%	0.11%	0.13%	0.07%	-0.14%	-0.31	0.09%	0.52	-0.04%	-0.17
Total skewness	0.19%	0.32%	0.14%	0.07%	0.00%	-0.19%	-1.21	0.08%	0.83	-0.11%	-1.08
Total kurtosis	0.26%	0.13%	0.13%	0.16%	0.00%	-0.26%	-1.63*	0.15%	1.45*	-0.11%	-1.22
Idiosyncratic standard deviation (market model)	0.19%	0.20%	0.21%	0.04%	-0.12%	-0.30%	-0.84	0.08%	0.62	-0.23%	-0.87
Idiosyncratic skewness (market model)	0.11%	0.14%	0.04%	0.11%	0.34%	0.23%	1.24	0.00%	-0.06	0.23%	1.80**
Idiosyncratic kurtosis (market model)	0.15%	0.03%	0.25%	0.21%	0.04%	-0.10%	-0.63	0.04%	0.37	-0.07%	-0.70
Idiosyncratic standard deviation (FF-3)	0.16%	0.13%	0.30%	0.09%	-0.11%	-0.26%	-0.80	0.05%	0.35	-0.22%	-0.94
Idiosyncratic skewness (FF-3)	0.05%	0.16%	0.19%	0.07%	0.23%	0.18%	1.27	-0.06%	-0.78	0.12%	1.30*
Idiosyncratic kurtosis (FF-3)	0.15%	0.18%	0.25%	-0.10%	0.18%	0.03%	0.23	0.04%	0.50	0.07%	0.71

Table 4.5: Average monthly return of portfolios sorted by realized values of risk variables in the U.S. market (estimation period of 12 months)

Table 4.5 reports the average monthly return of equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B) in the U.S. market. Quintile portfolios are formed every month from January 2007 to December 2019 (156 months) by sorting stocks based on the risk variable computed over the previous 12 months. The risk variables are total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3. We also report the average return difference between quintile 5 and quintile 1 (5-1), between quintile 1 and full sample portfolio (1-FS) and between quintile 5 and full sample portfolio (5-FS). The full sample portfolio is composed by all stocks traded in the market index. t-statistics of the null hypothesis that the mentioned differences are zero are also reported. ∗∗∗, ∗∗, ∗ indicate significance at the 1%, 5%, and 10% levels, respectively.

Panel A: Equal-weighted portfolios

Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.85%	0.94%	0.91%	0.78%	0.82%	-0.03%	-0.06	-0.01%	-0.04	-0.04%	-0.16
Total skewness	0.90%	0.80%	0.92%	0.90%	0.77%	-0.13%	-0.71	0.04%	0.36	-0.09%	-0.89
Total kurtosis	0.92%	0.88%	0.82%	0.69%	0.99%	0.07%	0.45	0.06%	0.55	0.13%	1.48*
Idiosyncratic standard deviation (market model)	0.88%	0.86%	0.96%	0.79%	0.82%	-0.07%	-0.16	0.02%	0.10	-0.04%	-0.19
Idiosyncratic skewness (market model)	0.91%	0.90%	0.82%	0.80%	0.87%	-0.05%	-0.30	0.05%	0.51	0.01%	0.03
Idiosyncratic kurtosis (market model)	0.73%	0.83%	0.85%	0.83%	1.05%	0.33%	1.79**	-0.13%	-1.26	0.19%	2.02**
Idiosyncratic standard deviation (FF-3)	0.87%	0.86%	0.90%	0.81%	0.86%	0.00%	0.00	0.01%	0.01	0.00%	0.00
Idiosyncratic skewness (FF-3)	0.92%	0.87%	0.78%	0.79%	0.94%	0.02%	0.15	0.06%	0.53	0.08%	0.95
Idiosyncratic kurtosis (FF-3)	0.68%	0.84%	0.89%	0.83%	1.07%	0.39%	2.20**	-0.18%	-1.75	0.21%	2.20**

Panel B: Value-weighted portfolios

Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.74%	0.89%	0.84%	0.71%	0.81%	0.07%	0.15	-0.06%	-0.34	0.01%	0.03
Total skewness	0.79%	0.76%	0.79%	0.82%	0.74%	-0.05%	-0.23	-0.01%	-0.04	-0.06%	-0.41
Total kurtosis	0.93%	0.74%	0.76%	0.67%	0.86%	-0.07%	-0.33	0.13%	0.95	0.06%	0.54
Idiosyncratic standard deviation (market model)	0.79%	0.77%	0.97%	0.60%	0.77%	-0.02%	-0.04	-0.01%	-0.07	-0.03%	-0.08
Idiosyncratic skewness (market model)	0.84%	0.79%	0.80%	0.70%	0.77%	-0.08%	-0.39	0.04%	0.44	-0.03%	-0.22
Idiosyncratic kurtosis (market model)	0.66%	0.69%	0.76%	0.87%	0.90%	0.24%	1.34*	-0.14%	-1.30	0.10%	0.96
Idiosyncratic standard deviation (FF-3)	0.74%	0.87%	0.83%	0.65%	0.95%	0.21%	0.52	-0.06%	-0.46	0.15%	0.52
Idiosyncratic skewness (FF-3)	0.74%	0.73%	0.86%	0.70%	0.86%	0.12%	0.63	-0.06%	-0.45	0.06%	0.59
Idiosyncratic kurtosis (FF-3)	0.57%	0.76%	0.85%	0.81%	0.92%	0.35%	1.93**	-0.23%	-2.26	0.12%	1.10

Table 4.6: Average monthly return of portfolios sorted by realized values of risk variables in the European market (estimation period of 12 months)

Table 4.6 reports the average monthly return of equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B) in the European market. Quintile portfolios are formed every month from January 2007 to December 2019 (156 months) by sorting stocks based on the risk variable computed over the previous 12 months. The risk variables are total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3. We also report the average return difference between quintile 5 and quintile 1 (5-1), between quintile 1 and full sample portfolio (1-FS) and between quintile 5 and full sample portfolio (5-FS). The full sample portfolio is composed by all stocks traded in the market index. t-statistics of the null hypothesis that the mentioned differences are zero are also reported. ∗∗∗, ∗∗, ∗ indicate significance at the 1%, 5%, and 10% levels, respectively.

Panel A: Equal-weighted portfolios

Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.39%	0.33%	0.22%	0.26%	0.28%	-0.11%	-0.25	0.11%	0.54	0.00%	0.03
Total skewness	0.18%	0.29%	0.39%	0.26%	0.36%	0.18%	1.18	-0.10%	-1.00	0.08%	1.01
Total kurtosis	0.31%	0.41%	0.25%	0.30%	0.21%	-0.10%	-0.61	0.03%	0.37	-0.07%	-0.69
Idiosyncratic standard deviation (market model)	0.27%	0.35%	0.29%	0.31%	0.27%	-0.01%	-0.02	-0.01%	-0.04	-0.01%	-0.06
Idiosyncratic skewness (market model)	0.13%	0.31%	0.24%	0.35%	0.46%	0.33%	2.37***	-0.15%	-1.57	0.18%	2.54***
Idiosyncratic kurtosis (market model)	0.27%	0.29%	0.35%	0.31%	0.27%	0.01%	0.05	-0.01%	-0.15	-0.01%	-0.06
Idiosyncratic standard deviation (FF-3)	0.25%	0.37%	0.35%	0.30%	0.23%	-0.02%	-0.08	-0.03%	-0.18	-0.05%	-0.28
Idiosyncratic skewness (FF-3)	0.17%	0.29%	0.20%	0.32%	0.51%	0.34%	2.63***	-0.11%	-1.24	0.23%	3.17***
Idiosyncratic kurtosis (FF-3)	0.28%	0.27%	0.29%	0.37%	0.28%	0.01%	0.04	0.00%	-0.02	0.00%	0.04

Panel B: Value-weighted portfolios

Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.26%	0.07%	0.10%	0.05%	0.06%	-0.21%	-0.42	0.15%	0.81	-0.05%	-0.18
Total skewness	0.21%	0.23%	0.14%	0.10%	0.17%	-0.04%	-0.23	0.10%	0.87	0.06%	0.43
Total kurtosis	0.09%	0.26%	0.08%	0.28%	0.09%	-0.01%	-0.05	-0.02%	-0.21	-0.02%	-0.23
Idiosyncratic standard deviation (market model)	0.14%	0.22%	0.00%	0.16%	0.11%	-0.03%	-0.07	0.03%	0.18	0.00%	-0.02
Idiosyncratic skewness (market model)	0.04%	0.18%	0.08%	0.18%	0.29%	0.26%	1.31*	-0.07%	-0.63	0.18%	1.55*
Idiosyncratic kurtosis (market model)	0.04%	0.22%	0.13%	0.20%	0.10%	0.06%	0.30	-0.07%	-0.73	-0.01%	-0.16
Idiosyncratic standard deviation (FF-3)	0.12%	0.23%	0.17%	0.05%	0.06%	-0.05%	-0.14	0.01%	0.03	-0.05%	-0.17
Idiosyncratic skewness (FF-3)	0.08%	0.12%	0.03%	0.19%	0.33%	0.25%	1.42*	-0.03%	-0.32	0.22%	1.99**
Idiosyncratic kurtosis (FF-3)	0.07%	0.19%	0.06%	0.23%	0.11%	0.04%	0.20	-0.04%	-0.40	0.00%	-0.06

Table 4.7: Average monthly return of portfolios sorted by estimated values of risk variables in the U.S. market

Table 4.7 reports the average monthly return of equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B) in the U.S. market. Quintile portfolios are formed every month from January 2007 to December 2019 (156 months) by sorting stocks based on the expected value of risk variables modeled by an EGARCH(1,1) for standard deviation and by an AR(1) for skewness and kurtosis. The risk variables are total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3. We also report the average return difference between quintile 5 and quintile 1 (5-1), between quintile 1 and full sample portfolio (1-FS) and between quintile 5 and full sample portfolio (5-FS). The full sample portfolio is composed by all stocks traded in the market index. t-statistics of the null hypothesis that the mentioned differences are zero are also reported. ∗∗∗, ∗∗, ∗ indicate significance at the 1%, 5%, and 10% levels, respectively.

