INSTITUTO
UNIVERSITÁRIO
DE LISBOA

Barrier Options and Dynamic Debt

João Miguel Mendes dos Reis

PhD in Finance

Supervisor:
Doutor José Carlos Gonçalves Dias, Professor Associado com Agregação,
ISCTE-IUL

December 2021

Department of Finance

Barrier Options and Dynamic Debt

João Miguel Mendes dos Reis

PhD in Finance

Jury:
Doutora Clementina Barroso, Professora Auxiliar
ISCTE-IUL
Doutor João Pedro Pereira, Professor Associado com Agregação
Nova SBE - Universidade Nova de Lisboa
Doutora Raquel Medeiros Gaspar, Professora Associada com Agregação
ISEG - Lisbon School of Economics and Managment
Doutor João Pedro Vidal Nunes, Professor Catedrático,
ISCTE-IUL
Doutor José Carlos Gonçalves Dias, Professor Associado com Agregação,
ISCTE-IUL

December 2021

Resumo

Esta tese foca-se na modelização de risco de crédito num modelo dinâmico, enfatisando o uso de opções barreira.

Começa por obter o preço de opções com multiplas barreiras sob o modelo JDCEV (*jump to default constant elasticity of variance*). As opções com multiplas barreiras partem das opções com uma barreira. No caso de uma barreira, o seu detentor tem uma opção *vanilla* contingente do preço do ativo subjacente atravessar ou não a dada barreira. No caso de barreiras multiplas, apresentado aqui tal como em Jun and Ku (2012), uma opção de uma barreira é ativada assim que um conjunto de barreiras é atravessado. A solução dessas opções sob o modelo JDCEV assenta na *stopping time approach* (ST) para opções barreira desenvolvida por Dias et al. (2015).

Depois, na segunda parte, o *payoff* das opções put com multiplas barreiras é utilizado para extender o modelo de dívida dinâmica de Das and Kim (2015) para o modelo CEV (*contant elasticity of variance*). O modelo de dívida dinâmica permite a uma firma aumentar ou diminuir o valor nominal da dívida numa dada sequência, desde que determinadas barreiras sejam atravessadas pelo valor total da firma. Esta caracteristica permite o uso de opções barreira para obter o valor da dívida e dos respetivos *spreads*.

Na terceira parte, um outro modelo de dívida dinâmica onde o valor da dívida pode ser alterado é explorado. Neste, o valor da dívida pode ser alterado, mas através de um *intensity process*. Este processo pode ser ligado ao valor da firma através da correlação de processos estocásticos. Através de mudanças de medida, as fórmulas fechadas para este modelo são obtidas no *geometric Brownian motion* (GBM) . Além de estender o modelo base de Merton (1974), também é feita uma extenção do modelo de dívida subordinada de Gorton and Santomero (1990).

Classificação JEL: G12, G32

Palavras chave: Modelo CEV, Modelo JDCEV, Opções barreira, Risco de crédito, Dívida dinâmica, *Spreads* de crédito, Modelo *intensity process*

Abstract

This thesis focuses on dynamic debt credit risk modeling, emphasizing the use of barrier options.

It starts by obtaining the price for multiple barrier options under the JDCEV (jump to default constant elasticity of variance) model. The multiple barrier options depart from the single barrier options. In the single barrier case, the owner has vanilla option contingent on the underlying asset price crossing or not the given barrier. In the multiple barrier options, presented here as in Jun and Ku (2012), a single barrier option is activated once a set of barriers is crossed. The solution of these options under the JDCEV model relies on the ST (stopping time) approach for barrier options developed by Dias et al. (2015).

Then, in the second part, the payoff of the multiple barrier put options is used to extend the dynamic debt model of Das and Kim (2015) to the CEV (constant elasticity of variance) model. The dynamic debt model allows a given firm to increase or decrease the face value of debt in a given sequence, provided that certain barriers are crossed by the total firm value. This feature allows the use of barrier options' formulae to solve the debt value and respective spreads.

In the third part, another dynamic debt model where the debt can change is explored. There, debt can change, but through an intensity process. This process can be linked to the firm value through correlation among the stochastic processes. Through measure changes, the closed formulae are obtained for the model under the geometric Brownian motion (GBM) setting. In addition to extending the baseline Merton (1974) model, there is also a an extension of the subordinated debt model from Gorton and Santomero (1990).

JEL Classification: G12, G32

Keywords: CEV model, JDCEV model, Barrier options, Credit risk, Dynamic debt, Credit spreads, Intensity based model

Contents

Acknowledgments

I would like to thank to ISCTE-IBS for the financial support during the time-span of the doctoral program and my advisor Professor José Carlos Dias, for the availability, feedback, support and suggestions provided during the development of this thesis.

Part I

Introduction

This PhD thesis focuses on three related parts, where various interlinked topics are covered. The starting point is the dynamic debt model of Das and Kim (2015) where multiple barrier options are used to build a dynamic debt structural risk model under a geometric Brownian motion (GBM) setting.

Structural risk models were introduced in Black and Scholes (1973) and Merton (1974) and are based on modeling the stochastic evolution of the balance sheet of the debtor firm. They depart from the basic idea that at the maturity date of debt, if the debtor firm value is less than the face value of the debt, default occurs. This allows the debt discount to be obtained with the same formula as a put option contract. Various extensions exist on the model, such as Geske (1977) on the valuation of coupon bonds as a group of compound options.

Black and Cox (1976) introduce models where default can occur once the firm's assets touch a low default boundary, Leland (1994) also uses a first passage debt pricing approach to value debt while showing that the boundary at which default is triggered can be chosen by the firm to maximize the value of the equity, while Zhou (2001) extends the possibility of the asset value suffering log-normally distributed jumps. A test on the performance of these models is done in Delianedis and Geske (1998) who compute risk neutral probabilities of default using the models of Merton (1974) and Geske (1977) and show that the models are able to predict not only the default events by the firms, but also the migrations in the ratings with months of advance.

The studies comparing the models to real world data go back to Jones et al. (1984) who use a dataset with both investment grade and noninvestment grade bonds and implement a rolling estimation approach. Notably, Eom et al. (2004) follow this approach to compare the corporate spreads for five different structural models empirically with cross-section data from 1986-1997 and conclude that the accuracy of structural models is an issue with the Merton model, as it produces spreads that are too low, while other models produce spreads that are generally too high. Other examples of this approach can be found in Hull et al. (2005) who propose an estimation method for the model's parameters from the implied volatilities of options on the company's equity and Arora et al. (2005) who contrast reduced-form and structural models.

As for different methods, Huang and Huang (2012) use a calibration with representative firms and show that many different structural models generate similar spreads once calibrated to the same default probabilities, recovery rates, and the equity premium. With a similar approach, the link between the Credit Spread Puzzle and the Equity Premium Puzzle is explored in Chen et al. (2008) to match historical spread levels and Schaefer and Strebulaev (2008) show that poor performances of credit risk models can be attributed to not capturing the interest rate sensitivity of corporate debt. For studies of the credit spreads with regression models, there are for instance Collin-Dufresne et al. (2001) with a study which points out that credit spread changes are principally explained by supply and demand shocks independent of credit-risk factors while Zhang et al. (2009) study credit default premium and calibrate a Merton-type model.

In more recent empirical studies, Huang et al. (2019) use a generalized method of moments (GMM) of Hansen (1982) which allows to estimate the parameters while allowing to precisely determine whether the model is rejected or not in the data, reaching the conclusions that the structural models under-perform capturing the dynamic behavior of both equity volatility and credit default swaps (CDS) spreads while performing well to explain the sensitivity of CDS spreads to equity returns. Furthermore, Du et al. (2019) first consider a model with priced stochastic asset risk which is afterwards estimated on firm-level data and Shi (2019) first develops a general equilibrium model where an ambiguity-averse agent applies a discount rate and performs an empirical verification afterwards.

In the market practice, these Merton style models are the base of the well known commercial Expected Default Frequency (EDF) model to forecast default probabilities which is owned by KMV Corporation, later acquired by Moody's. The KMV model instead of using the Merton (1974) framework estimates the probability frequency by a proprietary procedure, built over real world data which includes various years. As for the variables used, KMV uses the current book value of debt, the face value of near-term debt plus a fraction of long-term debt, and obtains the probability of default within a year. Volatility is also used, through an estimate based on the equity volatility. As in the usual structural models, the estimated level of assets is the sum of the debt value and the value of equity in the market.

Das and Kim (2015) depart from one of the Merton (1974) assumptions - the constant level of nominal debt. Firms holding debt tend to actively manage its debt structure and levels, and the face total debt — the D variable in the models presented here, which takes the place of the strike when using the option formulae — is often changed, and this can result in changes to the market value of debt and the corresponding yield.

The approach tries to emulate the idea that if the firm value increases to a certain point in the total value, the firm is able to use the new collateral to support more debt. Therefore, debt is increased and there is an impact over the initial debt value, and hence on its spreads. The counterpart is a swap-down that is triggered when there is a substantial drop in firm value. As the firm tries to avoid default, the lenders can swap debt principal for equity, reducing the total nominal amount of debt, making the default less likely and reducing the debt credit spreads. The assumption of the changes in debt being reflected by the opposite move by the amount of equity has the analytical advantage of preserving the total firm value.

There are several studies that point out to links between the firm value and the evolution of the debt structure. For instance, Roberts and Sufi (2009) find that a very big share of long-term debt contracts suffer renegotiations over terms such as the amount, maturity, and pricing of the loan, Nini et al. (2012) point out an active role played in the governance of the firms by creditors, even when default is not a close scenario and Flannery et al. (2012) observe that the contemporaneous leverage alone cannot explain the bond yields alone while the expected future leverage plays a part.

Das and Kim (2015) obtain closed-form solutions for the ex-ante value of the debt discount, debt values and corresponding credit spread term structure, with the sequence of debt changes being explicitly set beforehand alongside with the firm values at which they occur.

This is done through an extensive use of barrier options, a class of exotic derivatives that are activated or de-activated upon the firm value accessing barrier levels, coming in the form of knock-in and knock-out options. For instance, a knock-out option is a plain-vanilla option with the additional clause that it becomes worthless — is knocked-out — if the underlying price touches a pre-specified barrier. These are named down-and-out options in the case where the barrier is lower than the initial underlying asset, and up-and-out if it is higher. On the other side, there are also knock-in options, where the clause specifies that the option remains worthless as long as the underlying asset does not touch the specified barrier. Once the barrier is crossed, its holder owns a vanilla option. In addition, it is possible to specify a rebate, a previously agreed (small) amount that is paid when the option is knocked-out, either at the event, or at another specified date such as the maturity of the option. In the case of the knock-in option, there can be a rebate, although only at maturity. Under the most basic framework, the sum of a knock-in and a knock-out option with the same barrier level corresponds to the vanilla option with the same parameters. These basic types of barrier option formulae are studied in Rubinstein and Reiner (1991) and Rich (1994) and summarized in Haug (2006), under a GBM modeling setup.

The barrier options used in Das and Kim (2015) go beyond the basic kind explained above. The model also uses "one touch double barrier binary options of the knock-out kind", which are deactivated once one of two barriers is reached and are studied for instance in Hui (1996) and "first-then barrier options", which have two barriers where a single barrier option (knock-in or knock-out) is obtained once a preceding barrier is crossed and are presented in Jun and Ku (2012).

With a weighted sum of these options, Das and Kim (2015) reach the value for various possible combinations of debt increases and decreases. For instance, the combination of an up-and-out option and an up-and-in option builds a scenario where there is an increase in debt. An up-and-out option with a first-up-in-then-down-and-out option and a first-up-then-down-and-in option build a scenario where there is first a debt increase followed by a debt decrease. The use of double barrier knock-out options allows the combination of two possible paths, where the order of crossing the barriers is crucial for the problem.

A total of six cases are studied alongside the Merton (1974) base model, leading to changes of not only in the magnitude of the credit spread curve but also in its shape, matching behaviors often observed in empirical studies. Thus, the study provides a valuable insight by indicating that the anticipation of the possibility of changes in the debt structure impacts the value of bonds and respective spreads.

This is done in the GBM setting and the aim in the second part of this thesis is to expand it to the constant elasticity of variance (CEV) model of Cox (1975). The limitations of the GBM are widely known and mostly centered in its log-normal distribution. This is reflected in two main effects. First, there is the *leverage effect*, reflected in the negative correlation between stock returns and realized volatility. Second, there is the *implied volatility skew*, which occurs in an option context and results in a negative correlation between the implied volatility and the strike price of the option.

The presence of the *leverage effect* can be traced back to the classical reasoning of Modigliani and Miller (1958), where the fundamental asset of a corporation is the firm in its totality, where the composition in terms of equity and debt are just different ways of splitting up the asset. This perspective is discussed by Black (1976) who argues that the volatility of the firm components, the equity and the debt, comes from the fluctuations in the total firm value. The different compositions of the firm differ in the claims they have to the value of the firm, with most variations being transmitted to the value of equity, except when the firm is close to default, where the debt is more affected. This asymmetry is revealed when a negative return causes the equity to decrease while the nominal value of debt remains fixed, which raises the firm leverage, thereby increasing the future equity volatility.

This theoretical argument finds empirical support in, for instance, Christie (1982), who concludes that the variance on a underlying asset has a strong positive relation with the financial leverage. This is further confirmed with studies such as Bekaert and Wu (2000), who study the phenomenon for the case of the Japanese market, Schwert (1989) analyses variations not only in the equity volatility over time but also its relation to other variables, Bollerslev et al. (1992) review the literature on using econometric models of the ARCH — autoregressive conditional heteroskedasticity — family to time-varying volatilities for financial variables, confirming the former empirical conclusions. Nelson (1991) presents the EGARCH — exponential GARCH — model which is able to accommodate the asymmetric behavior of volatility within a GARCH-family framework.

As for the *implied volatility skew*, when observing the market price of options under the GBM model, it is common to find different implied volatility levels for different prices. A "volatility smile skew" curve emerges, as market investors tend to be willing to overpay for out-of-the-money and in-the-money options when compared to at-the-money-options, with the overpricing tending to be stronger for the out-of-the-money cases. This implies a deficiency in the constant volatility and log-normal distribution of the underlying asset return assumptions in the GMB based model. This is linked to the previously mentioned *leverage effect* and the higher volatility at lower asset levels, which produce kurtosis in the asset return distributions. Several studies exist on this phenomenon. Rubinstein (1994) shows that when the miss-specifications for the BSM framework are corrected with more precise models, the volatility smile turns into a horizontal line. Jackwerth and Rubinstein (1996) study the case of options on stock prices, finding that smiles have a largely skewed form in the cases where the volatility decreases for increasing strike values. Buraschi and Jackwerth (2001) show that the deterministic volatility assumed by the GBM does not take into account all the dynamics that determine option prices while models which include stochastic volatility are more accurate. Dennis and Mayhew (2002) provide an extensive study on the factors that impact skewness, finding in particular it to be more intense in periods of high market volatility. In addition, it is shown by Dennis et al. (2004) that changes in the implied volatility of individual stocks are negatively related to the stock returns.

While most literature focuses on the equity, as it serves as the underlying asset of the option contracts, the presence of higher volatilities when the price of assets is negatively impacted can be extended to debt. Through the Modigliani and Miller (1958) argument on the composition of the total firm value, the firm value is set as the underlying asset in the debt literature. Empirical studies on its presence in the specific context of credit risk can be found, for instance, in Cremers et al. (2008) who find a link between implied volatilities and credit spreads and Hilscher (2007) that observes that the future volatility is predicted by corporate bond yields.

Several attempts exist to tackle this. The Heston (1993) aims to capture the increased volatility at lower underlying asset levels through a mean reverting process for the volatility, whose stochastic process can be correlated to the underlying asset value, thereby allowing the volatility to increase by setting negative values for the correlation. The model presented in Merton (1976) emulates the "fat tail" dynamics by introducing a jump-diffusion model that complements the GBM diffusion with a jump component which follows a log-normal distribution driven by a Poisson process, allowing sudden changes in the underlying asset price.

The alternative focused here is the CEV model. Its central feature is the control of the relationship between the underlying asset price and its volatility, such that with the choice of the β parameter, the volatility increases or decreases as the price changes. In the cases where $\beta < 2$, the volatility increases as price falls, if $\beta > 2$, the volatility increases when prices increase while $\beta = 2$ nests the original GBM diffusion.

There is a stream in the literature that obtains solutions to option pricing under the CEV diffusion expressed in terms of the complementary gamma distribution and the complementary chi-square distribution. Closed-form solutions in this form for vanilla options for $\beta < 2$ were found by Cox (1975), by Emanuel and MacBeth (1982) for $\beta > 2$ and Schroder (1989) also presents solutions. An alternative form approach that relies on the eigenfunction expansion is presented by Davydov and Linetsky (2003). Beyond the vanilla options, the CEV model is also explored to obtain solutions for lookback options by Davydov and Linetsky (2001) through the use of the Laplace transform and by Linetsky (2004) using the spectral expansion approach, in Andersen and Andreasen (2000) there are extensions for interest rate markets which present high volatility skews in option prices and there is also the derivation of closed-form expressions for interest rate cap and swaption prices, and in Nunes (2009) the CEV model is used to price American-style options. As for the barrier options which are of the interest of this study, its solutions have been presented in Boyle and Tian (1999) through a trinomial lattice, Davydov and Linetsky (2001) do so by the Laplace transform, Davydov and Linetsky (2003) also obtain the value of the barrier options by the same eigenfunction expansion and also in Mijatović and Pistorius (2013) through the construction of an approximating continuous-time Markov chain. Dias and Nunes (2011) explore the CEV model in the field of real options through the payoff of a perpetual American-style option. In terms of applying the CEV model to credit risk, Campi et al. (2009) model the observable equity value using a CEV model with the possibility of default linked to the CEV parameter, and Chen (2015) explores the equity volatility in a structural model that assumes the CEV model for the diffusion of the firm value.

Most crucially, Dias et al. (2015) develop two novel methodologies for pricing and hedging European-style barrier option contracts under the CEV model, while allowing the possibility of a jump to default. The inclusion of this possibility is what sets up the jump to default extended CEV (JDCEV) of Carr and Linetsky (2006) which is presented with more detail in a later part of this document. Numerical methods play a crucial role to researchers and practitioners problems and Dias et al. (2015) generalize two numerical methods, the *stopping time approach* (ST) and the *static hedging portfolio approach*

(SHP). The stopping time approach is developed by Kuan and Webber (2003) to price single and double barrier European options through the recovery of the first passage time density of the underlying asset price to the barrier level(s) via numerically solving an integral equation. This technique of conditioning on the first time the spot level hits a barrier is often used for solving level crossing problems. In the probability literature, Park and Schuurmann (1976), Nobile et al. (1985), Giorno et al. (1989), Buonocore et al. (1990) and Gutiérrez et al. (1997) provide examples where conditioning is used to characterize the law of the first passage time in the class of one-dimensional diffusions through integral equations.

Dias et al. (2015) show that the ST can be extended to obtain exact pricing solutions under a more general stochastic process, not only allowing it to yield the JDCEV model but also providing more efficient pricing solutions. The second approach, the SHP, is a stream of literature that has been widely used to value European-style barrier options. One of the methods to do this is also called *variable-strike fixed-maturity static replication method* and is presented in Bowie and Carr (1994), Carr and Chou (1997) and Carr (1998). It prices barrier options through the hedging of static positions of European-style plain-vanilla options for a continuum of strikes while holding the same maturity date as the barrier option. The other method is the *fixed-strike variable-maturity static replication method* which uses a set of standard European-style options with a sequence of maturities and strikes equaling the known boundary until the maturity date of the barrier option. Here, the value of the static replication portfolio, composed by vanilla options with various maturities at n evenly spaced time points, matches the zero value of the barrier option where the underlying asset price equals the barrier, ensuring that the value of a knock-out option is zero whenever the knock-out event is triggered. It is presented by Derman et al. (1995) (thus named DEK method) and afterwards Chung et al. (2010) modify the method to hedge continuous up-and-out call options by constructing a portfolio of standard options and cash-or-nothing binary options with varying maturities. This portfolio not only matches the zero value but also the zero theta — the sensitivity of the value of the option to the passage of time — allowing a smaller replication error than that of the DEK portfolio and significantly improving the pricing and the hedging performance. Dias et al. (2015) extend this method to the valuation of double barrier option contracts with time-dependent barrier levels under the JDCEV model. This is done by first deriving the theta sensitivity measures for both plain-vanilla and cash-or-nothing European-style options under the JDCEV model and then hedging upper barriers through the addition of call options with varying maturities and strike equal to the upper barrier, while lower barriers are hedged through put option contracts. The more complex cases of double barrier options require a unit of a plain-vanilla European-style option conditional on no default during the time-span of the evaluated option for the knock-out case, whereas a recovery component with the value of the strike price is used for the knock-in case.

The first of the two explored methodologies in Dias et al. (2015), the *stopping time approach*, is the one extended in this thesis in order to provide the solutions to the options used to replicate the Das and Kim (2015) model, yielding the needed results for the debt discounts and the yield curves under the CEV model.

In Part II of this thesis, the focus goes to the crucial component of Das and Kim (2015) - the barrier options. As previously mentioned, there are more kinds of barrier options beyond the basic one barrier knock-in and knock-out kinds. In particular, there are the "first-then" options. These are studied in Jun and Ku (2012) under the GBM diffusion, who not only present the closed formulae for two barrier first-then call options, but also solve the three barrier case. Jun and Ku (2013) do so for curved barriers. As an example of these two barrier first-then options, we can have a first-up-then-down-and-in barrier option, that is, first, a barrier greater than the underlying asset starting value must be crossed, then a lower barrier must also be crossed, and only then the option is knocked-in. Or, as an alternative explanation, it is a down-and-in option, that first must be knocked-in by crossing an upper barrier. As for the knock-out case, one can have for instance a first-up-in-then-down-and-out option, which is active after the the underlying asset price crosses the upper barrier, and is deactivated if afterwards a lower barrier is crossed. Or alternatively, a down-and-out option that is knocked-in once the asset price crosses an upper barrier. In the GBM setting, the sum of the two options mentioned above results in a up-and-in barrier option.

Theoretically, the amount of added barriers is limitless, and in the case of Jun and Ku (2012) the amount reaches three barriers. This leads to, for instance, the case of a first-up-then-down-then-up-and-in option, which as the name indicates is knocked-in after an upper barrier is crossed, followed by a lower barrier, followed by an upper barrier. In other words, a first-down-then-up-and-in barrier option contract which is knocked-in by crossing an upper barrier.

In Jun and Ku (2012) closed formulae are obtained through successive applications of the reflection principle, an important result that allows often to simplify Brownian motion problems. Haug (2001) also presents formulae for the double barrier first-then options when the cost of carry is zero, that is, the interest rate is equal to the dividend yield, with the formulae being obtained through the put call barrier symmetry.

Part II of this thesis takes the option formulae presented by Jun and Ku (2012) and applies them to the JDCEV model from Carr and Linetsky (2006) using again the *stopping time approach* developed by Dias et al. (2015) to obtain the results.

The JDCEV model was introduced and solved in Carr and Linetsky (2006). It departs from the CEV diffusion - able to control the variance of the underlying asset by movements of its price - and complements it with the possibility of a jump to default (that is, the value of the underlying asset becomes zero, and remains so, given it being an absorbing barrier) ruled by an hazard process which is set up as an affine increasing function on the underlying asset's instantaneous volatility. It nests the CEV model, which is yielded by setting the affine function parameters to zero. This form of the hazard process is supported by evidence of the link between the asset volatility and the probability of default. For instance, Campbell and Taksler (2003) find that the volatility has as much explanatory power as credit ratings in explaining bond yields. Cremers et al. (2008) show that the implied volatility of stock options is also able to predict credit spreads and options are able to explain rating migrations and also establish a positive link between CDS rates and both the implied volatility levels and its slope. Hilscher (2007) is able to predict volatility with the credit yield spread. Consigli (2004) is able to conclude that implied volatility movements drive significant CDS spread movements for firms with a sufficient degree of risk. Carr and Wu (2010) find links between market risk and credit risk with CDS spreads being explained by stock option implied volatilities. The explanations presented before for the *leverage effect* and the *implied volatility skew* also contribute to justify the link between the possibility of jump to default and volatility.

Carr and Linetsky (2006) obtain a solution to the JDCEV model that unifies the valuation of corporate liabilities, credit derivatives and equity derivatives. This solution is obtained by time changes, scale changes, and measure changes relying on the theory of Bessel processes - a type of stochastic processes explored in dept for instance in Revuz and Yor (1999). The solution comes in the form of known functions such as the Gamma function and the Kummer confluent hypergeometric function.

Besides Dias et al. (2015) for the barrier options, Nunes (2009) and Ruas et al. (2013) price American-style options for the JDCEV model.

Therefore, given the advantageous properties of the JCDEV model, the first-then barrier options with two and three barriers have their solutions obtained in this thesis, once again expanding the work of Dias et al. (2015).

As for the third and final part of this thesis, the model of Das and Kim (2015) is explored again. In this analysis, the level at which the debt increase or decrease is triggered is not deterministic, although the firm leverage at which it happens is set beforehand, therefore, the only source of uncertainty on a debt increase happening is reduced to the path of the firm level. The aim of this part is to achieve a framework where the firm value increases still raise the probability of triggering debt increases and its decreases do so for the probability of the debt decreases, but without the need of setting beforehand the levels at which it does so, while allowing other factors to have an impact.

The approach taken is to set a hazard rate to trigger the debt increases and decreases, and in order to be able to link those to the path of the firm level, a correlation is defined for both stochastic movements. Hazard rates are often used in various models, for instance the jump to default in the JDCEV model of Carr and Linetsky (2006) is ruled by one. These are a key part of a class of credit risk models named the intensity models.

These models present a counterpart to the structural credit models and are able to incorporate factors beyond the total firm's asset value into the debt pricing models while maintaining the default-free term-structure modeling. The externally specified intensity process may or may not be related to the asset value and, therefore, the default can be treated as an unexpected event. Among the popular diffusions to model the hazard rate we can note the Vasicek (1977) and the Cox et al. (1985) which are also popular to model interest rates and the affine jump process as presented for instance in Duffie and Gârleanu (2001). They yield the probability of not defaulting in a given time period, which allows to easily obtain the probability of a default on the debt. The literature on these goes back to Jarrow and Turnbull (1995) who consider two types of risk on a derivative, the one coming from the underlying asset and the other coming from the writer of the derivative and Madan and Unal (1998) who also decompose the risk into two kinds, timing risk and recovery risk. Duffie and Singleton (1997) use an intensity model of default with fractional recovery to analyze the term structure of swap spreads and Duffee (1999) specifies a two-factor model for the treasury-bond and a one-factor firm-specific intensity process, and conclude that the intensity models appear to be capable of fitting term-structure changes for corporate bonds. As for empirical studies that rely on this class of models, one can cite Driessen (2005) who analyses the event risk premia by comparing the empirical default rates with corporate bond spreads and Duffie et al. (2003) who analyze with an intensity-based model the Russian bonds around the crisis in the year 1998.

Another literature stream that uses intensity processes and show a payoff structure similar to the one in this thesis is the one on vulnerable options. These emerge due to the fact that those who hold derivatives contracts often cannot ignore counterparty risk. The fact that those on the other side of a contract can default and not be able to pay their obligations can reduce the realized values of that contract. To do this, the possibility of the default by the counterpart is defined, and it can be so through an intensity model. There are early works with the approach that default occurs at option expiry and assume stochastic processes for both the firm value of the option writer and the underlying asset value of the option. Johnson and Stulz (1987) derive options with the possibility of correlation between the option's underlying asset and the credit risk of the counterparty with the possibility of an option writer defaulting, Klein (1996) does a similar exercise, including the possibility of counterparty and the option writer to have other liabilities and Klein and Inglis (2001) do a combination of the features of the two other papers which is solved through a numerical solution. Follow up works include, for instance, Hung and Liu (2004) who extend Klein's work in incomplete markets under the stochastic interest rate, Klein and Yang (2010) study the American-style vulnerable options by extending the previous works and Yang et al. (2014) incorporate the volatility of the underlying asset following a mean-reverting Ornstein–Uhlenbeck process.

More recent literature such as Fard (2015) and Koo and Kim (2017) explore intensity based models for vulnerable options where the default of the firm is ruled by a process similar to the one that rules the debt changes in this study. The obtained framework for the debt studied in the third part of this thesis adds the possibility of increasing and decreasing debt to the Merton model, although the flexible framework can be applied to other cases. In part IV, the case of the presence of subordinated debt is also studied. In Black and Cox (1976) the possibility of multiple debt claimants is explored and Gorton and Santomero (1990) present a formula where debt is separated into two categories upon default: the senior debt which has the priority in receiving its nominal debt value and the junior debt that only receives the nominal face value once the senior debt has been fully paid. They observe that under the presence of subordinated debt, the risk preference of debtholders may change, with the junior debtors preferring a higher amount of risk in certain circumstances. With the proper adaptations, the value of the junior debt when there is the possibility of increasing the senior debt is obtained.

The remainder of the thesis is organized as follows. Part II prices first-then-options under the JDCEV model. Part III studies credit spreads with dynamic debt under the CEV model. Part IV analyzes dynamic debt with intensity-ruled debt jumps. Finally, Part V presents the main conclusions of the research problems discussed in this thesis.

Part II

First-then-options under the JDCEV model

1 Introduction

Barrier options are a highly popular kind of path dependent exotic options, particularly in the over-the-counter markets and foreign exchange markets.

As the standard options, these have an underlying asset price, a strike price and a maturity. The addition comes in the form of one or more barriers of either knock-in or knock-out types. For example, a single barrier option with a knock-out barrier becomes worthless if that barrier is crossed by the asset price before the maturity date, while a single barrier option with a knock-in barrier is worthless unless the barrier is crossed. Given these constraints, these options are cheaper than the corresponding vanilla options, while allowing market participants to better fit their needs. The study of these options is well documented, especially for the geometric Brownian motion (GBM) setup. Merton (1973) reaches the price for a down-and-out call, while Rubinstein and Reiner (1991) and Rich (1994) derive the price of knock-in and knock-out put and call options for the remaining single barrier option contracts.

The complexity of these barrier options can be increased alongside the number of barriers. Double barrier contracts are popular in various forms. For instance, the double barrier knock-out options where there are two barriers and when either one is crossed, the option becomes worthless. This kind of contracts is well studied under the GBM assumption, as it can be seen in Kunitomo and Ikeda (1992), Geman and Yor (1996), Sidenius (1998), Pelsser (2000), Schröder (2000) or Buchen and Konstandatos (2009).

In this part of the thesis, a particular family of barrier options is studied, the first-then-barrier options. These rely on barriers starting their monitorization period after another barrier is crossed. For instance, a first-up-then-down-and-in call is worthless before the asset price crosses an upper barrier, and in addition, afterwards, also crosses a second lower barrier. In another case, one might have a first-up-in-then-down-and-out call, where an investor obtains a down-and-out call after an upper barrier is crossed.

The first-then options were studied in the GBM setting by Haug (2006), with closed formulae for the case where the cost-of-carry is null, that is, the interest rate equals the dividend rate. Furthermore, Jun and Ku (2012) also derive closed formulae for the call options, while allowing flexibility in the cost-of-carry parameter, and also including the case of triple barriers. For instance, a first-up-then-down-then-up-and-in option becomes active only after three barriers are crossed. In Jun and Ku (2013), options with multiple curved barriers are explored.

Given that all these results are presented for the GBM setting, they are exposed to its log-normal distribution assumption. This fails to capture various empirical facts, as for instance Jackwerth and Rubinstein (1996) explore. These limitations can be anchored in two effects: firstly, as shown in Bekaert and Wu (2000), there is a negative correlation between stock returns and realized volatility, the *leverage effect*; and secondly, as Dennis and Mayhew (2002) document, the negative correlation between the implied volatility and the strike of the stock, the *implied volatility skew*.

To address these issues, Cox (1975) introduces the constant elasticity of variance (CEV) model. This model departs from the GBM, allowing the volatility to be a function of the underlying asset price, thus addressing the mentioned limitations if one sets the price and volatility to be negatively connected. A further step is taken by Carr and Linetsky (2006), who include the possibility of a jump to default, where default is based on an affine function of equity and volatility, and so arriving at the jump to default extended CEV (JDCEV) model.

In this part of the thesis, we price the first-then style options presented in Jun and Ku (2012) in the CEV and JDCEV frameworks. To do so, the *stopping time approach* based on the work of Park and Schuurmann (1976) and developed in Dias et al. (2015) is crucial, adapting its methodology to the first-then options. The latter paper develops numerical methods capable of yielding accurate results for the double barrier options under the *JDCEV* model. In Dias et al. (2021), the authors explore more types of barrier options.

Here, the developed methodology, in addition to extending to the *JDCEV* model to the case of three barriers, allows for the first and the third barriers to hold different values, a possibility Jun and Ku (2012) do not contemplate, as these must hold the same value.

2 Markovian diffusion process with killing

Assume a general one-dimensional Markovian diffusion process with killing, under which the underlying asset price can diffuse to zero, where zero is an absorbing barrier,

$$\tau_0 := \inf \{t > t_0 : S_t = 0\}, \tag{1}$$

or, alternatively, the asset price can also jump to zero at the first time $\tilde{\zeta}$ of a doubly-stochastic Poisson process with intensity $\lambda(S,t) \in \mathbb{R}_+$.

The random time of default will occur when the first of two events occurs, that is,

$$\zeta = \tau_0 \wedge \tilde{\zeta}. \tag{2}$$

Therefore, at time ζ, the process heads towards zero, remaining there afterwards, emulating a default time. We will evaluate the options' prices assuming that the default is yet to occur.

Before ζ, at time-t, the price of the underlying asset under the martingale measure $\mathbb{Q}$, associated to the numéraire "money market account", follows the stochastic differential equation

$$\frac{dS_t}{S_t} = [r(t) - q(t) + \lambda(S,t)]dt + \sigma(S,t)dW_t^{\mathbb{Q}}. \tag{3}$$

As usual, $r(t)$ represents the interest rate, $q(t)$ stands for the dividend yield, $\sigma(S,t)$ is the instantaneous volatility of the asset returns at time-t given the asset price S and $\left\{W_t^{\mathbb{Q}}, t \geq t_0\right\}$ is a standard Brownian motion defined under measure $\mathbb{Q}$, generating the filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq t_0\}$.

Let $\mathbb{D} = \{\mathcal{D}_t : t \geq t_0\}$ denote the filtration denoted by the default indicator process $\mathcal{D}_t = \mathbb{1}_{\{t<\zeta\}}$. With the two filtrations, we can denote the enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.

The volatility is given with a CEV-type specification which accommodates the leverage effect. Therefore

$$\sigma(t,S) = a_t S_t^{\bar{\beta}}, \tag{4}$$

where $\bar{\beta} < 0$ is the volatility elasticity parameter and $a_t > 0$ is a deterministic volatility scale function.

As for the default intensity process, the Carr and Linetsky (2006) specification is used, with the default intensity being an affine function in terms of the instantaneous underlying asset variance. So we have

$$\lambda(S,t) = b_t + c\sigma(S,t)^2, \tag{5}$$

with $c \geq 0$ and $b_t \geq 0$ being a deterministic function of time.

Overall, equations (1) - (5) compose the JDCEV model which is used in this part of the thesis.

3 Barrier options contract payoffs and first passage times

In this section, the formal definitions for the option contracts are presented, alongside with the formal definitions of the first passage times. We start with the plain-vanilla options, which are the base of the barrier options.

3.1 Single barrier options

At time t_0, the price of a plain-vanilla European-style call (if $\phi = -1$) or put (if $\phi = 1$) with the underlying asset price S, with strike K and maturity at T is composed by the sum of the option in the case of no default and the recovery value in the case of default, that is:

$$v_{t_0}\left(S_{t_0}, K, T; \phi\right) = v_{t_0}^{0}\left(S_{t_0}, K, T; \phi\right) + v_{t0}^{D}\left(S_{t_0}, K, T; \phi\right). \tag{6}$$

The first component corresponds to the value of the option, although conditional on the survival of the underlying asset until the maturity date, T.

The second component corresponds to what is "recovered" in the case of the default event, ζ, happens before T. Given the options' payoff, it is null in the case of the call options.

3.2 First passage times

The first passage times are used to signal when the barriers are crossed, so let

$$\tau_B := \inf \{u > t_0 : S_u = B(t)\}, \tag{7}$$

be the first time that the underlying asset price hits the finite and deterministic barrier level $B(t)$, with $B \in \{L, U\}$, that is, it can be either a lower barrier (below the staring asset value) or an upper barrier (above the staring asset value).

3.3 One-touch single barrier options

We first borrow the single barriers valuation approach from Dias et al. (2021). These contracts represent plain-vanilla options that are activated (knocked-in) or deactivated (knocked-out) once a given barrier level is crossed, a lower barrier - designated by $L(t)$ - or an upper barrier - designated by $U(t)$.

These contracts are the most simple barrier options, which were studied by Rubinstein and Reiner (1991) and Rich (1994) under the GBM framework. First, a single barrier knock-in option only results in a payoff, if, before maturity, a given time dependent barrier level $L(t)$ - in case of a down-and-in option - or a barrier level $U(t)$ - for up-and-in options - is crossed. In the cases where the barrier is not crossed, the option contract is worthless. Second, a single barrier knock-out option only results in a payoff if a given time dependent barrier level $L(t)$ - in case of a down-and-out option - or a barrier level $U(t)$ - for up-and-out options - is not crossed. In the cases where the barrier is crossed, the option is worthless. In both the knock-in and knock-out cases where the option becomes worthless, a cash rebate may also be received until the maturity date (although it is not approached here).

In addition, the possibility of default must be taken into account when evaluating these options. For the case of a call option, when the price hits the absorbing barrier of zero, the option becomes worthless. The case of the put option is more nuanced, as one must regard that in the case of default, the value is equal to the strike, K, but only if the option is active, and this depends on the underlying asset's price path until the default time.

The following four definitions borrowed from Dias et al. (2021) summarize the contractual features of one-touch knock-in and knock-out single barrier options with no rebate for the case of time-dependent barriers.

Definition 1 ***Up-and-in options.*** *The time-T price of a unit face value and zero rebate European-style up-and-in single barrier option on the asset price S, with strike K, barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned} EUI_T\left(S_T, K, U, T; \phi, \tau_U\right) &= EUI_T^0\left(S_T, K, U, T; \phi, \tau_U\right) + EUI_T^D\left(S_T, K, U, T; \phi, \tau_U\right) \\ &= \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_U \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_U \leq \zeta \leq T\}}, \end{aligned} \tag{8}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. We note that there is no recovery component for an up-and-in call and, therefore, $EUI_T^D\left(S_T, K, U, T; -1, \tau_U\right) = 0$.

Definition 2 ***Down-and-in options.*** *The time-T price of a unit face value and zero rebate European-style down-and-in single barrier option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned} EDI_T\left(S_T, K, L, T; \phi, \tau_L\right) =& EDI_T^0\left(S_T, K, L, T; \phi, \tau_L\right) + EDI_T^D\left(S_T, K, L, T; \phi, \tau_L\right) \\ =& \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_L \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_L \leq \zeta \leq T\}} \\ =& \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_L \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\zeta \leq T\}} \\ & - (\phi K)^+ \mathbb{1}_{\{\zeta < \tau_L\}} \\ =& \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_L \leq T, \zeta > T\}} + v_T^D(S, K, T; \phi), \end{aligned} \tag{9}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. We note that there is no recovery component for a down-and-in call and, therefore, $EDI_T^D\left(S_T, K, L, T; -1, \tau_L\right) = 0$.