Panel A: Equal-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.81%	0.83%	0.90%	0.93%	0.84%	0.03%	0.07	-0.05%	-0.31	-0.02%	-0.12
Total skewness	0.80%	0.90%	0.78%	0.87%	0.93%	0.13%	0.86	-0.06%	-0.62	0.07%	0.89
Total kurtosis	0.70%	0.77%	0.87%	0.88%	1.07%	0.37%	1.95**	-0.16%	-1.31	0.21%	2.28**
Idiosyncratic standard deviation (market model)	0.85%	0.90%	0.90%	0.83%	0.82%	-0.03%	-0.08	-0.01%	-0.09	-0.04%	-0.20
Idiosyncratic skewness (market model)	0.81%	0.85%	0.86%	0.88%	0.91%	0.10%	0.71	-0.05%	-0.62	0.05%	0.63
Idiosyncratic kurtosis (market model)	0.65%	0.79%	0.91%	0.92%	1.03%	0.38%	2.03**	-0.21%	-1.65	0.17%	1.87**
Idiosyncratic standard deviation (FF-3)	0.87%	0.78%	0.82%	0.89%	0.92%	0.05%	0.16	0.01%	0.08	0.06%	0.30
Idiosyncratic skewness (FF-3)	0.75%	0.98%	0.92%	0.77%	0.88%	0.13%	0.91	-0.11%	-1.33	0.02%	0.17
Idiosyncratic kurtosis (FF-3)	0.67%	0.73%	0.88%	0.96%	1.06%	0.39%	2.19**	-0.19%	-1.69	0.20%	2.28**
Panel B: Value-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.73%	0.78%	0.96%	0.88%	0.76%	0.02%	0.06	-0.07%	-0.47	-0.04%	-0.15
Total skewness	0.74%	0.70%	0.69%	0.68%	1.02%	0.28%	1.67**	-0.06%	-0.47	0.22%	3.07***
Total kurtosis	0.64%	0.67%	0.81%	0.81%	1.16%	0.52%	2.54***	-0.16%	-1.24	0.36%	2.95***
Idiosyncratic standard deviation (market model)	0.83%	0.72%	0.83%	0.83%	0.66%	-0.17%	-0.50	0.03%	0.28	-0.14%	-0.55
Idiosyncratic skewness (market model)	0.68%	0.77%	0.75%	0.84%	0.83%	0.14%	0.86	-0.12%	-1.07	0.03%	0.38
Idiosyncratic kurtosis (market model)	0.61%	0.62%	0.75%	0.92%	1.18%	0.57%	2.78***	-0.19%	-1.46	0.38%	3.18***
Idiosyncratic standard deviation (FF-3)	0.66%	0.82%	0.91%	0.88%	0.75%	0.09%	0.29	-0.14%	-1.39	-0.05%	-0.22
Idiosyncratic skewness (FF-3)	0.59%	0.70%	0.79%	0.88%	0.90%	0.31%	1.94**	-0.21%	-2.23	0.10%	1.08
Idiosyncratic kurtosis (FF-3)	0.58%	0.79%	0.71%	0.81%	1.24%	0.65%	3.50***	-0.22%	-2.11	0.44%	3.79***

Table 4.8: Average monthly return of portfolios sorted by estimated values of risk variables in the European market

Table 4.8 reports the average monthly return of equal-weighted portfolios (Panel A) and value-weighted portfolios (Panel B) in the European market. Quintile portfolios are formed every month from January 2007 to December 2019 (156 months) by sorting stocks based on the expected value of risk variables modeled by an EGARCH(1,1) for standard deviation and by an AR(1) for skewness and kurtosis. The risk variables are total standard deviation, skewness, and kurtosis, and idiosyncratic risk variables estimated by the market model and by the FF-3. We also report the average return difference between quintile 5 and quintile 1 (5-1), between quintile 1 and full sample portfolio (1-FS) and between quintile 5 and full sample portfolio (5-FS). The full sample portfolio is composed by all stocks traded in the market index. t-statistics of the null hypothesis that the mentioned differences are zero are also reported. ***, **, * indicate significance at the 1%, 5%, and 10% levels, respectively.

Panel A: Equal-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.30%	0.35%	0.37%	0.28%	0.20%	-0.10%	-0.28	0.02%	0.11	-0.08%	-0.40
Total skewness	0.17%	0.34%	0.25%	0.35%	0.38%	0.22%	1.90**	-0.11%	-1.59	0.10%	1.63*
Total kurtosis	0.30%	0.20%	0.25%	0.32%	0.42%	0.13%	0.79	0.02%	0.26	0.14%	1.63*
Idiosyncratic standard deviation (market model)	0.33%	0.35%	0.34%	0.33%	0.15%	-0.18%	-0.67	0.05%	0.43	-0.13%	-0.79
Idiosyncratic skewness (market model)	0.20%	0.22%	0.32%	0.29%	0.46%	0.27%	2.18**	-0.08%	-1.11	0.18%	2.65***
Idiosyncratic kurtosis (market model)	0.25%	0.19%	0.30%	0.38%	0.37%	0.12%	0.93	-0.03%	-0.39	0.09%	1.24
Idiosyncratic standard deviation (FF-3)	0.30%	0.36%	0.35%	0.35%	0.13%	-0.17%	-0.67	0.02%	0.21	-0.15%	-0.96
Idiosyncratic skewness (FF-3)	0.20%	0.29%	0.26%	0.28%	0.45%	0.25%	2.03**	-0.08%	-1.02	0.17%	2.54***
Idiosyncratic kurtosis (FF-3)	0.32%	0.18%	0.34%	0.20%	0.46%	0.14%	1.11	0.04%	0.55	0.18%	2.25**

Panel B: Value-weighted portfolios											
Variable used to form portfolios	Quintile					5 - 1 portfolio		1 - FS portfolio		5 - FS portfolio	
	Low (1)	2	3	4	High (5)	Estimate	t-stat	Estimate	t-stat	Estimate	t-stat
Total standard deviation	0.20%	0.21%	0.17%	0.08%	-0.04%	-0.24%	-0.61	0.09%	0.53	-0.15%	-0.62
Total skewness	0.17%	0.03%	0.04%	0.21%	0.19%	0.02%	0.13	0.06%	0.56	0.08%	0.55
Total kurtosis	0.07%	0.06%	0.15%	0.22%	0.32%	0.25%	1.39*	-0.04%	-0.56	0.21%	1.71**
Idiosyncratic standard deviation (market model)	0.23%	0.12%	0.21%	0.12%	-0.18%	-0.42%	-1.20	0.12%	0.95	-0.29%	-1.25
Idiosyncratic skewness (market model)	0.08%	0.02%	0.12%	0.25%	0.37%	0.28%	1.53*	-0.03%	-0.33	0.26%	1.84**
Idiosyncratic kurtosis (market model)	0.08%	-0.04%	0.21%	0.24%	0.28%	0.20%	1.20	-0.03%	-0.33	0.17%	1.59*
Idiosyncratic standard deviation (FF-3)	0.10%	0.19%	0.17%	0.16%	0.05%	-0.05%	-0.17	-0.01%	-0.15	-0.06%	-0.33
Idiosyncratic skewness (FF-3)	0.11%	0.11%	-0.06%	0.23%	0.35%	0.23%	1.31*	0.00%	0.00	0.24%	1.67**
Idiosyncratic kurtosis (FF-3)	0.08%	0.00%	0.22%	0.15%	0.35%	0.27%	1.78**	-0.03%	-0.46	0.24%	2.24**

Figure 4.1: Total skewness and idiosyncratic skewness measured by the market model and by FF-3 in the U.S. market

Figure 4.1 shows the monthly values of total skewness (Panel A), idiosyncratic skewness measured by the market model (Panel B) and measured by the FF-3 (Panel C) in the U.S. market. The sample period runs from January 2002 to the end of 2019.

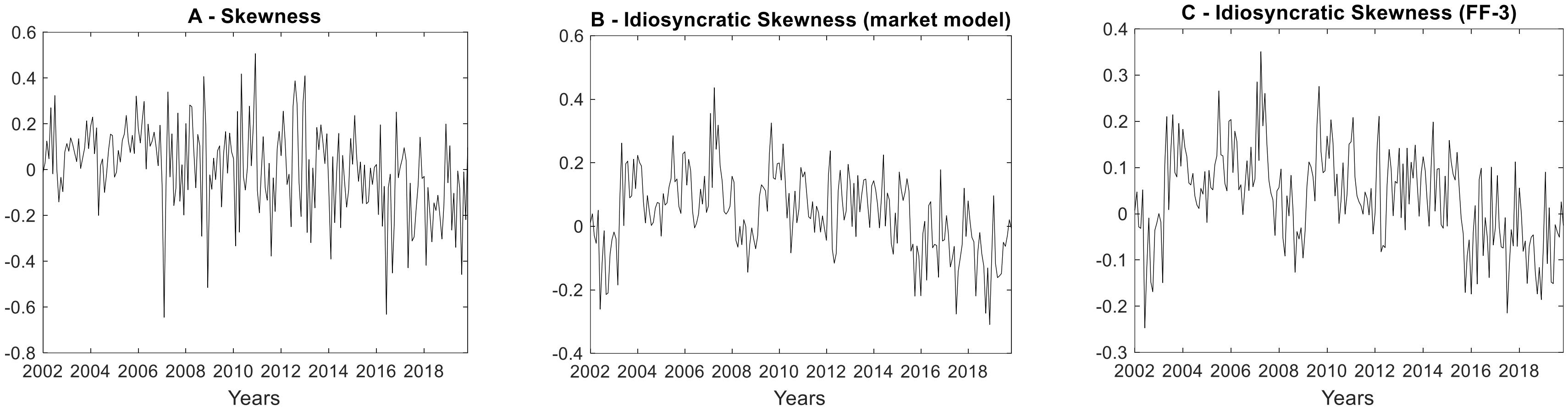


Figure 4.2: Total skewness and idiosyncratic skewness measured by the market model and by FF-3 in the European market

Figure 4.2 shows the monthly values of total skewness (Panel A), idiosyncratic skewness measured by the market model (Panel B) and measured by the FF-3 (Panel C) in the European market. The sample period runs from January 2002 to the end of 2019.

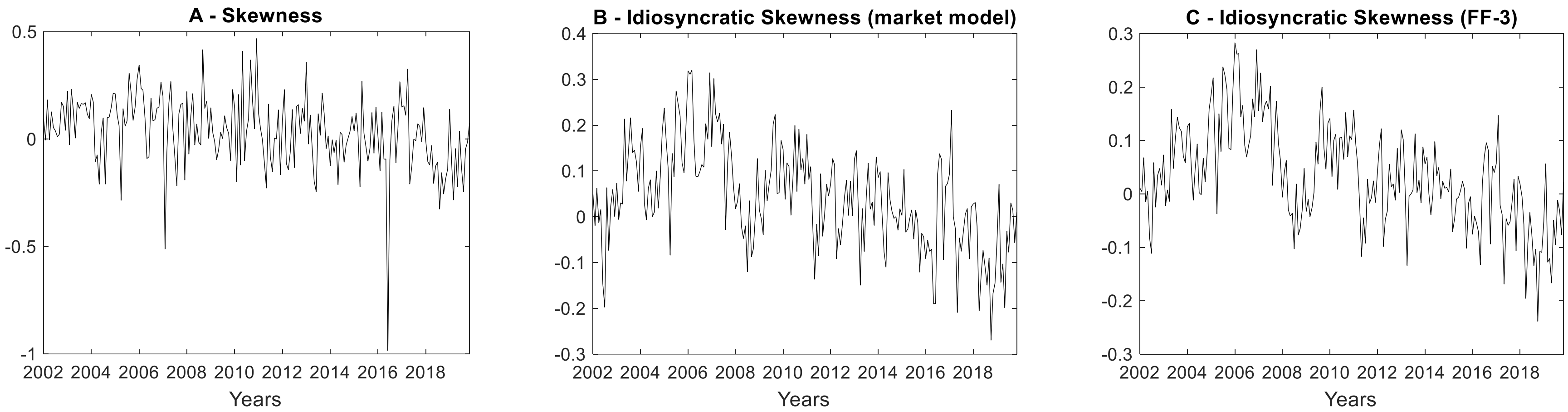


Figure 4.3: Total kurtosis and idiosyncratic kurtosis measured by the market model and by FF-3 in the U.S. market

Figure 4.3 shows the monthly values of total kurtosis (Panel A), idiosyncratic kurtosis measured by the market model (Panel B) and measured by the FF-3 (Panel C) in the U.S. market. The sample period runs from January 2002 to the end of 2019.