Moreover, since the default event cannot precede the knock-in event, then $\mathbb{1}_{\{\zeta<\tau_L\}}=0$ *and, hence,* $EDI_T^D(S_T,K,L,T;1,\tau_L)=v_T^D(S_T,K,T;1)$*, which is the recovery component of a vanilla put option, as defined in Carr and Linetsky (2006) and Dias et al. (2015).*

Definition 3 ***Up-and-out options.*** *The time-*T *price of a unit face value and zero rebate European-style up-and-out single barrier option on the asset price* S*, with strike* K*, barrier levels* $U:\mathcal{T}\to\mathbb{R}_+$ *(with* $U(t_0)>S_{t_0}$*), and maturity at time* $T\,(\geq t_0)$ *is equal to*

$$\begin{aligned}
EUO_T(S_T,K,U,T;\phi,\tau_U)=&EUO_T^0(S_T,K,U,T;\phi,\tau_U)+EUO_T^D(S_T,K,U,T;\phi,\tau_U)\\
=&(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_U>T,\zeta>T\}}+(\phi K)^+\mathbb{1}_{\{\zeta\leq T\wedge\tau_U\}}\\
=&v_T^0(S_T,K,T;\phi)-EUI_T^0(S_T,K,U,T;\phi,\tau_U)\\
&+(\phi K)^+\mathbb{1}_{\{\zeta\leq T\wedge\tau_U\}},
\end{aligned}\tag{10}$$

where $\phi=1$ *for a put option and* $\phi=-1$ *for a call option. We note that there is no recovery component for an up-and-out call and, therefore,*
$EUO_T^D(S_T,K,U,T;-1,\tau_U)=0.$

Definition 4 ***Down-and-out options*** *The time-*T *price of a unit face value and zero rebate European-style down-and-out single barrier option on the asset price* S*, with strike* K*, barrier levels* $L:\mathcal{T}\to\mathbb{R}_+$ *(with* $L(t_0)<S_{t_0}$*), and maturity at time* $T\,(\geq t_0)$ *is equal to*

$$\begin{aligned}
EDO_T(S_T,K,L,T;\phi,\tau_L)=&EDO_T^0(S_T,K,L,T;\phi,\tau_L)+EDO_T^D(S_T,K,L,T;\phi,\tau_L)\\
=&(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L>T,\zeta>T\}}+(\phi K)^+\mathbb{1}_{\{\zeta\leq T\wedge\tau_L\}}\\
=&v_T^0(S_T,K,T;\phi)-EDI_T^0(S_T,K,L,T;\phi,\tau_L),
\end{aligned}\tag{11}$$

where $\phi=1$ *for a put option and* $\phi=-1$ *for a call option. We note that there is no recovery component for a down-and-out call and, therefore,*
$EDO_T^D(S_T,K,L,T;-1,\tau_L)=0$*. Furthermore, since the default event cannot precede the knock-out event,* $\mathbb{1}_{\{\zeta\leq\tau_L\}}=0$ *and, hence,* $EDO_T^D(S_T,K,L,T;1,\tau_L)=0.$

3.4 First-then-barrier options

First-then-barrier options are barrier options contracts whose barrier monitoring starts when another barrier is crossed.

For instance, we can have a knock-in option than in addition to crossing a first upper barrier, it must also cross a second lower barrier in order to be activated. One can also have the knock-out option cases, where, for instance, first an upper barrier must be crossed to start the monitoring of a second lower knock-out barrier.

These contracts were studied by Jun and Ku (2012) and Jun and Ku (2013) under the GBM assumption for the call options with closed formulae through successive applications of the reflection principle. In this section, the contractual features of the knock-out and knock-in options are studied. In general, these are the previous four payoffs, but with the feature that beforehand an additional barrier must be crossed in order to start the barrier's monitorization.

Definition 5 ***First-down-then-up-and-in options.*** *The time-T price of a unit face value and zero rebate European-style first-down-then-up-and-in option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
& EDUI_T\left(S_T, K, L, U, T; \phi, \tau_L, \tau_U\right) \\
=& EDUI_T^0\left(S_T, K, L, U, T; \phi, \tau_L, \tau_U\right) + EDUI_T^D\left(S_T, K, L, U, T; \phi, \tau_L, \tau_U\right) \\
=& \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq \zeta \leq T\}},
\end{aligned} \tag{12}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, τ_L activates a European-style up-and-in barrier option with barrier level U. We note that there is no recovery component for the first-down-then-up-and-in call and, therefore,
$EDUI_T^D\left(S_T, K, L, U, T; -1, \tau_L, \tau_U\right) = 0$.

Definition 6 ***First-up-then-down-and-in options.*** *The time-T price of a unit face value and zero rebate European-style first-up-then-down-and-in barrier option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with*

$U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to

$$
\begin{aligned}
& EUDI_T\left(S_T, K, L, U, T; \phi, \tau_L, \tau_U\right) \\
= & EUDI_T^0\left(S_T, K, L, U, T; \phi, \tau_L, \tau_U\right) + EUDI_T^D\left(S_T, K, L, U, T; \phi, \tau_L, \tau_U\right) \\
= & \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_U < \tau_L \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_U < \tau_L \leq \zeta \leq T\}}
\end{aligned}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, τ_U activates a European-style down-and-in barrier option with barrier level L. We note that there is no recovery component for a first-up-then-down-and-in call and, therefore, $EUDI_T^D\left(S_T, K, L, U, T; -1, \tau_L, \tau_U\right) = 0$.

Definition 7 ***First-down-in-then-up-and-out options.*** *The time-T price of a unit face value and zero rebate European-style first-down-in-then-up-and-out barrier option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
& EDI, UO_T(S_T, K, L, U, T; \phi, \tau_L, \tau_U) \\
= & EDI, UO_T^0(S_T, K, L, U, T; \phi, \tau_L, \tau_U) + EDI, UO_T^D(S_T, K, L, U, T; \phi, \tau_L, \tau_U) \\
= & (\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_L \leq T < \tau_U, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_L \leq \zeta \leq T \wedge \tau_U\}},
\end{aligned}
\tag{13}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, τ_L activates a European-style up-and-out barrier option with barrier level U. We note that there is no recovery component for the first-down-in-then-up-and-out call, and, therefore $EDI, UO_T(S_T, K, L, U, T; -1, \tau_U, \tau_L) = 0$.

Definition 8 ***First-up-in-then-down-and-out-options.*** *The time-T price of a unit face value and zero rebate European-style first-up-in-then-down-and-out barrier option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
& EUI, DO_T(S_T, K, L, U, T; \phi, \tau_L, \tau_U) \\
= & EUI, DO_T^0(S_T, K, L, U, T; \phi, \tau_L, \tau_U) + EUI, DO_T^D(S_T, K, L, U, T; \phi, \tau_L, \tau_U) \\
= & (\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_U \leq T < \tau_L, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_U < \zeta \leq T \wedge \tau_L\}},
\end{aligned}
\tag{14}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, τ_U activates a European-style down-and-out barrier option with barrier level L.

In the previous Propositions, a new barrier is added to a single barrier option, thus originating two barriers. More barriers can be added to the option indefinitely, and here, the case of the three barriers will be explored. For instance, one can have a first-up-then-down-then-up-and-in barrier option which is knocked-in after three barriers are crossed in sequence: first an upper barrier, then a lower barrier and, finally, an upper barrier.

Jun and Ku (2012) derive the closed formulae for this kind of barrier option for the call case, although with the limitation of the first barrier to being equal to the third barrier. That is, they assume that $L_1 = L_2$ in the case of two lower barriers and $U_1 = U_2$ in the case of two upper barriers. Here the possibility of the first and third barriers being different is considered, that is, $U_1 \neq U_2$ and $L_1 \neq L_2$ are contemplated.

Definition 9 ***First-up-then-down-then-up-and-in options.*** *The time-T price of a unit face value and zero rebate European-style first-up-then-down-then-up-and-in option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), $U_1 : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), $U_2 : \mathcal{T} \to \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$) and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned}
&EUDUI_T\left(S_T, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right)\\
=&EUDUI_T^0\left(S_T, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right)\\
&+ EUDUI_T^D\left(S_T, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right)\\
=&\left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_{U_1} < \tau_L < \tau_{U_2} \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_{U_1} < \tau_L < \tau_{U_2} \leq \zeta \leq T\}},
\end{aligned} \tag{15}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, if τ_{U_1} has occurred, τ_L activates a European-style up-and-in barrier option with the barrier level at U_2. We note that there is no recovery component for a first-up-then-down-then-up-and-in call and, therefore, $EUDUI_T^D\left(S_T, K, L, U_1, U_2, T; -1, \tau_L, \tau_{U_1}, \tau_{U_2}\right) = 0$.

Definition 10 ***First-down-then-up-then-down-and-in options.*** *The time-T price of a unit face value and zero rebate European-style first-down-then-up-then-down-and-in barrier option on the asset price S, with strike K, barrier levels $L_1 : \mathcal{T} \to \mathbb{R}_+$ (with*

$L_1(t_0) < S_{t_0}$), $L_2 : \mathcal{T} \to \mathbb{R}_+$ (with $L_2(t_0) < S_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$) and maturity at time $T(\geq t_0)$ is equal to

$$
\begin{aligned}
&EDUDI_T\left(S_T, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\
=&EDUDI_T^0\left(S_T, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\
&+ EDUDI_T^D\left(S_T, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\
=&\left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_{L_1} < \tau_U < \tau_{L_2} \leq T, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_{L_1} < \tau_U < \tau_{L_2} \leq \zeta \leq T,\}},
\end{aligned}
\tag{16}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, if τ_{L_1} has occurred, τ_U activates a European-style down-and-in barrier option with the barrier level at L_2. We note that there is no recovery component for the first-down-then-up-then-down-and-in call and, therefore, $EDUDI_T^D\left(S_T, K, L_1, U, L_2, T; -1, \tau_{L_1}, \tau_U, \tau_{L_2}\right) = 0$.

Definition 11 ***First-up-then-down-in-then-up-and-out options.*** *The time-T price of a unit face value and zero rebate European-style first-up-then-down-in-then-up-and-out barrier option on the asset price S, with strike K, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), $U_1 : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), $U_2 : \mathcal{T} \to \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
&EUDI,UO_T\left(S_T, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right) \\
=&EUDI,UO_T^0\left(S_T, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right) \\
&+ EUDI,UO_T^D\left(S_T, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right) \\
=&\left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_{U_1} < \tau_L < T < \tau_{U_2}, \zeta > T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_{U_1} < \tau_L \leq \zeta \leq T \wedge \tau_{U_2}\}}
\end{aligned}
\tag{17}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, if τ_{U_1} has occurred, τ_L activates a European-style up-and-out barrier option with the barrier level at U_2.

Definition 12 ***First-down-then-up-in-then-down-and-out options.*** *The time-T price of a unit face value and zero rebate European-style first-down-then-up-in-then-down-and-out barrier option on the asset price S, with strike K, barrier levels $L_1 : \mathcal{T} \to \mathbb{R}_+$ (with $L_1(t_0) < S_{t_0}$), $L_2 : \mathcal{T} \to \mathbb{R}_+$ (with $L_2(t_0) < S_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
& EDUI, DO_T\left(S_T, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\
=& EDUI, DO_T^0\left(S_T, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\
& + EDUI, DO_T^D\left(S_T, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\
=& \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_{L_1} < \tau_U < T < \tau_{L_2}, \zeta > T\}} + (\phi K)^+ 1_{\{\tau_{L_1} < \tau_U < \zeta \leq T \wedge \tau_{L_2}\}}
\end{aligned}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, if τ_{L_1} has occurred, τ_U activates a European-style down-and-out barrier option with the barrier level at L_2. We note that there is no recovery component for the first-down-then-up-in-then-down-and-out call and, therefore, $EDUI, DO_T^D\left(S_T, K, L_1, L_2, U, T; -1, \tau_{L_1}, \tau_{L_2}, \tau_U\right) = 0$.

4 Valuation under the JDCEV model

With the barrier option payoffs derived, the standard options and various barrier cases results are computed for the conditional on no default components and the corresponding recovery values.

4.1 Plain-vanilla options

The results for the plain-vanilla options are obtained in Carr and Linetsky (2006) and the results are as follows.

$$
v_{t_0}^0\left(S_{t_0}, K, T; \phi\right) := \mathbb{E}_Q\left[e^{-\int_{t_0}^T r(t)dl}\left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\zeta > T\}} \mid \mathcal{G}_{t_0}\right] \tag{18}
$$

and

$$
v_{t_0}^D\left(S_{t_0}, K, T; \phi\right) := \mathbb{E}_Q\left[e^{-\int_{t_0}^T r(l)dl}(\phi K)^+ \mathbb{1}_{\{\zeta \leq T\}} \mid \mathcal{G}_{t_0}\right]. \tag{19}
$$

As mentioned before, in the case of the recovery component, its value is null for European-style calls, i.e., $v_{t_0}^D\left(S_{t_0}, K, T; -1\right) = 0$. In the case of the put, it is given by the probability of default multiplied by the strike level to be paid at the maturity date T. That is,

$$v_{t_0}^{D}\left(S_{t_0}, K, T; \phi\right) = (\phi K)^{+} e^{-\int_{t_0}^{T} r(l)dl}\left[1 - SP\left(S_{t_0}, t_0; T\right)\right], \tag{20}$$

where we have the risk-neutral survival probability beyond time $T(> t_0)$ to be

$$SP\left(S_{t_0}, t_0; T\right) := \mathbb{E}_{Q}\left[\mathbb{1}_{\{\zeta > T\}} \mid \mathcal{G}_{t_0}\right] = \mathbb{1}_{\{\zeta > t_0\}}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T} \lambda(S_l, l)dl}\mathbb{1}_{\{\tau_0 > T\}} \mid \mathcal{F}_{t_0}\right], \tag{21}$$

as defined in Carr and Linetsky (2006, Equation 3.1).

4.2 Single barrier results

First, the components conditional on no default formulae borrowed from Dias et al. (2021) allow the computation of both up and down kinds of single barrier options.

Proposition 1 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 (conditional on no default) value of a unit face value and zero rebate European-style knock-in call (if $\phi = -1$) or put (if $\phi = 1$) on the asset price S, with strike K, barrier levels $B : \mathcal{T} \to \mathbb{R}_{+}$ (with $B(t_0) > S_{t_0}$ for up barrier contracts and $B(t_0) < S_{t_0}$ for down barrier contracts), $\tau_B \in \{\tau_L, \tau_U\}$, and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned}
&EKI_{t_0}^{0}(S_{t_0}, K, B, T; \phi, \tau_B) \\
&= \int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} SP\left(S_{t_0}, t_0; u\right) v_u^0(B(u), K, T; \phi)\mathbb{Q}\left(\tau_B \in du \mid \mathcal{F}_{t_0}\right),
\end{aligned} \tag{22}$$

where $v_u^0(B(u), K, T; \phi)$ is the conditional on no default plain-vanilla option, $SP\left(S_{t_0}, t_0; u\right)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_B \in du \mid \mathcal{F}_{t_0}\right)$ represents the density function of the first passage time τ_B.

Proof. See Dias et al. (2021, Proposition 1).

Remark 1 *The conditional on no default value formulae for the knock-out options are* *obtained as*

$$EKO_{t_0}^0(S_{t_0}, K, B, T; \phi, \tau_B) = v_{t_0}^0(S_{t_0}, K, T; \phi) - EKI_{t_0}^0(S_{t_0}, K, B, T; \phi, \tau_B). \tag{23}$$

Next, we discuss how to compute the recovery components in case of default for the knock-in options and knock-out options. Again, the value is null for all kinds of call options, because when the underlying asset value remains at zero, the payoff becomes null.

As for the down-and-out puts, its value is also zero, as the lower knock-out barrier is crossed before the underlying asset becomes zero, thus, as mentioned in Dias et al. (2021, Definition 6):

$$EDO_{t_0}^D(S_{t_0}, K, L, T; 1, \tau_L) = 0. \tag{24}$$

In the case of the down-and-in puts, the jump to default automatically triggers the lower barrier, therefore the recovery value is the same as of a standard put, as shown in Dias et al. (2021, Definition 4):

$$EDI_{t_0}^D(S_{t_0}, K, U, T; 1, \tau_L) = v_{t_0}^D\left(S_{t_0}, K, T; 1\right). \tag{25}$$

Still, there are two missing cases that require the two following Propositions, the up-and-in and the up-and-out puts, that are also borrowed from Dias et al. (2021).

Proposition 2 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style up-and-in put on the asset price S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned}&EUI_{t_0}^D(S_{t_0}, K, U, T; 1, \tau_U)\\ =&Ke^{-\int_{t_0}^T r(l)dl}\left[\int_{t_0}^T SP\left(S_{t_0}, t_0; v\right)\left[1 - SP(U(v), v; T)\right]\mathbb{Q}\left(\tau_U \in dv \mid \mathcal{F}_{t_0}\right)\right],\end{aligned} \tag{26}$$

where $SP(U(v), v; T)$ is the risk-neutral survival probability and $\mathbb{Q}(\tau_U \in du \mid \mathcal{F}_{t_0})$ represents the density function of the first passage time τ_U.

Proof. See Dias et al. (2021, Proposition 2).

Proposition 3 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style up-and-out put on the asset price S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$EKO_{t_0}^D(S_{t_0}, K, U, T; 1, \tau_U) = v_{t_0}^D(S_{t_0}, K, T; 1) - EUI_{t_0}^D(S_{t_0}, K, U, T; 1, \tau_U), \tag{27}$$

where $v_{t_0}^D(S_{t_0}, K, T; 1)$ is the recovery component of a standard put and $EUI_{t_0}^D(S_{t_0}, K, U, T; 1)$ is the recovery value of an up-and-in put.

Proof. See Dias et al. (2021, Proposition 3).

4.3 First-then-barrier results

Now, by conditioning the previous one-barrier formulae with respect to the filtration that represents the crossing of other barriers, we obtain the two-barrier cases. We start with the conditional on no default components for the first two options.

Proposition 4 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 value of a conditional on no default unit face value and zero rebate European-style first-down-then-up-and-in call (if $\phi = -1$) or put (if $\phi = 1$) on the asset price S, with strike K, lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$) and maturity at time T $(\geq t_0)$ is equal to*

$$\begin{aligned} &EDUI_{t_0}^0(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) \\ &= \int_{t_0}^{T} e^{-\int_{t_0}^{l} r(l)dl} SP(S_{t_0}, t_0; l) EUI_l^0(L, K, U, T; \phi, \tau_U)\, \mathbb{Q}(\tau_L \in dl | \mathcal{F}_{t_0}), \end{aligned} \tag{28}$$

where $EUI_l^0(L, K, U, T; \phi, \tau_U)$ is the conditional on no default time-l price of a European-style down-and-in call (if $\phi = -1$) or put (if $\phi = 1$), $SP(S_{t_0}, t_0; l)$ is the risk-neutral *survival probability and* $\mathbb{Q}\left(\tau_L \in dl \middle| \mathcal{F}_{t_0}\right)$ *represents the probability density function of the first passage time* τ_L.

Proof. The time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-down-then-up-and-in barrier option is defined as

$$EDUI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) = \mathbb{1}_{\{\zeta > t_0\}} e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq T, \zeta > T\}} \mid \mathcal{G}_{t_0}\right]. \tag{29}$$

By using the tower law and Dias et al. (2015, equation (19)),

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq T, \zeta > T\}} \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_L < \tau_U\}} \mathbb{1}_{\{\tau_U \leq T\}} \mathbb{1}_{\{T < \zeta\}} \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L < \tau_U\}} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_U \leq T\}} (\phi K - \phi S_T)^+ \mathbb{1}_{\{T < \zeta\}} \mid \mathcal{G}_{\tau_U}\right] \mid \mathcal{G}_{\tau_L}\right] \mid \mathcal{G}_{t_0}\right].
\end{aligned} \tag{30}$$

Through successive uses of Dias et al. (2015, equation (19)), we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq T, \zeta > T\}} \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L < \tau_U\}} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_U < T\}} (\phi K - \phi S_T)^+ e^{-\int_{t_0}^{T} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0 \leq v \leq T\}}(S_v) > 0\}} \mid \mathcal{F}_{\tau_U}\right] \mid \mathcal{G}_{\tau_L}\right] \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{1}_{\{\tau_L < \tau_U\}} e^{-\int_{t_0}^{\tau_U} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0 \leq v \leq \tau_U\}}(S_v) > 0\}} \\
&\times \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_U < T\}} (\phi K - \phi S_T)^+ e^{-\int_{\tau_U}^{T} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{\tau_U \leq v \leq T\}}(S_v) > 0\}} \mid \mathcal{F}_{\tau_U}\right] \Bigg| \mathcal{F}_{\tau_L}\Bigg] \Bigg| \mathcal{G}_{t_0}\Bigg] \\
=&\mathbb{1}_{\{\zeta > t_0\}} \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_{t_0}^{\tau_L} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0 \leq v \leq \tau_L\}}(S_v) > 0\}} \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_{\tau_L}^{\tau_U} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{\tau_L \leq v \leq \tau_U\}}(S_v) > 0\}} \\
&\times \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ e^{-\int_{\tau_U}^{T} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{\tau_U \leq v \leq T\}}(S_v) > 0\}} \mid \mathcal{F}_{\tau_U}\right] \Bigg| \mathcal{F}_{\tau_L}\Bigg] \Bigg| \mathcal{F}_{t_0}\Bigg].
\end{aligned} \tag{31}$$

Given the underlying asset process behaves as a pure Markovian diffusion process with respect to the restricted filtration $\mathcal{F}$,

$$\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq T, \zeta > T\}} \mid \mathcal{G}_{t_0}\right]$$

$$
\begin{aligned}
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{l}^{T}\mathbb{E}_{\mathrm{Q}}\left[e^{-\int_{t_0}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le l\}}(S_v)>0\}}\mathbb{E}_{\mathrm{Q}}\left[e^{-\int_{l}^{u}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{l\le v\le u\}}(S_v)>0\}}\right.\right.\\
&\left.\left.\times\,\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}e^{-\int_{u}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u\le v\le T\}}(S_v)>0\}}\mid S_u=U(u)\right]\mid S_l=L(l)\right]\mid\mathcal{F}_{t_0}\right]\\
&\times\,\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right).
\end{aligned}
\tag{32}
$$

Combining equations (29) and (32), plus using again Dias et al. (2015, equation (19)),

$$
\begin{aligned}
&EDUI_{t_0}^{0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{T}r(l)dl}\int_{t_0}^{T}SP(S_{t_0},t_0;l)\\
&\times\int_{l}^{T}SP(L(l),l;u)\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}\mathbb{1}_{\{\zeta>T\}}\mid S_u=U(u)\right]\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\\
&\times\,\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{u}r(l)dl}\int_{t_0}^{T}SP(S_{t_0},t_0;l)\\
&\times\int_{l}^{T}SP(L(l),l;u)e^{-\int_{u}^{T}r(l)dl}\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}\mathbb{1}_{\{\zeta>T\}}\mid S_u=U(u)\right]\\
&\times\,\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right).
\end{aligned}
\tag{33}
$$

Since, by using Carr and Linetsky (2006, equation (3.8)), we can observe that

$$
e^{-\int_{u}^{T}r(l)dl}\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}\mathbb{1}_{\{\zeta>T\}}\mid S_u=U(u)\right]:=v_u^{0}(U(u),K,T;\phi),
\tag{34}
$$

therefore we have

$$
\begin{aligned}
&EDUI_{t_0}^{0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{u}r(l)dl}\int_{t_0}^{T}SP(S_{t_0},t_0;l)\\
&\times\int_{l}^{T}SP(L(l),l;u)v_u^{0}(U(u),K,T;\phi)\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right).
\end{aligned}
\tag{35}
$$

Furthermore,

$$
\begin{aligned}
&EDUI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) \\
=&\mathbb{1}_{\{\zeta > t_0\}} e^{-\int_{t_0}^{l} r(l)dl} \int_{t_0}^{T} SP(S_{t_0}, t_0; l) \\
&\times e^{-\int_{l}^{u} r(l)dl} \int_{l}^{T} SP(L(l), l; u) v^0_u(U, K, T; \phi) \mathbb{Q}\left(\tau_U \in du \mid S_l = L(l)\right) \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right),
\end{aligned}
\tag{36}
$$

and by Dias et al. (2021, equation (34)), we observe

$$
\begin{aligned}
&e^{-\int_{l}^{u} r(l)dl} \int_{l}^{T} SP(L, l; u) v^0_l(U(l), K, T; \phi) \mathbb{Q}\left(\tau_U \in du \mid S_l = L(l)\right) \\
:=&EUI^0_l(L(l), K, U(u); \phi, \tau_U),
\end{aligned}
$$

thus we obtain the intended result

$$
\begin{aligned}
&EDUI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) \\
=&\int_{t_0}^{T} e^{-\int_{t_0}^{l} r(l)dl} SP(S_{t_0}, t_0; l) EUI^0_l(L(l), K, U(l); \phi, \tau_U) \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right).
\end{aligned}
\tag{37}
$$

■

Proposition 5 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 conditional on no default value of a unit face value and zero rebate European-style first-up-then-down-and-in call (if $\phi = -1$) or put (if $\phi = 1$) on the asset price S, with strike K, lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), maturity at time T ($\geq t_0$) is equal to*

$$
\begin{aligned}
&EUDI^0_{t_0}\left(S_{t_0}, K, U, L, T; \phi, \tau_L, \tau_U\right) \\
=&\int_{t_0}^{T} e^{-\int_{t_0}^{u} r(l)dl} SP(S_{t_0}, t_0; u) EDI^0_u\left(U, K, L(l), T; \phi, \tau_L\right) \mathbb{Q}\left(\tau_U \in du \middle| \mathcal{F}_{t_0}\right),
\end{aligned}
\tag{38}
$$

where $EDI^0_u\left(U, K, L, T; \phi, \tau_L\right)$ is the conditional on no default price of a down-and-in call *(if $\phi = -1$) or put (if $\phi = 1$), $SP(S_{t_0}, t_0; u)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_U \in du \middle| \mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time τ_U.*

Proof. The time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-up-then-down-and-in barrier option is defined as

$$EUDI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) = e^{-\int_{t_0}^T r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_U < \tau_L \leq T, \zeta > T\}} \mid \mathcal{G}_{t_0}\right].$$

It can be observed that the roles of the upper barrier level, U, and the lower barrier level, L, are reversed in relation to Proposition 4. So, by having L in the place of U and U in the place of L in this Proposition, the steps are the same as in the previous Proposition. Hence, the proof is omitted. ■

Now, we compute the recovery value of the two former options. Again, the recovery value is null for call options, leaving the cases of the put options to be computed.

Proposition 6 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-down-then-up-and-in put on the asset S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$\begin{aligned}&EDUI^D_{t_0}(S_{t_0}, K, L, U, T; 1, \tau_L, \tau_U)\\ &= \int_{t_0}^T e^{-\int_{t_0}^l r(l)dl} SP(S_{t_0}, t_0; l) EUI^D_l(L, K, U, T; 1, \tau_U)\mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right),\end{aligned} \tag{39}$$

where $EUI^D_l\left(U, K, L, T; 1, \tau_U\right)$ is the recovery component of a up-and-in put, $SP(S_{t_0}, t_0; l)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_L \in dl \middle| \mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time τ_L.

Proof. The time-t_0 risk-neutral expectation of the recovery value of a first-down-then-up-and-in barrier put option is defined as

$$EDUI^D_{t_0}(S_0, K, L, U, T; 1, \tau_L, \tau_U) = Ke^{-\int_{t_0}^T r(l)dl}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau_L < \tau_U \leq \zeta \leq T\}} \mid \mathcal{G}_{t_0}\Big]. \tag{40}$$

The expectation can be written as

$$
\begin{aligned}
&\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U<T\leq\zeta\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U,\tau_U<\zeta,\zeta\leq T\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U\}}\mathbb{1}_{\{\tau_U<\zeta\}}\mathbb{1}_{\{\zeta\leq T\}}\mid\mathcal{G}_{t_0}\right],
\end{aligned}
\tag{41}
$$

and given $\mathbb{1}_{\{\zeta\leq T\}}=1-\mathbb{1}_{\{\zeta>T\}}$ we have

$$
\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U<T\leq\zeta\}}\mid\mathcal{G}_{t_0}\right]=\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U\}}\mathbb{1}_{\{\tau_U<\zeta\}}(1-\mathbb{1}_{\{\zeta>T\}})\mid\mathcal{G}_{t_0}\right],
\tag{42}
$$

and by the tower law,

$$
\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U<T\leq\zeta\}}\mid\mathcal{G}_{t_0}\right]=\mathbb{E}_{\mathrm{Q}}\left[\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U\}}\mathbb{1}_{\{\tau_U<\zeta\}}\mathbb{E}_{\mathrm{Q}}\left[(1-\mathbb{1}_{\{\zeta>T\}})\mid\mathcal{G}_{\tau_U}\right]\mid\mathcal{G}_{\tau_L}\right]\mid\mathcal{G}_{t_0}\right].
\tag{43}
$$

Through successive uses of Dias et al. (2015, equation (19)), we obtain

$$
\begin{aligned}
&\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U<T\leq\zeta\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathrm{Q}}\left[\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U\}}\mathbb{1}_{\{\tau_U<\zeta\}}\mathbb{E}_{\mathrm{Q}}\left[(1-e^{-\int_{\tau_U}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_U\leq v\leq T\}}(S_v)>0\}})\mid\mathcal{F}_{\tau_U}\right]\mid\mathcal{G}_{\tau_L}\right]\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathrm{Q}}\Bigg[\mathbb{1}_{\{\tau_L<\tau_U,\tau_L<\zeta\}}\mathbb{E}_{\mathrm{Q}}\Bigg[e^{-\int_{\tau_L}^{\tau_U}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_L\leq v\leq\tau_U\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathrm{Q}}\left[(1-e^{-\int_{\tau_U}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_U\leq v\leq T\}}(S_v)>0\}})\mid\mathcal{F}_{\tau_U}\right]\mid\mathcal{F}_{\tau_L}\Bigg]\mid\mathcal{G}_{t_0}\Bigg]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\mathbb{E}_{\mathrm{Q}}\Bigg[e^{-\int_{t_0}^{\tau_L}\lambda(S,i)di}\mathbb{1}_{\{\tau_L<\tau_U,\inf_{\{0\leq v\leq\tau_L\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathrm{Q}}\Bigg[e^{-\int_{\tau_L}^{\tau_U}\lambda(S,i)di}\mathbb{1}_{\inf_{\{\tau_L\leq v\leq\tau_U\}}(S_v)>0}\\
&\times\mathbb{E}_{\mathrm{Q}}\left[(1-e^{\int_{\tau_U}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_U\leq v\leq T\}}(S_v)>0\}})\mid\mathcal{F}_{\tau_U}\right]\mid\mathcal{F}_{\tau_L}\Bigg]\mid\mathcal{F}_{t_0}\Bigg].
\end{aligned}
\tag{44}
$$

Given the asset price process behaves as a pure Markovian diffusion process with respect to the restricted filtration $\mathbb{F}$, the equation above can be restated as

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U<T\leq\zeta\}}\mid\mathcal{G}_{t_0}\right]\\
&=\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{t_0}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{0\leq v\leq l\}}(S_v)>0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{l}^{u}\lambda(S,i)di}\mathbb{1}_{\inf_{\{l\leq v\leq u\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[(1-e^{-\int_{u}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u\leq v\leq T\}}(S_v)>0\}})\mid S_u=U(u)\Big]\mid S_l=L(l)\Big]\mid\mathcal{F}_{t_0}\Big]\\
&\times\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right).
\end{aligned}\tag{45}
$$

Again, through Dias et al. (2021, equation (19))

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\tau_U<T\leq\zeta\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{t_0}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{0\leq v\leq l\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{l}^{u}\lambda(S,i)di}\mathbb{1}_{\inf_{\{l\leq v\leq u\}}(S_v)>0\}}(1-SP(U(u),u;T))\mid S_l=L(l)\Big]\mid\mathcal{F}_{t_0}\Big]\\
&\times\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{t_0}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{0\leq v\leq l\}}(S_v)>0\}}\\
&\times SP(L(l),l;u)\mid\mathcal{F}_{t_0}\Big](1-SP(U(u),u;T))\\
&\times\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{l}^{T}SP(S_{t_0},t_0;l)SP(L(l),l;u)\Big](1-SP(U(u),u;T))\\
&\times\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right).
\end{aligned}\tag{46}
$$

Therefore, combining equations (40) and (46)

$$
\begin{aligned}
&EDUI_{t_0}^{D}(S_0,K,L,U,T;1,\tau_L,\tau_U)\\
=&\mathbb{1}_{\{\zeta>t_0\}}Ke^{-\int_{t_0}^{T}r(l)dl}\int_{t_0}^{T}\int_{l}^{T}SP(S_{t_0},t_0;l)SP(L(l),l;u)(1-SP(U(u),u;T))\\
&\times\mathbb{Q}\left(\tau_U\in du\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid\mathcal{F}_{t_0}\right)\\
&EDUI_{t_0}^{D}(S_0,K,L,U,T;1,\tau_L,\tau_U)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{l}r(l)dl}\int_{t_0}^{T}\int_{l}^{T}Ke^{-\int_{l}^{T}r(l)dl}SP(S_{t_0},t_0;l)SP(L(l),l;u)(1-SP(U(u),u;T))
\end{aligned}
$$

$$\times \mathbb{Q}\left(\tau_U \in du \mid S_l = L(l)\right) \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right). \tag{47}$$

With Dias et al. (2021, equation (36)), we can observe that

$$EUI_l^D(L, K, U, T; 1, \tau_U)$$
$$:= \int_l^T e^{-\int_l^T r(l)dl} SP(S_{t_0}, t_0, l) SP(L(l), l, u)\big] (1 - SP(U(u), u, T)) \times \mathbb{Q}\left(\tau_U \in du \mid S_l = L(l)\right), \tag{48}$$

thus we obtain the intended result.

■

Proposition 7 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-up-then-down-and-in put on the asset S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$EUDI_{t_0}^D(S_0, K, L, U, T; 1, \tau_L, \tau_U) = EUI_T^D\left(S_T, K, U, T; 1, \tau_U\right) \tag{49}$$

where $EUI_T^D\left(S_T, K, U, T; 1, \tau_U\right)$ is the recovery component of an up-and-in put.

Proof. The time-t_0 risk-neutral expectation of the recovery value of a first-down-then-up-and-in barrier put is defined as

$$EUDI_{t_0}^D(S_0, K, L, U, T; 1, \tau_L, \tau_U) = \mathbb{1}_{\{\zeta > t_0\}} K e^{-\int_{t_0}^T r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_U < \tau_L \leq \zeta \leq T\}}\right]. \tag{50}$$

The indicator function can be written as

$$\mathbb{1}_{\{\tau_U < \tau_L \leq \zeta \leq T\}} = \mathbb{1}_{\{\tau_U < \zeta \leq T\}} - \mathbb{1}_{\{\tau_U < \zeta < \tau_L\}} \tag{51}$$

Since the default event cannot precede the lower barrier knock-in event, $\mathbb{1}_{\{\tau_U < \zeta < \tau_L\}} =$

0, therefore

$$\mathbb{1}_{\{\tau_U<\tau_L\leq\zeta\leq T\}} = \mathbb{1}_{\{\tau_U<\zeta\leq T\}} \tag{52}$$

Combining equations (50) and (52),

$$EUDI_{t_0}^{D}(S_0,K,L,U,T;1,\tau_L,\tau_U) = \mathbb{1}_{\{\zeta>t_0\}}Ke^{-\int_{t_0}^{T}r(l)dl}\mathbb{E}_{\mathrm{Q}}\left[\mathbb{1}_{\{\tau_U<\zeta\leq T\}}\right]. \tag{53}$$

By Dias et al. (2021, equation (37)),

$$EUI_{T}^{D}\left(S_T,K,U,T;1,\tau_U\right) := \mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{T}r(l)dl}\mathbb{E}_{\mathrm{Q}}\left[(\phi K)^{+}\mathbb{1}_{\{\tau_U<\zeta\leq T\}}\right]. \tag{54}$$

Therefore, we reach the intended result. ■

Proposition 8 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-down-in-then-up-and-out on call (if $\phi = -1$) or put (if $\phi = 1$) on the asset S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$\begin{aligned} &EDI,UO_{t_0}^{0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U) \\ =&EDI_{t_0}^{0}(S_{t_0},K,L,T;\phi,\tau_L) - EDUI_{t_0}^{0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U). \end{aligned} \tag{55}$$

where $EDI_{t_0}^{0}(S_{t_0},K,L,T;\phi,\tau_L)$ is the conditional on no default first-down-and-in option and $EDUI_{t_0}^{0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U)$ is the conditional on no default first-down-then-up-and-in option.

Proof.

The time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-down-then-up-and-in option is defined as

$$\begin{aligned}
&EDI,UO^0_{t_0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U)\\
&=\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^T r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T<\tau_U,\zeta>T\}}\mid \mathcal{G}_{t_0}\right].
\end{aligned}\tag{56}$$

The expected value can be written using the tower law

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T<\tau_U,\zeta>T\}}\mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T<\tau_U\}}\mathbb{1}_{\{\zeta>T\}}\mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T\}}\mathbb{1}_{\{T<\tau_U\}}\mathbb{1}_{\{\zeta>T\}}\mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T\}}\mathbb{1}_{\{T<\tau_U\}}\mathbb{1}_{\{\zeta>T\}}\mid \mathcal{G}_{\tau_L}\right]\mid \mathcal{G}_{t_0}\right]
\end{aligned}\tag{57}$$

Using equation Dias et al. (2015, equation(19)) plus the fact that the asset price process behaves as a pure Markovian diffusion process,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T<\tau_U,\zeta>T\}}\mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{\tau_L\le T\}}\mathbb{1}_{\{T<\tau_U\}}e^{-\int_{t_0}^T\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\le s\le T}(S_s)>0\right\}}\mid \mathcal{G}_{\tau_L}\right]\mid \mathcal{F}_{t_0}\right]\\
=&\int_{t_0}^T\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\{T<\tau_U\}}e^{-\int_{t_0}^T\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\le s\le T}(S_s)>0\right\}}\mid S_l=L(l)\right]\mid \mathcal{F}_{t_0}\right]\mathbb{Q}\left(\tau_L\in dl\mid \mathcal{F}_{t_0}\right).
\end{aligned}\tag{58}$$

Given that $\mathbb{1}_{\{T<\tau_U\}}=1-\mathbb{1}_{\{T\ge\tau_U\}}$, and combining equations (56) and (58),

$$\begin{aligned}
&EDI,UO^0_{t_0}(S_{t_0},K,L,U,T;\phi,\tau_L,\tau_U)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^T r(l)dl}\\
&\times\int_{t_0}^T\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,(1-\mathbb{1}_{\{T\ge\tau_U\}})e^{-\int_{t_0}^T\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\le s\le T}(S_s)>0\right\}}\mid S_l=L(l)\right]\mid \mathcal{F}_{t_0}\right]\\
&\times\mathbb{Q}\left(\tau_L\in dl\mid \mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^T r(l)dl}\\
&\times\int_{t_0}^T\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,e^{-\int_{t_0}^T\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\le s\le T}(S_s)>0\right\}}\mid S_l=L(l)\right]\mid \mathcal{F}_{t_0}\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right)\\
&- \mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^{T} r(l)dl}\\
&\times \int_{t_0}^{T} \mathbb{E}_{\mathrm{Q}}\left[\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}\, \mathbb{1}_{\{T\geq\tau_U\}} e^{-\int_{t_0}^{T}\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\leq s\leq T}(S_s)>0\right\}} \mid S_l = L(l)\right] \mid \mathcal{F}_{t_0}\right]\\
&\times \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right).
\end{aligned} \tag{59}
$$

From equations (30) and Dias et al. (2021, equation (35)), we can observe, respectively, that

$$
\begin{aligned}
&EDI^0_{t_0}(S_{t_0}, K, L, T; \phi, \tau_L)\\
:=&\mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^{T} r(l)dl}\\
&\times \int_{t_0}^{T} \mathbb{E}_{\mathrm{Q}}\left[\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}\, e^{-\int_{t_0}^{T}\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\leq s\leq T}(S_s)>0\right\}} \mid S_l = L(l)\right] \mid \mathcal{F}_{t_0}\right]\\
&\times \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right)
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
&EDUI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U)\\
:=&\mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^{T} r(l)dl}\\
&\times \int_{t_0}^{T} \mathbb{E}_{\mathrm{Q}}\left[\mathbb{E}_{\mathrm{Q}}\left[(\phi K-\phi S_T)^{+}\, \mathbb{1}_{\{T\geq\tau_U\}} e^{-\int_{t_0}^{T}\lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\leq s\leq T}(S_s)>0\right\}} \mid S_l = L(l)\right] \mid \mathcal{F}_{t_0}\right]\\
&\times \mathbb{Q}\left(\tau_L \in dl \mid \mathcal{F}_{t_0}\right),
\end{aligned} \tag{61}
$$

therefore,

$$
\begin{aligned}
&EDI,UO^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U)\\
=&EDI^0_{t_0}(S_{t_0}, K, L, T; \phi, \tau_L) - EDUI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U).
\end{aligned} \tag{62}
$$

■

Proposition 9 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style* *first-down-in-then-up-and-out on call (if $\phi = -1$) or put (if $\phi = 1$) the asset S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$\begin{aligned} &EUI, DO^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) \\ =&EUI^0_{t_0}(S_{t_0}, K, L, T; \phi, \tau_L) - EUDI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U). \end{aligned} \tag{63}$$

where $EUI^0_{t_0}(S_{t_0}, K, L, T; \phi, \tau_L)$ is the conditional on no default first-up-and-in option and $EUDI^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U)$ is the conditional on no default first-up-then-down-and-in option.