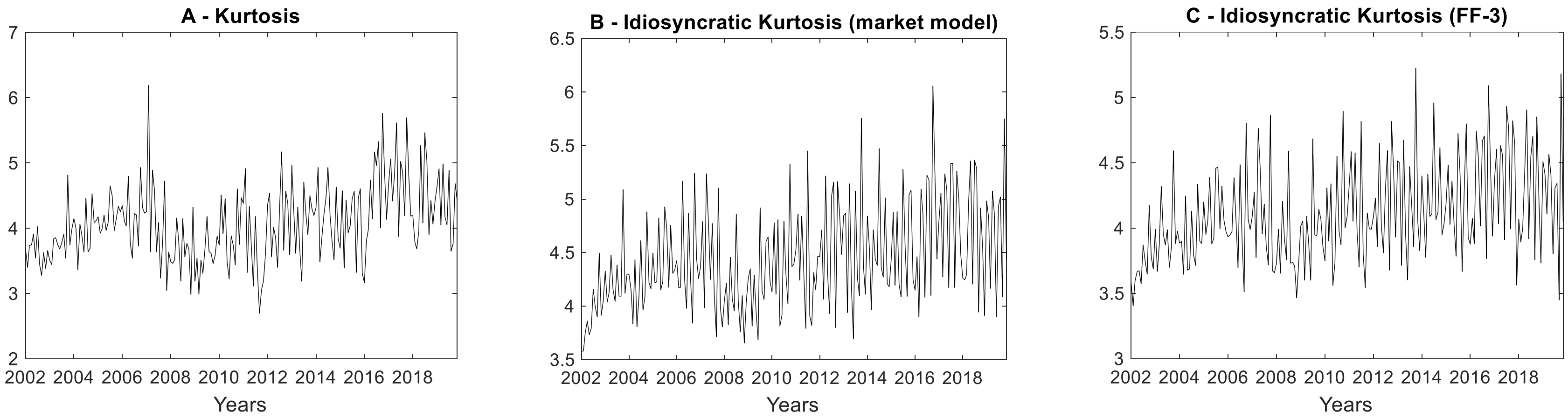
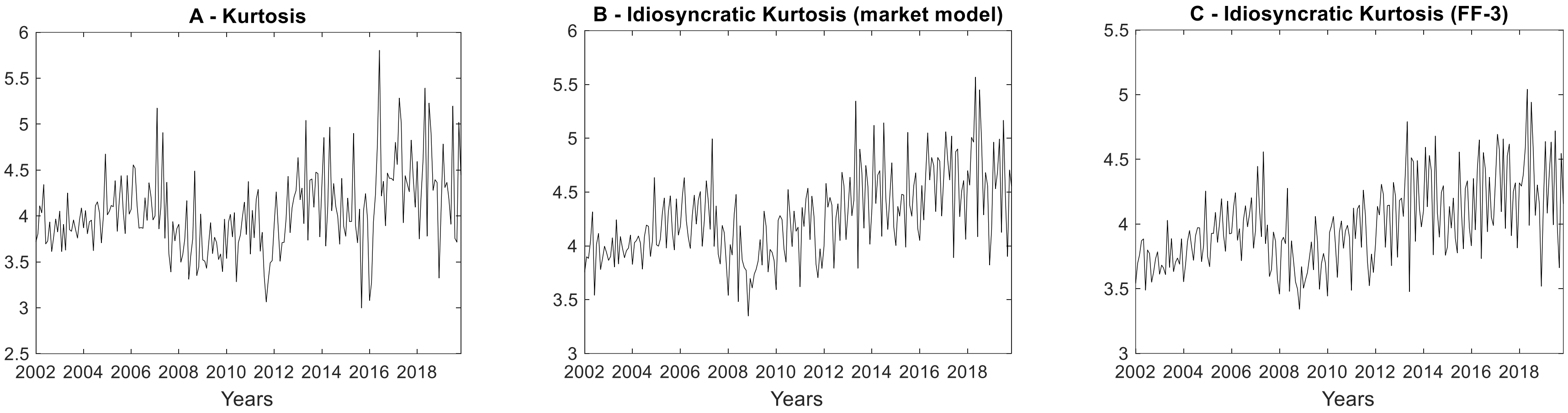


Figure 4.4: Total kurtosis and idiosyncratic kurtosis measured by the market model and by FF-3 in the European market

Figure 4.4 shows the monthly values of total kurtosis (Panel A), idiosyncratic kurtosis measured by the market model (Panel B) and measured by the FF-3 (Panel C) in the European market. The sample period runs from January 2002 to the end of 2019.



Chapter 5. Active versus Passive Strategies: Treynor-Black Model Empirically Revisited

5.1 Introduction

Passive or indexed strategies follow the Capital Asset Pricing Model idea that investors should hold a replica of the market portfolio (Sharpe, 1964). Since then, indexed mutual funds and exchange-traded funds become popular investment strategies and one of the fastest-growing investments in the world. The pioneering work of Markowitz (1952) motivates a handful of studies in the mean–variance framework, variations or generalizations, proving that it could be possible to improve substantially the performance of actively managed portfolios. Active portfolio management is based on the notion that stocks can temporarily deviate from their intrinsic prices, usually determined through some equilibrium asset pricing model. The goal of fund managers is to spot those mispricing opportunities and take advantage of them. However, consensus on which strategy achieves the best results or whether active strategies outperform, in terms of risk-adjusted return, passive strategies is far from being found. Additionally, historical returns of mutual funds show that most active managers do not achieve higher risk-adjusted return than passive strategies. Despite the drawbacks on mean–variance framework and the doubts that active strategies perform better than passive strategies, Treynor and Black (1973) (hereafter TB) model did not become popular in the investment industry although it provides an efficient and simple way of implementing an active investment strategy. Kane, Kim and White (2010) argue that TB model is not the standard operating procedure of active managers due to the poor track record of active strategies and Bernstein (2004) expresses concern on the difficulties and risks involved in generating alphas.

Frankfurter, Phillips and Seagle (1971), assuming a world where only three stocks exist, demonstrate that error in estimating the required parameters for selecting portfolios according to the mean-variance criteria is, potentially, of sufficient importance to bring into question the usefulness of models that ignore it. They suggest that portfolios selected according to the mean-variance criteria are not more efficient than portfolios selected randomly. Merton (1980) acknowledges the estimation error of using realized returns and variances as estimators, suggesting a nonnegative restriction on expected returns and an adjustment for heteroscedasticity in realized return time series. Chopra and Ziemba (1993) show that estimation error in means is much higher than estimation error in variances or covariances. These estimation errors lead to a suboptimal portfolio composition and thus to a poor portfolio out-of-sample performance.

Studies on TB model are considerably less compared with studies about mean-variance framework. He (2007) develops an integrated Bayesian framework to account for alpha uncertainty in TB model. In his proposed model, the ordinary least squares estimates of alpha converges to zero as the beliefs in market efficiency reduce. Using monthly returns of 48 industry portfolios from Kenneth French's data library during the period between July 1963 and December 2005, the author constructs a set of 10 active assets. The set is obtained by ranking all of the 48 industry portfolios by their information ratios, selecting the top five industries with the highest ratios and the bottom five industries with the lowest ratios. With this set of active assets, the author characterizes the active portfolio, the overall portfolio, and the portfolio weights as a function of different levels of active risk budget. The author concludes that active risk budget could be important to limit active management investments according to the confidence in their ability to generate alphas.

Kane et al. (2010) investigate the connection between the abnormal returns forecast precision and the superior performance of portfolios constructed optimally based on their predictions. Through a simulation where the mean and standard deviation of the benchmark portfolio return, as well as the betas and abnormal returns of the stocks follow a normal distribution and the correlation coefficient is 0.1 between any pair of stocks, they show the potential of TB model to be used as a tool to exploit benchmark inefficiency. Despite TB model is not superior than the benchmark portfolio when less than 10 stocks are used, TB model becomes superior to the benchmark portfolio when the prediction power and/or the number of stocks is sufficiently large. The authors also argue that the performance of TB model improves as the length of the estimation window increases. An estimation window with 60 months achieves better results opposed to 36 and 48 months. Finally, the authors show that TB model portfolio outperforms the equal-weighted portfolio in terms of Sharpe ratio, especially when the portfolio has a large number of stocks.

Allen, Lizieri and Satchell (2019) replicate the simulation research design of Kane et al. (2010) using an out-of-sample period of 120 months instead of 60 months. They show that TB model was superior to the equal-weighted portfolio strategy, in terms of Sharpe ratio, when investors have modest forecasting ability, and the investment universe size is of 100 or 500 assets.

Ramírez-Hassan and Guerra-Urzola (2020) propose a decision theory framework based on a Bayesian approach to mitigate estimation risk by minimizing the posterior expected loss function rather than the risk function, with a loss function based on the trading strategy rather than the utility function. Using weekly returns from June 2009 to June 2017 of 21 MSCI international equity indices to construct randomly portfolios with three different sizes (5, 10 and 15 indexes), they show that the minimum expected loss approach is the best statistical strategy when minimum-variance or tangency portfolios are used as the trading strategy. Concerning TB model, the naive approach got better out-of-sample results than the minimum expected loss approach, but the equal-weighted portfolio outperformed TB model for every approach considered in their study.

Studies about TB model, to the best of our knowledge, do not test the model using real data on assets and on the efficiency of the benchmark. In addition, studies on portfolio selection techniques, generally, fail to point out what are the eventual causes for the inferior return of the active strategy compared to the passive strategy. Finally, although most studies claim that survivorship bias should not affect the comparison between the strategies, datasets with large negative returns associated with delisted stocks, give disadvantages to the equal-weighted portfolio strategy. These datasets do prevent other strategies to put large weights of investment in assets with large negative returns while equal-weighted portfolio strategy will only have a small investment. The motivation of this Chapter is to incorporate and test these issues in the empirical analysis and answer to our investigation question if an active strategy based on TB model is superior to a passive strategy.

The main contributions of this Chapter are as follows. First, we have large dataset with 500 stocks of the U.S. market and 600 stocks of the European market, which is larger than most studies on the topic of portfolio selection. Second, since the majority of empirical studies focus on the mean-variance framework, we analyze a portfolio allocation strategy based on TB model, assuming inefficiently priced stocks. Third, we highlight the principal drawbacks of using TB model empirically. Finally, we use a sampling technique that deals with delisted stocks over the period to avoid survivorship bias.

This Chapter proceeds as follows. Section 5.2 describes the methodology used to implement and evaluate TB model. Section 5.3 reports the results of the empirical analysis. Section 5.4 highlights the downsides of implementing TB model empirically and Section 5.5 concludes.

5.2 Methodology

Treynor and Black (1973) developed an optimizing model that combines the weights of a passive portfolio and an active portfolio (the optimal portfolio). In TB model, investors use a limited number of stocks that are not efficiently priced to form an active portfolio. Thus, stocks that are not included in the active portfolio are considered efficiently priced. TB model identifies the portfolio of inefficiently priced stocks (active portfolio) that can be mixed with the passive portfolio to obtain the optimal portfolio. In the following subsections, we describe how we implement TB model empirically, in general, and how we use efficient portfolios as market portfolios in particular.

5.2.1 Treynor-Black (1973) Model Implementation

We use a rolling window estimation period of 60 months of returns to estimate all the necessary parameters to obtain the active portfolio. The rolling window procedure is widely used in the literature (see, for example, MacKinlay and Pastor, 2000; Larsen and Resnick, 2001; Clarke, Silva and Thorley, 2006; DeMiguel and Nogales, 2009; Kirby and Ostdiek, 2012; Kourtis, Dotsis and Markellos, 2012; Bessler, Opfer and Wolff, 2017; Allen et al., 2019). For all the stocks traded on the market index at the beginning of each month, we consider the previous 60 months of returns to calculate the portfolio weights. These portfolio weights are used to compute risk and return of our investment strategy. At the end of the month, we repeat the procedure by moving the estimation window forward: we include the data for the next month and drop the oldest month until we reach the end of the dataset. The performance of each strategy is measured with the returns observed in the month (out-of-sample test procedure) following the formation of each portfolio during the period between January 2007 and December 2019 (156 months). Investment transactions do not influence the price or dividend of any stock and taxes and transaction costs are not considered.

To calculate active portfolio weights, we need to estimate alphas, betas, and idiosyncratic variances, measured through the variance of the residuals, for each stock. The estimates of alpha and beta using ordinary least squares are given by the market model defined in equation (2.4) where α_i represents the extra expected return (abnormal return) and β_i measures the extent to which returns on the stock and the market portfolio move together. Following Treynor and Black (1973), the weights of each stock *i* in the active portfolio are calculated as:

$$w_i = \frac{\alpha_i / \varepsilon_i^2}{\sum_{j=1}^{m} \alpha_j / \varepsilon_j^2} \quad (5.1)$$

where ε_i^2 is the variance of the residuals of stock *i*, representing the nonsystematic variance, and *m* is the number of stocks in the active portfolio. Using the estimates for the parameters of equation (2.4) and the weights of equation (5.1), we estimate the alpha (α_A), beta (β_A) and idiosyncratic variance (ε_A^2) of the active portfolio as:

$$\alpha_A = \sum_{j=1}^{m} w_j \times \alpha_j$$
$$\beta_A = \sum_{j=1}^{m} w_j \times \beta_j \qquad (5.2)$$
$$\varepsilon_A^2 = \sum_{j=1}^{m} w_j^2 \times \varepsilon_j^2$$

With equation (5.2), we determine the fraction invested in the active portfolio (w_A), which is as follows:

$$w_A = \frac{\alpha_A / \varepsilon_A^2}{(r_M - r_f) / \sigma_M^2} \qquad (5.3)$$

where r_M is the monthly return of the market portfolio, σ_M is the monthly standard deviation of market portfolio returns and r_f is the risk-free rate. The active portfolio weight is adjusted for the beta exposure of the active portfolio, resulting in:

$$w_A{}^* = \frac{w_A}{1 + (1 - \beta_A) w_A} \qquad (5.4)$$

The remaining optimal investment proportion is allocated to the passive portfolio is given by:

$$w_P = 100\% - w_A{}^* \qquad (5.5)$$

In order to implement TB model, from all stocks with alpha and beta estimates statistically significant at a 10% level, we rank them by alpha. We select the top five stocks with the highest alpha and the bottom five stocks with the lowest alpha to construct our active portfolio.

Regarding the benchmark portfolio, we consider three different portfolios: tangency portfolio (hereafter TP-Unrest), tangency portfolio with short selling restriction (hereafter TP-Rest) and market index (hereafter TB-MI). Each benchmark portfolio corresponds to one different investment strategy. Following Lim, Durand and Yang (2014), for the U.S. market, we use S&P 500 as market index and for the European market, we use STOXX 600 under the same rational.

The tangency portfolio (hereafter TP) is the portfolio with the highest excess return per unit of risk, satisfying the constraint that portfolio weights (positive or negative) sum to 1. Therefore, we solve a mathematically problem formally written as:

$$\max_{w_i} \frac{r_p - r_f}{s_p} \quad s.t. \quad \sum_{i=1}^{m} w_i = 1 \tag{5.6}$$

The TP with short selling restriction satisfies the constraint that portfolio weights are all nonnegative. Therefore, we solve a mathematically problem formally written as:

$$\max_{w_i} \frac{r_p - r_f}{s_p} \quad s.t. \quad \sum_{i=1}^{m} w_i = 1 \quad and \quad w_i \geq 0 \tag{5.7}$$

We compare the difference between out-of-sample performance of the different active TB model strategies and passive portfolios on three different criteria: (i) portfolio returns; (ii) portfolio risk; and (iii) Sharpe measure.

Sharpe measure (hereafter SR) is the excess return divided by the standard deviation of excess return defined as:

$$SR = \frac{r_p - r_f}{s_p} \tag{5.8}$$

SR is probably the most widely used metric among the studies about portfolio selection and portfolio performance assessment (see, for example, Larsen and Resnick, 2001; DeMiguel and

Nogales, 2009; Kirby and Ostdiek, 2012; Kourtis et al., 2012; Bessler et al., 2017; Allen et al., 2019). Large positive values are an indication of superior portfolio performance and SR is invariant to the relative weights on the riskless asset and the risky portfolio.

5.2.2 Efficient Portfolios as Market Portfolios

Tables 5.1 and 5.2 report the number of stocks in the U.S. and European markets, respectively, segregated by their coefficient's signs and significance, for three different market portfolios: tangency portfolio, tangency portfolio with short selling restriction and market index.

The results show that stocks with positive and statistically significant coefficient estimates, at the 10% level, are only found when a market index is used as market portfolio. Thus, no significant positive relation and significant positive alpha estimates are found between stocks of the U.S. and European markets and the tangency portfolio, restricted or not. The application of TB model in these circumstances points to a null weight in the active portfolio and, therefore, the strategy results in a strictly passive strategy invested in the tangency portfolio. In general, a significant negative relationship between restricted and non-restricted tangency portfolios are the dominant characteristic. Since tangency portfolio returns are, generally, positive in every period, the absence of positive relations (and the existence of negative relations) suggests that stocks have consistently negative returns. The results also suggest that stocks with a superior performance, in terms of risk and return, are not related with the efficient portfolio, however, stocks with a poor performance could be negatively related with the efficient portfolio.

Moreover, for stocks with estimates of alpha and beta statistically significant, we observe that, when the non-restricted tangency portfolio is the benchmark portfolio, most stocks have a positive alpha and a negative beta. When the restricted tangency portfolio is the benchmark portfolio, most stocks have a negative alpha and a positive beta. These very curious results suggest that short selling restriction allows a positive relationship between stocks and efficient portfolio. Despite the positive relationship, on average, stocks underperform the market portfolio in terms of return due to the negative alpha.

When the market index is the market portfolio, stocks with a statistically significant negative beta are rare and stocks with a statistically significant positive beta are almost equally divided between stocks with positive and negative alpha. This result suggests that some stocks have higher return than the market index, while others have a lower return, which is not surprisingly since the market index return is an average return of all stocks traded on that market index.

In this context, TP-Unrest and TP-Rest strategies consist in a strictly passive strategy invested in non-restricted tangency portfolio and restricted tangency portfolio, respectively, since inefficiently priced stocks are not identified in these strategies. Regarding TB-MI strategy, there is a mixed investment between a portfolio of inefficiently priced stocks (active portfolio) and the market index (passive portfolio).

5.3 Empirical Results

Table 5.3 reports the average return, risk and SR for the U.S. and European markets and for three strategies: TP-Unrest, TP-Rest and TB-MI. From Table 5.3, we observe that the highest average return in the U.S. and European markets occurs when the tangency portfolio is the market portfolio (TP-Unrest strategy). In the U.S. market, the average return is 5.5% for TP-Unrest strategy, compared with 0.7% and 0.9% of the TP-Rest and TB-MI strategies, respectively. In the European market, the average return is 13.7% for TP-Unrest strategy, compared with 0.1% and 0.7% of the TP-Rest and TB-MI strategies, respectively.

We notice from the average risk that these high returns are associated with high risk. The average risk of TP-Unrest strategy is also the highest in both markets among the strategies analyzed (107.0% in the U.S. market and 120.3% in the European market). The high level of return and risk of TP-Unrest strategy is due to the large positions (long and short) of the tangency portfolio weights that arise when equation (5.6) is solved. Thus, if some stocks of the active portfolio with large long positions have large positive returns and some stocks with large short positions have large negative returns, this strategy return will be high. However, if stocks with large long positions have low returns or if stocks with large short positions have high returns, the strategy return will, generally, be negative, and in some cases could be inferior to -100%. In the U.S. and European markets, the TP-Unrest strategy has a return inferior to -100% in 10 months and 12 months, respectively, out of 156 months (13 years). Hence, this strategy is very risky and difficult to implement in the long-term since it loses all the investment almost, on average, in one month of each year.

The strategies that do not allow short-selling (TP-Rest and TB-MI strategies) have a significant decrease in the average risk compared to TP-Unrest strategy. In the U.S. market, TP-Rest and TB-MI strategies have an average risk of 5.3% and 9.8%, respectively. In the European market, TP-Rest and TB-MI strategies have an average risk of 4.2% and 11.3%, respectively. These levels of risk are significantly lower than TP-Unrest strategy, where average risk is 107.0% and 120.3% in the U.S. and European markets, respectively. However, the decrease in average risk also decreases the average return.

Focusing the analysis on TP-Rest and TB-MI strategies, we observe in Table 5.3 that TB-MI strategy has higher average return, higher average risk and higher SR when compared with TP-Rest strategy in the U.S. and European markets. The SR indicates that the average risk increase of using the market index as market portfolio, opposed to the tangency portfolio with short selling restriction, is compensated by the increase in average return. Thus, there is evidence of superior performance of TB-MI strategy over the TP-Rest strategy in terms of SR. The increase of average risk when the market index is used as the market portfolio comes from the short positions of the active portfolio. In TP-Rest strategy no short positions are taken.

Since stocks with positive and statistically significant coefficients estimates, at the 10% level, are only found when a market index is used as benchmark portfolio, the strategies using tangency portfolio and tangency portfolio with short selling restriction as market portfolios in TB model are simply the weights of the referred portfolios invested in the next period. This results from the impossibility to construct an active portfolio, and so, the strategy allocates all the investment in the passive portfolio: tangency portfolio or tangency portfolio with short selling restriction.

When a market index is the market portfolio, a comparison between the performance of TB model and the passive portfolio is possible since an active portfolio is formed. Table 5.4 reports the average return, risk and SR for the TB-MI strategy, the respective passive portfolio and the percentage of months that TB-MI strategy was superior to the passive portfolio (market index) on each criterion, for the U.S. and European markets. In both markets, the TB-MI strategy is superior to the passive portfolio in more than 50% of the months in terms of return. Furthermore, a small superiority of 0.1 p.p. and 0.2 p.p. in the U.S. market and the European market, respectively, is observed in the average return of TB-MI strategy over the market index.