Proof.

The time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-up-then-down-and-in option is defined as

$$\begin{aligned} &EUI, DO^0_{t_0}(S_{t_0}, K, L, U, T; \phi, \tau_L, \tau_U) \\ &= \mathbb{1}_{\{\zeta > t_0\}} e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}} \left[(\phi K - \phi S_T)^+ \mathbb{1}_{\{\tau_U \leq T < \tau_L, \zeta > T\}} \mid \mathcal{G}_{t_0} \right]. \end{aligned}$$

It can be observed that the roles of the upper barrier level, U, and the lower barrier level, L, are reversed in relation to Proposition 8. So, by having L in the place of U and U in the place of L in this Proposition, the steps are similar to those of the previous Proposition. Hence, the proof is omitted.

■

Proposition 10 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-down-in-then-up-and-out put on the asset S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$EDI, UO^D_{t_0}(S_T, K, L, U, T; \phi, \tau_L, \tau_U) = EUO^D_{t_0}(S_T, K, U, T; \phi, \tau_U) \tag{64}$$

where $EUO^D_{t_0}(S_T, K, U, T; \phi, \tau_U)$ is the recovery value of a up-and-out put.

Proof.

The time-t_0 risk-neutral expectation of the recovery value payoff of a first-down-and-in-then-up-and-out put is defined as

$$EDI,UO_{t_0}^{D}(S_T,K,L,U,T;\phi,\tau_L,\tau_U)=\mathbb{1}_{\{\zeta>t_0\}}Ke^{-\int_{t_0}^{T}r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L\leq\zeta\leq T\wedge\tau_U\}}\right] \tag{65}$$

The indicator function can be written as

$$\mathbb{1}_{\{\tau_L\leq\zeta\leq T\wedge\tau_U\}}=\mathbb{1}_{\{\zeta\leq T\wedge\tau_U\}}-\mathbb{1}_{\{\zeta<\tau_L\}}. \tag{66}$$

Given crossing the lower barrier cannot precede the option's default, $\mathbb{1}_{\{\zeta<\tau_L\}}=0$,

$$\mathbb{1}_{\{\tau_L\leq\zeta\leq T\wedge\tau_U\}}=\mathbb{1}_{\{\zeta\leq T\wedge\tau_U\}}. \tag{67}$$

Combining equations (65) and (67),

$$EDI,UO_{t_0}^{D}(S_T,K,L,U,T;\phi,\tau_L,\tau_U)=\mathbb{1}_{\{\zeta>t_0\}}Ke^{-\int_{t_0}^{T}r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\zeta\leq T\wedge\tau_U\}}\right]. \tag{68}$$

By Dias et al. (2021, equation (44)),

$$EUO_{T}^{D}(S_T,K,U,T;\phi,\tau_U):=\mathbb{1}_{\{\zeta>t_0\}}Ke^{-\int_{t_0}^{T}r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\zeta\leq T\wedge\tau_U\}}\right]. \tag{69}$$

Therefore, we reach the intended result.

■

Proposition 11 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta>t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-up-in-then-down-and-out put on the asset S, with strike K, upper barrier levels $U:\mathcal{T}\to\mathbb{R}_+$ (with $U(t_0)>S_{t_0}$), lower barrier levels $L:\mathcal{T}\to\mathbb{R}_+$ (with $L(t_0)<S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$EUI, DO_{t_0}^{D}(S_T, K, L, U, T; 1, \tau_L, \tau_U) = 0 \tag{70}$$

Proof.

The time-t_0 risk-neutral expectation of the recovery value payoff of a first-up-and-in-then-down-and-out put is defined as

$$EUI, DO_{t_0}^{D}(S_T, K, L, U, T; 1, \tau_L, \tau_U) = \mathbb{1}_{\{\zeta > t_0\}} K e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}} \left[\mathbb{1}_{\{\tau_U < \zeta \leq T \wedge \tau_L\}} \right]. \tag{71}$$

The indicator function can be written as

$$\mathbb{1}_{\{\tau_U < \zeta \leq T \wedge \tau_L\}} = \mathbb{1}_{\{\zeta \leq T \wedge \tau_L\}} - \mathbb{1}_{\{\zeta \leq \tau_U\}}. \tag{72}$$

Given $\mathbb{1}_{\{\zeta \leq T \wedge \tau_L\}} = 0$, by the arguments used in Dias et al. (2021, Definition 6), plus $\mathbb{1}_{\{\zeta \leq \tau_U\}} = 0$, as crossing the upper barrier cannot precede the option's default. Therefore, the value of equation (72) is equal to zero, which yields:

$$EUI, DO_{t_0}^{D}(S_T, K, L, U, T; 1, \tau_L, \tau_U) = 0. \tag{73}$$

■

Now, the three-barrier formulae are obtained in a similar fashion.

Proposition 12 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 value unit face value of a conditional on no default and zero rebate European-style first-up-then-down-then-up-and-in call (if $\phi = -1$) or put (if $\phi = 1$) on the asset price S, with strike K, lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), first upper barrier levels $U_1 : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), second upper barrier levels $U_2 : \mathcal{T} \to \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$), maturity at time T ($\geq t_0$) is equal to*

$$EUDUI_{t_0}^{0}\left(S_{t_0}, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}\right)$$

$$=\int_{t_0}^{T} e^{-\int_{t_0}^{u_1} r(l)dl} SP(S_{t_0},t_0;u_1) EDUI^0_{u_1}\left(U_1,K,L,U_2,T;\phi,\tau_L,\tau_{U_2}\right)\mathbb{Q}\left(\tau_{U_1}\in du_1\middle|\,\mathcal{F}_{t_0}\right), \tag{74}$$

where $EDUI^0_u\left(L_1,K,U,L_2,T;\phi,\tau_L,\tau_{U_2}\right)$ is the conditional on no default price of a first-down-then-up-and-in call (if $\phi=-1$) or put (if $\phi=1$), $SP(S_{t_0},t_0;u_1)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_{U_1}\in du_1\middle|\,\mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time τ_{U_1}.

Proof. Assuming that $\zeta > t_0$, the time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-up-then-down-then-up-and-in barrier option is defined as

$$\begin{aligned}
&EUDUI^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_{U_1},\tau_{U_2})\\
=&e^{-\int_{t_0}^{T} r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\left\{\tau_{U_1}<\tau_L<\tau_{U_2}\le T,\zeta>T\right\}}\mid\mathcal{G}_{t_0}\right]\\
=&e^{-\int_{t_0}^{T} r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\left\{\tau_{U_1}<\tau_L\right\}}\mathbb{1}_{\left\{\tau_L<\tau_{U_2}\right\}}\mathbb{1}_{\left\{\tau_{U_2}\le T\right\}}\mathbb{1}_{\{\zeta>T\}}\mid\mathcal{F}_{t_0}\right].
\end{aligned}\tag{75}$$

Using the lower law and Dias et al. (2015, equation (19)) the expectation becomes

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\left\{\tau_{U_1}<\tau_L\right\}}\mathbb{1}_{\left\{\tau_L<\tau_{U_2}\right\}}\mathbb{1}_{\left\{\tau_{U_2}\le T\right\}}\mathbb{1}_{\{\zeta>T\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_1}<\tau_L\right\}}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_L<\tau_{U_2}\right\}}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_2}\le T\right\}}(\phi K-\phi S_T)^+\mathbb{1}_{\{T<\zeta\}}\mid\mathcal{G}_{\tau_{U_2}}\right]\mid\mathcal{G}_{\tau_L}\right]\mid\mathcal{G}_{\tau_{U_1}}\right]\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{t_0}^{\tau_{U_1}}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le\tau_{U_1}\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{U_1}}^{\tau_L}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_{U_1}\le v\le L\}}(S_v)>0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_L}^{\tau_{U_2}}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{L\le v\le\tau_{U_2}\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{U_2}}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_{U_2}\le v\le T\}}(S_v)>0\}}(\phi K-\phi S_T)^+\mid\mathcal{F}_{\tau_{U_2}}\Big]\mid\mathcal{F}_{\tau_L}\Big]\mid\mathcal{F}_{\tau_{U_1}}\Big]\mid\mathcal{F}_{t_0}\Big].
\end{aligned}\tag{76}$$

Given the asset price process behaves as a pure Markovian diffusion process with respect to the restricted filtration $\mathbb{F}$, the equation above can be restated as

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\left\{\tau_{U_1}<\tau_L\right\}}\mathbb{1}_{\left\{\tau_L<\tau_{U_2}\right\}}\mathbb{1}_{\left\{\tau_{U_2}\le T\right\}}\mathbb{1}_{\{\zeta>T\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{t_0}^{u_1}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le u_1\}}(S_v)>0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{u_1}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u_1\le v\le l\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{l}^{u_2}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{l\le v\le u_2\}}(S_v)>0\}}\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{u_2}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u_2\le v\le T\}}(S_v)>0\}}\\
&\times(\phi K-\phi S_T)^+\mid S_{u_2}=U_2(u_2)\Big]\mid S_l=L(l)\Big]\mid S_{u_1}=U_1(u_1)\Big]\mid\mathcal{F}_{t_0}\Big]
\end{aligned}$$

$$\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U(u_1)\right)\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right). \tag{77}$$

Using again Dias et al. (2015, equation (19)) we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+\,\mathbb{1}_{\left\{\tau_{U_1}<\tau_L\right\}}\mathbb{1}_{\left\{\tau_L<\tau_{U_2}\right\}}\mathbb{1}_{\left\{\tau_{U_2}\leq T\right\}}\mathbb{1}_{\{\zeta>T\}}\mid\mathcal{G}_{t_0}\right]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}SP(S_{t_0},t_0;u_1)\\
&\times SP(S(u_1),u_1;l)SP(S(l),l;u_2)(\phi K-\phi S_T)^+\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U(u_1)\right)\\
&\times\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right).
\end{aligned}\tag{78}$$

Combining equations (75) and (78), we obtain

$$\begin{aligned}
&EUDUI^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_{U_1},\tau_{U_2})\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{T}r(l)dl}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}SP(S_{t_0},t_0;u_1)\\
&\times SP(S(u_1),u_1;l)SP(S(l),l,u_2)(\phi K-\phi S_T)^+\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U(u_1)\right)\\
&\times\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{t_0}^{u_1}r(l)dl}\int_{t_0}^{T}\\
&\times e^{-\int_{u_1}^{T}r(l)dl}\int_{u_1}^{T}\int_{l}^{T}SP(S_{t_0},t_0;u_1)\\
&\times SP(S(u_1),u_1;l)SP(S(l),l;u_2)(\phi K-\phi S_T)^+\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U(u_1)\right)\\
&\times\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right).
\end{aligned}\tag{79}$$

We can observe that

$$\begin{aligned}
&EDUI^0_{u_1}\left(U_1,K,L,U_2,T;\phi,\tau_L,\tau_{U_2}\right):=\\
&\mathbb{1}_{\{\zeta>t_0\}}e^{-\int_{u_1}^{T}r(l)dl}\int_{u_1}^{T}\int_{l}^{T}SP(S_{t_0},t_0;u_1)\\
&\times SP(S(u_1),u_1;l)SP(S(l),l;u_2)(\phi K-\phi S_T)^+\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U(u_1)\right).
\end{aligned}\tag{80}$$

Thus we obtain the intended result. ■

Proposition 13 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 value of a unit face value conditional on no default and zero rebate European-style first-down-then-up-then-down-and-in call (if $\phi = -1$) or put (if $\phi = 1$) on the asset price S, with strike K, first lower barrier levels $L_1 : \mathcal{T} \to \mathbb{R}_+$ (with $L_1(t_0) < S_{t_0}$), second lower barrier levels $L_2 : \mathcal{T} \to \mathbb{R}_+$ (with $L_2(t_0) < S_{t_0}$), upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U(t_0) > S_{t_0}$), maturity at time T ($\geq t_0$) is equal to*

$$\begin{aligned} & EDUDI^0_{t_0}\left(S_{t_0}, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\ = & \int_{t_0}^{T} e^{-\int_{t_0}^{l_1} r(l)dl} SP(S_{t_0}, t_0; l_1) EUDI^0_{l_1}\left(L_1, K, L_2, U, T; \phi, \tau_{L_2}, \tau_U\right) \mathbb{Q}\left(\tau_{L_1} \in dl_1 \middle| \mathcal{F}_{t_0}\right), \end{aligned} \tag{81}$$

where $EUDI^0_{l_1}\left(L_1, K, L_2, U, T; \phi, \tau_{L_2}, \tau_U\right)$ is the conditional on no default price of a first-up-then-down-and-in call (if $\phi = -1$) or put (if $\phi = 1$), $SP(S_{t_0}, t_0; l_1)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_{L_1} \in dl_1 \middle| \mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time τ_{L_1}.

Proof. The time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-down-then-up-then-down-and-in barrier option is defined as

$$\begin{aligned} & EDUDI^0_{t_0}\left(S_{t_0}, K, L_1, L_2, U, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U\right) \\ = & e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[\left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\left\{\tau_{L_1} < \tau_U < \tau_{L_2} \leq T, \zeta > T\right\}} \mid \mathcal{G}_{t_0}\right], \end{aligned}$$

By switching the roles of the barriers L_1, L_2, U of this Proposition with those of U_1, U_2, L of Proposition 12, respectively, this proof follows the same steps, and is therefore omitted. ■

Proposition 14 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-up-then-down-then-up-and-in put on the asset S, with strike K, lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), first upper barrier levels $U_1 : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), second upper barrier levels $U_2 : \mathcal{T} \to \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$) and maturity* *at time T $(\geq t_0)$ is equal to*

$$
\begin{aligned}
&EUDUI^D_{t_0}(S_{t_0}, K, L, U_1, U_2, T; 1, \tau_L, \tau_{U_1}, \tau_{U_2}) \\
=&\int_{t_0}^{T} e^{-\int_{t_0}^{u_1} r(l)dl} SP(S_{t_0}, t_0; u_1) EDUI^D_{u_1}(U_1, K, L, U_2, T; 1, \tau_L, \tau_{U_2}) \mathbb{Q}\left(\tau_{U_1} \in du_1 \mid \mathcal{F}_{t_0}\right),
\end{aligned} \tag{82}
$$

where $EDUI^D_{u_1}(U_1, K, L, U_2, T; 1, \tau_L, \tau_{U_2})$ is the time-u_1 recovery value of a first-down-then-up-and-in put, $SP(S_{t_0}, t_0; u_1)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_{U_1} \in du_1 | \mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time τ_{U_1}.

Proof. The time-t_0 risk-neutral expectation of the recovery value of a first-up-then-down-then-up-and-in barrier put is defined as

$$
EUDUI^D_{t_0}(S_0, K, L, U_1, U_2, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U) = \mathbb{1}_{\{\zeta > t_0\}} K e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{U_1} < \tau_L < \tau_{U_2} \leq \zeta \leq T\}} \mid \mathcal{G}_{t_0}\right]. \tag{83}
$$

Using the tower law, we can write the expectation as

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L < \tau_{U_2} < \zeta \leq T\right\}} \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L\right\}} \mathbb{1}_{\left\{\tau_L < \tau_{U_2}\right\}} \mathbb{1}_{\left\{\tau_{U_2} < \zeta\right\}} \mathbb{1}_{\{\zeta \leq T\}} \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L\right\}} \mathbb{1}_{\left\{\tau_L < \tau_{U_2}\right\}} \mathbb{1}_{\left\{\tau_{U_2} < \zeta\right\}} (1 - \mathbb{1}_{\{\zeta > T\}}) \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L\right\}} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_L < \tau_{U_2}\right\}} \mathbb{1}_{\left\{\tau_{U_2} < \zeta\right\}} \mathbb{E}_{\mathbb{Q}}\left[(1 - \mathbb{1}_{\{\zeta > T\}}) \mid \mathcal{G}_{\tau_{U_2}}\right] \mid \mathcal{G}_{\tau_L}\right] \mid \mathcal{G}_{\tau_{U_1}}\right] \mid \mathcal{G}_{t_0}\right].
\end{aligned} \tag{84}
$$

Using Dias et al. (2015, equation (19))

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L < \tau_{U_2} < \zeta \leq T\right\}} \mid \mathcal{G}_{t_0}\right] \\
=&\mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L\right\}} \mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{1}_{\left\{\tau_L < \tau_{U_2}\right\}} \mathbb{1}_{\left\{\tau_{U_2} < \zeta\right\}} \\
&\times \mathbb{E}_{\mathbb{Q}}\left[(1 - e^{-\int_{\tau_{U_2}}^{T} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{\tau_{U_2} \leq v \leq T\}}(S_v) > 0\}}) \mid \mathcal{F}_{\tau_{U_2}}\right] \mid \mathcal{G}_{\tau_L}\Bigg] \mid \mathcal{G}_{\tau_{U_1}}\Bigg] \mid \mathcal{G}_{t_0}\Bigg] \\
=&\mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{E}_{\mathbb{Q}}\Bigg[\mathbb{1}_{\left\{\tau_{U_1} < \tau_L\right\}} \mathbb{1}_{\left\{\tau_L < \tau_{U_2}\right\}} \mathbb{1}_{\left\{\tau_{U_2} < \zeta\right\}} \mathbb{E}_{\mathbb{Q}}\Bigg[e^{-\int_{\tau_L}^{\tau_{U_2}} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{\tau_L \leq v \leq \tau_{U_2}\}}(S_v) > 0\}} \\
&\times \mathbb{E}_{\mathbb{Q}}\left[(1 - e^{-\int_{\tau_{U_2}}^{T} \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{\tau_{U_2} \leq v \leq T\}}(S_v) > 0\}}) \mid \mathcal{F}_{\tau_{U_2}}\right] \mid \mathcal{F}_{\tau_L}\Bigg] \mid \mathcal{G}_{\tau_{U_1}}\Bigg] \mid \mathcal{G}_{t_0}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau_{U_1}<\tau_L\}}\mathbb{1}_{\{\tau_L<\tau_{U_2}\}}\mathbb{1}_{\{\tau_{U_2}<\zeta\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{U_1}}^{\tau_L}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_{U_1}\le v\le\tau_L\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_L}^{\tau_{U_2}}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_L\le v\le\tau_{U_2}\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[(1-e^{-\int_{\tau_{U_2}}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_{U_2}\le v\le T\}}(S_v)>0\}})\mid\mathcal{F}_{\tau_{U_2}}\Big]\mid\mathcal{F}_{\tau_L}\Big]\mid\mathcal{F}_{\tau_{U_1}}\Big]\mid\mathcal{G}_{t_0}\Big]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau_{U_1}<\tau_L\}}\mathbb{1}_{\{\tau_L<\tau_{U_2}\}}e^{-\int_{\tau_{t_0}}^{\tau_{U_1}}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le\tau_{U_1}\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{U_1}}^{\tau_L}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_{U_1}\le v\le\tau_L\}}(S_v)>0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_L}^{\tau_{U_2}}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_L\le v\le\tau_{U_2}\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[(1-e^{-\int_{\tau_{U_2}}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{\tau_{U_2}\le v\le T\}}(S_v)>0\}})\mid\mathcal{F}_{\tau_{U_2}}\Big]\mid\mathcal{F}_{\tau_L}\Big]\mid\mathcal{F}_{\tau_{U_1}}\Big]\mid\mathcal{F}_{t_0}\Big].
\end{aligned}
\tag{85}
$$

Given the asset price process behaves as a pure Markovian diffusion process with respect to the restricted filtration $\mathbb{F}$, the equation above can be restated as

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau_{U_1}<\tau_L<\tau_{U_2}<\zeta\le T\}}\mid\mathcal{G}_{t_0}\Big]\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{t_0}}^{u_1}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le u_1\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{u_1}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u_1\le v\le l\}}(S_v)>0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{l}^{u_2}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{l\le v\le u_2\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[(1-e^{-\int_{u_2}^{T}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u_2\le v\le T\}}(S_v)>0\}})\mid S_{u_2}=U_2(u_2)\Big]\mid S_l=L(l)\Big]\mid S_{u_1}=U_1(u_1)\Big]\mid\mathcal{F}_{t_0}\Big]\\
&\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U_1(u_1)\right)\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right).
\end{aligned}
\tag{86}
$$

Again, through Dias et al. (2015, equation (19))

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau_{U_1}<\tau_L<\tau_{U_2}<\zeta\le T\}}\mid\mathcal{G}_{t_0}\Big]=\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{t_0}}^{u_1}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le u_1\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{u_1}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u_1\le v\le l\}}(S_v)>0\}}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{l}^{u_2}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{l\le v\le u_2\}}(S_v)>0\}}\\
&\times\left(1-SP(U_2(u_2),u_2,T)\right)\Big]\mid S_l=L(l)\Big]\mid S_{u_1}=U_1(u_1)\Big]\mid\mathcal{F}_{t_0}\Big]\\
&\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U_1(u_1)\right)\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{t_0}}^{u_1}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le u_1\}}(S_v)>0\}}\\
&\times\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{u_1}^{l}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{u_1\le v\le l\}}(S_v)>0\}}SP(L(l),l,u_2)\\
&\times\left(1-SP(U_2(u_2),u_2,T)\right)\Big]\mid S_{u_1}=U_1(u_1)\Big]\mid\mathcal{F}_{t_0}\Big]\\
&\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U_1(u_1)\right)\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}\mathbb{E}_{\mathbb{Q}}\Big[e^{-\int_{\tau_{t_0}}^{u_1}\lambda(S,i)di}\mathbb{1}_{\{\inf_{\{t_0\le v\le u_1\}}(S_v)>0\}}\\
&\times SP(U_1(u_1),u_1,l)SP(L(l),l,u_2)\\
&\times(1-SP(U_2(u_2),u_2,T))\Big]\Big]\mid\mathcal{F}_{t_0}\Big]\\
&\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U_1(u_1)\right)\mathbb{Q}\left(\tau_{U_1}\in dl\mid\mathcal{F}_{t_0}\right)\\
=&\mathbb{1}_{\{\zeta>t_0\}}\int_{t_0}^{T}\int_{u_1}^{T}\int_{l}^{T}SP(S_0,t_0,u_1)SP(U_1(u_1),u_1,l)SP(L(l),l,u_2)(1-SP(U_2(u_2),u_2,T))\\
&\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U_1(u_1)\right)\mathbb{Q}\left(\tau_{U_1}\in du_1\mid\mathcal{F}_{t_0}\right).
\end{aligned}\tag{87}
$$

Therefore, by combining equations (83) and (87), and by taking into account equation (47), we observe that

$$
\begin{aligned}
&EDUI^D_{u_1}(U_1,K,L,U_2,T;1,\tau_L,\tau_{U_2})\\
:=&\mathbb{1}_{\{\zeta>t_0\}}Ke^{-\int_{u_1}^{T}r(l)dl}\int_{u_1}^{T}\int_{l}^{T}SP(U_1(u_1),u_1,l)SP(L(l),l,u_2)(1-SP(U_2(u_2),u_2,T))\\
&\times\mathbb{Q}\left(\tau_{U_2}\in du_2\mid S_l=L(l)\right)\mathbb{Q}\left(\tau_L\in dl\mid S_{u_1}=U_1(u_1)\right),
\end{aligned}\tag{88}
$$

thus we reach the intended result. ■

Proposition 15 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-down-then-up-then-down-and-in put on the asset S, with strike K, lower barrier levels $L:\mathcal{T}\to\mathbb{R}_+$ (with $L(t_0)<S_{t_0}$), first upper barrier levels $U_1:\mathcal{T}\to\mathbb{R}_+$ (with $U_1(t_0)>S_{t_0}$), second upper barrier levels $U_2:\mathcal{T}\to\mathbb{R}_+$ (with $U_2(t_0)>S_{t_0}$) and maturity at time T ($\ge t_0$) is equal to*

$$
EDUDI^D_{t_0}(S_{t_0},K,L_1,U,L_2,T;1;\tau_{L_1},\tau_{L_2},\tau_U)=EDUI^D_{t_0}(L_1,K,U,T;1,\tau_{L_1},\tau_U)\tag{89}
$$

where $EDUI^D_{u_1}(U_1,K,L,U_2,T;1,\tau_L,\tau_{U_2})$ is the time-u_1 recovery value of a first-down-then-up-and-in put.

Proof.

The time-t_0 risk-neutral expectation of the recovery value of a first-up-then-down-then-up-and-in barrier put is defined as

$$\begin{aligned}&EDUDI_{t_0}^{D}(S_{t_0}, K, L_1, U, L_2, T; 1; \tau_{L_1}, \tau_{L_2}, \tau_U)\\ =&\mathbb{1}_{\{\zeta>t_0\}} K e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{L_1}<\tau_U<\tau_{L_2}\leq\zeta\leq T,\}}\right]. \end{aligned} \tag{90}$$

The indicator function can be written as

$$(\phi K)^{+}\mathbb{1}_{\{\tau_{L_1}<\tau_U<\tau_{L_2}\leq\zeta\leq T,\}} = (\phi K)^{+}\mathbb{1}_{\{\tau_{L_1}<\tau_U\leq\zeta\leq T\}} - (\phi K)^{+}\mathbb{1}_{\{\tau_{L_1}<\tau_U<\zeta<\tau_{L_2}\}}. \tag{91}$$

Since the default event cannot precede the knock-in event, we have $\mathbb{1}_{\{\tau_{L_1}<\tau_U<\zeta<\tau_{L_2}\}} = 0$, therefore

$$(\phi K)^{+}\mathbb{1}_{\{\tau_{L_1}<\tau_U<\tau_{L_2}\leq\zeta\leq T,\}} = (\phi K)^{+}\mathbb{1}_{\{\tau_{L_1}<\tau_U\leq\zeta\leq T\}} \tag{92}$$

Combining equations (90) and (92),

$$EDUDI_{t_0}^{D}(S_{t_0}, K, L_1, U, L_2, T; 1; \tau_{L_1}, \tau_{L_2}, \tau_U) = \mathbb{1}_{\{\zeta>t_0\}} K e^{-\int_{t_0}^{T} r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{L_1}<\tau_U\leq\zeta\leq T,\}}\right]. \tag{93}$$

And from equation (40), we can observe that

$$EDUI_{t_0}^{D}(L_1, K, U, T; 1, \tau_{L_1}, \tau_U) := (\phi K)^{+}\mathbb{1}_{\{\tau_{L_1}<\tau_U\leq\zeta\leq T\}}, \tag{94}$$

Thus, the result is obtained. ■

Proposition 16 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-up-then-down-in-then-up-and-out on call (if $\phi = -1$) or put (if $\phi = 1$) the asset S, with strike K, first upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), second upper* *barrier levels $U : \mathcal{T} \rightarrow \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$), lower barrier levels $L : \mathcal{T} \rightarrow \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$
\begin{aligned}
&EUDI,UO^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_{U_1},\tau_{U_2})\\
=&EUDI^0_{t_0}(S_{t_0},K,L,U_1,T;\phi,\tau_L,\tau_{U_1}) - EUDUI^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_{U_1},\tau_{U_2}).
\end{aligned} \tag{95}
$$

where $EUDI^0_{t_0}(S_{t_0},K,L,T;\phi,\tau_L,\tau_{U_1})$ is the conditional on no default first-up-then-down-and-in option and $EUDUI^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_{U_1},\tau_{U_2})$ is the conditional on no default first-up-then-down-then-up-and-in option.

Proof. The time-t_0 risk-neutral expectation of the conditional on no default payoff of a first-up-then-down-then-up-and-in option is defined as

$$
\begin{aligned}
&EUDI,UO^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_U) \\
=&\mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^T r(l)dl} \mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\left\{\tau_{U_1}<\tau_L\leq T<\tau_{U_2},\zeta>T\right\}} \mid \mathcal{G}_{t_0}\right].
\end{aligned} \tag{96}
$$

Using the tower law, the expected value can be written as

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\left\{\tau_{U_1}<\tau_L\leq T<\tau_{U_2},\zeta>T\right\}} \mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\left\{\tau_{U_1}\leq\tau_L\right\}}\mathbb{1}_{\left\{\tau_L\leq T\right\}}\mathbb{1}_{\left\{T<\tau_{U_2}\right\}}\mathbb{1}_{\left\{\zeta>T\right\}} \mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mathbb{1}_{\left\{\tau_{U_1}\leq\tau_L\right\}}\mathbb{1}_{\left\{\tau_L\leq T\right\}}\mathbb{1}_{\left\{T<\tau_{U_2}\right\}}\mathbb{1}_{\left\{\zeta>T\right\}} \mid \mathcal{G}_{\tau_L}\right] \mid \mathcal{G}_{\tau_{U_1}}\right] \mid \mathcal{G}_{t_0}\right]
\end{aligned} \tag{97}
$$

Using equation Dias et al. (2015, equation(19))

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\Big[(\phi K - \phi S_T)^+ \mathbb{1}_{\left\{\tau_{U_1}<\tau_L\leq T<\tau_{U_2},\zeta>T\right\}} \mid \mathcal{G}_{t_0}\Big]\\
=&\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[(\phi K - \phi S_T)^+ \mathbb{1}_{\left\{\tau_{U_1}\leq\tau_L\right\}}\mathbb{1}_{\left\{\tau_L\leq T\right\}}\mathbb{1}_{\left\{T<\tau_{U_2}\right\}}\\
&\times e^{-\int_{t_0}^T \lambda(S,i)di}\mathbb{1}_{\left\{\inf_{t_0\leq s\leq T}(S_s)>0\right\}} \mid \mathcal{F}_{\tau_L}\Big] \mid \mathcal{G}_{\tau_{U_1}}\Big] \mid \mathcal{G}_{t_0}\Big].
\end{aligned} \tag{98}
$$

Combining equations (96) - (98), given the underlying asset process behaves as a pure Markovian diffusion process with respect to the restricted filtration $\mathcal{F}$,

$$
\begin{aligned}
& EUDI,UO^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_U) \\
& = \mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^T r(l)dl} \\
& \times \int_{t_0}^T \int_{u_1}^T \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[(\phi K-\phi S_T)^+ \mathbb{1}_{\left\{T<\tau_{U_2}\right\}} e^{-\int_{t_0}^T \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0\le v\le T\}}(S_v)>0\}} \mid \mathcal{F}_{\tau_L}\right] \mid \mathcal{F}_{\tau_{U_1}}\right] \mid \mathcal{F}_{t_0}\right] \\
& \times \mathbb{Q}\left(\tau_L\in dl \mid S_{u_1}=U_1(u_1)\right) \mathbb{Q}\left(\tau_{U_1}\in dl \mid \mathcal{F}_{t_0}\right)
\end{aligned}
\tag{99}
$$

Given $\mathbb{1}_{\{T<\tau_U\}} = 1-\mathbb{1}_{\{T\ge\tau_U\}}$

$$
\begin{aligned}
& EUDI,UO^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_U) \\
& = \mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^T r(l)dl} \int_{t_0}^T \int_{u_1}^T \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[(\phi K-\phi S_T)^+ (1-\mathbb{1}_{\{T\ge\tau_U\}}) e^{-\int_{t_0}^T \lambda(S,i)di} \\
& \times \mathbb{1}_{\{\inf_{\{t_0\le v\le T\}}(S_v)>0\}} \mid \mathcal{F}_{\tau_L}\Big] \mid \mathcal{F}_{\tau_{U_1}}\Big] \mid \mathcal{F}_{t_0}\Big] \\
& \times \mathbb{Q}\left(\tau_L\in dl \mid S_{u_1}=U_1(u_1)\right) \mathbb{Q}\left(\tau_{U_1}\in dl \mid \mathcal{F}_{t_0}\right).
\end{aligned}
\tag{100}
$$

Therefore,

$$
\begin{aligned}
& EUDI,UO^0_{t_0}(S_{t_0},K,L,U_1,U_2,T;\phi,\tau_L,\tau_U) \\
=& \mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^T r(l)dl} \\
& \times \int_{t_0}^T \int_{u_1}^T \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[(\phi K-\phi S_T)^+ e^{-\int_{t_0}^T \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0\le v\le T\}}(S_v)>0\}} \times \mid \mathcal{F}_{\tau_L}\Big] \mid \mathcal{F}_{\tau_{U_1}}\Big] \mid \mathcal{F}_{t_0}\Big] \\
& \times \mathbb{Q}\left(\tau_L\in dl \mid S_{u_1}=U_1(u_1)\right) \mathbb{Q}\left(\tau_{U_1}\in dl \mid \mathcal{F}_{t_0}\right) \\
& - \mathbb{1}_{\{\zeta>t_0\}} e^{-\int_{t_0}^T r(l)dl} \\
& \times \int_{t_0}^T \int_{u_1}^T \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[(\phi K-\phi S_T)^+ \mathbb{1}_{\{T\ge\tau_U\}} e^{-\int_{t_0}^T \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0\le v\le T\}}(S_v)>0\}} \times \mid \mathcal{F}_{\tau_L}\Big] \mid \mathcal{F}_{\tau_{U_1}}\Big] \mid \mathcal{F}_{t_0}\Big] \\
& \times \mathbb{Q}\left(\tau_L\in dl \mid S_{u_1}=U_1(u_1)\right) \mathbb{Q}\left(\tau_{U_1}\in dl \mid \mathcal{F}_{t_0}\right)
\end{aligned}
$$

With the result from Proposition 5 and equation (86), respectively, we can observe

$$
\begin{aligned}
&EUDI^0_{t_0}(S_{t_0}, K, L, U_1, T; \phi, \tau_L, \tau_{U_1}) \\
:=&\mathbb{1}_{\{\zeta > t_0\}} e^{-\int_{t_0}^T r(l)dl} \\
&\times \int_{t_0}^T \int_{u_1}^T \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[(\phi K - \phi S_T)^+ \, e^{-\int_{t_0}^T \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0 \leq v \leq T\}}(S_v) > 0\}} \times \mid \mathcal{F}_{\tau_L}\Big] \mid \mathcal{F}_{\tau_{U_1}}\Big] \mid \mathcal{F}_{t_0}\Big] \\
&\times \mathbb{Q}\left(\tau_L \in dl \mid S_{u_1} = U_1(u_1)\right) \mathbb{Q}\left(\tau_{U_1} \in dl \mid \mathcal{F}_{t_0}\right) \text{ and}
\end{aligned} \tag{101}
$$

$$
\begin{aligned}
&EUDUI^0_{t_0}(S_{t_0}, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2}) \\
&= \mathbb{1}_{\{\zeta > t_0\}} e^{-\int_{t_0}^T r(l)dl} \\
&\times \int_{t_0}^T \int_{u_1}^T \mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{E}_{\mathbb{Q}}\Big[(\phi K - \phi S_T)^+ \, \mathbb{1}_{\{T \geq \tau_U\}} e^{-\int_{t_0}^T \lambda(S,i)di} \mathbb{1}_{\{\inf_{\{t_0 \leq v \leq T\}}(S_v) > 0\}} \times \mid \mathcal{F}_{\tau_L}\Big] \mid \mathcal{F}_{\tau_{U_1}}\Big] \mid \mathcal{F}_{t_0}\Big] \\
&\times \mathbb{Q}\left(\tau_L \in dl \mid S_{u_1} = U_1(u_1)\right) \mathbb{Q}\left(\tau_{U_1} \in dl \mid \mathcal{F}_{t_0}\right),
\end{aligned} \tag{102}
$$

therefore,

$$
\begin{aligned}
&EUDI, UO^0_{t_0}(S_{t_0}, K, L, U_1, U_2, T; \phi, \tau_L, \tau_U) \\
=&EUDI^0_{t_0}(S_{t_0}, K, L, U_1, T; \phi, \tau_L, \tau_{U_1}) - EUDUI^0_{t_0}(S_{t_0}, K, L, U_1, U_2, T; \phi, \tau_L, \tau_{U_1}, \tau_{U_2})
\end{aligned} \tag{103}
$$

■

Proposition 17 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-down-then-up-in-then-down-and-out on call (if $\phi = -1$) or put (if $\phi = 1$) the asset S, with strike K, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U_(t_0) > S_{t_0}$), first lower barrier levels $L_1 : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), second lower barrier levels $L_2 : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$
EDUI, DO^0_{t_0}(S_{t_0}, K, L, U_1, U_2, T; \phi, \tau_{L_1}, \tau_{L_2}, \tau_U)
$$

$$=EDUI^0_{t_0}(S_{t_0},K,L_1,U,T;\phi,\tau_L,\tau_{U_1}) - EDUDI^0_{t_0}(S_{t_0},K,L_1,L_2,U,T;\phi,\tau_{L_1},\tau_{L_2},\tau_U). \tag{104}$$

where $EDUI^0_{t_0}(S_{t_0},K,L_1,U,T;\phi,\tau_L,\tau_{U_1})$ is the conditional on no default first-down-then-up-and-in option and $EDUDI^0_{t_0}(S_{t_0},K,L_1,L_2,U,T;\phi,\tau_{L_1},\tau_{L_2},\tau_U)$ is the conditional on no default first-up-then-down-then-up-and-in option.

Proof. By switching the roles of the barriers L_1, L_2, U of this Proposition with those of U_1, U_2, L of Proposition 16, respectively, this proof follows the same steps, and is therefore omitted. ■

Proposition 18 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-up-then-down-in-then-up-and-out put on the asset S, with strike K, lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), first upper barrier levels $U_1 : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), second upper barrier levels $U_2 : \mathcal{T} \to \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$) and maturity at time T ($\geq t_0$) is equal to*

$$\begin{aligned} &EUDI,UO^D_{t_0}(S_{t_0},K,L,U_1,U_2,T;1,\tau_L,\tau_{U_1},\tau_{U_2}) \\ &= \int_{t_0}^{T} e^{-\int_{t_0}^{u_1} r(l)dl} SP(S_{t_0},t_0;u_1) EDI,UO^D_{u_1}(U_1,K,L,U_2,T;1,\tau_L,\tau_{U_2})\, \mathbb{Q}\left(\tau_{U_1} \in du_1 \middle| \mathcal{F}_{t_0}\right), \end{aligned} \tag{105}$$

where $EDI,UO^D_{u_1}(U_1,K,L,U_2,T;1,\tau_L,\tau_{U_2})$ is the recovery value of a first-down-in-then-up-and-out put and $SP(S_{t_0},t_0;u_1)$ is the risk-neutral survival probability and $\mathbb{Q}\left(\tau_{U_1} \in du_1 \middle| \mathcal{F}_{t_0}\right)$ represents the probability density function of the first passage time over the lower barrier τ_{U_1}.