The active portfolio is a portfolio with a substantial level of risk since the risk of TB-MI strategy more than double the risk of passive portfolio. The increase in risk of adopting the TB-MI strategy is compensated by the increase in average return in the European market but is not compensated in the U.S. market. In fact, the average SR of TB-MI strategy is 0.43 in the U.S. market while the average SR of the passive portfolio is 0.47. In the European market, the average SR of TB-MI strategy is 0.37 while the average SR of the passive portfolio is 0.32.

Figure 5.1 shows the cumulative return of TP-Rest strategy, TB-MI strategy and market index for the U.S. and European markets. Since we already observed that TP-Unrest strategy is not feasible in the long-term due to monthly returns lower than -100% this strategy was not included in Figure 5.1. In addition, due to the inability to identify stocks with alphas estimates statistically different from zero when the tangency portfolio with short selling restriction is the market portfolio, the investment is always 100% in the passive portfolio. Thus, the TP-Rest strategy do not have a passive portfolio to be compared.

From Table 5.3 is possible to verify that TB-MI strategy has a superior average return compared to TP-Rest strategy in both markets. Figure 5.1 indicates the same superiority in terms of cumulative return. In the U.S. market, the TB-MI strategy has a cumulative return of 193% opposed to 133% of cumulative return of the TP-Rest strategy. In the European market, the TB-MI strategy has a cumulative return of 68% opposed to 11% of cumulative return of the TP-Rest strategy. Despite the referred superiority, between January 2007 and December 2019, the TP-Rest strategy had superior cumulative return compared to TB-MI strategy in 51 and 52 months in the U.S. and European markets, respectively. For example, in the U.S. market between January 2007 and June 2018, the TP-Rest strategy had 170% of cumulative return while TB-MI strategy had 163%. In the case of the European market, between January 2007 and November 2014 the TP-Rest strategy had -23% of cumulative return while TB-MI strategy had -24%. These outcomes support the conclusion that the superiority of TB-MI strategy over TP-Rest strategy depends on the periods that are considered, i.e. the superiority of the TB-MI strategy is not systematic.

Figure 5.1 shows distinct results on each market when the cumulative return of TB-MI strategy is compared with the respective passive portfolio (market index). In the U.S. market, despite the higher cumulative return of market index in the end of the period (200% versus 193% of the TB-MI strategy), the market index had only superior cumulative return in 19 out of 156 months. In the European market, the market index is substantially superior in terms of cumulative return. The cumulative return of the market index in the end of the period was 79%, opposed to the cumulative return of 68% of the TB-MI strategy and the market index had superior cumulative returns in 139 out of 156 months.

Table 5.5 reports the cumulative performance from January 2007 to December 2019 of TP-Rest strategy, TB-MI strategy and market index in annualized return, risk and SR in the U.S. and European markets. In the U.S. market, the market index had 8.8% of annualized return while TP-Rest and TB-MI strategies had 6.7% and 8.6%, respectively. In the European market, the market index had 4.6% of annualized return while TP-Rest and TB-MI strategies had 0.8% and 4.1%, respectively. In addition, we observe that SR of market index is also superior over the other strategies in both markets.

The findings of this section call into question the superiority of TB model over the passive portfolio in terms of risk and return. In the markets and time series studied in this Chapter, there is no evidence of a consistent superior risk-adjusted average return over the market index by pursuing a strategy based on TB model using the market index as market portfolio.

5.4 Major Downsides of Implementing Treynor-Black (1973) Model Empirically

To better understand how the TB model implementation could be improved to allow a better risk-adjusted return, in this section we highlight the main reasons that lead TB model to have not a consistently superior risk-adjusted average return over the market index.

5.4.1 Alpha Estimation Error

The effectiveness of any investment strategy is highly dependent on the inputs, mostly returns. The failure in estimating accurately these inputs could lead to poor out-of-sample performance because the expected values will be different from the realized values. Table 5.6 shows the number of stocks segregated by alpha estimation error (hereafter EE) intervals for positive and negative alphas in the U.S. and European markets. Alpha EE is given by the difference between stocks realized excess return over the market index and alpha estimation. In our TB model implementation, we select the top five stocks with the highest alpha and the bottom five stocks with the lowest alpha estimates to calculate the weights of the active portfolio. Since our out-of-sample period have 156 months, we calculated 156 active portfolios with 780 stocks with positive alpha estimates and 780 stocks with negative alpha estimates for each market. In the U.S. market, we only have 764 stocks with negative alpha estimates since some months do not have five stocks with a negative and an alpha statistically different from zero.

The EE could not be problematic if the error is in a favorable direction. For example, if the expected alpha is 2% and the realized alpha is 4%. Thus, positive EE for positive alphas is not the major problem. On the other hand, if the expected alpha is -2%, and accordingly we take a short position on that stock, and the realized alpha is -4%, the EE will also not be a problem. EE is a problem when positive alphas have a negative EE and negative alphas have a positive EE. This implies long positions with realized values inferior to the expected values and short positions with realized values superior to expected values. Table 5.6 indicates that 486 and 504 stocks with positive alpha, in the U.S. and European markets, respectively, had a negative EE.

Hence, 62% and 65% of the stocks with positive alpha used to construct the active portfolio had a lower alpha than our estimation. Regarding the negative alpha stocks, we observe that, in the U.S. and European markets, 426 and 439 stocks, respectively, had a positive EE. Consequently, 56% of the stocks with negative estimate for alpha of each market that were included in the active portfolio had a higher alpha than our estimated result.

Despite most stocks have an EE that is not favorable to our investment strategy, this would be mitigated if the average EE, in the favorable direction, is higher than the average EE in the unfavorable direction. Table 5.7 shows the average alpha EE divided in positive and negative EE for positive and negative alphas in the U.S. and European markets. In both markets, the average EE in absolute terms of positive alphas is lower for stocks with positive EE compared to stocks with negative EE. In addition, the average EE in absolute terms of negative alphas is higher for stocks with positive EE compared to stocks with negative EE.

The lack of evidence of superiority of TB model over the passive portfolio in terms of return could be partially explained by alpha EE, not by the EE itself but by the EE in an unfavorable direction. We recall that negative EE for positive alpha and positive EE for negative alpha are harmful to return when we implement TB model. The results suggest that passive portfolio has higher risk-adjusted average return compared to TB model because most stocks have an unfavorable deviation in EE and a higher EE average in the unfavorable direction.

5.4.2 Allocation of Investment Between Active and Passive Portfolios

Despite the estimation error mentioned in the previous subsection leads to an active portfolio with lower return than the passive portfolio in several periods, this downside could be mitigated if the weight of investment on the active portfolio is large when the active portfolio return is larger than the passive portfolio return, and small when the inverse occurs.

Table 5.8 reports the average weight of active portfolio when the active portfolio return is larger or smaller than the passive portfolio return in the U.S. and European markets, as well the average weight in active portfolio after the beta adjustment of our sensitivity analysis when this portfolio return is higher or lower than the passive portfolio return.

We observe that average weight of investment in the active portfolio is not substantial different when the active portfolio return is larger or smaller than the passive portfolio. In the U.S. market, the average weight of the active portfolio is 16.3% when the active portfolio return is higher than the passive portfolio return while it is 19.2% when the passive portfolio return is superior to the return of active portfolio. We highlight that this situation is the least desirable since this strategy invests more on the active portfolio when the active portfolio has a lower return than the passive portfolio. On the other hand, in the European market, the average weight of the active portfolio is 27.1% when the active portfolio return is higher than the passive portfolio return, while it is 25.6% when the passive portfolio return is superior to the return of active portfolio. In this case, the TB-MI strategy invests more, on average, on the active portfolio when the active portfolio return is higher than the passive portfolio return. However, this was not sufficient to produce higher cumulative return than the market index (the passive portfolio). As shown in Figure 5.1, the cumulative return in the end of the out-of-sample period of the market index was 79%, opposed to the cumulative return of 68% of the TB-MI strategy.

To evaluate the change in the cumulative return given a variation of beta estimates, we perform a sensitivity analysis on this parameter. Our sensitivity analysis is an increase of 0.25 or 0.5 in beta estimates when the active portfolio return is higher than the passive portfolio return and a decrease by the same amount when the active portfolio return is lower than the passive portfolio return. This way, we increase the investment weight in active portfolio when the active portfolio has a higher return than the passive portfolio and decrease the investment weight in active portfolio when the opposite happens.

The average weight of active portfolio has a positive relation with the variation of beta. However, the average increase in the active portfolio weight is not very expressive when beta changes 0.25 or 0.5. We observe, for a beta adjustment of 0.5, that when the active portfolio return is higher than the passive portfolio return the average weight in active portfolio increases from 16.3% to 18.4% in the U.S. market and from 27.1% to 33.4% in the European market. Nevertheless, the cumulative return suffers a significant impact due to the beta adjustment.

Table 5.9 reports the cumulative return of TB-MI strategy, with and without our beta adjustment, in the U.S. and European markets. Despite the reduced increase in the average weight of active portfolio when the active portfolio return is higher than the passive portfolio return, the cumulative return changes considerably. An increase of 0.25 in beta when the active portfolio has a higher return than the passive portfolio and a decrease by the same amount in beta when the opposite happens results in an increase of cumulative return from 192.9% to 281.1% in the U.S. market and from 67.9% to 188.6% in the European market. Therefore, even with alpha estimation error, there is evidence that TB model could achieve a constant superiority over the passive portfolio with an accurate estimation of beta. By beta accuracy, we assume that beta will be large when the active portfolio has higher return than the passive portfolio and will be small when the opposite is true.

5.4.3 Extreme Positions in the Active Portfolio

In this subsection, we focus our analysis on risk rather than on return, which was the focus of the previous subsections. In Table 5.4, we observe that TB-MI strategy risk is substantially higher than the passive portfolio. This is due to the extreme long and short positions of the active portfolio. These extreme positions arise, essentially, when the denominator of equation (5.1) is close to zero, which is true when the positive and negative alphas divided by the variance of the residuals offset each other.

Table 5.10 reports estimates of alpha, variance of residuals, alpha divided by the variance of residuals and the weight of each stock in the active portfolio in July 2008 for the U.S. market and in February 2008 for the European market. These months were selected as examples where extreme weights that turn the active portfolio very risky are present. For example, in the U.S. market, we have a long position of 2204% and a short position of -1579%, leading to 709% of risk. In the European market, we have long and short positions of 845% and -814%, respectively, resulting in 121% of risk. These extreme positions can be avoided if we restrict the positions in the active portfolio to long positions.

5.5 Conclusion

This Chapter analyzes the empirical implementation of TB model in the U.S. and European market using three different market portfolios: tangency portfolio, tangency portfolio with short selling restriction and market index. We use a rolling window approach with an estimation window of 60 months of returns to compute all the necessary parameters to obtain the active portfolio. The strategies are evaluated each month of the out-of-sample period between January 2007 and December 2019 (156 months).

The results suggest an absence of a statistically significant positive relation between stock returns and unrestricted tangency portfolio returns. Also, there is no evidence of a statistically significant positive alpha estimate when tangency portfolio with short selling restriction is the market portfolio. In this context, the weight of investment in the active portfolio of TB model is zero, and so, strategies based on these portfolios as market portfolios have a weight of 100% in the passive portfolio: restricted and non-restricted tangency portfolios.

The TB model strategy that uses unrestricted tangency portfolio as market portfolio has returns inferior to -100% in several months. Hence, this strategy is very risky and difficult to implement in the long-term since it can lose all investment in just one month. Regarding strategies that use tangency portfolio with short selling restriction and market index as market portfolios, we observe higher return, higher risk and higher Sharpe measure when the market index is the market portfolio in both markets.

When we compare TB model, using the market index as market portfolio, with the passive portfolio (the market index), we conclude that market index had a higher Sharpe measure in both markets, showing no advantages of pursuing an active strategy based on TB model.

To understand how the TB model implementation could be improved to allow a better risk-adjusted average return, we highlight the main reasons that lead TB model to have not a consistently superior risk-adjusted average return over the market index: alpha estimation error, allocation of investment between the active and passive portfolios and extreme positions of the active portfolio.

Alpha estimation error is only a problem when positive alphas have a negative estimation error and negative alphas have a positive estimation error, meaning that our long positions had a lower realized return than our expectations and short positions had a higher realized return. We observe that average estimation error in absolute terms of positive alphas is lower for stocks with positive estimation error compared to stocks with negative estimation error. In addition, average estimation error in absolute terms of negative alphas is higher for stocks with positive estimation error compared to stocks with negative estimation error.

Even with alpha estimation error, TB model will probability have a better risk-adjusted average return than the passive portfolio if the weight of investment on the active portfolio is large when the active portfolio return is larger than the passive portfolio return, and small then the inverse occurs. Despite the average weight of investment in the active portfolio is not substantial different when the active portfolio return is larger or smaller than the passive portfolio in both markets, we find evidence that TB model could achieve a constant superiority over the passive portfolio with an accurate estimation of beta. By beta accuracy, we assume that beta is large when the active portfolio has higher average return than the passive portfolio and is small when the opposite is true.

Finally, we observe that active portfolio is very risky due to the extreme weights generated by TB model when the weights of the active portfolio are calculated. These extreme positions arise, essentially, when the positive and negative alphas divided by the variance of the residuals offset each other. Extreme weights are eliminated if the active portfolio is only composed by long positions.

Since the reduction of uncertainty in alpha and betas estimates appears to improve the risk-adjusted performance of TB model, our results may be useful as a starting point to study the effect of new estimation procedures of alphas and betas to serve as inputs of TB model. Despite studies on these subjects go beyond the scope of our study, they seem interesting for future research.

Table 5.1: Alphas and betas in the U.S. market

Table 5.1 reports the number of stocks in the U.S. market, segregated by their coefficient's signs and significances, for three different market portfolios: tangency portfolio, tangency portfolio with short selling (SS) restriction and market index. Alphas and betas are calculated using 60 monthly returns through an ordinary least squares (OLS) regression. The number of stocks reported in the last row as not calculated respect to stocks that did not have returns in the full period of 60 months used as estimation period.

OLS regression coefficient signs	Alpha significance	Beta significance	Tangency portfolio	Tangency portfolio with SS restrictions	Market index
Positive alpha and positive beta	Significant	Significant	0	0	5,352
	Significant	Not signif.	0	0	1,135
	Not signif.	Significant	0	1	31,273
	Not signif.	Not signif.	7,486	9,623	2,294
Positive alpha and negative beta	Significant	Significant	20,141	400	1
	Significant	Not signif.	4,494	1,147	66
	Not signif.	Significant	2,033	110	2
	Not signif.	Not signif.	26,658	11,045	124
Negative alpha and positive beta	Significant	Significant	523	9,473	4,162
	Significant	Not signif.	105	342	21
	Not signif.	Significant	708	7,204	27,571
	Not signif.	Not signif.	10,162	32,898	746
Negative alpha and negative beta	Significant	Significant	0	0	0
	Significant	Not signif.	0	0	3
	Not signif.	Significant	0	0	0
	Not signif.	Not signif.	477	544	37
Not calculated			5,499	5,499	5,499

Table 5.2: Alphas and betas in the European market

Table 5.2 reports the number of stocks in the European market, segregated by their coefficient's signs and significances, for three different market portfolios: tangency portfolio, tangency portfolio with short selling (SS) restriction and market index. Alphas and betas are calculated using 60 monthly returns through an ordinary least squares (OLS) regression. The number of stocks reported in the last row as not calculated respect to stocks that did not have returns in the full period of 60 months used as estimation period.

OLS regression coefficient signs	Alpha significance	Beta significance	Tangency portfolio	Tangency portfolio with SS restrictions	Market index
Positive alpha and positive beta	Significant	Significant	0	0	6,386
	Significant	Not signif.	0	2	679
	Not signif.	Significant	3	4	34,377
	Not signif.	Not signif.	8,517	7,676	1,536
Positive alpha and negative beta	Significant	Significant	15,559	373	25
	Significant	Not signif.	2,960	471	102
	Not signif.	Significant	3,511	110	0
	Not signif.	Not signif.	31,057	9,484	169
Negative alpha and positive beta	Significant	Significant	904	19,882	6,202
	Significant	Not signif.	98	1,358	53
	Not signif.	Significant	1,105	10,143	35,342
	Not signif.	Not signif.	20,539	35,337	971
Negative alpha and negative beta	Significant	Significant	0	0	0
	Significant	Not signif.	0	0	4
	Not signif.	Significant	0	0	0
	Not signif.	Not signif.	1,646	1,059	53
Not calculated			7,704	7,704	7,704

Table 5.3: Average return, risk and Sharpe measure of each strategy

Table 5.3 reports the average monthly return, risk and Sharpe measure (SR) for three strategies: tangency portfolio (TP-Unrest), tangency portfolio with short selling restriction (TP-Rest) and market index (TB-MI). Panel A reports the results for the U.S. market while Panel B reports the results for the European market. The sample period runs from January 2007 to December 2019 (156 months).

	TP-Unrest	TP-Rest	TB-MI
Panel A: U.S. market			
Average return	5.5%	0.7%	0.9%
Average risk	107.0%	5.3%	9.8%
Average Sharpe measure	0.47	0.36	0.43
Panel B: European market			
	TP-Unrest	TP-Rest	TB-MI
Average return	13.7%	0.1%	0.7%
Average risk	120.3%	4.2%	11.3%
Average Sharpe measure	0.67	0.21	0.37

Table 5.4: Average return, risk and Sharpe measure of TB-MI strategy and the respective passive portfolio

Table 5.4 reports the average monthly return, risk and Sharpe measure (SR) for the TB-MI strategy and the respective passive portfolio, as well as the percentage of months that TB-MI strategy was superior to the passive portfolio (market index) on each criterion. Panel A reports the results for the U.S. market while Panel B reports the results for the European market. The sample period runs from January 2007 to December 2019 (156 months).

Panel A: U.S. market			
	TB-MI	Market index (MI)	TB-MI superior to MI
Average return	0.9%	0.8%	56.4%
Average risk	9.8%	4.6%	90.4%
Average Sharpe measure	0.43	0.47	49.4%
Panel B: European market			
	TB-MI	Market index (MI)	TB-MI superior to MI
Average return	0.7%	0.5%	55.8%
Average risk	11.3%	4.8%	92.3%
Average Sharpe measure	0.37	0.32	55.1%

Table 5.5: Annualized return, risk and Sharpe measure of TP-Rest, TB-MI and market index strategies

Table 5.5 reports the cumulative performance from January 2007 to December 2019 (156 months) of TP-Rest, TB-MI and market index (MI) strategies in annualized return, risk and Sharpe measure. Panel A reports the results for the U.S. market while Panel B reports the results for the European market.

Panel A: U.S. market			
	TP-Rest	TB-MI	MI
Annualized return	6.7%	8.6%	8.8%
Annualized risk	16.6%	22.5%	14.6%
Sharpe measure	0.39	0.37	0.58
Panel B: European market			
	TP-Rest	TB-MI	MI
Annualized return	0.8%	4.1%	4.6%
Annualized risk	13.8%	27.9%	14.5%
Sharpe measure	0.03	0.13	0.29

Table 5.6: Number of stocks segregated by alpha estimation error

Table 5.6 shows the number of stocks segregated by alpha estimation error (EE) intervals for positive and negative alphas in the U.S. and European markets. Alpha EE is given by the difference between stocks realized excess return over the market index and alpha estimation. The sample period runs from January 2007 to December 2019.

	U.S. market		European market	
	Positive alphas	Negative alphas	Positive alphas	Negative alphas
<= -30%	4	19	15	25
]-30%; -20%]	21	21	30	24
]-20%; -15%]	34	31	34	32
]-15%; -10%]	74	40	83	48
]-10%; -8%]	62	31	51	31
]-8%; -6%]	69	30	65	38
]-6%; -4%]	72	59	72	43
]-4%; -2%]	76	54	74	53
]-2%; 0%[	74	53	80	47
Number of stocks with negative EE	486	338	504	341
]0%; 2%]	79	59	70	51
]2%; 4%]	56	57	57	53
]4%; 6%]	48	58	40	43
]6%; 8%]	35	27	29	33
]8%; 10%]	18	30	28	44
]10%; 15%]	35	62	36	72
]15%; 20%]	15	49	6	49
]20%; 30%]	5	41	8	40
> 30%	3	43	2	54
Number of stocks with positive EE	294	426	276	439
Total number of stocks with EE	780	764	780	780

Table 5.7: Average alpha estimation error

Table 5.7 shows the average alpha estimation error (EE) for positive and negative alphas in the U.S. and European markets. The sample period runs from January 2007 to December 2019.

	U.S. market		European market	
	Positive alphas	Negative alphas	Positive alphas	Negative alphas
Average EE of stocks with negative EE	-7.9%	-9.9%	-8.8%	-11.0%
Average EE of stocks with positive EE	6.0%	14.7%	6.4%	14.9%
Average EE of all stocks	-2.6%	3.8%	-3.4%	3.6%

Table 5.8: Average weight of active portfolio of TB-MI strategy

Table 5.8 reports the average weight of active portfolio (AP) of TB-MI strategy, in the period between January 2007 and December 2019, when the AP return is larger or smaller than the passive portfolio (PP) return in the U.S. market (Panel A) and in the European market (Panel B). The average weight of AP with no beta adjustment respects to the average weight of the AP calculated accordingly to the methodology described in subsection 5.2.1. The beta adjustments respect to an increase in the beta when the AP return is higher than the PP return, and a decrease when the AP return is lower than the PP return.

Panel A: U.S. market		
	AP return > PP return	AP return < PP return
Average weight of AP (no beta adjustment)	16.3%	19.2%
Average weight of AP (beta adjustment of 0.25)	17.3%	18.1%
Average weight of AP (beta adjustment of 0.5)	18.4%	17.1%
Panel B: European market		
	AP return > PP return	AP return < PP return
Average weight of AP (no beta adjustment)	27.1%	25.6%
Average weight of AP (beta adjustment of 0.25)	29.9%	23.5%
Average weight of AP (beta adjustment of 0.5)	33.4%	21.7%

Table 5.9: Cumulative return of TB-MI strategy

Table 5.9 reports the cumulative return of TB model when the market index is the market portfolio (TB-MI strategy) in the U.S. and European markets. The beta adjustments respect to an increase in beta when the active portfolio (AP) return is higher than the passive portfolio (PP) returns, and a decrease in beta when the AP return is lower than the PP return. The sample period runs from January 2007 to December 2019.