Proof. The time-t_0 risk-neutral expectation of the recovery value of first-up-the-down-then-up-out put is defined by

$$EUDI,UO^D_{t_0}(S_{t_0},K,L,U_1,U_2,T;1,\tau_L,\tau_{U_1},\tau_{U_2}) = Ke^{-\int_{t_0}^{T} r(l)dl}\mathbb{E}_{\mathbb{Q}}\Big[\mathbb{1}_{\{\tau_{U_1}<\tau_L<\zeta\leq T\wedge\tau_{U_2}\}} \mid \mathcal{G}_{t_0}\Big]. \tag{106}$$

Using the tower law and Dias et al. (2015, Equation (19)), the expected value becomes

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{U_1}<\tau_L<\zeta\leq T\wedge\tau_{U_2}\}} \mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{U_1}<\tau_L<\zeta\leq T\wedge\tau_{U_2}\}} \mid \mathcal{G}_{\tau_{U_1}}\right] \mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{U_1}<\tau_L\}}\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{U_2}\}} \mid \mathcal{G}_{\tau_{U_1}}\right] \mid \mathcal{G}_{t_0}\right]\\
=&\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_{L_1}}\lambda(S,i,)di}\mathbb{1}_{\left\{\inf_{t_0\leq s\leq\tau_{L_1}}(S_s)>0\right\}}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{U_2}\}} \mid \mathcal{F}_{\tau_{U_1}}\right] \mid \mathcal{F}_{t_0}\right]. \qquad (107)
\end{aligned}
$$

Given the asset price with respect to the restricted filtration $\mathbb{F}$, the expectation can be written as

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{L_1}<\tau_L<\zeta\leq T\wedge\tau_{L_2}\}} \mid \mathcal{G}_{t_0}\right]\\
=&\int_{t_0}^{T}\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\tau_{L_1}}\lambda(S,i,)di}\mathbb{1}_{\left\{\inf_{t_0\leq s\leq\tau_{L_1}}(S_s)>0\right\}}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{L_2}\}} \mid S_{u_1}=U(u_1)\right] \mid \mathcal{F}_{t_0}\right]\\
&\times\mathbb{Q}\left(\tau_{U_1}\in du_1 | \mathcal{F}_{t_0}\right). \qquad (108)
\end{aligned}
$$

And using again Dias et al. (2015, Equation (19))

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{L_1}<\tau_L<\zeta\leq T\wedge\tau_{L_2}\}} \mid \mathcal{G}_{t_0}\right]\\
=&\int_{t_0}^{T}SP(S_{t_0},t_0,u_1)\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{L_2}\}} \mid S_{u_1}=U(u_1)\right]\mathbb{Q}\left(\tau_{U_1}\in du_1 | \mathcal{F}_{t_0}\right). \qquad (109)
\end{aligned}
$$

By combining equations(106) and (109),

$$
\begin{aligned}
&EUDI,UO_{t_0}^{D}\left(S_{t_0},K,L,U_1,U_2,T;1,\tau_L,\tau_{U_1},\tau_{U_2}\right)\\
=&Ke^{-\int_{t_0}^{T}r(l)dl}\int_{t_0}^{T}SP(S_{t_0},t_0,u_1)\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{L_2}\}} \mid S_{u_1}=U(u_1)\right]\mathbb{Q}\left(\tau_{U_1}\in du_1 | \mathcal{F}_{t_0}\right)\\
=&e^{-\int_{t_0}^{u_1}r(l)dl}\int_{t_0}^{T}Ke^{-\int_{u_1}^{T}r(l)dl}SP(S_{t_0},t_0,u_1)\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{L_2}\}} \mid S_{u_1}=U(u_1)\right]\mathbb{Q}\left(\tau_{U_1}\in du_1 | \mathcal{F}_{t_0}\right).\\
&\qquad (110)
\end{aligned}
$$

Given equation (65), it can be observed that

$$Ke^{-\int_{u_1}^{T} r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_L<\zeta\leq T\wedge\tau_{U_2}\}} \mid S_{u_1}=U(u_1)\right] := EDI,UO^{D}_{u_1}(S_{u_1},K,L,U_2,T;1,\tau_L,\tau_{U_2}),$$

thus we achieve the intended result ■

Proposition 19 *Under the financial model defined by equations (1) to (5) and assuming that $\zeta > t_0$, the time-t_0 recovery value of a unit face value and zero rebate European-style first-down-then-up-in-then-down-and-out put on the asset S, with strike K, first lower barrier levels $L_1 : \mathcal{T} \to \mathbb{R}_+$ (with $L(t_0) < S_{t_0}$), second lower barrier levels $L_2 : \mathcal{T} \to \mathbb{R}_+$ (with $U_1(t_0) > S_{t_0}$), upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $U_2(t_0) > S_{t_0}$) and maturity at time T $(\geq t_0)$ is equal to*

$$EDUI,DO^{D}_{t_0}\left(S_{t_0},K,L,U_1,U_2,T;1,\tau_{L_1},\tau_{L_2},\tau_U\right)=0 \tag{111}$$

Proof. The time-t_0 risk-neutral expectation of the recovery value of first-up-the-down-then-up-out put is defined by

$$EDUI,DO^{D}_{t_0}\left(S_{t_0},K,L,U_1,U_2,T;1,\tau_{L_1},\tau_{L_2},\tau_U\right)=Ke^{-\int_{t_0}^{T} r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{L_1}<\tau_U<\zeta\leq T\wedge\tau_{L_2}\}} \mid \mathcal{G}_{t_0}\right]. \tag{112}$$

By switching the roles of the barriers L_1, L_2, U of this Proposition with those of U_1, U_2, L of Proposition 18, respectively, following the same steps until equation (110), we obtain

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_{L_1}<\tau_U<\zeta\leq T\wedge\tau_{L_2}\}} \mid \mathcal{G}_{t_0}\right]\\ &=\int_{t_0}^{T} SP(S_{t_0},t_0,l_1)\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_U<\zeta\leq T\wedge\tau_{U_2}\}} \mid S_{l_1}=U(l_1)\right]\mathbb{Q}\left(\tau_{L_1}\in dl_1 \middle| \mathcal{F}_{t_0}\right).\end{aligned} \tag{113}$$

We can observe that

$$Ke^{-\int_{t_0}^{T} r(l)dl}\mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_U<\zeta\leq T\wedge\tau_{U_2}\}} \mid S_{l_1}=U(l_1)\right] =: EUI, DO_{t_0}^{D}(S_T, K, L, U_2, T; 1, \tau_L, \tau_{U_2}), \tag{114}$$

which is turn $EUI, DO_{t_0}^{D}(S_T, K, L, U_2, T; 1, \tau_L, \tau_{U_2}) = 0$ as stated in Proposition 11, thus equation (114) is equal to zero. ■

5 Numerical results

In this section, we start by testing the accuracy of the ST approach as in Dias et al. (2015) numerical results by comparing those with the closed formula of Jun and Ku (2012) and afterwards we explore the results with various JDCEV model parameters.

Although Jun and Ku (2012) derive the options' values under GBM assumptions, the JDCEV model encompasses it as a special case, when, as presented in Definition 1 Nunes et al. (2015), $r(t) = r$, $q(t) = q$, $\lambda = 0$ $(b = c = 0)$, $\delta(t, S) = \sigma$. The recovery value is absent, as it is not relevant when studying the GBM setting. Afterwards, the first-then-options are studied under various parameter sets.

The strike level is presented for $K = \{95, 100, 105\}$, the initial underlying asset price is $S_{t_0} = 100$, the time period is $T - t_0 = 0.5$, the interest rate $r = 0.1$, the dividend yield $q = 0$, the parameter a is such that $\sigma(S_{t_0}, t_0) = 0.25$, the upper barrier $U = 110$, and the lower barriers are all equal to 90, that is, $L = L_1 = L_2 = 90$.

In Table 1, we test the accuracy of the ST approach to obtain the value of the two barrier first-then-barrier options. The values for call options are obtained with the closed formulae provided by Jun and Ku (2012) under GBM assumptions. As mentioned in Definition 1 Nunes et al. (2015), using the JDCEV model, we find the GBM when $\lambda = 0$ $(b = c = 0)$ and $\bar{\beta} = 0$. Given the solutions obtained for the JDCEV model are only valid when $\bar{\beta} < 0$, small values approaching zero from negative values are used, therefore $\bar{\beta} = \{-0.02, -0.05, -0.1\}$. As it can be observed, the difference between the two approaches is small, and the closer $\bar{\beta}$ is to zero, the smaller it is. When $\bar{\beta} = -0.02$, the average of the absolute value of the relative difference between the closed formulae and the ST approach is 0.0282, that is, less than 3%. The same analysis is performed for the three barrier first-then calls in Table 2 and the conclusions are similar, with the case of $\bar{\beta} = -0.02$ yielding the average value for the absolute value of the relative difference between the closed formulae and the ST approach at 0.0265.

As for studying the values while exploring the JDCEV parameters, three different JDCEV parameter sets are explored for $\bar{\beta}$, and as known, the first set $b = c = 0$ corresponds to the CEV model and the two other sets are $b = 0$ and $c = 1$ where $\lambda(S, t)$ is equal to the instantaneous volatility plus $b = 0.02$ and $c = 0.5$ where λ is a affine function of the volatility. The put contracts are studied, given these allow to observe the recovery values upon default. The default free component for each option will be designated by $V^0_{t_0}$, the recovery-value upon default will be $V^D_{t_0}$, and their sum V_{t_0}.

In Table 3 and Table 4, respectively, the results for various first-up-then-down-and-in puts and first-down-then-up-then-down-and-in puts are presented as the sum of the conditional on no default and the recovery value.

Several observations arise from the results. The values of the put options decrease from the two-barrier cases to the three barrier cases. As one would expect, due to the reduced probability of the required barriers being crossed and activating the barrier option, the values of the options tend to decrease.

In addition, overall, the increases in the jump-to-default parameters, b and c, lead to decreases in the conditional on no default components, $V^0_{t_0}$, and an increase of the recovery values, $V^D_{t_0}$, and an increase in the puts' total values, V_{t_0}. The recovery value in the case of default tends relatively low values the CEV model, while being often greater than the conditional on no default value in the other cases. Again, an expected result, given the JDCEV parameters reduce the survival probability as the possibility of a jump to default is included.

In Table 5 and Table 6, the results are respectively presented for the first-down-in-then-up-and-out puts and first-up-down-in-then-up-and-out puts. The set of parameters is the same, except that $U = U_1 = U_2 = 110$ and $L = 90$. Regarding the addition of barriers, the expected result of decreasing in value as more barriers are added is observed again.

In addition, it can be observed that increases in the $\bar{\beta}$, parameter although tending to decrease the conditional on no default component and decrease the recovery value, do not always do so. The outcome of changes $\bar{\beta}$ has mixed effects due to the knock-in and knock-out barriers.

Table 1: Jun and Ku (2012) closed form two barrier first-then call option solutions compared with the ST approach based on Dias et al. (2015)

			Closed solution	ST	Rel. Dif.	ST	Rel. Dif.	ST	Rel. Dif.
$\bar{\beta}$				-0.02		-0.05		-0.1	
K	95	EDUI	1,4691	1,4665	0,0018	1,4622	0,0047	1,4550	0,0096
K	95	EUDI	0,1623	0,1639	0,0102	0,1664	0,0253	0,1705	0,0508
K	95	DI,UO	0,4881	0,4951	0,0143	0,5056	0,0357	0,5231	0,0716
K	95	UI,DO	11,8849	11,9892	0,0088	11,9884	0,0087	11,9870	0,0086
K	100	EDUI	1,0688	1,0665	0,0022	1,0628	0,0057	1,0565	0,0115
K	100	EUDI	0,0794	0,0804	0,0125	0,0818	0,0307	0,0843	0,0615
K	100	DI,UO	0,1436	0,1488	0,0361	0,1566	0,0902	0,1695	0,1806
K	100	UI,DO	9,3361	9,3196	0,0018	9,3180	0,0019	9,3153	0,0022
K	105	EDUI	0,7096	0,7074	0,0031	0,7038	0,0081	0,6980	0,0163
K	105	EUDI	0,0377	0,0383	0,0144	0,0391	0,0349	0,0404	0,0697
K	105	DI,UO	0,0178	0,0215	0,2097	0,0271	0,5227	0,0364	1,0435
K	105	UI,DO	7,0406	6,8744	0,0236	6,8713	0,0240	6,8662	0,0248
Avg. dif.					**0,0282**		**0,0661**		**0,1292**

This table compares the closed form solution values obtained for the call options in Jun and Ku (2012) for the two barrier first-then options under the GBM with the values obtained by the Stopping Time approach for the same options under the JDCEV model. In order to compare the GBM with the JDCEV, $b = c = 0$ and while the GBM model is found when $\bar{\beta} = 0$, that value cannot be used as a solution for the JDCEV model given $\bar{\beta} < 0$, therefore values close to zero are explored, with $\bar{\beta} = \{-0.02, -0.05, -0.1\}$. The remainder of the parameters are as mentioned above: $K = \{95, 100, 105\}$, $S_{t_0} = 100$, $T - t_0 = 0.5$, $r = 0.1$, $q = 0$, a is such that $\sigma(S_{t_0}, t_0) = 0.25$, all the upper barriers are 110 and all the lower barriers are 90. The relative difference is calculated through dividing the absolute value of the difference between the two approaches by the closed formula value. The average difference is the simple average of the differences for each $\bar{\beta}$.

Table 2: Jun and Ku (2012) closed form three barrier first-then call option solutions compared with the ST approach based on Dias et al. (2015)

			Closed solution	ST	Rel. Dif.	ST	Rel. Dif.	ST	Rel. Dif.
$\bar{\beta}$				-0.02		-0.05		-0.1	
K	95	UDUI	0,0826	0,0827	0,0020	0,0829	0,0038	0,0831	0,0067
K	95	DUDI	0,0035	0,0036	0,0127	0,0036	0,0276	0,0037	0,0526
K	95	UDI,UO	0,0797	0,0812	0,0188	0,0835	0,0476	0,0874	0,0964
K	95	DUI,DO	1,4656	1,4629	0,0018	1,4585	0,0048	1,4513	0,0098
K	100	UDUI	0,0578	0,0580	0,0018	0,0580	0,0033	0,0582	0,0057
K	100	DUDI	0,0013	0,0013	0,0163	0,0013	0,0350	0,0014	0,0666
K	100	UDI,UO	0,0216	0,0224	0,0413	0,0238	0,1042	0,0261	0,2113
K	100	DUI,DO	1,0675	1,0652	0,0022	1,0614	0,0057	1,0551	0,0116
K	105	UDUI	0,0352	0,0352	0,0012	0,0353	0,0015	0,0353	0,0019
K	105	DUDI	0,0005	0,0005	0,0194	0,0005	0,0408	0,0005	0,0769
K	105	UDI,UO	0,0025	0,0030	0,1970	0,0038	0,4973	0,0051	1,0100
K	105	DUI,DO	0,7091	0,7069	0,0031	0,7034	0,0081	0,6975	0,0164
Avg. dif.					**0,0265**		**0,0650**		**0,1305**

This table compares the closed form solution values obtained for the call options in Jun and Ku (2012) for the three barrier first-then options under the GBM with the values obtained by the Stopping Time approach for the same options under the JDCEV model. In order to compare the GBM with the JDCEV, $b = c = 0$ and while the GBM model is found when $\bar{\beta} = 0$, that value cannot be used as a solution for the JDCEV model given $\bar{\beta} < 0$, therefore values close to zero are explored, with $\bar{\beta} = \{-0.02, -0.05, -0.1\}$. The remainder of the parameters are as mentioned above: $K = \{95, 100, 105\}$, $S_{t_0} = 100$, $T - t_0 = 0.5$, $r = 0.1$, $q = 0$, a is such that $\sigma(S_{t_0}, t_0) = 0.25$, all the upper barriers are 110 and all the lower barriers are 90. The relative difference is calculated through dividing the absolute value of the difference between the two approaches by the closed formula value. The average difference is the simple average of the differences for each $\bar{\beta}$.

Table 3: European style first-up-then-down-and-in put options under the JDCEV model

		$b = c = 0$			$b = 0$ and $c = 1$			$b = 0.02$ and $c = 0.5$		
K	$\bar{\beta}$	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}
95	-0.5	0.5101	0.0000	0.5101	0.3819	1.2708	1.6528	0.4086	1.0398	1.4485
95	-1	0.5800	0.0000	0.5800	0.4215	1.2617	1.6832	0.4603	1.0370	1.4973
95	-2	0.7236	0.0139	0.7375	0.4758	1.3030	1.7788	0.5504	1.0706	1.6210
95	-3	0.7043	0.2222	0.9265	0.4235	1.4951	1.9186	0.5152	1.2676	1.7827
95	-4	0.5144	0.6457	1.1602	0.2987	1.8019	2.1006	0.3711	1.6172	1.9884
100	-0.5	0.8370	0.0000	0.8370	0.6444	1.3377	1.9821	0.6843	1.0946	1.7789
100	-1	0.9195	0.0000	0.9195	0.6907	1.3281	2.0188	0.7460	1.0916	1.8375
100	-2	1.0751	0.0146	1.0898	0.7448	1.3716	2.1163	0.8425	1.1269	1.9694
100	-3	1.0388	0.2339	1.2726	0.6688	1.5738	2.2426	0.7882	1.3343	2.1225
100	-4	0.7996	0.6797	1.4793	0.5002	1.8967	2.3969	0.5998	1.7024	2.3022
105	-0.5	1.2106	0.0000	1.2106	0.9496	1.4046	2.3542	1.0036	1.1493	2.1529
105	-1	1.3110	0.0000	1.3110	1.0074	1.3945	2.4019	1.0800	1.1461	2.2261
105	-2	1.4882	0.0154	1.5035	1.0696	1.4402	2.5097	1.1916	1.1833	2.3749
105	-3	1.4437	0.2455	1.6893	0.9771	1.6525	2.6296	1.1260	1.4010	2.5270
105	-4	1.1636	0.7137	1.8773	0.7697	1.9915	2.7612	0.8996	1.7875	2.6870

The table obtains the values for the first-up-then-down-and-in put options under the JDCEV model. The value of the default-free component and the recovery value are presented and their sum are presented. The sets of parameters are $b = c = 0$, $b = 0$ and $c = 1$ plus $b = 0.02$ and $c = 0.5$.

Table 4: European style first-down-then-up-then-down-and-in put options under the JDCEV model

		$b = c = 0$			$b = 0$ and $c = 1$			$b = 0.02$ and $c = 0.5$		
K	$\bar{\beta}$	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}
95	-0.5	0.0202	0.0000	0.0202	0.0157	0.0741	0.0898	0.0167	0.0609	0.0777
95	-1	0.0218	0.0000	0.0218	0.0167	0.0683	0.0849	0.0181	0.0566	0.0747
95	-2	0.0237	0.0000	0.0238	0.0173	0.0574	0.0747	0.0197	0.0481	0.0678
95	-3	0.0224	0.0017	0.0241	0.0154	0.0482	0.0636	0.0185	0.0411	0.0596
95	-4	0.0167	0.0059	0.0226	0.0111	0.0407	0.0518	0.0139	0.0363	0.0502
100	-0.5	0.0359	0.0000	0.0359	0.0285	0.0780	0.1065	0.0302	0.0641	0.0944
100	-1	0.0377	0.0000	0.0377	0.0296	0.0719	0.1015	0.0319	0.0596	0.0915
100	-2	0.0391	0.0000	0.0391	0.0296	0.0604	0.0901	0.0332	0.0506	0.0838
100	-3	0.0358	0.0017	0.0375	0.0259	0.0508	0.0766	0.0303	0.0432	0.0736
100	-4	0.0270	0.0062	0.0332	0.0190	0.0429	0.0619	0.0232	0.0382	0.0614
105	-0.5	0.0533	0.0000	0.0533	0.0429	0.0819	0.1248	0.0453	0.0674	0.1126
105	-1	0.0556	0.0000	0.0556	0.0443	0.0755	0.1198	0.0475	0.0626	0.1101
105	-2	0.0566	0.0001	0.0566	0.0439	0.0635	0.1073	0.0487	0.0531	0.1018
105	-3	0.0513	0.0018	0.0531	0.0383	0.0533	0.0916	0.0443	0.0454	0.0897
105	-4	0.0394	0.0065	0.0459	0.0288	0.0450	0.0738	0.0346	0.0401	0.0747

The table obtains the values for the first-down-then-up-then-down-and-in put options under the JDCEV model. The value of the default-free component and the recovery value are presented and their sum are presented. The sets of parameters are $b = c = 0$, $b = 0$ and $c = 1$ plus $b = 0.02$ and $c = 0.5$.

Table 5: European style first-down-in-then-up-and-out put options under the JDCEV model

		$b=c=0$			$b=0$ and $c=1$			$b=0.02$ and $c=0.5$		
K	$\bar{\beta}$	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}	$V^0_{t_0}$	$V^D_{t_0}$	V_{t_0}
95	-0.5	2.9921	0.0000	2.9921	2.1387	1.5560	3.6948	2.2964	1.2732	3.5696
95	-1	3.0796	0.0000	3.0796	2.1349	1.6446	3.7795	2.3313	1.3212	3.6524
95	-2	3.1809	0.0856	3.2665	1.9995	1.9580	3.9575	2.2938	1.5343	3.8281
95	-3	2.6175	0.8459	3.4634	1.5361	2.6114	4.1475	1.8253	2.1915	4.0168
95	-4	1.7474	1.9371	3.6844	1.0056	3.3505	4.3561	1.2048	3.0245	4.2293
100	-0.5	4.4859	0.0000	4.4859	3.3078	1.6379	4.9457	3.5235	1.3402	4.8638
100	-1	4.5242	0.0000	4.5242	3.2526	1.7312	4.9837	3.5105	1.3907	4.9012
100	-2	4.5102	0.0901	4.6003	2.9964	2.0611	5.0574	3.3599	1.6151	4.9750
100	-3	3.7752	0.8904	4.6656	2.3779	2.7489	5.1268	2.7372	2.3068	5.0440
100	-4	2.6905	2.0390	4.7295	1.6747	3.5269	5.2015	1.9352	3.1837	5.1188
105	-0.5	6.2067	0.0000	6.2067	4.6850	1.7198	6.4048	4.9618	1.4072	6.3690
105	-1	6.2041	0.0000	6.2041	4.5857	1.8177	6.4035	4.9076	1.4602	6.3678
105	-2	6.0919	0.0946	6.1865	4.2235	2.1641	6.3876	4.6573	1.6959	6.3532
105	-3	5.2036	0.9349	6.1386	3.4648	2.8863	6.3511	3.8941	2.4221	6.3162
105	-4	3.9243	2.1410	6.0653	2.6007	3.7032	6.3039	2.9233	3.3428	6.2661

The table obtains the values for the first-down-then-up-and-out put options under the JDCEV model. The value of the default-free component and the recovery value are presented and their sum are presented. The sets of parameters are $b=c=0$, $b=0$ and $c=1$ plus $b=0.02$ and $c=0.5$.

Table 6: European style first-up-then-down-in-then-up-and-out put options under the JDCEV model

		$b = c = 0$			$b = 0$ and $c = 1$			$b = 0.02$ and $c = 0.5$		
K	$\bar{\beta}$	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}	$V_{t_0}^0$	$V_{t_0}^D$	V_{t_0}
95	-0.5	0.5098	0.0000	0.5098	0.3817	0.0100	0.3916	0.4083	0.0083	0.4167
95	-1	0.5797	0.0000	0.5797	0.4213	0.0096	0.4309	0.4601	0.0081	0.4682
95	-2	0.7235	0.0007	0.7242	0.4757	0.0100	0.4857	0.5503	0.0086	0.5589
95	-3	0.7043	0.0054	0.7097	0.4235	0.0123	0.4358	0.5151	0.0117	0.5268
95	-4	0.5144	0.0098	0.5242	0.2988	0.0138	0.3125	0.3711	0.0141	0.3852
100	-0.5	0.8358	0.0000	0.8358	0.6434	0.0105	0.6539	0.6833	0.0088	0.6921
100	-1	0.9186	0.0000	0.9186	0.6900	0.0101	0.7001	0.7452	0.0085	0.7537
100	-2	1.0747	0.0007	1.0754	0.7444	0.0106	0.7550	0.8421	0.0091	0.8512
100	-3	1.0386	0.0057	1.0443	0.6687	0.0130	0.6816	0.7881	0.0123	0.8004
100	-4	0.7996	0.0103	0.8099	0.5002	0.0145	0.5147	0.5998	0.0149	0.6147
105	-0.5	1.2067	0.0000	1.2067	0.9461	0.0110	0.9571	1.0000	0.0092	1.0092
105	-1	1.3079	0.0000	1.3079	1.0046	0.0106	1.0152	1.0771	0.0090	1.0861
105	-2	1.4864	0.0007	1.4871	1.0680	0.0111	1.0791	1.1900	0.0095	1.1996
105	-3	1.4429	0.0060	1.4489	0.9764	0.0136	0.9900	1.1252	0.0129	1.1382
105	-4	1.1634	0.0108	1.1742	0.7695	0.0152	0.7847	0.8994	0.0156	0.9149

The table obtains the values for the first-up-then-down-then-up-and-out put options under the JDCEV model. The value of the default-free component and the recovery value are presented and their sum are presented. The sets of parameters are $b = c = 0$, $b = 0$ and $c = 1$ plus $b = 0.02$ and $c = 0.5$.

6 Conclusions

In this part of the thesis, we have extended the JDCEV model presented in Carr and Linetsky (2006) to the multiple barrier options that are developed in Jun and Ku (2012). This was achieved through extensive use of the stopping time methodologies developed in Dias et al. (2015) and Dias et al. (2021), who extend the Kuan and Webber (2003) model.

This leads to formulae that rely on the first-hitting times of crossing barriers, and the survival probabilities - probabilities of not defaulting - before doing so. As in Dias et al. (2015), these first hitting time densities are recovered by solving integral equations that involve truncated moments of a noncentral chi-square law.

In total, the results for eight different kinds of options - that can either be call options or put options - are obtained. When reduced to the GBM setting, the values are found to be accurate. In the case of the put options, this takes into account the recovery value received upon default - either through the jump to default or the diffusion process.

This was mostly achieved through preconditioning options that are received upon crossing a given barrier with respect to the filtrations that represent the crossing of those barriers, while relying on the Markov property of the underlying asset price.

Many of the results in this part of the thesis are widely instrumental to the credit risk model explored in the second part, which relies on the CEV model, nested in the JDCEV model explored here.

Part III

Credit spreads with dynamic debt under the CEV model

7 Introduction

A key factor always played a role when economic agents engage in the lending and borrowing: the likelihood of default by the debtor. Since Black and Scholes (1973) and Merton (1974), a wide class of models emerged: the structural models. These use financial options' formulation, considering the firm's total value as the underlying asset, and the nominal value of the debt as the strike price.

As Eom et al. (2004) find, however, the classical Merton (1974) model is not able to properly capture the observed empirical observations. The Merton model spreads are too low when compared to the real data, while other structural models predict spreads that are to high as for instance Collin-Dufresne and Goldstein (2001) do. A key assumption in the standard structural models is the immutability of the liabilities level through time, although debt rarely remains fixed and firms actively manage it. One expects the firm value to influence the debt level, with higher levels in the former being associated with increases of the latter. There are several papers that try to tackle this issue, for instance, Collin-Dufresne and Goldstein (2001) study debt with a continuous variation, which follows the total value of the firm, while Eisenthal-Berkovitz et al. (2020) consider the possibility of a leveraged buyout event on the firm, which increases the value of debt by a lump sum.

An approach based on barrier options is used in Das and Kim (2015), modeling the changes in the debt level through discrete changes at prescribed thresholds of the firm's value.

As for the justifications on why to allow the debt to be changed, there are several studies that contest the assumption of a fixed amount of debt. Roberts and Sufi (2009) find that most of long-term debt contracts suffer renegotiations over the amount, maturity, and pricing of the contract, Nini et al. (2012) highlight the active role of creditors through informal channels in the governance of firms (even when default is a far scenario), and Flannery et al. (2012) point a direct impact on the credit risk, as it is affected by the expected future leverage.

To do so, the standard options in the original Merton model are replaced by various kinds of barrier options, and as the corresponding barriers are reached, the liability values are updated. Doing so, Das and Kim (2015) are able to match features that would be expected empirically: the credit spreads are increased in anticipation to debt increases and decreased when expecting debt decreases. They obtain closed-form solutions for the ex-ante debt discount and corresponding debt spreads. When compared to other models, the authors are able to replicate the upward slope of the spread curve, not only for investment grade debt but also for low-grade debt.

There is an improvement in comparison to Merton (1974), but the model still depends on the geometric Brownian motion (GBM) to reach the asset price. The GBM formulation has long been recognized as limited in mathematical finance, given its log-normal distribution assumption, as for instance Jackwerth and Rubinstein (1996) explore. The limitation can be summed up in two effects. Bekaert and Wu (2000) study the negative correlation between stock returns and realized volatility - the *leverage effect*; and in option context, Dennis and Mayhew (2002) approach the negative correlation between the implied volatility and strike of stock options - the *implied volatility skew*. In the context of credit risk, several studies exist, such as Cremers et al. (2008), which find a link between implied volatilities and credit spreads, while Hilscher (2007) finds that corporate bond yields can predict future volatility.

To address this kind of issues, Cox (1975) introduced the CEV model, standing for *constant elasticity of variance*. This well known model departs from the GBM, allowing the volatility to be a function of the underlying asset price, and with the proper calibration, increasing the volatility at lower underlying asset values. This part of the thesis attempts to extend the original dynamic debt model, making it so that the firm's assets follow a *CEV* process, thus aspiring to observe how the inclusion of non-constant volatility affects the dynamic model's credit spreads. Therefore, here, the CEV model is used to evaluate well known barrier options: the single barrier kind which is presented for instance in Rubinstein and Reiner (1991), the double barrier options in works such as Geman and Yor (1996) and the first-then options as presented in Jun and Ku (2012).

To do this, the work of Dias et al. (2015) is instrumental. Dias et al. (2015) develop two alternative accurate methodologies to price European-style barrier options under the *JDCEV* model - an extension of the *CEV* model which nests it - adding the possibility of a jump to default to the asset's process. In Dias et al. (2021) the methodology is expanded to more kinds of barrier options. Those two studies are instrumental here, as this part of the thesis uses the first of the two methodologies, the *stopping time approach* (ST), to price double barrier options under the *CEV* model. Part II of this thesis expands on these two studies to obtain more barrier options which are used here. With those option contract formulae, we are able to adapt the payoffs and reach all the required components to set up the intended dynamic debt model cases.

In the literature, the *stopping time approach* is a well known applied probability method for solving level-crossing problems, as for instance Park and Schuurmann (1976) do for the standard Wiener process and Kuan and Webber (2003) are able to price single and double barrier options under the GBM assumptions.

8 The firm value under the CEV model

First, the CEV model is introduced. Let the process for the firm value V_t be defined by:

$$\frac{dV_t}{V_t} = (r - q)dt + \sigma(V_t)dW_t^{\mathbb{Q}}, \tag{115}$$

As usual, r represents the interest rate, q stands for the total payout to debt and equity-holders, $\sigma(V_t)$ is the instantaneous volatility of the asset returns at time-t given the asset price V, which is defined by the expression

$$\sigma(V_t) = \delta V_t^{\frac{\beta}{2}-1}, \tag{116}$$

and $\left\{W_t^{\mathbb{Q}}, t \geq t_0\right\}$ is a standard Brownian motion defined under measure $\mathbb{Q}$, thus generating the filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq t_0\}$.

In this part of the thesis, we will explore the CEV process with $\beta < 2$ - the cases where volatility is increased at lower underlying asset levels - which implies that the firm value hits zero with positive probability. See Davydov and Linetsky (2001) for additional details on the boundary zero. Therefore, there is a possibility of default by diffusion, which is represented by

$$\tau_0 := \inf\{t > t_0 : V_t = 0\}. \tag{117}$$

9 The barrier options

9.1 First passage times

In order to solve the options that compose the dynamic debt model, this valuation method conditions the first time the spot level hits a (constant) barrier. The two following definitions are analogous to Dias et al. (2021, Definitions 1 and 2).

Definition 13 *For a barrier option contract, we define*

$$\tau_L := \inf\{l > t_0 : V_l = L\} \tag{118}$$

as the first hitting time of the lower barrier L by the underlying asset, V_l, for $l \in \mathcal{T}$, with $\mathcal{T} := [t_0, T]$. For the upper barriers, we denote

$$\tau_U := \inf\{u > t_0 : V_u = U\} \tag{119}$$

as the first hitting time of the upper barrier U by the firm value, V_u, for $u \in \mathcal{T}$.

In the dynamic debt model, there will be cases where not only a barrier must be crossed in order to obtain a payoff, but if a second barrier is crossed, the payoff is not obtained. Therefore, we need the passage times with two trigger clauses.

These are used in Dias et al. (2015) to obtain the value for one-touch double barrier options. Here, they will be used for those options, but not only. So, the definitions are as follow.

Definition 14 *For barrier option contracts with two trigger clauses,*

$$\bar{\tau}_L := \inf \left\{ l > t_0 : V_l = L, \sup_{t_0 \le v \le l} (V_v - U) < 0 \right\}, \tag{120}$$

denotes the first passage time of the asset price to the lower barrier before ever crossing the upper barrier while

$$\bar{\tau}_U := \inf \left\{ u > t_0 : V_u = U, \inf_{t_0 \le v \le u} (V_v - L) > 0 \right\}. \tag{121}$$

represents the first hitting time of the upper barrier by the firm value before touching the lower barrier level. Finally, the first hitting time

$$\tau_{LU} = \bar{\tau}_L \wedge \bar{\tau}_U, \tag{122}$$

represents the first passage time of the underlying asset price to one of the two barriers, with $L < U$.

9.2 First passage time densities

The implementation of the pricing solutions requires the knowledge of the two optimal hitting times densities, as in Definition 13, contained in the respective equations. Based on Park and Schuurmann (1976, Theorem 2), both densities can be recovered as the implicit solution of the integral equations that simply involve the transition density function of the underlying asset price. The following Proposition uses Dias et al. (2015, equations (52) and (53)).

Proposition 20 *The first passage time densities $\mathbb{Q}\left(\tau_L \in du \middle| \mathcal{F}_{t_0}\right)$ and $\mathbb{Q}\left(\tau_U \in du \middle| \mathcal{F}_{t_0}\right)$ are, respectively, the implicit solutions of*

$$F\left(t_0, V_{t_0}; u, L\right) = \int_{t_0}^{u} F\left(v, L; u, L\right) \mathbb{Q}\left(\tau_L \in dv \middle| \mathcal{F}_{t_0}\right) \tag{123}$$

and

$$F\left(t_0, V_{t_0}; u, U\right) = \int_{t_0}^{u} F\left(v, U; u, U\right) \mathbb{Q}\left(\tau_U \in dv \middle| \mathcal{F}_{t_0}\right) \tag{124}$$

and given $\beta < 2$, with

$$F\left(v, E_v; u, E\right) := Q_{\chi^2\left(\frac{2}{2-\beta}, 2\kappa_{v,u}E^{2-\beta}\right)}\left(2\kappa_{v,u}E_v^{2-\beta}e^{(2-\beta)(r-q)(u-v)}\right) \tag{125}$$

representing the transition density function $\mathbb{Q}\left(V_u \leq E \middle| V_v = E\right)$ of the CEV process, as given by Schroder (1989, Equation 1) for $\beta < 2$,

$$\kappa_{v,u} := \frac{2\left(r-q\right)}{\left(2-\beta\right)\delta^2\left[e^{(2-\beta)(r-q)(u-v)} - 1\right]}, \tag{126}$$

and where $Q_{\chi^2(v,\lambda)}\left(x\right)$ represents the complementary distribution function of a noncentral chi-square law with v degrees of freedom and noncentrality parameter λ.

As in Dias et al. (2015), these are solved with the algorithm offered by Benton and Krishnamoorthy (2003).

In addition, the two trigger densities, as in Definition 14, are obtained using simultaneously equations (123) and (124) but with $\bar{\tau}_L$ and $\bar{\tau}_U$.

With the density solutions presented, we define the options that will be needed to solve the dynamic debt model.

9.3 Single barrier options

Here, four definitions for various single barriers borrowed from Dias et al. (2021) are provided, adapted for the CEV model, where there is no jump to default. These will be used to obtain the results for the dynamic debt model.

These are the most common barrier option contracts, studied in Rubinstein and Reiner (1991) and Rich (1994) under the GBM setting. A knock-in single barrier option has the payoff of a vanilla option if a given barrier (upper or lower) is touched during its lifetime. A knock-out single barrier option, has the payoff of vanilla option if a given barrier (upper or lower) is not touched during its lifetime.

In both cases, there is a possibility of a rebate if the underlying asset does not touch the knock-in barrier or touches the knock-out barrier. The definitions provided next do not take into account the possibility of a rebate. The strike price here is represented by D, which later will take the place of the nominal debt value.

Definition 15 ***Up-and-in options.*** *The time-T price of a unit face value and zero rebate European-style up-and-in single barrier option on the asset price V, with strike D, barrier level $U : \mathcal{T} \to \mathbb{R}_+$ (with $U > V_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned} EUI_T\left(V_T, D, U, T; \phi, \tau_U\right) &= EUI_T^0\left(V_T, D, U, T; \phi, \tau_U\right) + EUI_T^D\left(V_T, D, U, T; \phi, \tau_U\right) \\ &= \left(\phi D - \phi V_T\right)^+ \mathbb{1}_{\{\tau_U \leq T, \tau_0 > T\}} + (\phi D)^+ \mathbb{1}_{\{\tau_U \leq \tau_0 \leq T\}}, \end{aligned} \tag{127}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. We note that there is no recovery component for an up-and-in call and, therefore, $EUI_T^D\left(V_T, D, U, T; -1, \tau_U\right) = 0$.

Definition 16 ***Down-and-in options.*** *The time-T price of a unit face value and zero rebate European-style down-and-in single barrier option on the asset price V, with strike D, barrier level $L : \mathcal{T} \to \mathbb{R}_+$ (with $L < V_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned} EDI_T\left(V_T, D, L, T; \phi, \tau_L\right) &= EDI_T^0\left(V_T, D, L, T; \phi, \tau_L\right) + EDI_T^D\left(V_T, D, L, T; \phi, \tau_L\right) \\ &= \left(\phi D - \phi V_T\right)^+ \mathbb{1}_{\{\tau_L \leq T, \tau_0 > T\}} + v_T^D(V_T, D, T; \phi), \end{aligned} \tag{128}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. We note that there is no recovery component for a down-and-in call and, therefore, $EDI_T^D\left(V_T, D, L, T; -1, \tau_L\right) = 0$. Moreover, since the default event cannot precede the knock-in event, then $\mathbb{1}_{\{\tau_0 < \tau_L\}} = 0$ and, hence, $EDI_T^D\left(V_T, D, L, T; 1, \tau_L\right) = v_T^D\left(V_T, D, T; 1\right)$, which is the recovery component of a vanilla put option.

Definition 17 ***Up-and-out options.*** *The time-T price of a unit face value and zero rebate European-style up-and-out single barrier option on the asset price V, with strike D, barrier level $U : \mathcal{T} \to \mathbb{R}_+ \left(\text{with } U > V_{t_0}\right)$, and maturity at time $T\left(\geq t_0\right)$ is equal to*

$$\begin{aligned}EUO_T(V_T,D,U,T;\phi,\tau_U) =&EUO_T^0(V_T,D,U,T;\phi,\tau_U)+EUO_T^D(S_T,K,U,T;\phi,\tau_U)\\ =&v_T^0(V_T,D,T;\phi)-EUI_T^0(V_T,D,U,T;\phi,\tau_U)\\ &+(\phi D)^+\mathbb{1}_{\{\tau_0\le T\wedge\tau_U\}},\end{aligned}\tag{129}$$

where $\phi=1$ *for a put option and* $\phi=-1$ *for a call option. We note that there is no recovery component for an up-and-out call and, therefore,* $EUO_T^D(V_T,D,U,T;-1,\tau_U)=0$.

Definition 18 ***Down-and-out options*** *The time-*T *price of a unit face value and zero rebate European-style down-and-out single barrier option on the asset price* V*, with strike* D*, barrier level* $L:\mathcal{T}\to\mathbb{R}_+$ *(with* $L<V_{t_0}$*), and maturity at time* $T\,(\ge t_0)$ *is equal to*

$$\begin{aligned}EDO_T(V_T,D,L,T;\phi,\tau_L)&=EDO_T^0(V_T,D,L,T;\phi,\tau_L)+EDO_T^D(V_T,D,L,T;\phi,\tau_L)\\ &=v_T^0(V_T,D,T;\phi)-EDI_T^0(V_T,D,L,T;\phi,\tau_L),\end{aligned}\tag{130}$$

where $\phi=1$ *for a put option and* $\phi=-1$ *for a call option. We note that there is no recovery component for a down-and-out call and, therefore,* $EDO_T^D(V_T,D,L,T;-1,\tau_L)=0$*. Furthermore, since the default event cannot precede the knock-out event,* $\mathbb{1}_{\{\tau_0\le\tau_L\}}=0$ *and, hence,* $EDO_T^D(V_T,D,L,T;1,\tau_L)=0$.

9.4 One touch double barrier options

Next, drawn from Dias et al. (2015), we have the one touch barrier options, from which the knock-out style one is part of the dynamic debt model. A one-touch knock-in double barrier option has the payoff of a vanilla option if, during the option's lifetime, one of these barriers is crossed. A one-touch knock-out double barrier has the payoff a vanilla option, if during the same period, none of the barriers is touched.

As the single barrier options, these may have a rebate, but its value is not explored in the two following definitions borrowed from Dias et al. (2021, Definitions 11 and 14).

Definition 19 *The time−*T *price of a unit face value and zero rebate European-style one-touch double barrier knock-in option on the asset price* V*, with strike* D*, lower barrier* *level $L : \mathcal{T} \to \mathbb{R}_+$, upper barrier level $U : \mathcal{T} \to \mathbb{R}_+$ (with $L < V_{t_0} < U$) and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
&EDBKI_T(V_T, D, L, U, T; \phi, \bar{\tau}_L, \bar{\tau}_U) \\
=&EDI_T^0(V_T, D, L, U, T; \phi, \bar{\tau}_L) + EUI_T^0(V_T, D, L, U, T; \phi, \bar{\tau}_U) + v_T^D(V, D, T; \phi),
\end{aligned}
\tag{131}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option, and because the sets $\{\bar{T}_L \leq T\}$ and $\{\bar{T}_U \leq T\}$ are disjoint, that is $\mathbb{1}_{\{\tau_{LU} \leq T\}} = \mathbb{1}_{\{\bar{\tau}_L \leq T\}} + \mathbb{1}_{\{\bar{\tau}_U \leq T\}}$. As expected, there is no recovery component for a one touch double barrier knock-in call. The last term is the recovery value associated with a down-and-in vanilla-put option.

Definition 20 *The time$-T$ price of a unit face value and zero rebate European-style one-touch double barrier knock-out option on the asset price V, with strike K, lower barrier levels $L : \mathcal{T} \to \mathbb{R}_+$, upper barrier levels $U : \mathcal{T} \to \mathbb{R}_+$ (with $L < V_{t_0} < U$) and maturity at time $T(\geq t_0)$ is equal to*

$$
\begin{aligned}
&EDBKO_T(V_T, D, L, U, T; \phi, \bar{\tau}_L, \bar{\tau}_U) \\
=&v_T^0(V_T, D, T; \phi) - EDBKI_T^0(V_T, D, L, U, T; \phi, \bar{\tau}_L, \bar{\tau}_U),
\end{aligned}
\tag{132}
$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option, and because the sets $\{\bar{\tau}_L \leq T\}$ and $\{\bar{\tau}_U \leq T\}$ are disjoint, that is $\mathbb{1}_{\{\tau_{LU} \leq T\}} = \mathbb{1}_{\{\bar{\tau}_L \leq T\}} + \mathbb{1}_{\{\bar{\tau}_U \leq T\}}$. As expected, there is no recovery component for a one-touch double barrier knock-out call. The recovery component is always zero because the default event forces the option knock-out (as long $L \in \mathbb{R}_+$).

9.5 First-then-barrier options

Now, four definitions are borrowed from the second part of this thesis and adapted for the CEV case. These are the first-then barrier option contracts, studied in Haug (2006), Jun and Ku (2012) and Jun and Ku (2013). These are options whose monitorization period starts after another barrier is crossed.

For instance, a first-up-then-down-and-in barrier option is knocked-in when the asset price crosses an upper barrier and afterwards a second lower barrier. Or, in alternative, it can be interpreted as a down-and-in barrier option whose monitorization starts when the upper barrier is crossed.

These can also be of the knock-out kind, as for instance, a first-up-in-then-down-and-out barrier option is first knocked-in when the asset price crosses an upper barrier, and knocked-out afterwards if a lower barrier is crossed. Or, in alternative, it can be interpreted as a down-and-out barrier option which is activated once the upper barrier is crossed.

Definition 21 ***First-down-then-up-and-in options.*** *The time-T price of a unit face value and zero rebate European-style first-down-then-up-and-in option on the asset price V, with strike D, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L < V_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U > V_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned} & EDUI_T\left(V_T, D, L, U, T; \phi, \tau_L, \tau_U\right) \\ =& EDUI_T^0\left(V_T, D, L, U, T; \phi, \tau_L, \tau_U\right) + EDUI_T^D\left(V_T, D, L, U, T; \phi, \tau_L, \tau_U\right) \\ =& \left(\phi K - \phi S_T\right)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq \tau_0 \leq T\}} + (\phi K)^+ \mathbb{1}_{\{\tau_L < \tau_U \leq T, \tau_0 > T\}}, \end{aligned} \tag{133}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, τ_L activates a European-style up-and-in barrier option with barrier level U. We note that there is no recovery component for the first-down-then-up-and-in call and, therefore, $EDUI_T^D\left(V_T, D, L, U, T; -1, \tau_L, \tau_U\right) = 0$.

Definition 22 ***First-up-then-down-and-in options.*** *The time-T price of a unit face value and zero rebate European-style first-up-then-down-and-in barrier option on the asset price V, with strike D, barrier levels $L : \mathcal{T} \to \mathbb{R}_+$ (with $L < V_{t_0}$), $U : \mathcal{T} \to \mathbb{R}_+$ (with $U > V_{t_0}$), and maturity at time $T(\geq t_0)$ is equal to*

$$\begin{aligned} & EUDI_T\left(V_T, D, L, U, T; \phi, \tau_L, \tau_U\right) \\ =& EUDI_T^0\left(V_T, D, L, U, T; \phi, \tau_L, \tau_U\right) + EUDI_T^D\left(V_T, D, L, U, T; \phi, \tau_L, \tau_U\right) \\ =& \left(\phi D - \phi V_T\right)^+ \mathbb{1}_{\{\tau_U < \tau_L \leq T, \tau_0 > T\}} + (\phi D)^+ \mathbb{1}_{\{\tau_U < \tau_L \leq \tau_0 \leq T\}}, \end{aligned} \tag{134}$$

where $\phi = 1$ for a put option and $\phi = -1$ for a call option. Hence, τ_U activates a

European-style down-and-in barrier option with barrier level L. We note that there is no recovery component for first-up-then-down-and-in the call and, therefore, $EUDI_T^D\left(V_T, D, L, U, T; -1, \tau_L, \tau_U\right) = 0$.

Definition 23 ***First-down-in-then-up-and-out options.*** *The time-T price of a unit face value and zero rebate European-style first-down-in-then-up-and-out barrier option on the asset price V, with strike D, barrier levels* $L : \mathcal{T} \to \mathbb{R}_+$ *(with* $L < V_{t_0}$*),* $U : \mathcal{T} \to \mathbb{R}_+$ *(with* $U > V_{t_0}$*), and maturity at time* $T(\geq t_0)$ *is equal to*

$$
\begin{aligned}
&EDI,UO_T(V_T, D, L, U, T; \phi, \tau_L, \tau_U)\\
=&EDI,UO_T^0(V_T, D, L, U, T; \phi, \tau_L, \tau_U) + EDI,UO_T^D(V_T, D, L, U, T; \phi, \tau_L, \tau_U)\\
=&(\phi D - \phi V_T)^+ \mathbb{1}_{\{\tau_L \leq T < \tau_U, \tau_0 > T\}} + (\phi D)^+ \mathbb{1}_{\{\tau_L \leq \tau_0 \leq T \wedge \tau_U\}}
\end{aligned}
\tag{135}
$$

where $\phi = 1$ *for a put option and* $\phi = -1$ *for a call option. Hence,* τ_L *activates a European-style up-and-out barrier option with barrier level U. We note that there is no recovery component for the first-down-in-then-up-and-out call,* $EDI,UO_T(V_T, D, L, U, T; -1, \tau_U, \tau_L) = 0$.

Definition 24 ***First-up-in-then-down-and-out-options.*** *The time-T price of a unit face value and zero rebate European-style first-up-in-then-down-and-out barrier option on the asset price S, with strike K, barrier levels* $L : \mathcal{T} \to \mathbb{R}_+$ *(with* $L < V_{t_0}$*),* $U : \mathcal{T} \to \mathbb{R}_+$ *(with* $U(t_0) > V_{t_0}$*), and maturity at time* $T(\geq t_0)$ *is equal to*

$$
\begin{aligned}
&EUI,DO_T(V_T, D, L, U, T; \phi, \tau_L, \tau_U)\\
=&EUI,DO_T^0(V_T, D, L, U, T; \phi, \tau_L, \tau_U) + EUI,DO_T^D(V_T, D, L, U, T; \phi, \tau_L, \tau_U)\\
=&(\phi D - \phi V_T)^+ \mathbb{1}_{\{\tau_U \leq T < \tau_L, \tau_0 > T\}} + (\phi D)^+ \mathbb{1}_{\{\tau_U < \tau_0 \leq T \wedge \tau_L\}},
\end{aligned}
\tag{136}
$$

where $\phi = 1$ *for a put option and* $\phi = -1$ *for a call option. Hence,* τ_U *activates a European-style down-and-out barrier option with barrier level L. We note that there is no recovery component for the first-up-in-then-down-and-out call, and, therefore* $EUI,DO_T(S_T, K, L, U, T; -1, \tau_U, \tau_L) = 0$.

10 Modified discounts

10.1 The model of debt changes

In this section, we obtain the debt discount formulae for each of the dynamic debt cases. In Das and Kim (2015), the dynamic debt model is presented in seven cases, the original Merton (1974) model, plus six combinations of barrier options.

Following the results of Roberts and Sufi (2009), Nini et al. (2012) and Flannery et al. (2012), the model tries to capture the mechanisms of changes in debt. By following the basic idea that as the leverage decreases (with increases in the firm value), the nominal debt is exposed to the increases, as the additional collateral allows to do so. On the other side leverage increases (with decreases in the firm value), the nominal debt becomes subject to debt write downs in order to counterweight the increased risk of default.

So, the models sets the parameters for the debt increases and decreases, both the amounts by how much the debt changes and the levels at which the firm value, V, triggers those debt changes. For the amount of the changes, $\delta(> 0)$ represents the proportion by how much debt increases, $d \in (0, 1)$ by how much it decreases. These debt changes occur when the firm value, D/V crosses certain barriers which are set in terms of leverage. $K(< 1)$ is the level which triggers the debt ratchet in nominal debt, while M is the value when the debt write down occurs.

There is also the possibility of not recovering the full value upon default, recovering only the recovery value, $\phi_{dwl}(\leq 1)$, supporting a dead-weight cost, thus receiving $V_T\phi_{dwl}$, which is incorporated into the put option formulae. The face value of debt is represented by D.

As in Das and Kim (2015), in order to hold the firm value constant, the changes in the face value of debt are assumed to be drawn from the equity value. Although debt changes may not operate accordingly to this assumption, the direction of the effects is the same (increases/decreases in debt increase/decrease the probability of default) and it is analytically convenient.

Below, we present and adapt, under the possibility of debt changes, the value of the debt, B, and the debt discount, G - the difference between the risky bond and risk-less debt - under the dynamic debt cases to the CEV model, which takes into consideration the possibility of reaching zero by diffusion, the cases where $\tau_0 < T$.

The baseline cases of debt increase and decrease are presented first and then combined. Then, the possibilities of having both an increase and decrease are developed and are also combined.

Afterwards, we compute the credit spreads of the risky debt in relation to the risk-less debt. These spreads, $\mathcal{S}$, follow the standard formula of the structural models, that is,

$$\mathcal{S} = -\frac{1}{T-t_0} \ln\left(\frac{B_0}{De^{-r(T-t_0)}}\right) = -\frac{1}{T-t_0} \ln\left(\frac{De^{-r(T-t_0)} - G_0}{De^{-r(T-t_0)}}\right). \tag{137}$$

10.2 Case 1: Original static debt

This is the standard well know Merton (1974) model, from where all the other cases depart. This model sets that a given firm has a single liability with a terminal payoff at T and a face value of D. At time T, this debt face value is redeemed from the total firm's total assets, V. Default can only occur at time T, and if $V_T < D$, that is, the firm does not have funds to repay the face value of debt. When such event occurs, the debt-holder takes whatever value remains of the firm value, weighted by the recovery value, ϕ_{dwl}. If there is a diffusion to zero before time-T, the firm no longer contains value, therefore, the bondholders receive nothing. So, when $\tau_0 > T$, the bond value payoff, B_T^1, can be written as

$$B_T^1(V_T, D, T; \phi_{dwl}) = \mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}}\right). \tag{138}$$

As for the discount on debt, G_T^1, it can be written by subtracting the risky debt value from the risk-free debt with the same face value.

Definition 25 *The time-T price of the debt discount on case 1 (the Merton Model) of a firm with value V, nominal debt D and maturity time $T(\geq t_0)$, assuming that $\tau_0 > t_0$, is given by*

$$
\begin{aligned}
&G_T^1(V_T, D, T; \phi_{dwl}) \\
=&D - \mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\
=&D \left(\mathbb{1}_{\{\tau_0 \leq T\}} + \mathbb{1}_{\{\tau_0 > T\}} \right) - \mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\
=&D\mathbb{1}_{\{\tau_0 > T\}} - \mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) + D\mathbb{1}_{\{\tau_0 \leq T\}} \\
=&\mathbb{1}_{\{\tau_0 > T\}} D(1 - \mathbb{1}_{\{D < V_T\}}) - \mathbb{1}_{\{\tau_0 > T\}} \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} + D\mathbb{1}_{\{\tau_0 \leq T\}} \\
=&\mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D \geq V_T\}} - \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) + D \left(1 - \mathbb{1}_{\{\tau_0 > T\}} \right) \\
=&v_T^0(V_T, D; 1, \phi_{dwl}) + v_T^D(V_T, D; 1, \phi_{dwl}), \\
=&v_T(V_T, D, T; \phi_{dwl}). \qquad (139)
\end{aligned}
$$

that is, a put option with the conditional on no default value, $v_T^0(V_T, D, T; 1, \phi_{dwl})$ and the recovery value $v_T^D(V_T, D, T; 1, \phi_{dwl})$. These are given in Dias et al. (2020, equations (21) and (22)), respectively.

10.3 Case 2: Discount with debt ratchet

Now, the possibility of an increase in the nominal debt held by the firm is contemplated. As the firm value, V, evolves, if it rises to an exogenous level D/K, the increased debt level to $D(1 + \delta)$, staying at that level until T.

For when the firm value does not diffuse to zero, the outcomes where the upper level D/K is not crossed before maturity, $\tau_{D/K} > T$, hold the same payoff as before

$$
\mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right), \qquad (140)
$$

while those where the upper barrier is crossed before maturity, $\tau_{D/K} \leq T$, will be given by

$$
\frac{1}{1 + \delta} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1 + \delta) \mathbb{1}_{\{D(1+\delta) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta) \geq V_T\}} \right). \qquad (141)
$$

Here, the default occurs when the firm value is below the (increased) face value of debt, $D(1+\delta)$. The whole expression is weighted by the ratio between the old and new face values of debt, $D/D(1+\delta) = 1/(1+\delta)$, and this weighting serves two purposes. First, in the case of no default, it guarantees that the original debt-holder receives the original amount of nominal debt, not the increased amount. Second, in the case of default, it reduces the amount of the remainder of the firm level to be received by the original debt-holder, as he must share it with the new debt-holder(s).

Therefore, the bond value of the original debt-holder will be given by:

$$\begin{aligned}
&B_T^2(V_{t_0}, D, K, \delta, T; \tau_{D/K}, \phi_{dwl}) \\
=&\mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right) \\
&+ \mathbb{1}_{\{\tau_{D/K}\leq T\}}\mathbb{1}_{\{\tau_0>T\}}\frac{1}{1+\delta}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right). \qquad (142)
\end{aligned}$$

As for the debt discount, again, we subtract the bond value to the price of the risk-less debt.

Definition 26 *The time-T price of the debt discount on case 2 (Discount with debt ratchet) of a firm with the value V, face value of debt D, debt increase level D/K, debt increase amount δ and maturity time $T(\geq t_0)$, assuming that $\tau_0 > t_0$, is given by*

$$\begin{aligned}
&G_T^2(V_T, D, K, \delta, T; \tau_{D/K}, \phi_{dwl}) \\
=&D - \mathbb{1}_{\left\{\tau_{D/K}>T\right\}}\mathbb{1}_{\{\tau_0>T\}}(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}) \\
&- \mathbb{1}_{\{\tau_{D/K}\leq T\}}\mathbb{1}_{\{\tau_0>T\}}\frac{1}{1+\delta}(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}) \\
=&D\mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0\leq T\}} + D\mathbb{1}_{\{\tau_{D/K}\leq T\}}\mathbb{1}_{\{\tau_0\leq T\}} \\
&+ D\mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0>T\}} + D\mathbb{1}_{\{\tau_{D/K}\leq T\}}\mathbb{1}_{\{\tau_0>T\}} \\
&- \mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0>T\}}(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}) \\
&- \mathbb{1}_{\{\tau_{D/K}\leq T\}}\mathbb{1}_{\{\tau_0>T\}}\frac{1}{1+\delta}(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}) \\
=&\ \mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D\geq V_T\}} - \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right) + D\mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0\leq T\}} \\
&+ \frac{1}{1+\delta}\mathbb{1}_{\{\tau_{D/K}\leq T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)\geq V_T\}} - \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right) \\
&+ \frac{1+\delta}{1+\delta}D\mathbb{1}_{\left\{\tau_{D/K}\leq T\right\}}\mathbb{1}_{\{\tau_0\leq T\}} \\
=&EUO_T^0(V_T, D, D/K; 1, \tau_{D/K}, \phi_{dwl}) + EUO_T^D(V_T, D, D/K; 1, \tau_{D/K}, \phi_{dwl})
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{1+\delta}\cdot EUI_T^0(V_T, D(1+\delta), D/K; 1, \tau_{D/K}, \phi_{dwl})\\
&+\frac{1}{1+\delta}\cdot EUI_T^D(V_T, D(1+\delta), D/K; 1, \tau_{D/K}, \phi_{dwl})\\
=&EUO_T(V_T, D, D/K; 1, \tau_{D/K}, \phi_{dwl}) + \frac{1}{1+\delta}\cdot EUI_T(V_T, D(1+\delta), D/K; 1, \tau_{D/K}, \phi_{dwl})
\end{aligned} \tag{143}
$$

where

$$
\tau_{D/K} = \inf\{u > 0 : V_u = D/K\}.
$$

that is, a weighted sum of two puts: an up-and-out put — as presented in Definition 17, while the conditional on no default component is solved as in Dias et al. (2021, Proposition 1) , through subtracting the value from a vanilla put, and the recovery component is solved as in Dias et al. (2021, Proposition 3) (not considering the possibility of a jump to zero) — and an up-and-in put — as presented in Definition 15, while the conditional on no default component is solved as in Dias et al. (2021, Proposition 1) and the recovery component is solved as in Dias et al. (2021, Proposition 2) (not considering the possibility of a jump to zero).

The first option has the same parameters as the one in case 1, but carrying a knock-out feature at the upper level, D/K, making it an up-and-out barrier option. So, the options is deactivated once this threshold is crossed, translating into the firm no longer holding the initial amount of debt, D, but a higher amount.

To transition to the higher amount of debt, a second element is added: an option that departs from the initial firm value as the previous one, but has a knock-in feature at the upper level D/K, the same level where the first option is deactivated. This knock-in option has the strike price of $D(1+\delta)$, the new debt level, increasing the original face value of debt by $100\cdot\delta\%$.

10.4 Case 3: Discount with a debt swap down

This case is similar to the previous one, with the debt increase replaced by a debt decrease. If V hits the value D/M, the liabilities drop to the level $D(1-d)$.

Now, the cases where the firm's value does not cross the lower threshold before maturity, $\tau_{D/M} > T$, will hold the same payoff as before

$$\mathbb{1}_{\{\tau_0 > T\}} \left(D \mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right), \tag{144}$$

while the cases where the lower barrier is crossed, $\tau_{D/M} \leq T$, it will be given by

$$\mathbb{1}_{\{\tau_0 > T\}} \frac{1}{1-d} \left(D(1-d) \mathbb{1}_{\{D(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right). \tag{145}$$

Now, default only occurs when the firm level is below the reduced amount of debt. The change in the debt level from the initial value is taken into account. So again, we weight the payoff of when the debt changes by the ratio between the old and the new face values of debt: $1/(1-d)$.

Therefore, the payoff of the bond at maturity is

$$\begin{aligned}
&B_T^3(V_T, D, M, d, T; \tau_{D/M}, \phi_{dwl}) \\
=&\mathbb{1}_{\left\{\tau_{D/M} > T\right\}} \mathbb{1}_{\{\tau_0 > T\}} \left(D \mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\
&+ \mathbb{1}_{\left\{\tau_{D/M} \leq T\right\}} \mathbb{1}_{\{\tau_0 > T\}} \frac{1}{1-d} \left(D(1-d) \mathbb{1}_{\{D(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right).
\end{aligned} \tag{146}$$

As for the debt discount, again, we subtract from the price of the risk-less debt.

Definition 27 *The time-T price of the debt discount on case 3 (Discount with a debt swap down) of a firm with value V, face value of debt D, debt decrease level D/M, debt decrease amount d and maturity time $T(\geq t_0)$, assuming that $\tau_0 > t_0$, is given by*

$$\begin{aligned}
&G_3(V_T, D, M, d, T; \tau_L, \phi_{dwl}) \\
=&D - \mathbb{1}_{\{\tau_{D/M} > T\}} \mathbb{1}_{\{\tau_0 > T\}} (D \mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}}) \\
&- \mathbb{1}_{\{\tau_{D/M} \leq T\}} \mathbb{1}_{\{\tau_0 > T\}} \frac{1}{1-d} (D(1-d) \mathbb{1}_{\{D(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}}) \\
=&D \mathbb{1}_{\{\tau_{D/M} > T\}} \mathbb{1}_{\{\tau_0 \leq T\}} + D \mathbb{1}_{\{\tau_{D/M} \leq T\}} \mathbb{1}_{\{\tau_0 \leq T\}}
\end{aligned}$$

$$
\begin{aligned}
&+ D\mathbb{1}_{\{\tau_{D/M}>T\}}\mathbb{1}_{\{\tau_0>T\}} + D\mathbb{1}_{\{\tau_{D/M}\leq T\}}\mathbb{1}_{\{\tau_0>T\}} \\
&- \mathbb{1}_{\{\tau_{D/M}>T\}}\mathbb{1}_{\{\tau_0>T\}}(D\mathbb{1}_{\{D>V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\leq V_T\}}) \\
&- \frac{1}{1-d}\mathbb{1}_{\{\tau_{D/M}\leq T\}}\mathbb{1}_{\{\tau_0>T\}}(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1-d)\geq V_T\}}) \\
=&EDO_T(V_T, D, D/M; 1, \tau_{D/M}, \phi_{dwl}) + \frac{1}{1-d}\cdot EDI_T(V_T, D(1-d), D/M; 1, \tau_{D/M}, \phi_{dwl})
\end{aligned}
\tag{147}
$$

where

$$
\tau_{D/M} = \inf\{u > 0 : V_u = D/M\}.
$$

that is, a weighted sum of two puts: a down-and-out put — as presented in Definition 18, while the conditional on no default component is solved as in Dias et al. (2021, Proposition 1), through subtracting the value from a vanilla put (not considering the possibility of a jump to zero), and the recovery component is always equal to zero — and a down-and-in put — as presented in Definition 16, while the conditional on no default component is solved as in Dias et al. (2021, Proposition 1) (not considering the possibility of a jump to zero) and the recovery component (the same as in a vanilla put) is solved by Dias et al. (2020, equation (22)).

Therefore, first, we have an option starting at the level V and the strike price that corresponds to the initial face value of debt level, D, and this option is knocked-out when V crosses the lower level D/M, a knock-out barrier option.

To complete the debt value transition, upon the knock-out event of the first option, a down-and-in option is knocked-in, holding the strike price that represents the lower debt level, $D(1-d)$, decreasing it by $100 \cdot d\%$.

10.5 Case 4: Discount with the option to either ratchet or swap down debt

Here, the two previous cases are combined, with one, and only one, of the adjustments on the debt level occurring. In the cases where the upper barrier, D/K, is crossed first, the debt level is ratcheted to $D(1+\delta)$. If the lower barrier, D/M, is crossed first, the debt level is swapped-down to $D(1-d)$.

Assuming the default does not occur before maturity, in the cases where the firm level touches neither the upper barrier, $\tau_{D/K} > T$, nor the lower barrier, $\tau_{D/M} > T$, that is, $\tau_{D/K,D/M} > T$, the face value of debt remains at the initial level

$$\mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right). \tag{148}$$

If the upper barrier was touched, and so before the upper barrier being, $\bar{\tau}_{D/K} \leq T$, the payoff of the debt-holder will take into account the increase in debt, while being multiplied by the appropriate weight

$$\frac{1}{1+\delta} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta) \geq V_T\}} \right), \tag{149}$$

whereas, in the cases where the lower barrier is touched before the upper barrier, $\bar{\tau}_{D/M} \leq T$, the payoff with the proper weight is

$$\frac{1}{1-d} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right). \tag{150}$$

Here, the use of the two trigger clauses is crucial, as we need the first passage time of one barrier without touching the other one.

Thus, the bond value of the original debt-holder will be given by

$$\begin{aligned}
&B_T^4(V_T, D, K, M, \delta, d, T; \bar{\tau}_{D/K}, \bar{\tau}_{D/M}, \phi_{dwl}) \\
=&\mathbb{1}_{\{\tau_{D/K,D/M} > T\}} \mathbb{1}_{\{\tau_0 > T\}} \left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\
&+ \mathbb{1}_{\{\bar{\tau}_{D/K} \leq T\}} \frac{1}{1+\delta} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta) \geq V_T\}} \right) \\
&+ \mathbb{1}_{\{\bar{\tau}_{D/M} \leq T\}} \frac{1}{1-d} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right).
\end{aligned} \tag{151}$$

The debt discount, once again, is obtained by subtracting the price of debt from the risk-less debt.

Definition 28 *The time-T price of the debt discount on case 4 (Discount with the option to either ratchet or swap down debt) of a firm with V, nominal debt D, debt increase level D/K, debt increase amount δ, debt decrease level D/M, debt decrease amount d and maturity time $T(\geq t_0)$, assuming that $\tau_0 > t_0$, is given by*

$$
\begin{aligned}
&G^4_T(V_T, D, K, M, \delta, d, T; \bar{\tau}_{D/K}, \bar{\tau}_{D/M}, \phi_{dwl})\\
=&D - \mathbb{1}_{\{\tau_{D/K,D/M}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right)\\
&- \mathbb{1}_{\{\bar{\tau}_{D/K}\leq T\}}\frac{1}{1+\delta}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right)\\
&- \mathbb{1}_{\{\bar{\tau}_{D/M}\leq T\}}\frac{1}{1-d}\mathbb{1}_{\{\tau_0>T\}}\left(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1-d)\geq V_T\}}\right)\\
=&D\mathbb{1}_{\{\tau_0\leq T\}}\mathbb{1}_{\{\tau_{D/K,D/M}>T\}} + D\mathbb{1}_{\{\tau_0\leq T\}}\mathbb{1}_{\{\bar{\tau}_{D/K}<T\}} + D\mathbb{1}_{\{\tau_0\leq T\}}\mathbb{1}_{\{\bar{\tau}_{D/M}<T\}}\\
&+ D\mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\tau_{D/K,D/M}>T\}} + D\mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\bar{\tau}_{D/K}<T\}} + D\mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\bar{\tau}_{D/M}<T\}}\\
&- \mathbb{1}_{\{\tau_{D/K,D/M}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right)\\
&- \mathbb{1}_{\{\bar{\tau}_{D/K}\leq T\}}\frac{1}{1+\delta}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right)\\
&- \mathbb{1}_{\{\bar{\tau}_{D/M}\leq T\}}\frac{1}{1-d}\mathbb{1}_{\{\tau_0>T\}}\left(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1-d)\geq V_T\}}\right)\\
=&EDBKO_T(V_T, D, D/M, D/K; 1, \tau_{D/K,D/M}, \phi_{dwl})\\
&+ \frac{1}{1+\delta}\cdot EUI_T(V_T, D(1+\delta), D/K; 1, \bar{\tau}_{D/K}, \phi_{dwl})\\
&+ \frac{1}{1-d}\cdot EDI_T(V_T, D(1-d), D/M; 1, \bar{\tau}_{D/M}, \phi_{dwl})
\end{aligned}
\tag{152}
$$

where

$$\bar{\tau}_{D/M} := \inf\left\{u > t_0 : V_u = D/M, \sup_{t_0\leq v\leq u}(V_v - D/K) < 0\right\},$$

$$\bar{\tau}_{D/K} := \inf\left\{u > t_0 : V_u = D/K, \inf_{t_0\leq v\leq u}(V_v - D/M) > 0\right\}$$

and

$$\tau_{D/K,D/M} = \bar{\tau}_{D/M} \wedge \bar{\tau}_{D/K}.$$

that is, a weighted sum of three puts: a double barrier knock-out put — as presented in Definition 20, while the conditional on no default component is solved as in (Dias et al., 2015, Proposition 3.2) (not considering the possibility of a jump to zero) and the recovery component is always zero — an up-and-in put - as presented in Definition 15, while the conditional on no default component is solved by Dias et al. (2021, Proposition 1) and the recovery component is solved by Dias et al. (2021, Proposition 2) (not considering the possibility of a jump to zero) — and a down-and-in put — as presented in Definition 16, while the conditional on no default component is solved as in Dias et al. (2021, Proposition 1) (not considering the possibility of a jump to zero) and the recovery component (the same as in a vanilla put) is solved by equation 3.10 with the survival probability from in Dias et al. (2020, equation 22).

Therefore, first, there is a double barrier knock-out option, which is knocked-out when one of its two barriers is reached. It has the original debt level, D, as the strike price, and the knock-out barriers are D/K and D/M.

For the debt ratch-up branch, there is an up-and-in barrier option with strike at the ratch-up level of debt, $D(1+\delta)$. The barrier is at D/K, but with a two trigger clause, and consequently the knock-in event only occurs if D/M is not crossed first.

As for the debt write down branch, there is a down-and-in barrier option, with the strike at the write down level of debt, $D(1-d)$. The barrier which activates the option is at D/M, provided that D/K is not crossed.

The setting of the knock-in options with a two trigger clause ensures that only one of the options is knocked-in. This allows only either the debt ratch-up or the debt write down to occur once the double barrier option is knocked-out, never both.

10.6 Case 5: Discount with the option to swap down after ratcheting

In this case, there is a possible sequence of two events, each with a debt change.

First, a debt ratchet after the asset price crosses the barrier D/K, increasing the debt to the ratch-up level, $D(1+\delta)$. After that, a swap down is possible if $D(1+\delta)/M$ is crossed, decreasing the debt level by $d \cdot 100\%$ to the final level, $D(1+\delta)(1-d)$.

As in case 2, in the case the upper barrier is not crossed, $\tau_{D/K} > T$, the payoff will consider the initial face value of debt

$$\mathbb{1}_{\{\tau_0 > T\}} \left(D \mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) . \tag{153}$$

Also, as in case 2, if the upper barrier, D/K, is crossed by the firm value, the face value of debt will be increased by δ. Although, as an additional condition, after crossing D/K, the firm level cannot cross $D(1+\delta)/M$ without triggering another debt change. Therefore, at $\tau_{D/K} \leq T < \tau_{D(1+\delta)/M}$:

$$\frac{1}{1+\delta} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta) \mathbb{1}_{\{D(1+\delta) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta) \geq V_T\}} \right) . \tag{154}$$

Finally, if the firm value has crossed the upper level, D/K, and afterwards crosses the lower level, $D(1+\delta)/M$, the face value of debt suffers a decrease of d, thus, it is set at $D(1+\delta)(1-d)$. So, when $\tau_{D/K} < \tau_{D(1+\delta)/M} \leq T$ the payoff on the debt is

$$\frac{1}{(1+\delta)(1-d)} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta)(1-d) \mathbb{1}_{\{D(1+\delta)(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta)(1-d) \geq V_T\}} \right) , \tag{155}$$

and the bond value of the original debt-holder is:

$$\begin{aligned} & B_T^5(V_T, D, K, M, \delta, d, T; \tau_{D(1+\delta)/M}, \tau_{D/K}, \phi_{dwl}) \\ = & \mathbb{1}_{\{\tau_{D/K} > T\}} \mathbb{1}_{\{\tau_0 > T\}} \left(D \mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\ & + \mathbb{1}_{\{\tau_{D/K} \leq T < \tau_{D(1+\delta)/M}\}} \frac{1}{1+\delta} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta) \mathbb{1}_{\{D(1+\delta) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta) \geq V_T\}} \right) \\ & + \mathbb{1}_{\{\tau_{D/K} < \tau_{D(1+\delta)/M} \leq T\}} \frac{1}{(1+\delta)(1-d)} \\ & \times \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta)(1-d) \mathbb{1}_{\{D(1+\delta)(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta)(1-d) \geq V_T\}} \right) \end{aligned} \tag{156}$$

And once again, to obtain the debt discount, we subtract the expression above from the risk-less debt.

Definition 29 *The time-T price of the debt discount on Case 5 (Discount with the option to swap down after ratcheting) of a firm with value V, nominal debt D, debt increase level D/K, debt increase amount δ, debt decrease level $D(1+\delta)/M$, debt decrease amount d and maturity time $T(\geq t_0)$, assuming that $\tau_0 > t_0$, is given by*

$$\begin{aligned}
&G_T^5(V_T, D, K, M, \delta, d, T; \tau_{D(1+\delta)/M}, \tau_{D/K}, \phi_{dwl}) \\
=&D - \mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right) \\
&- \mathbb{1}_{\{\tau_{D/K}\leq T<\tau_{D(1+\delta)/M}\}}\frac{1}{1+\delta}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right) \\
&- \mathbb{1}_{\{\tau_{D/K}<\tau_{D(1+\delta)/M}\leq T\}}\frac{1}{(1+\delta)(1-d)} \\
&\times \mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right) \\
=&D\mathbb{1}_{\{\tau_0\leq T\}}\mathbb{1}_{\{\tau_{D/K}>T\}} + D\mathbb{1}_{\{\tau_0\leq T\}}\mathbb{1}_{\{\tau_{D/K}\leq T<\tau_{D(1+\delta)/M}\}} + D\mathbb{1}_{\{\tau_0\leq T\}}\mathbb{1}_{\{\tau_{D/K}<\tau_{D(1+\delta)/M}\leq T\}} \\
&+ D\mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\tau_{D/K}>T\}} + D\mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\tau_{D/K}\leq T<\tau_{D(1+\delta)/M}\}} + D\mathbb{1}_{\{\tau_0>T\}}\mathbb{1}_{\{\tau_{D/K}<\tau_{D(1+\delta)/M}\leq T\}} \\
&- \mathbb{1}_{\{\tau_{D/K}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right) \\
&- \mathbb{1}_{\{\tau_{D/K}\leq T<\tau_{D(1+\delta)/M}\}}\frac{1}{1+\delta}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right) \\
&- \mathbb{1}_{\{\tau_{D/K}<\tau_{D(1+\delta)/M}\leq T\}}\frac{1}{(1+\delta)(1-d)} \\
&\times \mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right) \\
=&EUO_T(V_T, D, D/K; 1, \tau_{D/K}, \phi_{dwl}) \\
&+ \frac{1}{1+\delta}\cdot EUI, DO_T(V_T, D(1+\delta), D(1+\delta)/M, D/K; 1, \tau_{D(1+\delta)/M}, \tau_{D/K}, \phi_{dwl}) \\
&+ \frac{1}{(1+\delta)(1-d)} \\
&\times EUDI_T(V_T, D(1+\delta)(1-d), D(1+\delta)/M, D/K; 1, \tau_{D(1+\delta)/M}, \tau_{D/K}, \phi_{dwl})
\end{aligned} \tag{157}$$

with

$$\tau_{D/K} = \inf\{u > 0 : V_u = D/K\},$$

and

$$\tau_{D(1+\delta)/M} = \inf\left\{u > \tau_{D/K} : V_u = D(1+\delta)/M\right\},$$

that is, a weighted sum of three puts: an up-and-out put — as presented in Definition 17, while the conditional on no default component is solved as in Dias et al. (2021, Proposition 1), through subtracting from a vanilla put, and the recovery component is solved as in Dias et al. (2021, Proposition 3) (not considering the possibility of a jump to zero) — a first-up-in-then-down-and-out put — as presented in Definition 24, while the conditional on no default component is solved by Proposition 9 from Part II (not considering the possibility of a jump to zero) and the recovery value is always zero given the result of Proposition 11 — and a first-up-then-down-and-in put — as presented in Definition 22, while the conditional on no default is also solved as in Proposition 5 from Part II and the recovery value is solved by Proposition 7 from Part II (not considering the possibility of a jump to zero).

This case starts as case 2, with an up-and-out put with the initial debt value, D, as the strike price.

As for the second option, it is also knocked-in when the firm value crosses the upper barrier D/K, but it is knocked out at the lower barrier $D(1+\delta)/M$, therefore, it is a first-up-in-then-down-and-out put option. As for the strike price, it has the second face value of debt, $D(1+\delta)$.

As for the third option, it is knocked-in when the firm value crosses the $D(1+\delta)/M$ barrier, provided it has crossed D/K before. Thus, we have a first-up-then-down-and-in put option with the strike price being the third possible level of face value of debt, $D(1+\delta)(1-d)$.

10.7 Case 6: Discount with the option to ratchet after swap down

This case is analogous to case 5, now extending from case 3. We have a possible sequence of two debt change events.

First, when the firm value, V, hits D/M, the debt level is reduced to $D(1-d)$, but now, in addition, if the firm value hits $D(1+\delta)/K$ afterwards, the debt value ratchets to $D(1+\delta)(1-d)$.