	U.S. market	European market
Cumulative return (no beta adjustment)	192.9%	67.9%
Cumulative return (beta adjustment of 0.25)	281.1%	188.6%
Cumulative return (beta adjustment of 0.5)	397.8%	409.3%

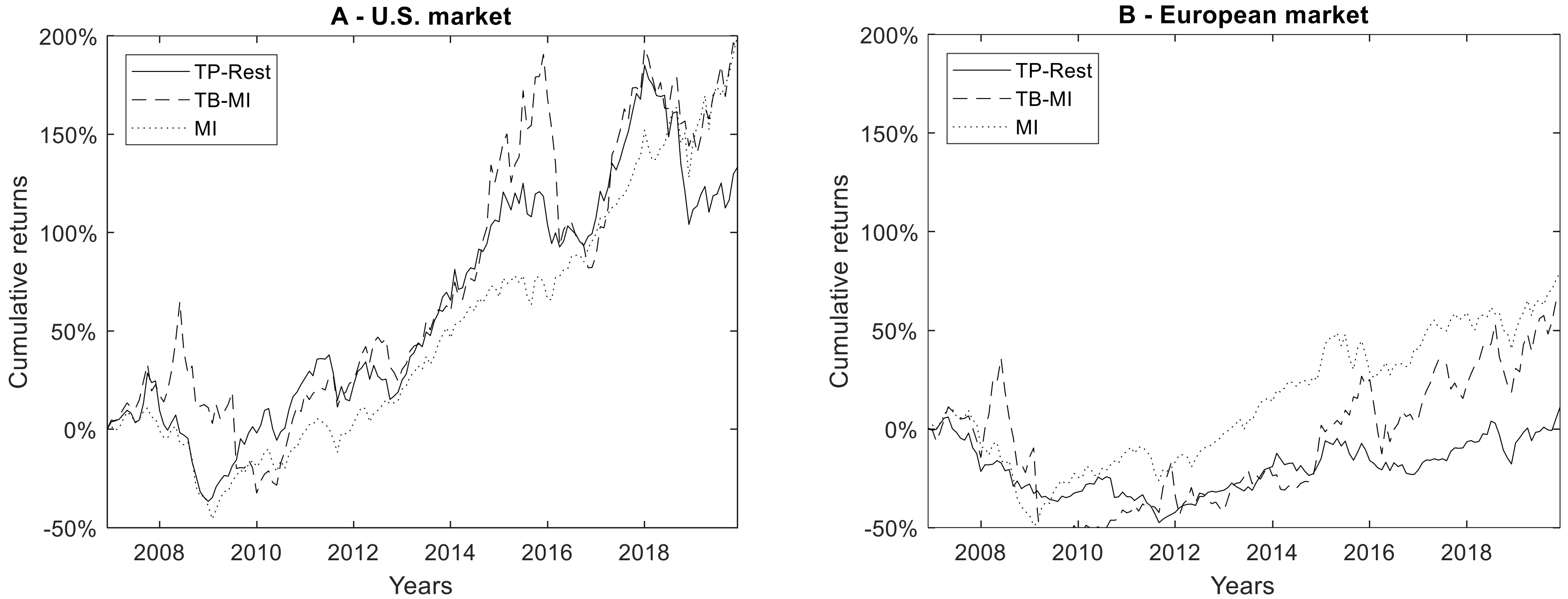
Table 5.10: Weights of the active portfolio of TB-MI strategy in July 2008 for the U.S. market and in February 2008 for the European market

Table 5.10 reports estimates of alpha, residuals variance, alpha divided by the residuals variance and the weight of each stock in the active portfolio (AP) of TB-MI strategy in July 2008 for the U.S. market (Panel A) and in February 2008 for the European market (Panel B). Stocks that compose the AP are the five stocks with the highest alpha and the five stocks with the lowest alpha.

Panel A: U.S. market				
	Alpha	Residuals variance	Alpha / Res. Var.	Weight in AP
Stock #1	-3.62%	0.60%	-6.05	-1360%
Stock #2	4.19%	1.94%	2.16	486%
Stock #3	-5.34%	1.43%	-3.73	-838%
Stock #4	3.57%	0.36%	9.81	2204%
Stock #5	-5.12%	1.67%	-3.07	-689%
Stock #6	4.10%	0.84%	4.91	1103%
Stock #7	4.78%	1.21%	3.97	892%
Stock #8	3.62%	1.98%	1.83	411%
Stock #9	-4.34%	1.84%	-2.36	-530%
Stock #10	-3.51%	0.50%	-7.03	-1579%
Total			0.44	100%
Panel B: European market				
	Alpha	Residuals variance	Alpha / Res. Var.	Weight in AP
Stock #1	3.44%	1.57%	2.20	288%
Stock #2	-3.62%	1.35%	-2.68	-350%
Stock #3	3.08%	0.80%	3.85	504%
Stock #4	-4.97%	3.79%	-1.31	-172%
Stock #5	-4.05%	0.69%	-5.86	-767%
Stock #6	-3.95%	1.23%	-3.21	-420%
Stock #7	3.00%	0.46%	6.45	845%
Stock #8	5.39%	2.03%	2.65	347%
Stock #9	3.25%	0.66%	4.88	639%
Stock #10	-3.09%	0.50%	-6.21	-814%
Total			0.76	100%

Figure 5.1: Cumulative return of TP-Rest strategy, TB-MI strategy and market index

Figure 5.1 shows the cumulative return between January 2007 and December 2019 of Treynor-Black (1973) model considering the tangency portfolio with short selling restriction as market portfolio (TP-Rest), Treynor-Black (1973) model considering the market index as market portfolio (TB-MI) and market index (MI). Panel A reports the results for the U.S. market while Panel B reports the results for the European market.



Chapter 6. Conclusions

Using adjusted daily prices of all stocks traded on Standard & Poor's 500 (S&P 500) and STOXX Europe 600 (STOXX 600), and their respective market indexes, during the period between January 2002 and December 2019, we examine three topics related with portfolio management. First, the minimum number of stocks that a portfolio should have to achieve the major benefits of diversification in terms of risk and return. We investigate how risk and return of a portfolio change as the number of stocks in equal-weighted and value-weighted portfolios increases. Second, the relationship between next month return and risk variables (standard deviation, skewness, and kurtosis). We analyze the mentioned relationship by forming value-weighted and equal-weighted quintile portfolios sorted by risk variables. Third, the implementation of three investment strategies based on Treynor-Black (1973) model: tangency portfolio, tangency portfolio with short selling restriction and market index as market portfolios. We investigate how average risk and average return of the referred investment strategies compare with a passive strategy. All relevant parameters are obtained through an ordinary least squares regression using a rolling window approach with an estimation period of 60 months.

We find that major benefits of diversification, in the U.S. and European markets, can be achieved with an equal-weighted portfolio with 50 stocks and a value-weighted portfolio with 64 stocks. These portfolios reduce, at least, 95% of diversifiable risk. Moreover, we observe that the increase of the number of stocks in equal-weighted portfolios has no significant impact on average end-of-period wealth, while the mentioned increase has a slight negative effect in value-weighted portfolios. Additionally, end-of-period wealth standard deviation decreases as the number of stocks in a portfolio increases in both markets and in both weighting schemes. Equal-weighted portfolios with 50 stocks and value-weighted portfolios with 64 stocks have an end-of-period wealth standard deviation lower than 0.05 per $1 or 1€ of investment in the U.S. and European markets, respectively.

Regarding the relationship between risk variables and next month return, generally, we see no clear increasing or decreasing monotonic relations. These relations are only present in the U.S. market. Moreover, we observe some statistically significant differences, at a 5% level, between the average return of the quintile portfolios formed stocks with lowest values and highest values of risk variables. These cases are important if a self-financing portfolio consisting in buy or sell a quintile portfolio of stocks with the lowest risk variables values and sell or buy a quintile portfolio of stocks with the highest risk variables values is implemented.

However, the differences in average return of extreme quintile portfolios are rarely statistically significant in both weighting schemes or in both markets, simultaneously. In addition, with respect to the difference of average return between extreme quintile portfolios and a portfolio composed by all the stocks in the market index, we observe that, generally, this difference yields lower average return than the self-financing portfolio average return. Given the lack of similarity between the results of the U.S. and European markets for the same risk variables, it appears that relations between risk variables and next month return are originated randomly rather than by economic significance. The results for both markets and both weighting schemes show that, at least one negative relation and one positive relation, can be found for standard deviation, skewness, and kurtosis estimates.

With respect to Treynor-Black (1973) model strategies, in both market, we observe higher average return, higher average risk and higher Sharpe measure when the market index is the market portfolio. When we compare the Treynor-Black (1973) model, using the market index as market portfolio, with the passive portfolio (the market index), we conclude that market index had higher average Sharpe measure in both markets. We point out three reasons that lead Treynor-Black (1973) model to have not a consistently superior risk-adjusted average return over the passive portfolio. Alpha estimation error (long positions with lower realized return than expectations and short positions with higher realized return than expectations), small weight of investment on the active portfolio when the active portfolio return is larger than the passive portfolio return, and large when the inverse occurs, and extreme weights of the active portfolio that lead to high levels of risk.

References

Allen, D., Lizieri, C., & Satchell, S. (2019). In Defense of Portfolio Optimization: What If We Can Forecast? *Financial Analysts Journal*, *75*(3), 20-38.

Ang, A., Hodrick, R., Xing, Y., & Zhang, X. (2006). The cross-section of volatility and expected returns, *The Journal of Finance, 61*(1), 259–299.

Ang, A., Hodrick, R., Xing, Y., & Zhang, X. (2009). High idiosyncratic volatility and low returns: International and further U.S. evidence, *Journal of Financial Economics, 91*(1), 1-23.

Ayadi, M., Cao, X., Lazrak, S., & Wang, Y. (2019). Do idiosyncratic skewness and kurtosis really matter? *The North American Journal of Economics and Finance*, *50*, 101008.

Bali, T., & Cakici, N. (2008). Idiosyncratic Volatility and the Cross Section of Expected Returns, *Journal of Financial and Quantitative Analysis, 43*(1), 29-58.

Bali, T., Hu, J., & Murray, S. (2019). Option Implied Volatility, Skewness, and Kurtosis and the Cross-Section of Expected Stock Returns, *Research Collection Lee Kong Chian School Of Business*, 1-67.

Beck, K., Perfect, S., & Peterson, P. (1996). The role of alternative methodology on the relation between portfolio size and diversification, *The Financial Review, 31*(2), 381–406.

Benjelloun, H. (2010). Evans and Archer – forty years later, *Investment Management and Financial Innovations, 7*(1), 98-104.

Bernstein, P. (2004). The Great Alpha Tree, *The Journal of Portfolio Management*, *30*(4), 1.