As in case 3, when the lower barrier is not crossed, $\tau_{D/M} > T$, the debt-holder will receive a payoff corresponding the original face value of debt

$$\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right). \tag{158}$$

Also, as case 3, if the lower barrier, D/M, is crossed by the firm value, the face value of debt will be decreased by d. Now, as an additional condition, after crossing D/M, if the firm level crosses $D(1-d)/K$ there is another debt change. So, at $\tau_{D/M} \leq T < \tau_{D(1-d)/K}$ the payoff is

$$\frac{1}{1-d}\mathbb{1}_{\{\tau_0>T\}}\left(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1-d)\geq V_T\}}\right). \tag{159}$$

In the last scenario, if after crossing the lower level D/M, the option crosses the upper level $D(1-d)/K$, the face value of debt suffers an increase, and is set at $D(1+\delta)(1-d)$. Thus, when $\tau_{D/M} < \tau_{D(1-d)/K} \leq T$, the payoff on the debt is

$$\frac{1}{(1+\delta)(1-d)}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right). \tag{160}$$

Thus, the bond value of the original debt-holder is given by:

$$\begin{aligned}
&B_T^6(V_T, D, K, M, \delta, d, T; \tau_{D/M}, \tau_{D(1-d)/K}, \phi_{dwl})\\
=&\mathbb{1}_{\{\tau_{D/M}<T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right)\\
&+\mathbb{1}_{\{\tau_{D/M}\leq T<\tau_{D(1-d)/K}\}}\frac{1}{1+\delta}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right)\\
&+\mathbb{1}_{\{\tau_{D/M}<\tau_{D(1-d)/K}\leq T\}}\frac{1}{(1+\delta)(1-d)}\\
&\times\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}} + \phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right).
\end{aligned} \tag{161}$$

And the debt discount, subtracted from the risk-less debt value, is defined as follows.

Definition 30 *The time-T price of the debt discount on case 6 (Discount with the option to ratchet after swap down) of a firm with value V, nominal debt D, debt increase level $D(1-d)/K$, debt increase amount δ, debt decrease level D/M, debt decrease amount d and maturity time $T(\geq t_0)$, assuming that $\tau_0 > t_0$, is given by*

$$
\begin{aligned}
& G_T^6(V_T, D, K, M, \delta, d, T; \tau_{D/M}, \tau_{D(1-d)/K}, \phi_{dwl}) \\
=& D - \mathbb{1}_{\{\tau_{D/M}>T\}} \mathbb{1}_{\{\tau_0>T\}} \left(D \mathbb{1}_{\{D<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\
& - \mathbb{1}_{\{\tau_{D/M} \leq T < \tau_{D(1-d)/K}\}} \frac{1}{1-d} \mathbb{1}_{\{\tau_0>T\}} \left(D(1-d) \mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right) \\
& - \mathbb{1}_{\{\tau_{D/M} < \tau_{D(1-d)/K} \leq T\}} \frac{1}{(1+\delta)(1-d)} \\
& \times \mathbb{1}_{\{\tau_0>T\}} \left(D(1+\delta)(1-d) \mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta)(1-d) \geq V_T\}} \right) \\
=& D \mathbb{1}_{\{\tau_0 \leq T\}} \mathbb{1}_{\{\tau_{D/M}>T\}} + D \mathbb{1}_{\{\tau_0 \leq T\}} \mathbb{1}_{\{\tau_{D/M} \leq T < \tau_{D(1-d)/K}\}} + D \mathbb{1}_{\{\tau_0 \leq T\}} \mathbb{1}_{\{\tau_{D/M} < \tau_{D(1-d)/K} \leq T\}} \\
& + D \mathbb{1}_{\{\tau_0>T\}} \mathbb{1}_{\{\tau_{D/M}>T\}} + D \mathbb{1}_{\{\tau_0>T\}} \mathbb{1}_{\{\tau_{D/M} \leq T < \tau_{D(1-d)/K}\}} + D \mathbb{1}_{\{\tau_0>T\}} \mathbb{1}_{\{\tau_{D/K} < \tau_{D(1-d)/K} \leq T\}} \\
& - \mathbb{1}_{\{\tau_{D/M}>T\}} \mathbb{1}_{\{\tau_0>T\}} \left(D \mathbb{1}_{\{D<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) \\
& - \mathbb{1}_{\{\tau_{D/M} \leq T < \tau_{D(1-d)/K}\}} \frac{1}{1-d} \mathbb{1}_{\{\tau_0>T\}} \left(D(1-d) \mathbb{1}_{\{D(1-d)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right) \\
& - \mathbb{1}_{\{\tau_{D/M} < \tau_{D(1-d)/K} \leq T\}} \frac{1}{(1+\delta)(1-d)} \\
& \times \mathbb{1}_{\{\tau_0>T\}} \left(D(1+\delta)(1-d) \mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta)(1-d) \geq V_T\}} \right) \\
=& EDO_T(V_T, D, D/M; 1, \tau_{D/M}, \phi_{dwl}) \\
& + \frac{1}{1-d} \cdot EDI,UO_T(V_T, D(1-d), D/M, D(1-d)/K; 1, \tau_{D/M}, \tau_{D(1-d)/K}, \phi_{dwl}) \\
& + \frac{1}{(1+\delta)(1-d)} \\
& \times EDUI_T(V_T, D(1+\delta)(1-d), D/M, D(1-d)/K; 1, \tau_{D/M}, \tau_{D(1-d)/K}, \phi_{dwl})
\end{aligned} \tag{162}
$$

with

$$
\tau_{D(1-d)/K} = \inf \left\{ u > \tau_{D/M} : V_u = D(1-d)/K \right\},
$$

and

$$\tau_{D/M} = \inf \{u > 0 : V_u = D/M\},$$

that is, a weighted sum of three puts: a down-and-out put — as presented in Definition 18, while the conditional on no default component is solved by from Dias et al. (2021, Proposition 1) (not considering the possibility of a jump to zero), through subtracting the value from a vanilla put, and the recovery component is always equal to zero — a first-down-in-then-up-and-out put — as presented in Definition 23, while the conditional on no default component is solved by Proposition 8 from Part II and the recovery value is solved by by Proposition 10 from Part II — and a first-down-then-up-and-in put - as presented in Definition 21, while the conditional on no default is solved by Proposition 4 from Part II and the recovery value is solved as in Proposition 6 from Part II (not considering the possibility of a jump to zero).

This case starts as case 3, with an up-and-out put with the initial debt value as the strike price, D.

As for the second option, it is also knocked-in when the firm value crosses the lower barrier D/M, but it is knocked-out at the upper barrier $D(1-d)/K$, therefore, it is a first-down-in-then-up-and-out put option. Its strike price is the second possibility for the value of debt, $D(1-d)$.

As for the third option, it is knocked-in after the firm value crosses $D(1-d)/K$, provided it has crossed the D/M before. Thus, it is a first-down-then-up-and-in put option with the strike price being the third possible level of face value of debt, $D(1+\delta)(1-d)$.

10.8 Case 7: Discount allowing ratchet after swap down or vice versa

Alike case 4, this is a combination of the previous two cases.

In the firm's value path, if the upper barrier, D/K, is crossed first, the debt is ratcheted to $D(1+\delta)$ and has the possibility of being swapped down if the lower barrier, $D(1+\delta)/M$, is reached afterwards. If the lower barrier, D/M, is crossed first, the debt is swapped-down to $D(1-d)$, and a possibility is open for a debt ratchet once the upper barrier, $D(1-d)/K$, is crossed. In both cases the final face value of debt is $D(1+\delta)(1-d)$.

The fist payoff is the same as in case 4: if the firm level touches neither the upper barrier, $\bar{\tau}_{D/K} > T$, nor the lower barrier, $\bar{\tau}_{D/M} > T$, that is, if $\tau_{D/K,D/M} > T$, the initial value for the face value of debt remains,

$$\mathbb{1}_{\{\tau_0 > T\}} \left(D \mathbb{1}_{\{D < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D \geq V_T\}} \right) . \tag{163}$$

Again, as in case 4, if the upper barrier was touched before the lower barrier being so, the debt level is increased to $D(1+\delta)$. As in case 5, this is not the final possible level of debt, it is only so if the lower barrier, $D(1+\delta)/M$, was not touched. Therefore the payoff in the case $\bar{\tau}_{D/K} \leq T < \tau_{D(1+\delta)/M}$ will be

$$\frac{1}{1+\delta} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta) \mathbb{1}_{\{D(1+\delta) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta) \geq V_T\}} \right) . \tag{164}$$

Afterwards, if the firm level crosses the lower barrier, $D(1+\delta)/M$, the debt level will be decreased to $D(1+\delta)(1-d)$. So when $\bar{\tau}_{D/K} < \tau_{D(1+\delta)/M} \leq T$ the payoff is

$$\frac{1}{(1+\delta)(1-d)} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1+\delta)(1-d) \mathbb{1}_{\{D(1+\delta)(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1+\delta)(1-d) \geq V_T\}} \right) . \tag{165}$$

As for the the cases where the lower barrier, D/M, is the first touched barrier, the debt level is decreased to $D(1-d)$, and as in case 6, this is not the final possible level of debt. If the upper barrier, $D(1-d)/K$, is crossed afterwards, the debt level is changed. Therefore, for $\bar{\tau}_{D/M} \leq T < \tau_{D(1-d)/K}$ the payoff is

$$\frac{1}{1-d} \mathbb{1}_{\{\tau_0 > T\}} \left(D(1-d) \mathbb{1}_{\{D(1-d) < V_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{D(1-d) \geq V_T\}} \right) . \tag{166}$$

If after crossing D/M, the firm value crosses the upper barrier, $D(1-d)/K$, the debt level is increased to $D(1+\delta)(1-d)$. Thus, at $\bar{\tau}_{D/M} < \tau_{D(1-d)/K} \leq T$, the payoff is given by

$$\frac{1}{(1+\delta)(1-d)}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}}+\phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right). \tag{167}$$

Therefore, the value of the bond in case 7 is going to be

$$\begin{aligned}
&B_T^7(V_T,D,K,M,\delta,d,T;\bar{\tau}_{D/M},\tau_{D(1+\delta)/M},\bar{\tau}_{D/K},\tau_{D(1-d)/K},\phi_{dwl})\\
=&\mathbb{1}_{\{\tau_{D/K,D/M}>T\}}\mathbb{1}_{\{\tau_0>T\}}\left(D\mathbb{1}_{\{D<V_T\}}+\phi_{dwl}V_T\mathbb{1}_{\{D\geq V_T\}}\right)\\
&+\mathbb{1}_{\{\bar{\tau}_{D/K}\leq T<\tau_{D(1+\delta)/M}\}}\frac{1}{1+\delta}\mathbb{1}_{\{\tau_0>T\}}\left(D(1+\delta)\mathbb{1}_{\{D(1+\delta)<V_T\}}+\phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)\geq V_T\}}\right)\\
&+\mathbb{1}_{\{\bar{\tau}_{D/K}<\tau_{D(1+\delta)/M}\leq T\}}\frac{1}{(1+\delta)(1-d)}\mathbb{1}_{\{\tau_0>T\}}\\
&\times\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}}+\phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right)\\
&+\mathbb{1}_{\{\bar{\tau}_{D/M}\leq T<\tau_{D(1-d)/K}\}}\frac{1}{1-d}\mathbb{1}_{\{\tau_0>T\}}\left(D(1-d)\mathbb{1}_{\{D(1-d)<V_T\}}+\phi_{dwl}V_T\mathbb{1}_{\{D(1-d)\geq V_T\}}\right)\\
&+\mathbb{1}_{\{\bar{\tau}_{D/M}<\tau_{D(1-d)/K}\leq T\}}\frac{1}{(1+\delta)(1-d)}\mathbb{1}_{\{\tau_0>T\}}\\
&\times\left(D(1+\delta)(1-d)\mathbb{1}_{\{D(1+\delta)(1-d)<V_T\}}+\phi_{dwl}V_T\mathbb{1}_{\{D(1+\delta)(1-d)\geq V_T\}}\right)
\end{aligned} \tag{168}$$

Once again, subtracting the bond value from the risk-free bond, we obtain the debt discount.

Definition 31 *The time-T price of the debt discount on case 7 (Discount allowing ratchet after swap down or vice versa) of a firm with value V, nominal debt D, debt increase levels D/K and $D(1-d)/K$, debt increase amount δ, debt decrease levels D/M and $D(1+\delta)/M$, debt decrease amount d and maturity time $T(\geq t_0)$, assuming that $\tau_0>t_0$, is given by*

$$\begin{aligned}
&G_7(V_T,D,K,M,\delta,d,T;\bar{\tau}_{D/M},\tau_{D(1+\delta)/M},\bar{\tau}_{D/K},\tau_{D(1-d)/K},\phi_{dwl})\\
=&EDBKO_T(V_T,D,D/M,D/K;1,\bar{\tau}_{D/K,D/M},\phi_{dwl})\\
&+\frac{1}{1+\delta}\cdot EUI,DO_T(V_T,D(1+\delta),D(1+\delta)/M,D/K;1,\tau_{D(1+\delta)/M},\bar{\tau}_{D/K},\phi_{dwl})\\
&+\frac{1}{(1+\delta)(1-d)}\\
&\times EUDI_T(V_T,D(1+\delta)(1-d),D(1+\delta)/M,D/K;1,\tau_{D(1+\delta)/M},\bar{\tau}_{D/K},\phi_{dwl})
\end{aligned}$$

$$+\frac{1}{1-d}\cdot EDI,UO_T(V_T, D(1-d), D/M, D(1-d)/K; 1, \bar{\tau}_{D/M}, \tau_{D(1-d)/K}, \phi_{dwl})$$
$$+\frac{1}{(1+\delta)(1-d)}$$
$$\times EDUI_T(V_T, D(1+\delta)(1-d), D/M, D(1-d)/K; 1, \bar{\tau}_{D/M}, \tau_{D(1-d)/K}, \phi_{dwl}) \tag{169}$$

with

$$\bar{\tau}_{D/M} := \inf\left\{u > t_0 : S_u = D/M, \sup_{t_0\leq v\leq u}(V_v - D/K) < 0\right\},$$

$$\bar{\tau}_{D/K} := \inf\left\{u > t_0 : S_u = D/K, \inf_{t_0\leq v\leq u}(V_v - D/M) > 0\right\},$$

$$\tau_{D(1+\delta)/M} = \inf\left\{u > \bar{\tau}_{D/K} : V_u = D(1+\delta)/M\right\},$$

$$\tau_{D(1-d)/K} = \inf\left\{u > \bar{\tau}_{D/M} : V_u = D(1-d)/K\right\},$$

and

$$\tau_{D/K,D/M} = \bar{\tau}_{D/M} \wedge \bar{\tau}_{D/K},$$

that is, a sum weighted of five puts: a double barrier knock-out put — as presented in Definition 20, while the conditional on no default component is solved as in Dias et al. (2015, Proposition 3.2) (not considering the possibility of a jump to zero) and the recovery component is always zero — a first-up-in-then-down-and-out put — as presented in Definition 24, while the conditional on no default component is solved by Proposition 9 from Part II (not considering the possibility of a jump to zero) and the recovery value is always *zero given the result of Proposition 11 — a first-up-then-down-and-in put — as presented in Definition 22, while the conditional on no default is also solved as in Proposition 5 from Part II and the recovery value is solved by Proposition 7 from Part II (not considering the possibility of a jump to zero) — a first-down-in-then-up-and-out put — as presented in Definition 23, while the conditional on no default component is solved by Proposition 8 from Part II and the recovery value is solved by by Proposition 10 from Part II — and a first-down-then-up-and-in put — as presented in Definition 21, while the conditional on no default is solved by Proposition 4 from Part II and the recovery value is solved as in Proposition 6 from Part II (not considering the possibility of a jump to zero).*

As V evolves, it can go up through the D/K barrier and trigger case 5, have its debt ratched-up to $D(1+d)$ and then have the possibility of being swapped-down to $D(1+d)(1-d)$ if $D(1+d)/M$ is crossed by V. Or, in alternative, V goes down through D/M and triggers case 6, with the debt being swapped down to $D(1-d)$, and having the possibility of being ratched-up to the level $D(1+\delta)(1-d)$ if V crosses $D(1-d)/K$ afterwards.

As in case 4, we start with a double knock-out option, which has D for the strike value and the knock-out barriers of D/K and D/M. Once one of the barriers is crossed by V and the option is knocked-out, the trigger clause corresponding to the crossed barrier activates one of the two branches - either the one that first increases debt and then decreases it, or vice-versa.

For the first branch, when the first barrier to be crossed is the upper one, D/K, a knock-out barrier option with strike $D(1+\delta)$ is activated, representing the debt increase. Then, if V crosses $D(1+\delta)/M$ the knock-out barrier is deactivated and a new option with strike $D(1+\delta)(1-d)$ is knocked-in, enabling the decrease in debt.

As for the second branch, in the case where the first barrier to be crossed is D/M, a knock-out barrier option with strike corresponding to the lower debt level $D(1-d)$ is activated. Afterwards, if V crosses $D(1-d)/K$, the former option is deactivated and the option corresponding to the debt level $D(1+\delta)(1-d)$ becomes active.

11 Double barrier first-then-options and results mismatch

In Das and Kim (2015), the results that arrive from first-then-barrier options are based on a result presented in Haug (2006), which departs from standard barrier options' formulae and option symmetry. Although, as mentioned in Haug (2006), this result is only valid when the cost-of-carry is zero, that is $r = q$. Despite this, the results presented in Das and Kim (2015) take on the parameters, $r = 2\%$ and $q = 0\%$, which leads to the belief that the results display incorrectness. This is explored in Table 7, where the results using the Das and Kim (2015) methodology are compared to the ones based on Dias et al. (2015) (the one presented in this part of the thesis, adapted for the GBM) and to the closed formulae reached by successive measure changes from Jun and Ku (2012). For the case of a first-up-then-down-and-in call option, the comparison is as follows.

Table 7: First-up-then-down-and-in call option, with the spot price $S = 100$, strike 100, upper barrier $U = 105$, lower barrier $L = 95$, $T = 1$ year and volatility $\sigma = 0.20$.

Contract number	r	q	Jun and Ku (2012)	Dias et al. (2015)	Das and Kim (2015)
1	1%	1%	1.642	1.642	1.642
2	1.5%	1%	1.675	1.675	1.620
3	2%	1%	1.708	1.708	1.599
4	2.5%	1%	1.741	1.741	1.576
5	3%	1%	1.773	1.773	1.553
6	3%	1.5%	1.732	1.732	1.568
7	3%	2%	1.691	1.691	1.583
8	3%	2.5%	1.650	1.650	1.596
9	3%	3%	1.609	1.609	1.609

As it can be observed, in contracts number 1 and number 9, when the cost-of-carry is null, all the three results match. However, the Das and Kim (2015) results which rely on Haug (2006) are not aligned with the other two when $r \neq q$. So, it is confirmed that there is a result mismatch, which implies the results where this kind of option are used (cases 5 and 6) will not have an exact match for the same parameters. In addition, we also confirm that the Dias et al. (2015) methodology extended to first-then options is accurate when compared to the closed formulae of Jun and Ku (2012).

12 Numerical Analysis

In this section, the debt discount values are studied alongside the credit spreads under the CEV model, in order to allow to see the impact of the β parameter over the results. The impact of β over some of the debt change parameters are also studied.

In Table 8, the results for the debt discounts, G, are presented in the seven debt cases along various initial β, initial leverage, D/V, and debt recovery value, ϕ_{dwl}, parameters. As for the remaining parameters, the values are the same as in Das and Kim (2015): the firm value $V_{t_0} = 1$, time to maturity $T = 15$ (for $t_0 = 0$), initial volatility $\sigma = 0.2$, interest rate $r = 2\%$, total payout to debt $q = 0\%$, debt ratchet amount $\delta = 30\%$, debt ratchet threshold $K = 0.4$, debt write down amount $d = 30\%$ and debt write down threshold $M = 1$.

Several observations arise from Table 8. First, there is a trend for the increase of the debt discount as the β parameter is increased. This is as expected, given that the put options that are summed to obtain the debt discounts increase in value as the β decreases. This will coincide with the increased spreads, a reflection of the additional volatility of the asset value, and thus the increased risk of default.

Another trend is the convergence of various cases' debt discounts, that is, as β is decreased, the values become more alike, converging to the same values in some cases. This is explained by the increased probability of V reaching very low values, making the default of the firm more frequent, withering the possibility of debt increases when these are possible.

Afterwards, the analysis is presented for the credit spreads curve for three cases in Figures 1 - 3: case 1 - the Merton (1974) model; case 4 - the case where the debt, can increase or decreases; and case 7 - the case where the debt, can increase and then decrease or vice-versa. This holds the same parameters as before, through various maturity dates. As expected, the debt spreads are increased when β is decreased and for instance in case 4, the credit spread curve gains the typical hump shape.

An analysis on the effects of the β parameter comes in Figures 4 and 5. Here, analogous cases are studied, with the difference between the spreads being highlighted.

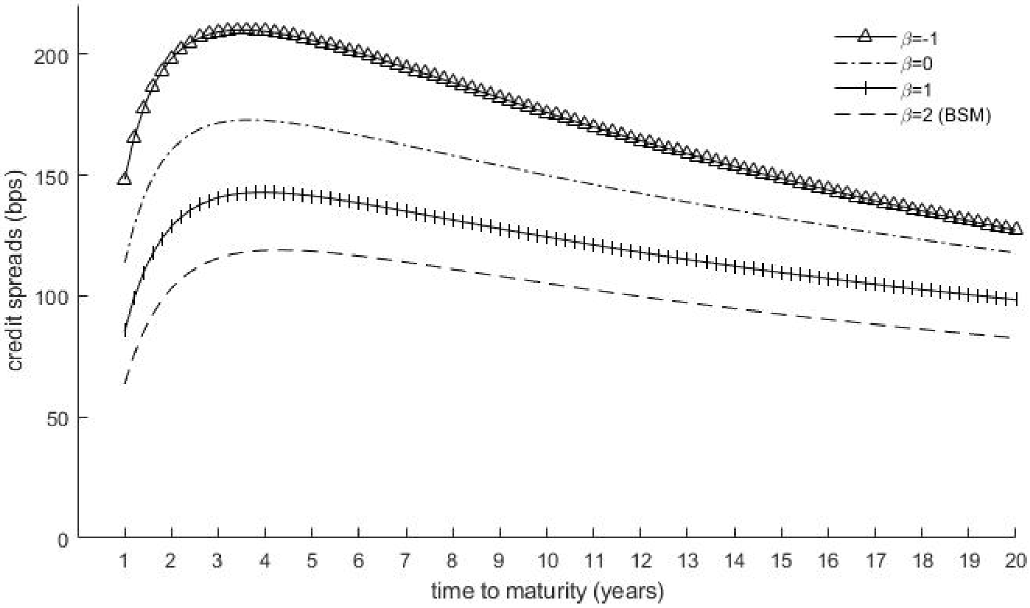

Figure 1: Case 1: Original static debt

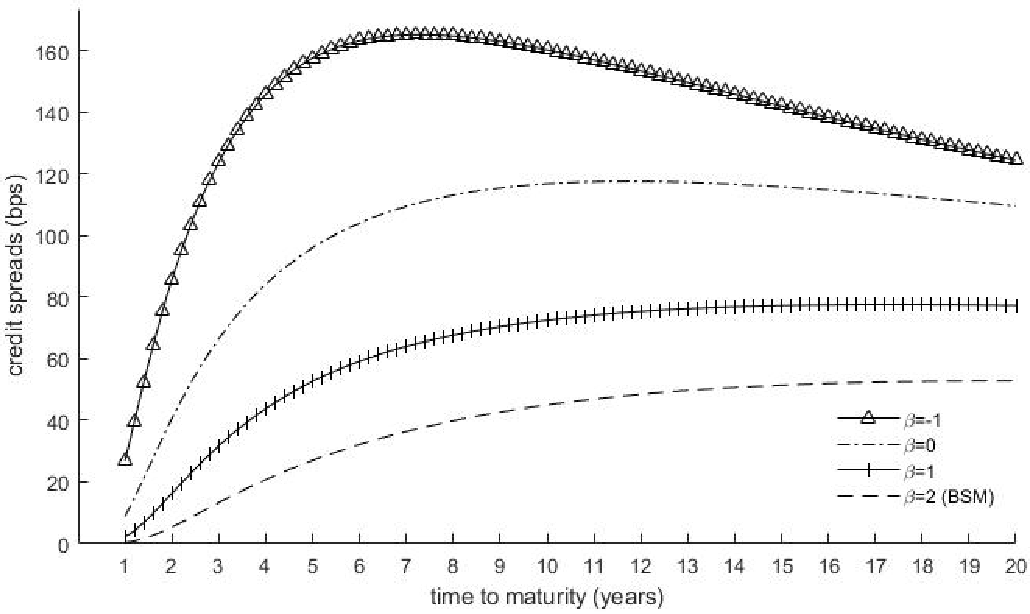

Figure 2: Case 4: Spread with the option to either ratchet or swap down debt

Table 8: The discount pricing for loan principle

	$D/V_0 = 0.75$		$D/V_0 = 0.5$		$D/V_0 = 0.75$		$D/V_0 = 0.5$	
	$G_{\phi_{dwl}=1}$	$G_{\phi_{dwl}=0.7}$	$G_{\phi_{dwl}=1}$	$G_{\phi_{dwl}=0.7}$	$G_{\phi_{dwl}=1}$	$G_{\phi_{dwl}=0.7}$	$G_{\phi_{dwl}=1}$	$G_{\phi_{dwl}=0.7}$
	$\beta = 1$				$\beta = 0$			
Case 1	0.0843	0.1109	0.0373	0.0481	0.1000	0.1141	0.0579	0.0622
Case 2	0.0860	0.1140	0.0426	0.0560	0.1005	0.1152	0.0607	0.0669
Case 3	0.0587	0.0759	0.0271	0.0339	0.0879	0.0950	0.0542	0.0563
Case 4	0.0609	0.0799	0.0375	0.0489	0.0887	0.0964	0.0716	0.0765
Case 5	0.0813	0.1066	0.0357	0.0439	0.0976	0.1104	0.0681	0.0593
Case 6	0.0596	0.0777	0.0274	0.0344	0.0869	0.0944	0.0559	0.0581
Case 7	0.0587	0.0766	0.0329	0.0407	0.0968	0.1036	0.0912	0.0898
	$\beta = -1$				$\beta = -2$			
Case 1	0.1111	0.1172	0.0708	0.0720	0.1146	0.1173	0.0752	0.0756
Case 2	0.1112	0.1175	0.0720	0.0743	0.1146	0.1173	0.0757	0.0766
Case 3	0.1065	0.1087	0.0699	0.0703	0.1129	0.1135	0.0750	0.0751
Case 4	0.1067	0.1090	0.0902	0.0921	0.1129	0.1136	0.0948	0.0955
Case 5	0.1099	0.1151	0.0877	0.0593	0.1151	0.1162	0.0933	0.0623
Case 6	0.1046	0.1069	0.0726	0.0731	0.1110	0.1148	0.0781	0.0782
Case 7	0.1242	0.1258	0.1248	0.1198	0.1347	0.1348	0.1369	0.1316

This table considers the cases where $\beta = \{1, 0, -1, -2\}$. As in Das and Kim (2015) $D = \{0.75, 0.5\}$, the firm value is $V = 1$, $T = 15$ years, $\sigma = 0.20$, $r = 2\%$, $q = 0\%$ $\delta = 30\%$, $K = 0.4$, $d = 30\%$, $M = 1$ and the recovery value is presented for $\phi_{dwl} = \{1, 0.7\}$.

Again, these hold the same parameters as before, but the values of debt increase and decrease (δ and d respectively) for $\beta \in [1, -1]$.

Firstly, case 1 is compared to case 2 in Figure 4. These are identical except the latter includes the possibility of increasing debt once the leverage, D/V, is low enough. The values reveal a widening of the spreads as δ increases, an expected result given the higher increases in debt of case 2. As β decreases, the CEV model volatility increases, V is steered towards smaller values, reducing the probability of the debt to increase and reducing the possible outcomes that make case 2 spreads higher. Overall, the increasingly resemblance between the two cases is confirmed, as the possible debt increase of case 2 becomes less likely.

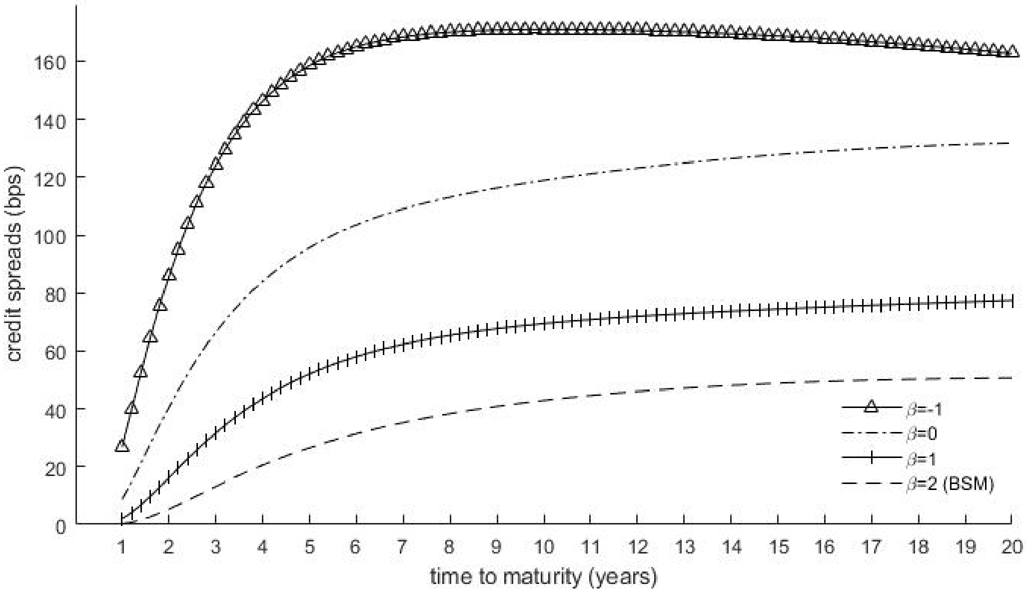

Figure 3: Case 7: Spread allowing ratchet after swap down or vice versa

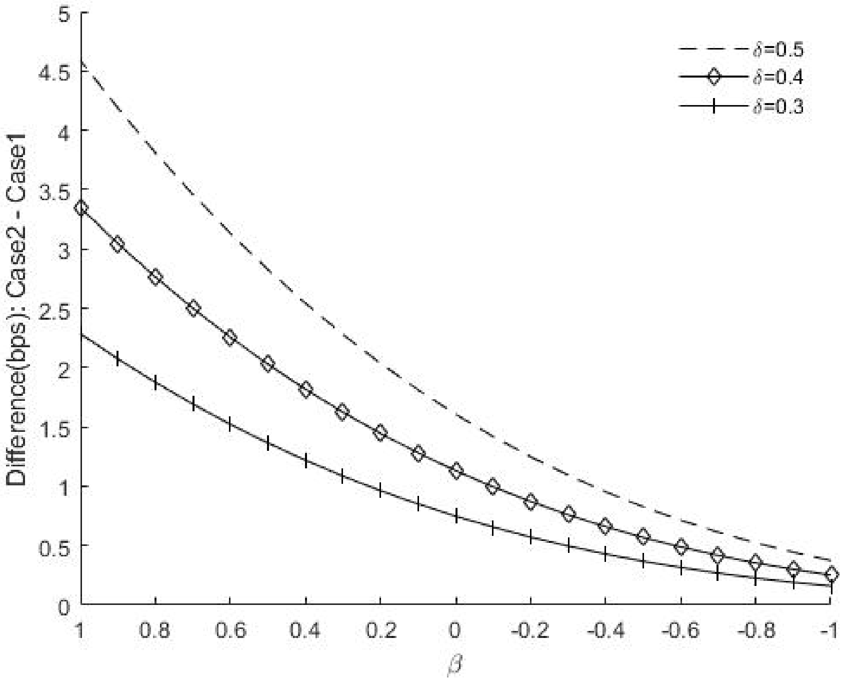

Figure 4: Spread difference between case 2 and case 1

Figure 5 compares case 1 to case 3. Case 3 is the addition to case 1 of the possibility of decreasing debt once V crosses lower a barrier. The parameter d represents by how much, and as it can be observed, it widens the spread between the cases, as they are decreased for case 3. The second observation is more ambiguous. As β is decreased and the volatility at lower levels increases, one could expect two ways the debt write down

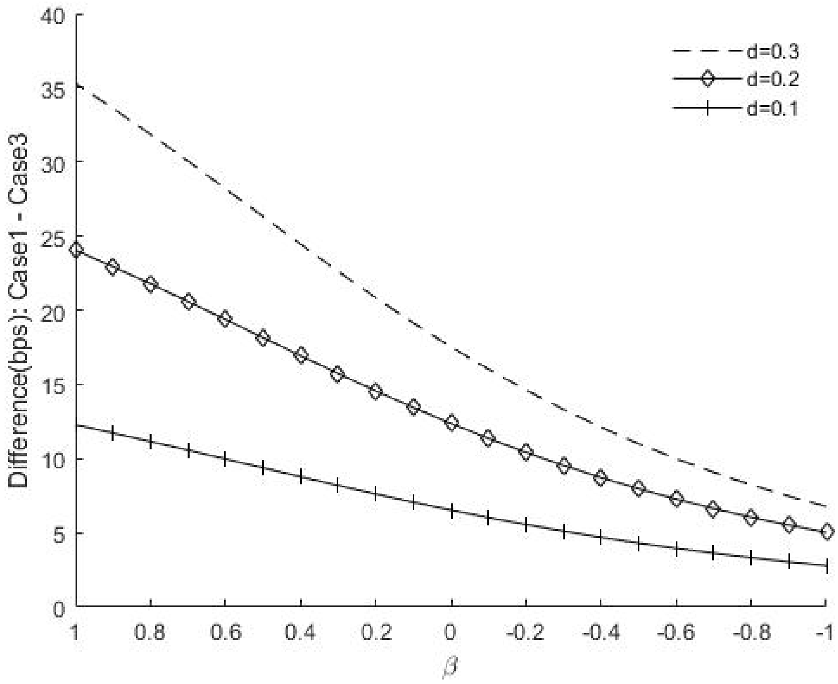

Figure 5: Spread difference between case 1 and case 3

possibility to be impacted. The first, as the firm value tends towards lower levels, the write down mechanism to reduce debt becomes more likely to occur and to impact the values, widening the spreads. The second, given V tends towards lower values more often, the reduced amount of debt tends to be of less relevance, converging the spreads. As it is observed, the second effect is more powerful, as the spreads become more alike.

A last analysis is performed on the impact of the M and K parameters, the parameters that indicate the leverage levels at which the debt is written-down and ratcheted, respectively.

As it can be seen in Figure 6, the increase in K leads to an increase between the spreads of case 1 and case 2, as the probability of the debt ratchet occurring in case 2 increases. In Figure 7 the impact of the increases of M in the difference between case 2 and case 5 is explored, displaying a decrease as the debt write down becomes less likely. In both cases, as before, the increases in β reduce the impact of the model parameters.

13 Conclusions

This part of the thesis applies many of the formulae option presented in Jun and Ku (2012) to obtain the dynamic debt model Das and Kim (2015) under CEV model. To to so, Part

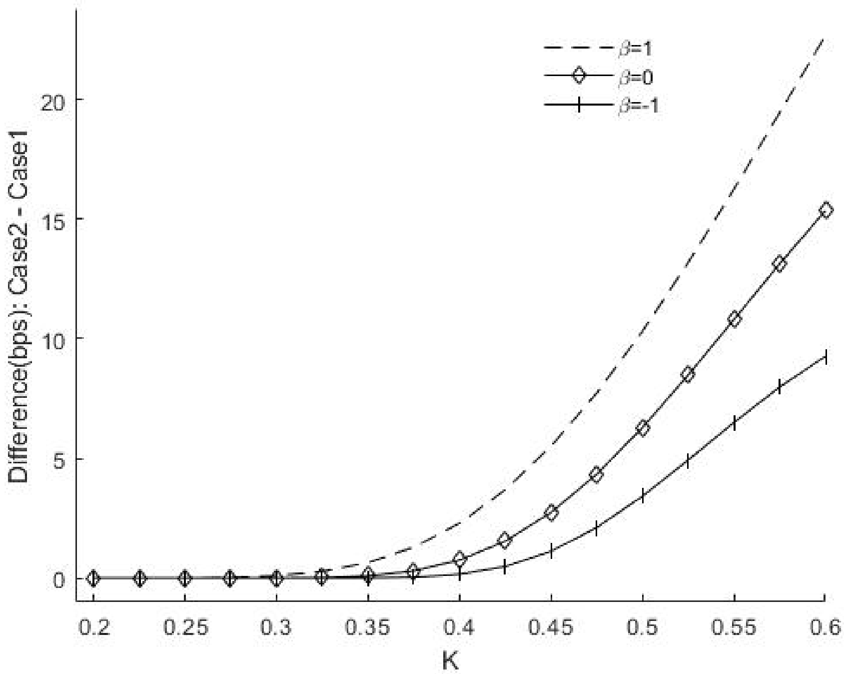

Figure 6: Spread difference between case 2 and case 1

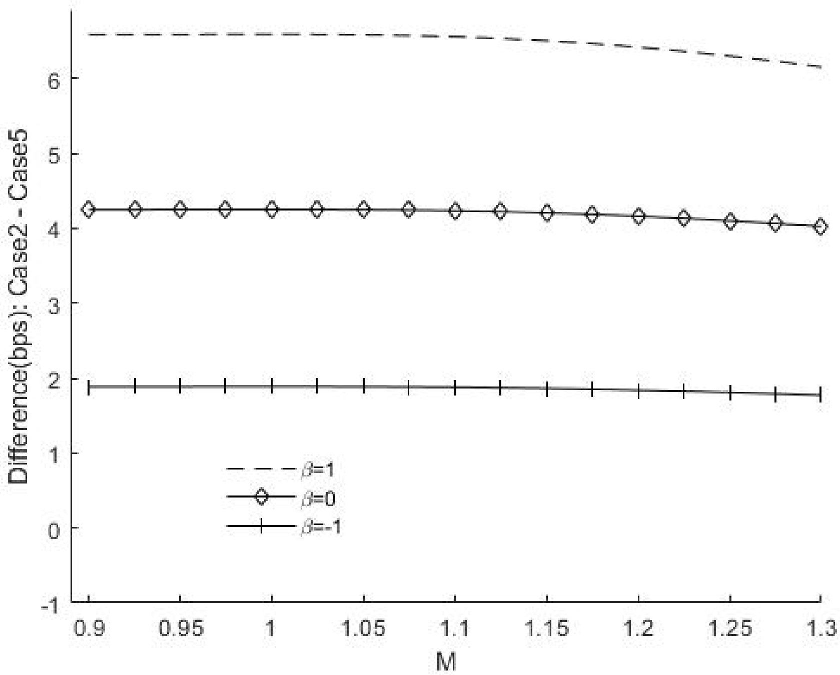

Figure 7: Spread difference between case 5 and case 2

II, that uses the work of Dias et al. (2015) and Dias et al. (2021), is instrumental, providing both the conditional on no default and recovery values of those contracts. Although Part II is under the JDCEV model, given the CEV model is nested in it, the results are a particular case.

First, the option contracts used to build dynamic debt model are presented. Afterwards, through the payoff structure of each of the possible case of the debt model, the options' combinations used in each of the cases are identified. Afterwards, the debt discounts are obtained, and studied with the same base parameterization as Das and Kim (2015), while studying different cases of the CEV model β parameter. Finally, various analysis on the impact of the model parameters on the dynamic of debt are provided, while also studying how β affects that impact.

Overall, increases in β increase the spreads as expected, while the resulting lower V values tend to make the dynamic debt cases more alike.

Part IV

Dynamic debt with intensity-ruled jumps

14 Introduction

The pricing of risky debt claims and corresponding credit spreads constitutes a rich field of research. The structural models begin with Black and Scholes (1973) and Merton (1974), where a fixed debt level is held and at the maturity date, the total value of the firm determines if there is a default event.

In order to deal with the limitations that arise from its assumptions, various extensions to the baseline model have occurred. Here, the challenged assumption is that of firms holding fixed amounts of debt. Contrary to what is assumed by the baseline model, firms tend to adjust it over time. One can expect more debt to be accrued when the value of the firm increases and less (or even reductions) when it decreases.

As examples of studies that point to the value of analyzing adjustments in the debt levels we highlight Roberts and Sufi (2009), that mention that most of long-term debt contracts suffer renegotiations over the amount, maturity, and pricing of the contract; Nini et al. (2012) find that creditors, through informal channels, play an active role in the governance of firms, even when default is a far scenario; and Flannery et al. (2012) who show that expected future leverage affects bond yields, above and beyond the effects of the observed leverage.

Among the attempts to introduce a dynamic behavior on the level of debt, there is for instance Collin-Dufresne and Goldstein (2001) setting that debt changes continuously, with a process for the debt level which is linked to the firm value and is steered towards a defined leverage. Notably, Das and Kim (2015) set a model where the debt level can increase or decrease when certain barriers are crossed by the firm value. By setting the knock-in and knock-out values for the barrier options — to indicate the firm values at which the debt amounts are changed — and adjusting the strike prices properly — to indicate how much debt changes — Das and Kim (2015) are able to simulate debt ratchets and write downs. They study a total of six cases of possible debt changes which are compared to the baseline Merton (1974) model. Although, the setup only allows for debt increases when the firm value increases and debt decreases when the firm value decreases.

Eisenthal-Berkovitz et al. (2020) explore the dynamics of the leveraged buyout (LBO) events and explore a structural model case where the firm can suffer a LBO which triggers an increase in the debt level. To do so, a Cox process attributes a probability to the event of the LBO, which is defined exogenously and is independent in relation to variations of the firm value.

This part of the thesis aims to emulate the dynamic debt concept of Das and Kim (2015), but with a hazard process triggering the debt events. It uses an intensity process that can be exogenous, although with the possibility of including correlation between this process and the firm's value, thus allowing variations in the firm value to influence the direction of the debt changes. By avoiding the barrier option setup, this method allows for greater flexibility - while the firm value increases (decreases) increase the probability of a debt increase (decrease), debt increases (decreases) can still occur when the firm value decreases (increases).

By using a Vasicek (1977) intensity process, this model allows to calibrate the debt changes under its own process, while keeping the possibility of correlation between the debt change process and the firm value. Therefore, one does not need to set beforehand values for the firm value to trigger the increase or decrease events. The debt increase and decrease events can be approach as random events, as, for instance, the default of a firm can be approached in the vulnerable options literature.

This kind of intensity processes is not a novelty in the credit risk literature, as various credit risk models use intensity processes. Often, these are used to allow exogenous factors beyond the firm-value to predict the default event, allowing the use of econometric specifications from term-structure modeling. Both Jarrow and Turnbull (1995) and Madan and Unal (1998) provide early examples of this approach when modeling two sources of risk simultaneously. In addition, the pricing of vulnerable options — contracts where in addition to the usual risk of the asset price, the issuers' default risk is also taken into consideration — often rely on intensity processes. Klein (1996) and Klein and Inglis (2001) provide early examples of this approach. More recently, Fard (2015) and Koo and

Kim (2017) provide studies where intensity based models are used in vulnerable options to simulate the default event.

Here, at maturity date, one checks the face value of debt which will depend on the debt increase and/or decrease events having occurred or not and the size of the debt variations. This amount of standing debt is what determines if the firm defaults. For instance, in the case of the debt increase, the value of the debt is not only influenced by the higher possibility of default, but also by the presence of the new debt, which, in the case of default, is entitled to receive a share of the recovered value.

The framework presented in this part of the thesis is flexible and adaptable to other existing models. Besides studying the spreads on increases over the baseline Merton (1974) model, the possibility of a debt increase is also studied in the case of the presence of subordinated debt. Gorton and Santomero (1990) present a model where debt is separated between senior debt and junior debt, with the latter being impacted by the amount of the former. They reach a formula that can be interpreted as the difference between two call options and which can be adapted to the possibility of debt changes. Here, we set the possibility of increasing the senior debt, and study how it impacts the credit spreads of the senior debt and the junior debt.

The remainder of the work is organized as follows. In Section 15, we present the extensions to the Merton (1974) model with the possibilities of increasing and decreasing the debt. In Section 16, we develop the extension to the particular case of the subordinated debt of Gorton and Santomero (1990). In Section 17, we obtain the closed formulae solutions for the bonds. In Section 18, we explore some numerical results. Finally, in Section 19, we present the conclusions.

15 The random dynamic debt model

15.1 The baseline Merton model

We begin by introducing the Merton (1974) model which is the baseline for all of the study. For a given firm, the face value of debt is represented by D and the maturity of the zero-coupon bond is T. The risk-free rate is represented by r, q stands for the firm's total payout to debt and equity holders and the firm value is V - the sum of the equity and the market value of debt. V will follow the usual (risk-neutral) geometric Brownian motion, therefore, the dynamics of V are given by

$$\frac{dV_t}{V_t} = (r-q)dt + \sigma_V dW_V(t), \tag{170}$$

which is solved by

$$V_t = V_0 \exp\left[\left(r - q - \frac{1}{2}\sigma_V^2\right)t + \sigma_V W_V(t)\right], \tag{171}$$

where σ_V represents the standard deviation of the firm value and $W_V(t)$ being a standard Brownian motion defined under the measure Q and generating the filtration $\mathbb{F} := \{\mathcal{F}_t, t \geq t_0\}$.

In this model, the firm issues a zero coupon bond. The default can only occur at maturity and does so when the firm value is below the face value of debt, $V_T < D$. When default occurs, the debt-holder obtains a fraction of the firm's value $V_T\phi_{dwl}$. The inclusion of $\phi_{dwl} \leq 1$ contemplates the possibility of a dead-weight loss, that is, not recovering the full firm's value upon default, only the fraction ϕ_{dwl}.

Following the known solution, the bond value at time zero will be given by

$$\begin{aligned} B_{t_0}(V, D, T) &= e^{-r(T-t_0)}\mathbb{E}\left[D\mathbb{1}_{\{V_T>D\}} + \phi_{dwl}V_T\mathbb{1}_{\{V_T\leq D\}}|\mathcal{F}_{t_0}\right] \\ &= De^{-r(T-t_0)}N\left(d_2\right) + \phi_{dwl}V_{t_0}e^{-q(T-t_0)}N\left(-d_1\right), \end{aligned} \tag{172}$$

where

$$d_1 = \frac{\ln(V_{t_0}/D) + \left(r - q + \sigma_V^2/2\right)(T-t_0)}{\sigma\sqrt{T-t_0}}, \; d_2 = d_1 - \sigma\sqrt{T}. \tag{173}$$

15.2 Debt with the possibility of increase - debt ratchet

Now, after time-zero and before maturity, there is the possibility of occurring only once a debt increase event — a debt ratchet. The debt ratchet event is assumed to follow a Vasicek (1977) process with intensity λ_t^u, that is, for a small Δ, the probability of a ratchet event occurring between t and $t+\Delta$ is approximately $\lambda_t^u \Delta$. We also have θ_u, the risk adjusted long term average of the process, and κ_u, the speed of the reversion of the process. The correlation between the debt increase and the firm value is defined as $dW_u(t)W_V(t) = \rho_{uV}dt$. By selecting $\rho_{uV} > 0$, the firm value and the likelihood of increasing the debt level become positively correlated.

As in Lando (1998) it will be assumed that $P(\tau_u > T) = \mathbb{E}[e^{-\int_{t_0}^{T} \lambda_t^u dt}]$ and the Vasicek process is given by

$$d\lambda_t^u = \kappa_u \left(\theta_u - \lambda_t^u\right) dt + \sigma_u dW_u(t), \tag{174}$$

and the known unique solution to the SDE given by [1]

$$\lambda_t^u = \lambda_{t_0}^u e^{-\kappa_u(t-t_0)} + \theta_u \left(1 - e^{-\kappa_u(t-t_0)}\right) + \sigma_u \int_{t_0}^{t} e^{-\kappa_u(t-s)} dW_u(s). \tag{175}$$

The total firm debt level starts at D_{t_0}, and if no ratchet occurs, $\tau_U > T$, it remains at that level. Thus, we have same payoff as in the basic Merton (1974) model:

$$D_{t_0}\mathbb{1}_{\{V_T > D_T\}} + \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_T\}} = D_{t_0}\mathbb{1}_{\left\{V_T > D_{t_0}\right\}} + \phi_{dwl} V_T \mathbb{1}_{\left\{V_T \leq D_{t_0}\right\}}. \tag{176}$$

When such distinction is opportune, we will use D_T to signal that, at maturity, the total value of the firm's debt can differ from D_{t_0}.

If the ratchet event occurs, the debt level is increased to $D_T = D_{t_0}e^U$, where $U > 0$. As assumed in Das and Kim (2015), the potential increases in debt are assumed to come from the equity, thus preserving the total firm value.

Now, with the debt increase, in the case of default, $V_T < D_T$, there is more debt with the right to receive the remainder of the firm value, therefore the share the that debt-holder obtains must be adjusted. Assuming the same maturity date and the same seniority among the new and the old debt, the amount to be received is weighted by $\frac{D_{t_0}}{D_{t_0}e^U} = e^{-U}$, that is, the original debt-holder will receive a percentage of the recovered V_t

[1] For details on the solution, see, for instance, Musiela and Rutkowski (2005, Chapter 10).

equal to his share of the whole debt.

So when $\tau_U \leq T$,

$$D_T e^{-U} \mathbb{1}_{\{V_T > D_{t_0} e^U\}} + \phi_{dwl} e^{-U} V_T \mathbb{1}_{\{V_T \leq D_{t_0} e^U\}},$$

which given, in this case, $D_T = D_{t_0} e^U$, the above can be written as

$$D_{t_0} \mathbb{1}_{\{V_T > D_{t_0} e^U\}} + \phi_{dwl} e^{-U} V_T \mathbb{1}_{\{V_T \leq D_{t_0} e^U\}}.$$

The event of debt increase is assumed to be able to occur only after t_0, and $\mathbb{D} := \{\mathcal{D}_t, t \geq t_0\}$ denotes the filtration generated by the indicator process $\mathcal{D}_t := \mathbb{1}_{\{t > \tau_U\}}$. In addition, $\mathbb{G} := \{\mathcal{G}_t : t \geq t_0\}$ will denote the enlarged filtration obtained as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.

Therefore, debt value at t_0 will be given by

$$\begin{aligned}
&B^U_{t_0}(V, D_{t_0}, T, \lambda_u, U) \\
=&e^{-r(T-t_0)} \mathbb{E}\Big[(D_{t_0} \mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}}) \mathbb{1}_{\{\tau_U > T\}} \\
&+ \big(D_{t_0} \mathbb{1}_{\{V_T > D_{t_0} e^U\}} + \phi_{dwl} e^{-U} V_T \mathbb{1}_{\{V_T \leq D_{t_0} e^U\}}\big) \mathbb{1}_{\{\tau_U \leq T\}} | \mathcal{G}_{t_0}\Big].
\end{aligned} \tag{177}$$

The equation above can be written as

$$B^U_{t_0}(V, D, T, \lambda_u) = e^{-r(T-t_0)} \big(A^U_1 + A^U_2 + B^U_1 - B^U_2 + B^U_3 - B^U_4\big). \tag{178}$$

The first set of terms is

$$\begin{aligned}
&\mathbb{E}\Big[(D_{t_0} \mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}}) \mathbb{1}_{\{\tau_U > T\}} | \mathcal{G}_{t_0}\Big] \\
=&\mathbb{E}\Big[(D_{t_0} \mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}}) e^{-\int_{t_0}^T \lambda^u_t dt} | \mathcal{F}_{t_0}\Big] \\
=&\underbrace{\mathbb{E}\Big[D_{t_0} \mathbb{1}_{\{V_T > D_{t_0}\}} e^{-\int_{t_0}^T \lambda^u_t dt} | \mathcal{F}_{t_0}\Big]}_{A^U_1} + \underbrace{\mathbb{E}\Big[\phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}} e^{-\int_{t_0}^T \lambda^u_t dt} | \mathcal{F}_{t_0}\Big]}_{A^U_2},
\end{aligned} \tag{179}$$

and the second set is

$$
\begin{aligned}
&\mathbb{E}\Big[\big(D_{t_0}\mathbb{1}_{\{V_T>D_{t_0}e^U\}}+\phi_{dwl}e^{-U}V_T\mathbb{1}_{\{V_T\leq D_{t_0}e^U\}}\big)\mathbb{1}_{\{\tau_U\leq T\}}|\mathcal{F}_{t_0}\Big]\\
=&\mathbb{E}\Big[\big(D_{t_0}\mathbb{1}_{\{V_T>D_{t_0}e^U\}}+\phi_{dwl}e^{-U}V_T\mathbb{1}_{\{V_T\leq D_{t_0}e^U\}}\big)(1-e^{-\int_{t_0}^T\lambda_t^u dt)}|\mathcal{F}_{t_0}\Big]\\
=&\underbrace{\mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T>D_{t_0}e^U\}}|\mathcal{F}_{t_0}\Big]}_{B_1^U}-\underbrace{\mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T>D_{t_0}e^U\}}e^{-\int_{t_0}^T\lambda_t^u dt}|\mathcal{F}_{t_0}\Big]}_{B_2^U}\\
&+\underbrace{\mathbb{E}\Big[\phi_{dwl}e^{-U}V_T\mathbb{1}_{\{V_T\leq D_{t_0}e^U\}}|\mathcal{F}_{t_0}\Big]}_{B_3^U}-\underbrace{\mathbb{E}\Big[\phi_{dwl}e^{-U}V_T\mathbb{1}_{\{V_T\leq D_{t_0}e^U\}}e^{-\int_{t_0}^T\lambda_t^u dt}|\mathcal{F}_{t_0}\Big]}_{B_4^U}. \qquad (180)
\end{aligned}
$$

This way, one can have the possibility of a debt increase which is more likely when the firm value increases, but nevertheless possible when the firm value decreases.

15.3 Debt with the possibility of decrease - debt write down

In the previous subsection, we considered the possibility of a debt increase. Now, there can be a debt decrease event — a debt write down. Again, we use a Vasicek (1977) process, now with intensity λ_t^l. We also have θ_l, the risk adjusted long term average of the process, and κ_l, the speed of the reversion of the process. The correlation with the firm value is again present and defined as $dW_l(t)W_V(t)=\rho_{lV}dt$. Now, with $\rho_{lV}<0$, the correlation between the firm value and the likelihood of debt decreasing is negative.

The process will follow the same kind of dynamic, that is

$$
d\lambda_t^l=\kappa_l\left(\theta_l-\lambda_t^l\right)dt+\sigma_l dW_l(t), \qquad (181)
$$

again with the known unique solution to the SDE given by

$$
\lambda_t^l=\lambda_{t_0}^l e^{-\kappa_l(t-t_0)}+\theta_l\left(1-e^{-\kappa_l(t-t_0)}\right)+\sigma_l\int_{t_0}^t e^{-\kappa_l(t-s)}dW_l(s). \qquad (182)
$$

In the debt decrease case, the debt level is changed to $D_T=D_{t_0}e^L$, with $L<0$ ensuring a decrease in the amount of debt. As in the ratch-up event, the firm value is preserved. The debt is exchanged for equity, increasing the latter.

Once again, when no debt write down occurs, $\tau_L > T$, the result is $D_T = D_{t_0}$ and thus

$$D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}}.$$

When the debt write down occurs, the debt level becomes $D_T = De^L$, while the assumptions on debt maturity again imply that, in case of default, the value received must be weighted by $\frac{D_{t_0}}{D_{t_0}e^L} = e^{-L}$.

Thus, in the cases where $\tau_L \leq T$:

$$\begin{aligned}
&D_{t_0}e^L e^{-L}\mathbb{1}_{\{V_T > D_{t_0}e^L\}} + \phi_{dwl}e^{-L} V_T \mathbb{1}_{\{V_T \leq D_{t_0}e^L\}} \\
=&D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}} + \phi_{dwl}e^{-L} V_T \mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}.
\end{aligned}$$

Therefore, with the possibility of a debt decrease, the debt value will be:

$$\begin{aligned}
&B^L_{t_0}(V, D_{t_0}, T, \lambda_l, L) \\
=&e^{-r(T-t_0)}\mathbb{E}\Big[\big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}}\big)\mathbb{1}_{\{\tau_L > T\}} \\
&+ \big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}} + \phi_{dwl}e^{-L} V_T \mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}\big)\mathbb{1}_{\{\tau_L \leq T\}}|\mathcal{G}_{t_0}\Big].
\end{aligned} \tag{183}$$

It can be written in similar terms to those of the debt increase, that is

$$B^L_T(V, D_{t_0}, T, \lambda_l, L) = e^{-r(T-t_0)}\big(A^L_1 + A^L_2 + B^L_1 - B^L_2 + B^L_3 - B^L_4\big), \tag{184}$$

with

$$A^L_1 = \mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}\}}e^{-\int_{t_0}^T \lambda^l_t dt}|\mathcal{F}_{t_0}\Big],\ A^L_2 = \mathbb{E}\Big[\phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_{t_0}\}}e^{-\int_{t_0}^T \lambda^l_t dt}|\mathcal{F}_{t_0}\Big],$$

and

$$B_1^L = \mathbb{E}\Bigg[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}}|\mathcal{F}_{t_0}\Bigg], \quad B_2^L = \mathbb{E}\Bigg[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}}e^{-\int_{t_0}^T \lambda_t^l dt}|\mathcal{F}_{t_0}\Bigg],$$
$$B_3^L = \mathbb{E}\Bigg[\phi_{dwl}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}|\mathcal{F}_{t_0}\Bigg], \quad B_4^L = \mathbb{E}\Bigg[\phi_{dwl}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}e^{-\int_{t_0}^T \lambda_t^l dt}|\mathcal{F}_{t_0}\Bigg].$$

15.4 Debt with the possibility of increase and decrease

Now, the two previous cases are combined. The firm can have its value increased, decreased or both. Any of the events can occur, with the probability being ruled by equations (174) and (181) plus the respective solutions. The order of the changes in debt is not relevant, as the decisive factor is if by the maturity date the face value of debt has increased and/or decreased, how much of it is standing, and if the firm value can match it. There are four possible debt events: no debt changes, $D_T = D_{t_0}$; debt increase, but no debt decrease, $D_T = D_{t_0}e^U$; debt decrease, but no debt increase, $D_T = D_{t_0}e^L$; both debt increase and debt decrease $D_T = D_{t_0}e^U e^L$.

Alongside the previous correlations, one can also include the correlation between the debt increase and decrease processes: $dW_u(t)dW_l(t) = \rho_{ul}dt$. So the debt value will be given by

$$\begin{aligned}
&B_{t_0}^{UL}(V, D_{t_0}, \lambda_u, U, \lambda_l, L)\\
=&e^{-r(T-t_0)}\mathbb{E}\Bigg[\big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl}V_T\mathbb{1}_{\{V_T \leq D_{t_0}\}}\big)\mathbb{1}_{\{\tau_U > T\}}\mathbb{1}_{\{\tau_L > T\}}\\
&+ (D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}} + \phi_{dwl}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}\big)\mathbb{1}_{\{\tau_U > T\}}\mathbb{1}_{\{\tau_L \leq T\}}\\
&+ (D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U\}} + \phi_{dwl}e^{-U}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^U\}}\big)\mathbb{1}_{\{\tau_U \leq T\}}\mathbb{1}_{\{\tau_L > T\}}\\
&+ (D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U e^L\}} + \phi_{dwl}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^U e^L\}}\big)\mathbb{1}_{\{\tau_U \leq T\}}\mathbb{1}_{\{\tau_L \leq T\}}|\mathcal{G}_{t_0}\Bigg],
\end{aligned}$$

which can be written as

$$\begin{aligned}
&B_{t_0}^{UL}(V, D_{t_0}, \lambda_u, U, \lambda_l, L)\\
=&e^{-r(T-t_0)}\mathbb{E}\Bigg[\big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}\}} + \phi_{dwl}V_T\mathbb{1}_{\{V_T \leq D_{t_0}\}}\big)e^{-\int_{t_0}^T \lambda_t^u dt}e^{-\int_{t_0}^T \lambda_t^l dt}|\mathcal{F}_{t_0}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ e^{-r(T-t_0)}\mathbb{E}\Big[\big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}} + \phi_{dwl}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}\big)\\
&\qquad \times e^{-\int_{t_0}^T \lambda_t^u dt}(1-e^{-\int_{t_0}^T \lambda_t^l dt})|\mathcal{F}_{t_0}\Big]\\
&+ e^{-r(T-t_0)}\mathbb{E}\Big[\big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U\}} + \phi_{dwl}e^{-U}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^U\}}\big)\\
&\qquad \times (1-e^{-\int_{t_0}^T \lambda_t^u dt})e^{-\int_{t_0}^T \lambda_t^l dt}|\mathcal{F}_{t_0}\Big]\\
&+ e^{-r(T-t_0)}\mathbb{E}\Big[\big(D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U e^L\}} + \phi_{dwl}e^{-U}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^U e^L\}}\big)\\
&\qquad \times (1-e^{-\int_{t_0}^T \lambda_t^u dt})(1-e^{-\int_{t_0}^T \lambda_t^l dt})|\mathcal{F}_{t_0}\Big]
\end{aligned}
\tag{185}
$$

and

$$
\begin{aligned}
B_{t_0}^{UL}(V, D_{t_0}, \lambda_u, U, \lambda_l, L) =& e^{-r(T-t_0)}\big[A_1^{UL} + A_2^{UL} + B_1^{UL} - B_2^{UL} + B_3^{UL} - B_4^{UL}\\
&+ C_1^{UL} - C_2^{UL} + C_3^{UL} - C_4^{UL}\\
&+ D_1^{UL} - D_2^{UL} - D_3^{UL} + D_4^{UL} + D_5^{UL} - D_6^{UL} - D_7^{UL} + D_8^{UL}\big],
\end{aligned}
\tag{186}
$$

with the terms

$$
\begin{aligned}
A_1^{UL} &= \mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l)dt}|\mathcal{F}_{t_0}\Big],\\
A_2^{UL} &= \mathbb{E}\Big[\phi_{dwl}V_T\mathbb{1}_{\{V_T \leq D_{t_0}\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l)dt}|\mathcal{F}_{t_0}\Big],\\
B_1^{UL} &= \mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}}e^{-\int_{t_0}^T \lambda_t^u dt}|\mathcal{F}_{t_0}\Big],\\
B_2^{UL} &= \mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^L\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l)dt}|\mathcal{F}_{t_0}\Big],\\
B_3^{UL} &= \mathbb{E}\Big[\phi_{dwl}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}e^{-\int_{t_0}^T \lambda_t^u dt}|\mathcal{F}_{t_0}\Big],\\
B_4^{UL} &= \mathbb{E}\Big[\phi_{dwl}e^{-L}V_T\mathbb{1}_{\{V_T \leq D_{t_0}e^L\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l)dt}|\mathcal{F}_{t_0}\Big],\\
C_1^{UL} &= \mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U\}}e^{-\int_{t_0}^T \lambda_t^l dt}|\mathcal{F}_{t_0}\Big],\\
C_2^{UL} &= \mathbb{E}\Big[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l)dt}|\mathcal{F}_{t_0}\Big],
\end{aligned}
$$

$$
\begin{aligned}
C_3^{UL} &= \mathbb{E}\left[\phi_{dwl}e^{-U}V_T\mathbb{1}_{\left\{V_T\le D_{t_0}e^U\right\}}e^{-\int_{t_0}^T\lambda_t^l dt}|\mathcal{F}_{t_0}\right],\\
C_4^{UL} &= \mathbb{E}\left[\phi_{dwl}e^{-U}V_T\mathbb{1}_{\left\{V_T\le D_{t_0}e^U\right\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l) dt}|\mathcal{F}_{t_0}\right],\\
D_1^{UL} &= \mathbb{E}\left[D_{t_0}\mathbb{1}_{\left\{V_T> D_{t_0}e^Ue^L\right\}}|\mathcal{F}_{t_0}\right],\\
D_2^{UL} &= \mathbb{E}\left[D_{t_0}\mathbb{1}_{\left\{V_T> D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T\lambda_t^l dt}|\mathcal{F}_{t_0}\right],\\
D_3^{UL} &= \mathbb{E}\left[D_{t_0}\mathbb{1}_{\left\{V_T> D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T\lambda_t^u dt}|\mathcal{F}_{t_0}\right],\\
D_4^{UL} &= \mathbb{E}\left[D_{t_0}\mathbb{1}_{\left\{V_T> D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l) dt}|\mathcal{F}_{t_0}\right],\\
D_5^{UL} &= \mathbb{E}\left[\phi_{dwl}e^{-U}e^{-L}V_T\mathbb{1}_{\left\{V_T\le D_{t_0}e^Ue^L\right\}}|\mathcal{F}_{t_0}\right],\\
D_6^{UL} &= \mathbb{E}\left[\phi_{dwl}e^{-U}e^{-L}V_T\mathbb{1}_{\left\{V_T\le D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T\lambda_t^l dt}|\mathcal{F}_{t_0}\right],\\
D_7^{UL} &= \mathbb{E}\left[\phi_{dwl}e^{-U}e^{-L}V_T\mathbb{1}_{\left\{V_T\le D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T\lambda_t^u dt}|\mathcal{F}_{t_0}\right] \text{ and}\\
D_8^{UL} &= \mathbb{E}\left[\phi_{dwl}e^{-U}e^{-L}V_T\mathbb{1}_{\left\{V_T\le D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T(\lambda_t^u+\lambda_t^l) dt}|\mathcal{F}_{t_0}\right].
\end{aligned}
$$

16 Debt increase and subordinated debt

In this section, the possibility of the debt increase is expanded for a particular case, the study of subordinated debt. The framework used will be that of Gorton and Santomero (1990), which is based on the work of Black and Cox (1976). It departs from the Merton (1974) base case and contemplates the presence of two kinds of debt: junior, D^J, and senior, D^S. In this study, the possibility of only increasing the senior debt and its impact over the junior debt value is contemplated.

The senior debt has the priority in the cases where the firm defaults, and this difference is highlighted in the payoffs at the different scenarios. At time T, if the firm value, V_T, is greater than the sum of both debts, $V_T \geq D_T^S + D_T^J$ the two kinds of debt are paid in full. If $D_T^S + D_T^J > V_T \geq D_T^S$, the senior debt-holder gets paid in full, while the junior debt-holder receives $V_T - D_T^S$, that is, the senior debt-holder has the priority to be paid in full, while the junior debt-holder receives the remainder of the firm value. When $D_T^S > V_T$, the senior debt-holder receives V_T while the junior debt-holder receives zero, as once again the senior debt-holder has the priority to be paid as much as possible, although not enough to be fully reimbursed, and the junior debt-holder finds no value to compensate his loan. As for the equity holders, they receive $V_T - D_T^S - D_T^J$ when the value is positive, zero otherwise.

The cases above are summarized in the following table.

Table 9: Realized asset values at maturity

	$V_T \geq D_T^S + D_T^J$	$D_T^S + D_T^J > V_T \geq D_T^S$	$D_T^S > V_T$
D_T^S (Senior Debt)	D_T^S	D_T^S	V_T
D_T^J (Junior Debt)	D_T^J	$V_T - D_T^S$	0
E_T (Equity)	$V_T - D_T^S - D_T^J$	0	0

This table summarizes the payoffs for the Senior Debt, the Junior Debt and the equity when we admit the two possible types of debt seniority

Recalling the model presented in Gorton and Santomero (1990), the value of the junior debt at time-zero is presented as

$$\begin{aligned}&JB_{t_0}(V, D_{t_0}^J, D_{t_0}^S)\\ =&e^{-q(T-t_0)}V_{t_0}(N(d_1) - N(\hat{d}_1)) - e^{-r(T-t_0)}D_T^S N(d_2) + e^{-r(T-t_0)}(D_T^S + D_T^J)N(\hat{d}_2),\end{aligned} \quad (187)$$

where d_1 and d_2 are as in (173), including the firm's total payout to debt, q as part of the drift and there are also the terms

$$\hat{d}_1 = \frac{\log(V_{t_0}/(D_T^S + D_T^J)) + (r - q + \frac{\sigma^2}{2})(T - t_0)}{\sigma\sqrt{(T - t_0)}} \text{ and } \hat{d}_2 = \hat{d}_1 - \sigma\sqrt{(T - t_0)}.$$

The value can also be noted to be the difference between two call options:

$$\begin{aligned}JB_{t_0}(V, D_{t_0}^J, D_{t_0}^S) =& \underbrace{e^{-q(T-t_0)}V_{t_0}N(d_1) - e^{-r(T-t_0)}D_T^S N(d_2)}_{\text{Call with Strike } D_T^S}\\ &- \underbrace{(e^{-q(T-t_0)}V_{t_0}N(\hat{d}_1) - e^{-r(T-t_0)}(D_T^S + D_T^J)N(\hat{d}_2))}_{\text{Call with Strike } D_T^S + D_T^J}.\end{aligned} \quad (188)$$

The expression above can be obtained from the payoffs and respective scenarios for the subordinated debt in Table 9. Below, the payoff is adapted to include the dead-weight default loss parameter, ϕ_{dwl}. Using the payoffs from Table 9, the junior debtor will receive

$$D_T^J \mathbb{1}_{\left\{V_T \geq D_T^S + D_T^J\right\}} + (\phi_{dwl} V_T - D_T^S) \mathbb{1}_{\left\{D_T^S + D_T^J > V_T \geq D_T^S\right\}}, \tag{189}$$

and the expression at t_0 can be written as:

$$\begin{aligned}
& JB_{t_0}(V, D_{t_0}^J, D_{t_0}^S) \\
=& e^{-r(T-t_0)} \mathbb{E}\left[D_T^J \mathbb{1}_{\left\{V_T \geq D_T^S + D_T^J\right\}} + (\phi_{dwl} V_T - D_T^S) \mathbb{1}_{\{D_T^S + D_T^J > V_T \geq D_T^S\}} | \mathcal{F}_{t_0} \right] \\
=& e^{-r(T-t_0)} \mathbb{E}\left[D_T^J \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}} + (\phi_{dwl} V_T - D_T^S)(\mathbb{1}_{\{V_T < D_T^S + D_T^J\}} - \mathbb{1}_{\{V_T \leq D_T^S\}}) | \mathcal{F}_{t_0} \right] \\
=& e^{-r(T-t_0)} \mathbb{E}\Big[D_T^S \mathbb{1}_{\{V_T \leq D_T^S\}} - \phi_{dwl} V_T \mathbb{1}_{\{V_T \leq D_T^S\}} + D_T^J \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}} \\
& \qquad + \phi_{dwl} V_T \mathbb{1}_{\{V_T < D_T^S + D_T^J\}} - D_T^S \mathbb{1}_{\{V_T < D_T^S + D_T^J\}} | \mathcal{F}_{t_0} \Big] \\
=& e^{-r(T-t_0)} \mathbb{E}\Big[D_T^S (1 - \mathbb{1}_{\{V_T > D_T^S\}}) - \phi_{dwl} V_T (1 - \mathbb{1}_{\{V_T > D_T^S\}}) + D_T^J \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}} \\
& \qquad + \phi_{dwl} V_T (1 - \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}}) - D_T^S (1 - \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}}) | \mathcal{F}_{t_0} \Big] \\
=& e^{-r(T-t_0)} \mathbb{E}\Big[\phi_{dwl} V_T \mathbb{1}_{\{V_T > D_T^S\}} - D_T^S \mathbb{1}_{\{V_T > D_T^S\}} \\
& \qquad - (\phi_{dwl} V_T \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}} - (D_T^J + D_T^S) \mathbb{1}_{\{V_T \geq D_T^S + D_T^J\}}) | \mathcal{F}_{t_0} \Big] \\
=& e^{-q(T-t_0)} \phi_{dwl} V_{t_0} N(d_1) - e^{-r(T-t_0)} D_T^S N(d_2) \\
& - (e^{-q(T-t_0)} \phi_{dwl} V_{t_0} N(\hat{d}_1) - e^{-r(T-t_0)} (D_T^S + D_T^J) N(\hat{d}_2)).
\end{aligned} \tag{190}$$

Now, the possibility of a senior debt increase is contemplated. Between t_0 and T, there is the possibility of an increase only in the senior debt through a process $d\lambda_t^u$ as defined in equation (174) which has the solution in equation (175). Again, to set a positive relation between the debt increase and the firm value, the correlation between the drifts of the firm value and the debt increases must be positive, $\rho_{uV} > 0$. When this event occurs, the senior debt to be reimbursed will be $D_{t_0}^S e^U$, where once again $U > 0$ represents a debt increase.

To reach the payoff of the junior debt under the possibility of a senior debt increase,

$JB_{t_0}^U$, when there is the possibility of the debt increase, the expression in equation (189) suffers two adaptations in the senior debt components:

$$D_T^J \mathbb{1}_{\{V_T \geq D_{t_0}^S e^U + D_T^J\}} + (\phi_{dwl} V_T - D_{t_0}^S e^U) \mathbb{1}_{\{D_{t_0}^S e^U + D_T^J > V_T \geq D_{t_0}^S e^U\}}, \tag{191}$$

First, represented by the change in the indicator functions, the V_T values at which there are defaults are impacted by the increased value of the senior debt at maturity. Second, in the case where the senior debt is paid in full while the junior debt is only partially reimbursed, the increased amount of the former reduces the amount received by the latter.

Again, there are two possible cases. In the case where the debt is not increased, $\tau_U > T$, equation (190) yields the payoff. In the case where there is the debt increase, the value is deduced from (191) with the same steps as in (190). Thus

$$\begin{aligned}
&JB_{t_0}^U(V, D_{t_0}^J, D_{t_0}^S, \lambda_u, U) \\
=& e^{-r(T-t_0)} \mathbb{E}\Bigg[\Big(\phi_{dwl} V_T \mathbb{1}_{\{V_T > D_{t_0}^S\}} - D_{t_0}^S \mathbb{1}_{\{V_T > D_{t_0}^S\}} \\
&\quad - \phi_{dwl} V_T \mathbb{1}_{\{V_T \geq D_{t_0}^S + D_{t_0}^J\}} + (D_{t_0}^S + D_{t_0}^J) \mathbb{1}_{\{V_T \geq D_{t_0}^S + D_{t_0}^J\}} \Big) \mathbb{1}_{\{\tau_U > T\}} \\
&\quad + \Big(\phi_{dwl} V_T \mathbb{1}_{\{V_T > D_{t_0}^S e^U\}} - D_{t_0}^S e^U \mathbb{1}_{\{V_T > D_{t_0}^S e^U\}} \\
&\quad - \phi_{dwl} V_T \mathbb{1}_{\{V_T \geq D_{t_0}^S e^U + D_{t_0}^J\}} + (D_{t_0}^S e^U + D_{t_0}^J) \mathbb{1}_{\{V_T \geq D_{t_0}^S e^U + D_{t_0}^J\}} \Big) \mathbb{1}_{\{\tau_U \leq T\}} | \mathcal{G}_{t_0} \Bigg],
\end{aligned} \tag{192}$$

which can be written as

$$\begin{aligned}
&JB_{t_0}^U(V, D_{t_0}^J, D_{t_0}^S, \lambda_u, U) \\
=& e^{-r(T-t_0)} \Big[JA_1^U - JA_2^U - JA_3^U + JA_4^U \\
&\quad + JB_1^U - JB_2^U - JB_3^U + JB_4^U - JB_5^U + JB_6^U + JB_7^U - JB_8^U \Big]
\end{aligned} \tag{193}$$

where

$$
\begin{aligned}
JA_1^U &= \mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T > D_{t_0}^S\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right], \\
JA_2^U &= \mathbb{E}\left[D_{t_0}^S \mathbb{1}_{\left\{V_T > D_{t_0}^S\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right], \\
JA_3^U &= \mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T \geq D_{t_0}^S + D_{t_0}^J\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right], \\
JA_4^U &= \mathbb{E}\left[(D_{t_0}^S + D_{t_0}^J) \mathbb{1}_{\left\{V_T \geq D_{t_0}^S + D_{t_0}^J\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right], \\
JB_1^U &= \mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T > D_{t_0}^S e^U\right\}} | \mathcal{F}_{t_0}\right], \\
JB_2^U &= \mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T > D_{t_0}^S e^U\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right], \\
JB_3^U &= \mathbb{E}\left[D_{t_0}^S e^U \mathbb{1}_{\left\{V_T > D_{t_0}^S e^U\right\}} | \mathcal{F}_{t_0}\right], \\
JB_4^U &= \mathbb{E}\left[D_{t_0}^S e^U \mathbb{1}_{\left\{V_T > D_{t_0}^S e^U\right\}} e^{-\int_{t_0}^T \lambda_t^u ds}) | \mathcal{F}_{t_0}\right], \\
JB_5^U &= \mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T \geq D_{t_0}^S e^U + D_{t_0}^J\right\}} | \mathcal{F}_{t_0}\right], \\
JB_6^U &= \mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T \geq D_{t_0}^S e^U + D_{t_0}^J\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right], \\
JB_7^U &= \mathbb{E}\left[(D_{t_0}^S e^U + D_{t_0}^J) \mathbb{1}_{\left\{V_T \geq D_{t_0}^S e^U + D_{t_0}^J\right\}} | \mathcal{F}_{t_0}\right], \\
JB_8^U &= \mathbb{E}\left[(D_{t_0}^S e^U + D_{t_0}^J) \mathbb{1}_{\left\{V_T \geq D_{t_0}^S e^U + D_{t_0}^J\right\}} e^{-\int_{t_0}^T \lambda_t^u ds} | \mathcal{F}_{t_0}\right].
\end{aligned}
$$

17 Solutions to the cases

In this section, the solutions to the debt discount cases are solved. The following two Propositions will be instrumental for the solutions.

Proposition 21 *Under the financial model presented by equation (170), with equations (174) and (181), assuming that $\tau_U > t_0$ and $\tau_L > t_0$, the following expected value has the solution*

$$
\begin{aligned}
&\mathbb{E}\left[\phi_{dwl} V_T \mathbb{1}_{\left\{V_T \leq D_{t_0} e^U e^L\right\}} e^{-\int_{t_0}^T \left(\lambda_t^u \alpha_u + \lambda_t^l \alpha_l\right) dt} | \mathcal{F}_{t_0}\right] \\
=&\phi_{dwl} V_{t_0} e^{(r-q)(T-t_0)} H(\alpha_u, \alpha_l) F(\alpha_u, \alpha_l) \left(1 - G(\alpha_u, \alpha_l, a_1(X))\right),
\end{aligned}
$$

where

$$G(\alpha_u, \alpha_l, a_1(X)) = N\left(\frac{a_1(X) - \alpha_u \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u} \int_{t_0}^{T} n_u(s,T)ds - \alpha_l \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l} \int_{t_0}^{T} n_l(s,T)ds}{\sigma_V \sqrt{(T-t_0)}}\right),$$

$$a_1(X) = \log(V_{t_0}/X) + (r - q + \frac{1}{2}\sigma_V^2)(T - t_0),$$

$$X = D_{t_0} e^U e^L,$$

$$\begin{aligned} F(\alpha_u, \alpha_l) = \exp\Big[&\alpha_u \left(-\frac{\lambda_{t_0}^u}{\kappa_u} n_u(t_0, T) - \theta_u \int_{t_0}^{T} n_u(s,T)ds + \frac{\sigma_u^2}{2\kappa_u^2} \int_{t_0}^{T} n_u^2(s,T)du\right) \\ &+ \alpha_l \left(-\frac{\lambda_{t_0}^l}{\kappa_l} n_l(t_0, T) - \theta_l \int_{t_0}^{T} n_l(s,T)ds + \frac{\sigma_l^2}{2\kappa_l^2} \int_{t_0}^{T} n_l^2(s,T)du\right) \\ &+ \alpha_u \alpha_l \rho_{ul} \frac{\sigma_u}{\kappa_u} \frac{\sigma_l}{\kappa_l} \int_{t_0}^{T} n_u(s,T) n_l(s,T)ds\Big], \end{aligned}$$

$$H(\alpha_u, \alpha_l) = \exp\Big[-\alpha_u \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u} \int_{t_0}^{T} n_u(s,T)ds - \alpha_l \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l} \int_{t_0}^{T} n_l(s,T)ds\Big],$$

and

$$n_x(t,T) = 1 - e^{\kappa_x (T-t)}.$$

Proof.