Bessler, W., Opfer, H., & Wolff, D. (2017). Multi-asset portfolio optimization and out-of-sample performance: an evaluation of Black–Litterman, mean-variance, and naïve diversification approaches, *The European Journal of Finance, 23*(1), 1-30.

Bird, R., & Tippett, M. (1986). Naive diversification and portfolio risk: A note, *Management Science, 32*(2), 244-251.

Bloomfield, T., Leftwich, R., & Long, J. (1977). Portfolio strategies and performance, *Journal of Financial Economics, 5*(2), 201-218.

Bollerslev, T., Chou, R., & Kroner, K. (1992). ARCH modeling in finance: A review of the theory and empirical evidence, *Journal of Econometrics, 52*(1-2), 5-59.

Boyer, B., Mitton, T., & Vorkink, K. (2010). Expected Idiosyncratic Skewness, *The Review of Financial Studies*, *23*(1), 169–202.

Bradfield, D., & Munro, B. (2017). The number of stocks required for effective portfolio diversification: the South African case, *South African Journal of Accounting Research*, *31*(1), 44-59.

Breitenfellner, B., & Wagner, N. (2012). Explaining aggregate credit default swap spreads, *International Review of Financial Analysis*, *22*(C), 18-29.

Campbell, J. (1987). Stock returns and the term structure, *Journal of Financial Economics*, *18*(2), 373-399.

Chang, B., Christoffersen, P., & Jacobs, K. (2013). Market skewness risk and the cross section of stock returns, *Journal of Financial Economics*, *107*(1), 46-68.

Chopra, K., & Ziemba, W. (1993). The effect of errors in means, variances, and covariances on optimal portfolio choice, *The Journal of Portfolio Management*, *19*(2), 6–11.

Clarke, R., Silva, H., & Thorley, S. (2006). Minimum-Variance Portfolios in the U.S. Equity Market, *Journal of Portfolio Management*, *33*(1), 10–24.

Conrad, J., Dittmar, R., & Ghysels, E. (2013). Ex ante skewness and expected stock returns, *Journal of Finance, 68*(1), 85–124.

Copp, D., & Cleary, S. (1999). Diversification with Canadian stocks: How much is enough? *Canadian Investment Review, 12*(3), 21–25.

DeMiguel, V., & Nogales, F. (2009). Portfolio Selection with Robust Estimation, *Operations Research*, *57*(3), 560-577.
Domian, D., Louton, D., & Racine, M. (2003). Portfolio diversification for long holding periods: How many stocks do investors need? *Studies in Economics and Finance*, *21*(2), 40–64.
Domian, D., Louton, D., & Racine, M. (2007). Diversification in Portfolios of Individual Stocks: 100 Stocks Are Not Enough, *The Financial Review*, *42*(4), 557-570.
Elton, E., & Gruber, M. (1977). Risk reduction and portfolio size: An analytical solution, *Journal of Business*, *50*(4), 415–437.
Elyasiani, E., Gambarelli, L., & Muzzioli, S. (2020). Moment risk premia and the cross-section of stock returns in the European stock market, *Journal of Banking & Finance*, *111*, 105732.
Engle, R., & Mustafa, C. (1992). Implied ARCH Models from Options Prices, *Journal of Econometrics*, *52*(1-2), 289-311.
Engle, R., & Ng, V. (1993). Measuring and Testing the Impact of News on Volatility, *The Journal of Finance*, *48*(5), 1749-1778.
Evans, J., & Archer, S. (1968). Diversification and the reduction of dispersion: An empirical analysis, *Journal of Finance*, *23*(5), 761–767.
Fabozzi, F., Cheng, X., & Chen, R. (2007). Exploring the components of credit risk in credit default swaps, *Finance Research Letters*, *4*, 10-18.
Fama, E., & French, K. (1993). Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics*, *33*(1), 3-56.
Fang, H., & Lai, T. (1997). Co-kurtosis and capital asset pricing, *Financial Review*, *32*(2), 293-307.
Fielitz, B. (1974). Indirect versus direct diversification, *Financial Management*, *3*(4), 54–62.
Fisher, L., & Lorie, J. (1970). Some studies of variability of returns on investments in common stocks, *Journal of Business*, *43*(2), 99-134.
Frankfurter, G., Phillips, H., & Seagle, J. (1971). Portfolio Selection: The Effects of Uncertain Means, Variances, and Covariances, *Journal of Financial and Quantitative Analysis*, *6*(5), 1251-1262.
Frankfurter, G. (1976). The Effect of "Market Indexes" on the Ex-Post Performance of the Sharpe Portfolio Selection Model, *The Journal of Finance*, *31*(3), 949-955.
Fu, F. (2009). Idiosyncratic Risk and the Cross-Section of Expected Stock Returns, *Journal of Financial Economics*, *91*(1), 24-37.
Gilbert, E., & Strugnell, D. (2010). Does survivorship bias really matter? An empirical investigation into its effects on the mean reversion of share returns on the JSE (1984–2007), *Investment Analysts Journal*, *39*(72), 31-42.
Glosten, R., Jagannathan, R., & Runkle, D. (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance*, *48*(5), 1779-1801.
Goyal, A., & Santa-Clara, P. (2003). Idiosyncratic risk matters! *The Journal of Finance*, *58*(3), 975–1007.
Guo, H., & Savickas, R. (2006). Idiosyncratic volatility, stock market volatility, and expected stock returns, *Journal of Business & Economic Statistics*, *24*(1), 43–56.
Haensly, P. (2020). Risk decomposition, estimation error, and naïve diversification, *North American Journal of Economics and Finance*, *52*, 101146.
Harvey, C., & Siddique, A. (2000). Conditional Skewness in Asset Pricing Tests, *The Journal of Finance*, *55*(3), 1263-1295.
He, Z. (2007). Incorporating Alpha Uncertainty into Portfolio Decisions: A Bayesian Revisit of the Treynor-Black Model, *Journal of Asset Management*, *8*(3), 161-175.

Huang, W., Liu, Q., Rhee, S., & Zhang, L. (2010). Return reversals, idiosyncratic risk, and expected returns, *The Review of Financial Studies*, *23*(1), 147–168.

Jennings, E. (1971). An empirical analysis of some aspects of common stock diversification, *Journal of Financial and Quantitative Analysis*, *6*(2), 797–813.

Jiang, X., & Lee, B. (2006). The dynamic relation between returns and idiosyncratic volatility, *Financial Management, 35*(2), 43–65.

Johnson, K., & Shannon, D. (1974). A note on diversification and the reduction of dispersion, *Journal of Financial Economics*, *1*(4), 365–372.

Kane, A., Kim, T., & White, H. (2010). Forecast Precision and Portfolio Performance, *Journal of Financial Econometrics, 8*(3), 265–304.

Kirby, C., & Ostdiek, B. (2012). It's All in the Timing: Simple Active Portfolio Strategies that Outperform Naïve Diversification, *The Journal of Financial and Quantitative Analysis*, *47*(2), 437-467.

Klemkosky, R., & Martin, J. (1975). The effect of market risk on portfolio diversification, *The Journal of Finance, 30*(1), 147–154.

Kourtis, A., Dotsis, G., & Markellos, R. (2012). Parameter Uncertainty in Portfolio Selection: Shrinking the Inverse Covariance Matrix, *Journal of Banking and Finance*, *36*(9), 2522–2531.

Kraus, A., & Litzenberger, R. (1976). Skewness preference and the valuation of risk assets, *The Journal of Finance*, *31*(4), 1085-1100.

Kryzanowski, L., & Singh, S. (2010). Should minimum portfolio sizes be prescribed for achieving sufficiently well-diversified equity portfolios? *Frontiers in Finance and Economics*, *7*(2), 1–37.

Larsen, G., & Resnick, B. (2001). Parameter Estimation Techniques, Optimization Frequency, and Portfolio Return Enhancement, *The Journal of Portfolio Management*, *27*(4), 27–34.

Latane, H., & Young, E. (1969). Test of portfolio building rules, *The Journal of Finance*, *24*(4), 595–612.

Levy, R. (1968). Measurement of Investment Performance, *The Journal of Financial and Quantitative Analysis*, *3*(1), 35-57.

Lim, D., Durand, R., & Yang, J. (2014). The microstructure of fear, the Fama–French factors and the global financial crisis of 2007 and 2008, *Global Finance Journal*, *25*(3), 169-180.

Lintner, J. (1965). The Valuation of Risky Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets, *Review of Economics and Statistics*, *47*(1), 13–37.

MacKinlay, A., & Pastor, L. (2000). Asset pricing models: implications for expected returns and portfolio selection, *Review of Financial Studies*, *13*(4), 883-916.

Malkiel, B., & Xu, Y. (2002). Idiosyncratic Risk and Security Returns, Working Paper, University of Texas at Dallas.

Markowitz, H. (1952). Portfolio Selection, *The Journal of Finance*, *7*(1), 77-91.

Massey, F. (1951). The Kolmogorov-Smirnov Test for Goodness of Fit, *Journal of the American Statistical Association*, *46*(253), 68-78.

Merton, R. (1980). On estimating the expected return on the market: An exploratory investigation, *Journal of Financial Economics*, *8*(4), 323–361.

Mossin, J. (1966). Equilibrium in a Capital Asset Market, *Econometrica*, *34*(4), 768-783.

Newbould, G., & Poon, P. (1996). Portfolio risk, portfolio performance, and the individual investor, *Journal of Investing*, *5*(2), 72–78.

Nelson, D. (1991). Conditional heteroscedasticity in asset returns: A new approach, *Econometrica*, *59*(2), 347-370.

Qadan, M. (2019). Risk appetite, idiosyncratic volatility and expected returns, *International Review of Financial Analysis*, *65*, 101372.

Ramírez-Hassan A., & Guerra-Urzola, R. (2020). Optimal portfolio choice: a minimum expected loss approach, *Mathematics and Financial Economics*, *14*(1), 97-120.
S&P Dow Jones Indices LLC (2021, January 31), *Factsheet of S&P 500*, Retrieved from https://www.spglobal.com/
Sharpe, W. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk, *The Journal of Finance*, *19*(3), 425–442.
Solnik, B. (1974). Why not diversify internationally rather than domestically? *Financial Analysts Journal*, *30*(4), 48-54.
Statman, M. (1987). How many stocks make a diversified portfolio? *Journal of Financial and Quantitative Analysis*, *22*(3), 353–364.
Statman, M. (2004). The diversification puzzle, *Financial Analysts Journal*, *60*(4), 44–53.
STOXX Limited (2021, January 31), *Factsheet of STOXX Europe 600*, Retrieved from https://www.stoxx.com/
Tang, G. (2004). How efficient is naive portfolio diversification? An educational note, *Omega Elsevier*, *32*(2), 155–160.
Tole, T. (1982). You can't diversify without diversifying, *The Journal of Portfolio Management*, *8*(2), 5–11.
Treynor, J., & Black, F. (1973). How to Use Security Analysis to Improve Portfolio Selection, *The Journal of Business*, *46*(1), 66-86.
Wagner, W., & Lau, S. (1971). The effect of diversification on risk, *Financial Analysts Journal*, *27*(6), 48–53.