With

$$n_u(t_0, T) := 1 - e^{-\kappa_u (T - t_0)}$$

We have equation (175)

$$\lambda_t^u = \lambda_{t_0}^u e^{-\kappa_u(t-t_0)} + \theta_u \left(1 - e^{-\kappa_u(t-t_0)}\right) + \sigma_u \int_{t_0}^{t} e^{-\kappa_u(t-s)} dW_u(s)$$

and following steps similar to those of Proposition 10.1.2 in Musiela and Rutkowski (2005), we start by integrating

$$\begin{aligned}
&\int_{t_0}^{T} \lambda_t^u dt \\
=&\int_{t_0}^{T} \lambda_{t_0}^u e^{-\kappa_u(t-t_0)} dt + \int_{t_0}^{T} \theta_u \left(1 - e^{-\kappa_u(t-t_0)}\right) dt + \sigma_u \int_{t_0}^{T} \int_{t_0}^{t} e^{-\kappa_u(t-s)} dW_u(t) dt.
\end{aligned} \tag{194}$$

We have

$$\int_{t_0}^{T} e^{-\kappa_u(t-t_0)} dt = \frac{1}{\kappa_u}(1 - e^{-\kappa_u(T-t_0)}) = \frac{1}{\kappa_u} n_u(t_0, T), \tag{195}$$

plus

$$\begin{aligned}
&\int_{t_0}^{T} \theta_u \left(1 - e^{-\kappa_u(t-t_0)}\right) dt \\
=&\theta_u \left(\int_{t_0}^{T} dt - \int_{t_0}^{T} e^{-\kappa_u(t-t_0)} dt \right) = \theta_u \left((T - t_0) - \frac{1}{\kappa_u} n_u(t_0, T) \right),
\end{aligned}$$

while noting that

$$\begin{aligned}
&\int_{t_0}^{T} n(s, T) ds = (T - t_0) - e^{-\kappa_u T} \int_{t_0}^{T} e^{\kappa_u s} ds \\
=&(T - t_0) - e^{-\kappa_u T} \frac{1}{\kappa_u} (e^{\kappa_u T} - e^{\kappa_u t_0}) = (T - t_0) - \frac{1}{\kappa_u}(1 - e^{-\kappa_u(T-t_0)}),
\end{aligned}$$

comparing the two former expressions, we get

$$\int_{t_0}^{T} \theta_u \left(1 - e^{-\kappa_u(t-t_0)}\right) dt = \theta_u \int_{t_0}^{T} n_u(s, T) ds, \tag{196}$$

and using the stochastic Fubini theorem[2],

$$
\begin{aligned}
&\int_{t_0}^{T}\int_{t_0}^{t} e^{-\kappa_u(t-s)}dW_u(s)dt = \int_{t_0}^{T}\int_{s}^{T} e^{-\kappa_u(T-t_0)}e^{\kappa_u s}dtdW_u(s)\\
=&\int_{t_0}^{T} e^{\kappa_u s}\int_{s}^{T} e^{-\kappa_u(T-t_0)}dtdW_u(s) = \frac{1}{\kappa_u}\int_{t_0}^{T}(1-e^{-\kappa_u(T-s)})dW_u(s)\\
=&\frac{1}{\kappa_u}\int_{t_0}^{T} n(s,T)dW_u(s) \qquad (197)
\end{aligned}
$$

Combining equations (194), (195), (196) and (197):

$$
\int_{t_0}^{T}\lambda_t^u dt = \frac{\lambda_{t_0}^u}{\kappa_u}n_u(t_0,T) + \theta_u\int_{t_0}^{T} n_u(s,T)ds + \frac{\sigma_u}{\kappa_u}\int_{t_0}^{T} n_u(s,T)dW_u(s).
$$

With similar steps for equation (182), with $n_l(t_0,T) = (1-e^{-\kappa_l(T-t_0)})$, we get

$$
\int_{t_0}^{T}\lambda_t^l dt = \frac{\lambda_{t_0}^l}{\kappa_l}n_l(t_0,T) + \theta_l\int_{t_0}^{T} n_l(s,T)ds + \frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_l(s,T)dW_l(s).
$$

Now, we define

$$
\xi_T \stackrel{\text{def}}{=} -\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt.
$$

The variable ξ_T has the expected value:

$$
\mathbb{E}\Big[\xi_T|\mathcal{F}_{t_0}\Big] = -\frac{\lambda_{t_0}^u}{\kappa_u}n_u(t_0,T) - \theta_u\int_{t_0}^{T} n_u(s,T)ds - \frac{\lambda_{t_0}^l}{\kappa_l}n_l(t_0,T) - \theta_l\int_{t_0}^{T} n_l(s,T)ds,
$$

and given the variance of the two separate processes is

$$
\frac{\sigma_u^2}{\kappa_u^2}\int_{t_0}^{T} n_u^2(s,T)ds
$$

[2]Presented, for instance, in Protter (1992, see Theorem IV.64).

and

$$\frac{\sigma_l^2}{\kappa_l^2}\int_{t_0}^{T} n_l^2(s,T)ds,$$

the variance of their sum, ξ_T, will be given by

$$\frac{\sigma_u^2}{\kappa_u^2}\int_{t_0}^{T} n_u^2(s,T)ds + \frac{\sigma_l^2}{\kappa_l^2}\int_{t_0}^{T} n_l^2(s,T)ds + 2\rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_u(s,T)n_l(s,T)ds.$$

With Musiela and Rutkowski (2005, Lemma10.1.1), we have that

$$\begin{aligned}
&\mathbb{E}\Big[e^{\xi_T}|\mathcal{F}_{t_0}\Big]\\
&=\exp\Big[-\frac{\lambda_{t_0}^u}{\kappa_u}n_u(t_0,T)-\theta_u\int_{t_0}^{T} n_u(s,T)ds-\frac{\lambda_{t_0}^l}{\kappa_l}n_l(t_0,T)-\theta_l\int_{t_0}^{T} n_l(s,T)ds\\
&\quad+\frac{\sigma_u^2}{2\kappa_u^2}\int_{t_0}^{T} n_u^2(s,T)ds+\frac{\sigma_l^2}{2\kappa_l^2}\int_{t_0}^{T} n_l^2(s,T)ds+\rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_u(s,T)n_l(s,T)ds\Big] \qquad (198)
\end{aligned}$$

With this, we define a new measure, Q, such that

$$\frac{dQ}{dP}=\frac{e^{-\int_{t_0}^{T}(\lambda_t^u+\lambda_t^l)dt}}{\mathbb{E}\Big[e^{-\int_{t_0}^{T}(\lambda_t^u+\lambda_t^l)dt}|\mathcal{F}_{t_0}\Big]}=\frac{e^{\xi_T}}{\mathbb{E}\Big[e^{\xi_T}|\mathcal{F}_{t_0}\Big]}.$$

After suppressing the common terms in the division,

$$\frac{dQ}{dP}=\frac{\exp\Big[-\frac{\sigma_u}{\kappa_u}\int_{t_0}^{T} n_u(s,T)dW_u(s)-\frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_l(s,T)dW_l(s)\Big]}{\exp\Big[\frac{\sigma_u^2}{2\kappa_u^2}\int_{t_0}^{T} n_u^2(s,T)ds+\frac{\sigma_l^2}{2\kappa_l^2}\int_{t_0}^{T} n_l^2(s,T)ds+\rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_u(s,T)n_l(s,T)ds\Big]},$$

that is

$$\begin{aligned}
\frac{dQ}{dP}&=\exp\Big[-\frac{\sigma_u}{\kappa_u}\int_{t_0}^{T} n_u(s,T)dW_u(s)-\frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_l(s,T)dW_l(s)\\
&\quad-\frac{\sigma_u^2}{2\kappa_u^2}\int_{t_0}^{T} n_u^2(s,T)du-\frac{\sigma_l^2}{2\kappa_l^2}\int_{t_0}^{T} n_l^2(s,T)du-\rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_{t_0}^{T} n_u(s,T)n_l(s,T)ds\Big].
\end{aligned}$$

By the multidimensional Girsanov theorem [3]

$$W_V^Q(t) = W_V(t) + \frac{\sigma_u \rho_{uV}}{\kappa_u} \int_{t_0}^{t} n_u(s,T)ds + \frac{\sigma_l \rho_{lV}}{\kappa_l} \int_{t_0}^{t} n_l(s,T)ds.$$

Thus, in measure Q, the value of V_t will be

$$\begin{aligned} V_t =& V_{t_0} \exp\Big[(r - q - \frac{1}{2}\sigma_V^2)(t - t_0) - \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u} \int_{t_0}^{t} n_u(s,T)ds \\ & - \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l} \int_{t_0}^{t} n_l(s,T)ds + \sigma_V W_V^Q(t)\Big]. \end{aligned} \tag{199}$$

So, the initial expression can be written as

$$\begin{aligned} & \mathbb{E}\Big[\phi_{dwl} V_T \mathbb{1}_{\{V_T \le D_{t_0}\}} e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt} | \mathcal{F}_{t_0}\Big] \\ =& \phi_{dwl} \mathbb{E}\Big[V_T \mathbb{1}_{\{V_T \le D_{t_0}\}} e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt} | \mathcal{F}_{t_0}\Big] \\ =& \phi_{dwl} \mathbb{E}\Big[e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt} | \mathcal{F}_{t_0}\Big] \mathbb{E}\Big[V_T \mathbb{1}_{\{V_T \le D_{t_0}\}} \frac{e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt}}{\mathbb{E}\Big[e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt} | \mathcal{F}_{t_0}\Big]} | \mathcal{F}_{t_0}\Big]. \end{aligned} \tag{200}$$

The solution of $\mathbb{E}\Big[e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt} | \mathcal{F}_{t_0}\Big]$is given in equation (198), and the remaining expected value will be under the equivalent measure Q:

$$\mathbb{E}\Big[V_T \mathbb{1}_{\{V_T \le D_{t_0}\}} \frac{e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt}}{\mathbb{E}\Big[e^{-\int_{t_0}^{T}(\lambda_t^u + \lambda_t^l)dt} | \mathcal{F}_{t_0}\Big]} | \mathcal{F}_{t_0}\Big] = \mathbb{E}^{\mathbb{Q}}\Big[V_T \mathbb{1}_{\{V_T \le D_{t_0}\}} | \mathcal{F}_{t_0}\Big]. \tag{201}$$

Then, we define the new equivalent measure, $\tilde{Q}$, such that

$$\frac{dQ}{d\tilde{Q}} = e^{-\frac{1}{2}\sigma_V^2 (T - t_0) + \sigma_V W_V^Q(T)}.$$

With V_t as in equation (199), equation (201) can be written as

---

[3]as presented, for instance, in Jeanblanc et al. (2009, see 1.7.4)

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Bigg[V_{t_0}\exp\big[(r-q-\frac{1}{2}\sigma_V^2)(T-t_0)\\
&-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^{T}n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^{T}n_l(s,T)ds+\sigma_V W_V^{Q}(T)\big]\mathbb{1}_{\{V_T\le D_{t_0}\}}|\mathcal{F}_{t_0}\Bigg]\\
=&V_{t_0}\exp\Big[(r-q)(T-t_0)\Big]\exp\Big[-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^{T}n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^{T}n_l(s,T)ds\Big]\\
&\times\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\big[-\frac{1}{2}\sigma_V^2(T-t_0)+\sigma_V W_V^{Q}(T)\big]\mathbb{1}_{\{V_T\le D_{t_0}\}}|\mathcal{F}_{t_0}\Bigg].
\end{aligned}
$$

Changing to the measure $\tilde{Q}$, the expression above is written as

$$
\begin{aligned}
&V_{t_0}\exp\big[(r-q)(T-t_0)\big]\exp\Big[-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^{T}n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^{T}n_l(s,T)ds\Big]\\
&\times\mathbb{E}^{\tilde{\mathbb{Q}}}\Bigg[\mathbb{1}_{\{V_T\le D_{t_0}\}}|\mathcal{F}_{t_0}\Bigg]\\
=&V_{t_0}\exp\big[(r-q)(T-t_0)\big]\exp\Big[-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^{T}n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^{T}n_l(s,T)ds\Big]\\
&\times\left(1-\mathbb{E}^{\tilde{\mathbb{Q}}}\Bigg[\mathbb{1}_{\{V_T> D_{t_0}\}}|\mathcal{F}_{t_0}\Bigg]\right),
\end{aligned}
\tag{202}
$$

and by the Girsanov theorem $W_V^{\tilde{Q}}(t)=W_V^{Q}(t)-\sigma_V(t-t_0)$, and under measure $\tilde{\mathrm{Q}}$, V_t is

$$
\begin{aligned}
V_t=&V_{t_0}\exp\big[(r-q+\frac{1}{2}\sigma_V^2)(t-t_0)\\
&-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^{t}n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^{t}n_l(s,T)ds+\sigma_V W_V^{\tilde{Q}}(t)\big].
\end{aligned}
$$

Therefore, $\log(V_T)$ is normally distributed with expected value

$$
(r-q+\frac{1}{2}\sigma_V^2)(T-t_0)-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^{t}n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^{t}n_l(s,T)ds
$$

and variance

$$
\sigma_V^2(T-t_0).
$$

Thus, we get

$$
\begin{aligned}
&\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\mathbb{1}_{\left\{V_T>D_{t_0}\right\}}|\mathcal{F}_{t_0}\right]=\mathbb{P}^{\tilde{\mathbb{Q}}}\left[V_T>D_{t_0}\right]\\
=&N\left(\frac{a_1(D_{t_0})-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^t n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^t n_l(s,T)ds}{\sigma_V\sqrt{(T-t_0)}}\right).
\end{aligned}
\tag{203}
$$

Therefore, combining equations (200), (201), (202) and (203):

$$
\begin{aligned}
&\mathbb{E}\left[\phi_{dwl}V_T\mathbb{1}_{\left\{V_T\leq D_{t_0}\right\}}e^{-\int_{t_0}^T\left(\lambda_t^u+\lambda_t^l\right)dt}|\mathcal{F}_{t_0}\right]\\
=&\phi_{dwl}V_{t_0}\exp\left[(r-q)(T-t_0)\right]\\
&\times\exp\Big[-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^T n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^T n_l(s,T)ds\Big]\\
&\times\exp\Big[-\frac{\lambda_{t_0}^u}{\kappa_u}n_u(t_0,T)-\theta_u\int_{t_0}^T n_u(s,T)ds-\frac{\lambda_{t_0}^l}{\kappa_l}n_l(t_0,T)-\theta_l\int_{t_0}^T n_l(s,T)ds\\
&+\frac{\sigma_u^2}{2\kappa_u^2}\int_{t_0}^T n_u^2(s,T)du+\frac{\sigma_l^2}{2\kappa_l^2}\int_{t_0}^T n_l^2(s,T)du+\rho_{ul}\frac{\sigma_u}{\kappa_u}\frac{\sigma_l}{\kappa_l}\int_{t_0}^T n_u(s,T)n_l(s,T)ds\Big]\\
&\times\left(1-N\left(\frac{a_1(D_{t_0})-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^T n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^T n_l(s,T)ds}{\sigma_V\sqrt{(T-t_0)}}\right)\right),
\end{aligned}
$$

where N represents the standard normal distribution.

Then, if we set $\alpha_u\in\{0,1\}$, $\alpha_l\in\{0,1\}$ and include the debt change parameters, U and L, such that the initial expression is

$$
\mathbb{E}\left[\phi_{dwl}V_T\mathbb{1}_{\left\{V_T\leq D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T\left(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l\right)dt}|\mathcal{F}_{t_0}\right],
$$

with the previous deduction, it is straightforward to see that the solution will be

$$
\begin{aligned}
&\mathbb{E}\left[\phi_{dwl}V_T\mathbb{1}_{\left\{V_T\leq D_{t_0}e^Ue^L\right\}}e^{-\int_{t_0}^T\left(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l\right)dt}|\mathcal{F}_{t_0}\right]\\
=&\phi_{dwl}V_{t_0}e^{(r-q)(T-t_0)}H(\alpha_u,\alpha_l)F(\alpha_u,\alpha_l)\left(1-G(\alpha_u,\alpha_l,a_1(X))\right).
\end{aligned}
$$

■

Proposition 22 *Under the financial model presented by equation (170), with equations (174) and (181), assuming that $\tau_U > t_0$ and $\tau_l > t_0$, the following expected value has the solution*

$$\mathbb{E}\left[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U e^L\}}e^{-\int_{t_0}^T(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l)dt}|\mathcal{F}_{t_0}\right] = D_{t_0}F(\alpha_u,\alpha_l)G(\alpha_u,\alpha_l,a_2(X)),$$

with

$$a_2(X) = (r-q-\frac{1}{2}\sigma_V^2)(T-t_0)+\log(V_{t_0}/X),$$

$$X = D_{t_0}e^U e^J,$$

and functions $G(\alpha_u,\alpha_l,x)$ and $F(\alpha_u,\alpha_l)$ as defined in Proposition 21.

Proof This proof is similar to that of Proposition 21. The key difference is not needing to use the measure $\tilde{Q}$.

The steps until equation (201) are identical. Thus, we have

$$\begin{aligned}&\mathbb{E}\left[D_{t_0}\mathbb{1}_{\{V_T > D_{t_0}e^U e^J\}}e^{-\int_{t_0}^T(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l)dt}|\mathcal{F}_{t_0}\right]\\ =&D_{t_0}\mathbb{E}\left[e^{-\int_{t_0}^T(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l)dt}|\mathcal{F}_{t_0}\right]\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{V_T > D_{t_0}e^U e^J\}}|\mathcal{F}_{t_0}\right]\end{aligned}$$

where the solution of $\mathbb{E}\left[e^{-\int_{t_0}^T(\lambda_t^u\alpha_u+\lambda_t^l\alpha_l)dt}|\mathcal{F}_{t_0}\right]$ is given in equation (198), by taking into consideration α_u and α_l, we obtain $F(\alpha_u,\alpha_l)$.

As for the remaining expected value, given now $\log(V_T)$ having the expected value of

$$(r-q-\frac{1}{2}\sigma_V^2)(T-t_0)-\frac{\sigma_V\sigma_u\rho_{uV}}{\kappa_u}\int_{t_0}^t n_u(s,T)ds-\frac{\sigma_V\sigma_l\rho_{lV}}{\kappa_l}\int_{t_0}^t n_l(s,T)ds,$$

and the variance, $\sigma_V(T-t_0)$, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{V_T > D_{t_0}e^U e^J\}}|\mathcal{F}_{t_0}\right] = \mathbb{P}^{\tilde{\mathbb{Q}}}\left[V_T > D_{t_0}e^U e^J\right] \\
=&N\left(\frac{a_2(D_{t_0}e^U e^J) - \alpha_u \frac{\sigma_V \sigma_u \rho_{uV}}{\kappa_u}\int_{t_0}^{t} n_u(s,T)ds - \alpha_l \frac{\sigma_V \sigma_l \rho_{lV}}{\kappa_l}\int_{t_0}^{t} n_l(s,T)ds}{\sigma_V\sqrt{(T-t_0)}}\right) \\
:=&G(\alpha_u, \alpha_l, a_2(D_{t_0}e^U e^J)).
\end{aligned}
$$

■

Remark 2 *The following integrated results, needed to obtain the debt cases payoffs, are presented.*

Where, $n_x(t,T) = 1 - e^{\kappa_x(T-t)}$*, we have:*

$$
\int_{t_0}^{T} n_x(s,T)ds = \int_{t_0}^{T}\left(1 - e^{-\kappa_x(T-s)}\right)ds = (T - t_0) - \frac{1}{\kappa_x}(1 - e^{-\kappa_x(T-t_0)}),
$$

$$
\begin{aligned}
&\int_{t_0}^{T} n_x^2(s,T)ds \\
=&\int_{t_0}^{T}\left(1 - e^{-\kappa_x(T-s)}\right)^2 ds = \frac{-3 - e^{-2\kappa_x(T-t_0)} + 4e^{-\kappa_x T(T-t_0)} + 2\kappa_x(T-t_0)}{2\kappa_x}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{t_0}^{T} n_u(s,T)n_l(s,T)ds \\
=&T + \frac{1}{\kappa_u}(e^{-\kappa_u(T-t_0)} - 1) + \frac{1}{\kappa_l}(e^{-\kappa_l(T-t_0)} - 1) + \frac{1}{\kappa_u + \kappa_l}(1 - e^{-(T-t_0)(\kappa_u+\kappa_l)}).
\end{aligned}
$$

The formula on the possibility of debt increase and or decrease is the first one to be solved. The remaining formulae follow similar steps.

Proposition 23 (Debt increase and or decrease) *Under the financial model presented by equation (170) and with the debt changes being ruled by equations, (174) and (181), assuming that* $\tau_U > t_0$ *and* $\tau_L > t_0$*, the value of a bond at* t_0 *on a firm under the possibility of suffering a debt increase and/or decrease is equal to equation (186), where*

$$
\begin{aligned}
A_1^{UL} &= D_{t_0} F(1,1) G(1,1,a_2(D_{t_0})) \\
A_2^{UL} &= \phi_{dwl} e^{-(r-q)(T-t_0)} V_{t_0} F(1,1) H(1,1) \left(1 - G(1,1,a_1(D_{t_0} e^U e^J)\right) \\
B_1^{UL} &= D_{t_0} F(1,0) G(1,0,a_2(D_{t_0} e^L)) \\
B_2^{UL} &= D_{t_0} F(1,1) G(1,1,a_2(D_{t_0} e^L)) \\
B_3^{UL} &= \phi_{dwl} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} F(1,0) H(1,0) \left(1 - G(1,0,a_1(D_{t_0} e^L))\right) \\
B_4^{UL} &= \phi_{dwl} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} F(1,1) H(1,1) \left(1 - G(1,1,a_1(D_{t_0} e^L))\right) \\
C_1^{UL} &= D_{t_0} F(0,1) G(0,1,a_2(D_{t_0} e^U)) \\
C_2^{UL} &= D_{t_0} F(1,1) G(1,1,a_2(D_{t_0} e^U)) \\
C_3^{UL} &= \phi_{dwl} e^{-U} V_{t_0} e^{-(r-q)(T-t_0)} F(0,1) H(0,1) \left(1 - G(0,1,a_1(D_{t_0} e^U))\right) \\
C_4^{UL} &= \phi_{dwl} e^{-U} V_{t_0} e^{-(r-q)(T-t_0)} F(1,1) H(1,1) \left(1 - G(1,1,a_1(D_{t_0} e^U))\right) \\
D_1^{UL} &= D_{t_0} G(0,0,a_2(D_{t_0} e^U e^J)) \\
D_2^{UL} &= D_{t_0} F(0,1) G(0,1,a_2(D_{t_0} e^U e^J)) \\
D_3^{UL} &= D_{t_0} F(1,0) G(1,0,a_2(D_{t_0} e^U e^J)) \\
D_4^{UL} &= D_{t_0} F(1,1) G(1,1,a_2(D_{t_0} e^U e^J)) \\
D_5^{UL} &= \phi_{dwl} e^{-U} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} \left(1 - G(0,0,a_1(D_{t_0} e^U e^J))\right) \\
D_6^{UL} &= \phi_{dwl} e^{-U} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} F(0,1) H(0,1) \left(1 - G(0,1,a_1(D_{t_0} e^U e^J))\right) \\
D_7^{UL} &= \phi_{dwl} e^{-U} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} F(1,0) H(1,0) \left(1 - G(1,0,a_1(D_{t_0} e^U e^J))\right) \\
D_8^{UL} &= \phi_{dwl} e^{-U} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} F(1,1) H(1,1) \left(1 - G(1,1,a_1(D_{t_0} e^U e^J))\right).
\end{aligned}
$$

and functions $F(\alpha_u, \alpha_l)$, $G(\alpha_u, \alpha_l, x)$ and $H(\alpha_u, \alpha_l)$ are defined in Propositions 21 and 22.

Proof.

For A_1^{UL}, it can be observed that the expected value matches Proposition 22 if $\alpha_u = 1$, $\alpha_l = 1$, $U = 0$ and $L = 0$. Therefore, the solution is

$$A_1^{UL} = D_{t_0} F(1,1) G(1,1,d_2(D_{t_0})).$$

In the case of A_2^{UL}, the expected value can be matched to Proposition 21 if $\alpha_u = 1$,

$\alpha_l = 1$. Therefore, the solution is

$$A_2^{UL} = \phi_{dwl} e^{(r-q)(T-t_0)} V_{t_0} H(1,1) F(1,1) \left(1 - G(1,1,a_1(D_{t_0} e^U e^L))\right).$$

As for B_1^{UL}, the expected value is matched by Proposition 22 if $\alpha_u = 1$, $\alpha_l = 0$ and $U = 0$. And therefore, the solution is

$$B_1^{UL} = D_{t_0} F(1,0) G(1,0,a_2(D_{t_0} e^L)).$$

The remainder of the elements follow the same rationale. ■

Proposition 24 (Debt increase) *Under the financial model presented by equation (170) and with the debt changes being ruled by equation (174), assuming that $\tau_U > t_0$, the value of a bond at t_0 on a firm under the possibility of suffering a debt increase is equal to equation (178), where*

$$\begin{aligned}
A_1^U &= D_{t_0} F(1,0) G(1,0,a_2(D_{t_0})) \\
A_2^U &= \phi_{dwl} e^{-(r-q)(T-t_0)} V_{t_0} F(1,0) H(1,0) \left(1 - G(1,0,a_1(D_{t_0}))\right) \\
B_1^U &= D_{t_0} G(0,0,a_2(D_{t_0} e^U)) \\
B_2^U &= D_{t_0} F(1,0) G(1,0,a_2(D_{t_0} e^U) \\
B_3^U &= \phi_{dwl} e^{-U} e^{-(r-q)(T-t_0)} V_{t_0} \left(1 - G(0,0,a_1(D_{t_0} e^U))\right) \\
B_4^U &= \phi_{dwl} e^{-U} e^{-(r-q)(T-t_0)} F(1,0) H(1,0) V_{t_0} \left(1 - G(1,0,a_1(D_{t_0} e^U))\right)
\end{aligned}$$

and functions $F(\alpha_u, \alpha_l)$, $G(\alpha_u, \alpha_l, x)$ and $H(\alpha_u, \alpha_l)$ are defined in Propositions 21 and 22.

Proof. This proof is similar to the one used in Proposition 23 and is therefore omitted.

Proposition 25 (Debt decrease) *Under the financial model presented by equation (170) and with the debt changes being ruled by equation (174), assuming that $\tau_L > t_0$, the value of a bond at t_0 on a firm under the possibility of suffering a debt decrease is equal to equation (184), where*

$$
\begin{aligned}
A_1^L &= D_{t_0} F(0,1) G(0,1,a_2(D_{t_0})) \\
A_2^L &= \phi_{dwl} e^{-(r-q)(T-t_0)} V_{t_0} F(0,1) H(0,1) \left(1 - G(0,1,a_1(D_{t_0}))\right) \\
B_1^L &= D_{t_0} G(0,0,a_2(D_{t_0} e^L)) \\
B_2^L &= D_{t_0} F(0,1) G(0,1,a_2(D_{t_0} e^L) \\
B_3^L &= \phi_{dwl} e^{-L} e^{-(r-q)(T-t_0)} V_{t_0} \left(1 - G(0,0,a_1(D_{t_0} e^L))\right) \\
B_4^U &= \phi_{dwl} e^{-L} e^{-(r-q)(T-t_0)} F(0,1) H(0,1) V_{t_0} \left(1 - G(0,1,a_1(D_{t_0} e^L))\right)
\end{aligned}
$$

and functions $F(\alpha_u, \alpha_l)$, $G(\alpha_u, \alpha_l, x)$ and $H(\alpha_u, \alpha_l)$ are defined as in Proposition 21.

Proof. This proof is similar to the one used in Proposition 23 and is therefore omitted.

Proposition 26 (Junior debt under senior debt increase) *Under the financial model presented by equation (170) and with the debt changes being ruled by equation (181), assuming that $\tau_L > t_0$, the value at t_0 of a junior bond on a firm under the possibility of suffering a debt decrease on the senior debt is equal to equation (184), where*

$$
\begin{aligned}
JA_1^U &= \phi_{dwl} e^{(r-q)(T-t_0)} V_{t_0} F(1,0) H(1,0) G(1,0,a_1(D_{t_0}^S)) \\
JA_2^U &= D_{t_0}^S F(1,0) G(1,0,a_2(D_{t_0}^S)) \\
JA_3^U &= \phi_{dwl} V_{t_0} e^{(r-q)(T-t_0)} F(1,0) H(1,0) G(1,0,a_1(D_{t_0}^S + D_{t_0}^J)) \\
JA_4^U &= (D_{t_0}^S + D_{t_0}^J) F(1,0) G(1,0,a_2(D_{t_0}^S + D_{t_0}^J)) \\
JB_1^U &= \phi_{dwl} e^{(r-q)(T-t_0)} V_{t_0} G(0,0,a_1(D_{t_0}^S e^U)) \\
JB_2^U &= \phi_{dwl} e^{(r-q)(T-t_0)} V_{t_0} F(1,0) H(1,0) G(1,0,a_1(D_{t_0}^S e^U)) \\
JB_3^U &= D_{t_0}^S e^U G(0,0,a_2(D_{t_0}^S e^U)) \\
JB_4^U &= D_{t_0}^S e^U F(1,0) G(1,0,a_2(D_{t_0}^S e^U)) \\
JB_5^U &= \phi_{dwl} e^{(r-q)(T-t_0)} V_{t_0} G(0,0,a_1(D_{t_0}^S e^U + D_{t_0}^J)) \\
JB_6^U &= \phi_{dwl} e^{(r-q)(T-t_0)} V_{t_0} F(1,0) H(1,0) G(1,0,a_1(D_{t_0}^S e^U + D_{t_0}^J)) \\
JB_7^U &= (D_{t_0}^S e^U + D_{t_0}^J) G(0,0,a_2(D_{t_0}^S e^U + D_{t_0}^J)) \\
JB_8^U &= (D_{t_0}^S e^U + D_{t_0}^J) F(1,0) G(1,0,a_2(D_{t_0}^S e^U + D_{t_0}^J)),
\end{aligned}
$$

and functions $F(\alpha_u, \alpha_l)$, $G(\alpha_u, \alpha_l, x)$ and $H(\alpha_u, \alpha_l)$ are defined in Propositions 21 and 22.

Proof. This proof is similar to the one used in Proposition 23 and is therefore omitted.

18 Numerical Analysis

The parameters used in this section are based on those used in Das and Kim (2015). The firm value is normalized to $V_{t_0} = 1$, while the two explored debt levels are $D = \{0.75, 0.5\}$. The firm value volatility is assumed to be $\sigma_V = 20\%$, the risk-free rate is $r = 2\%$, the total payout on debt and equity holders is set at zero, and the studied levels of the recovery values are $\phi_{dwl} = \{1, 0.7\}$. The debt increase is $U = \log(1.3)$, which corresponds to an increase of 30% over the initial amount of debt and the debt decrease is $L = \log(1/1.3)$, which corresponds to a decrease of approximately 23.1%. As for the increase and decrease events, the volatility of both processes is given by $\sigma_u = \sigma_l = 0.3$. The initial and mean reverting value of the processes is the same in both cases $\lambda_u = \lambda_l = \theta_u = \theta_l = 0.4$. Thus, both the initial and the long term average imply that in a short time interval Δ, the possibility of either an increase or a decrease occurring on the debt is approximately 0.4Δ. The speed of reversion of both processes is $\kappa_u = \kappa_l = 1$. Perfect correlations are assumed between the firm value and the debt increase and debt decrease processes, positive in the debt increase $\rho_{uV} = 1$, and negative in the debt decrease $\rho_{lV} = -1$. As for the correlation between both processes, it is assumed to be zero, $\rho_{ul} = 0$. In the subordinated debt case, the total debt is equally distributed among the junior debt and the senior debt, such that $D^J = D^S = 0.3750$ in one case and $D^S = D^J = 0.25$ in the other.

First, in Table 10 the various values of the bonds and the corresponding spreads are presented. The spreads in all these cases are computed as $\mathcal{S} = -\frac{1}{T-t_0}\log(\frac{B_{t_0}}{De^{-r(T-t_0)}})$.

In the base case, B_{t_0}, it is possible to observe that without the possibility of debt changes, the bond values are smaller and the spreads are higher for greater amounts of initial debt and lower recovery values. This pattern holds when the debt can change.

Comparing with the base case, the possibility of increasing debt, $B^U_{t_0}$, translates into a spread increment of 54 basis points to 146 and the possibility of decreasing debt, $B^L_{t_0}$, into a decline of 38 basis points to 54. These results are expected, given increases in debt lead to an increased probability of default, and when those defaults occur, the amount to

Table 10: Bond Values and Spreads for the studied cases at $T = 15$

	$D/V = 0.75$				$D/V = 0.5$			
	$G_{\phi=1}$		$G_{\phi=0.7}$		$G_{\phi=1}$		$G_{\phi=0.7}$	
	Bond Value	Spread	Bond Value	Spread	Bond Value	Spread	Bond Value	Spread
B_{t_0}	0.4842	92	0.4464	146	0.3492	39	0.3350	67
$B_{t_0}^U$	0.4466	146	0.3982	222	0.3340	69	0.3129	113
$B_{t_0}^L$	0.5124	54	0.4855	90	0.3592	21	0.3504	37
$B_{t_0}^{UL}$	0.4844	91	0.4466	146	0.3494	39	0.3351	67
$JB_{t_0}^L$	0.1934	241	0.1566	382	0.1570	110	0.1405	184

This table presents the bond values and the spreads for the baseline Merton model, the debt under a possibility of increase, the debt under the possibility of decrease, the debt under the possibility of increase and/or decrease and the junior debt under the possibility of a senior debt increase. The parameters are $V_{t_0} = 1$, $D = \{0.75, 0.5\}$, $\sigma_V = 20\%$, $r = 2\%$, $q = 0$, $\phi_{dwl} = \{1, 0.7\}$, $U = \log(1.3)$, $L = \log(1/1.3)$, $\sigma_u = \sigma_l = 0.3$, $\lambda_u = \lambda_l = \theta_u = \theta_l = 0.4$, $\kappa_u = \kappa_l = 1$, $\rho_{uV} = 1$, $\rho_{lV} = -1$, $\rho_{ul} = 0$. In the subordinated debt case, the debt levels are different with $D^J = D^S = 0.3750$ and $D^S = D^J = 0.25$.

be received from the original debt must be shared with the new debt-holder(s).

As for how the debt value reacts under the presence of the possibility of both the increase and the decrease in debt, the change is close to null, with the spread decreasing by one point. Given the studied time period is 15 years, by then, this is expected, as with high likelihood, the increases and decreases in the debt have returned it to its original value. In the subordinated debt case, although the bond values have a difficult comparison given the face value of debt being different, the high spreads highlight the increased risk of the junior debt in relation to the base case.

Figure 8 plots the credit spreads over the span of 20 years for the first four cases. It is confirmed that the possibility of debt increases augments the spreads, while the possibility of decreases reduces them. As for the shape of the curves, the possibility of debt increases leads to a greater magnitude of the hump-shaped curve that comes from the Merton (1974) base case, while in the case where there is the possibility of a debt decrease, a close to flat curve is produced.

As for the comparison between the baseline Merton (1974) case and the increase and/or decrease possibilities for the debt, we note that over the short term, the uncertainty

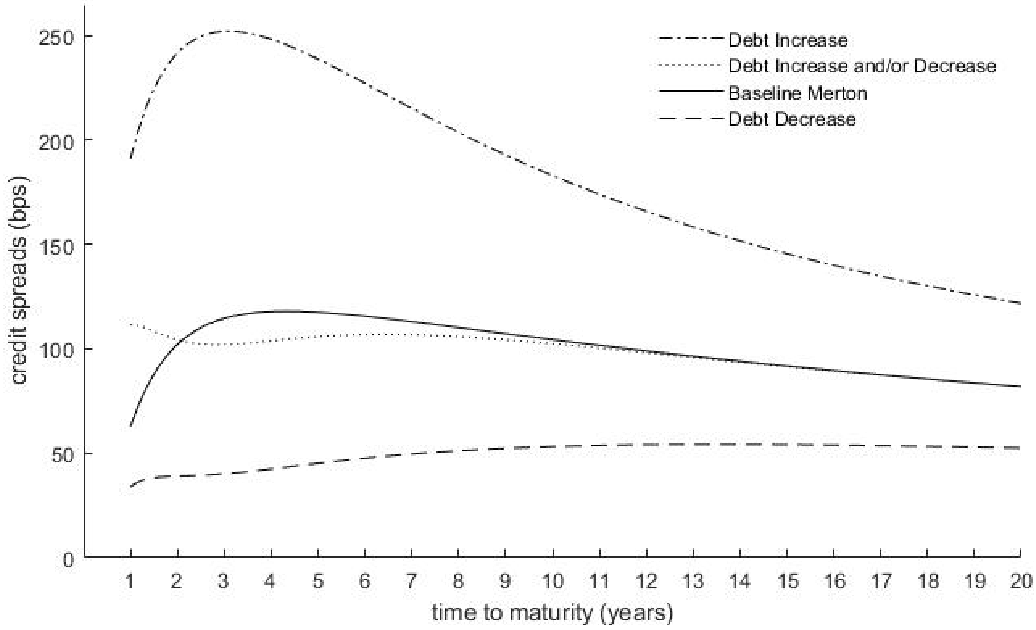

Figure 8: Credit Spreads of the baseline Merton (1974) model, the possibility to increase debt, the possibility to decrease debt and the possibility to increase and/or decrease debt with the standard parameters.

over the debt level leads to an increased spread. Over the medium term, the spreads are lower, with the impact of the possibility of the debt decrease having a greater impact than the possibility of the debt increase. Over the long term, the spreads tend to converge, as the possibility of both the increase and the decrease having occurred is very high.

In Figure 9, the impact of the magnitude of the debt increase is explored over different starting debt levels. As expected, and as observed before, higher debt increases lead to higher spreads, given the higher debt levels increasing the possibility of default and reducing the share of the original debt-holder when default occurs. As for the different debt levels, the impact of the (proportional) debt increase is bigger at the greater initial debt levels, which is expected, given it departs from higher absolute values. The same analysis for the debt decrease is also explored in the figure, again with the two expected results of bigger debt reductions always reducing the debt spreads and the impact being greater for higher debt levels given the proportionality of the debt decrease.

In Figure 10, the effect of the correlation between the firm value and possibility of debt decrease is studied. This is done trough the spread difference between the cases where the correlation is null, $\rho_{lV} = 0$, and perfectly negative, $\rho_{ul} = -1$. In the latter case, as the firm decreases in value, the likelihood of a debt decrease is increased, counterbalancing

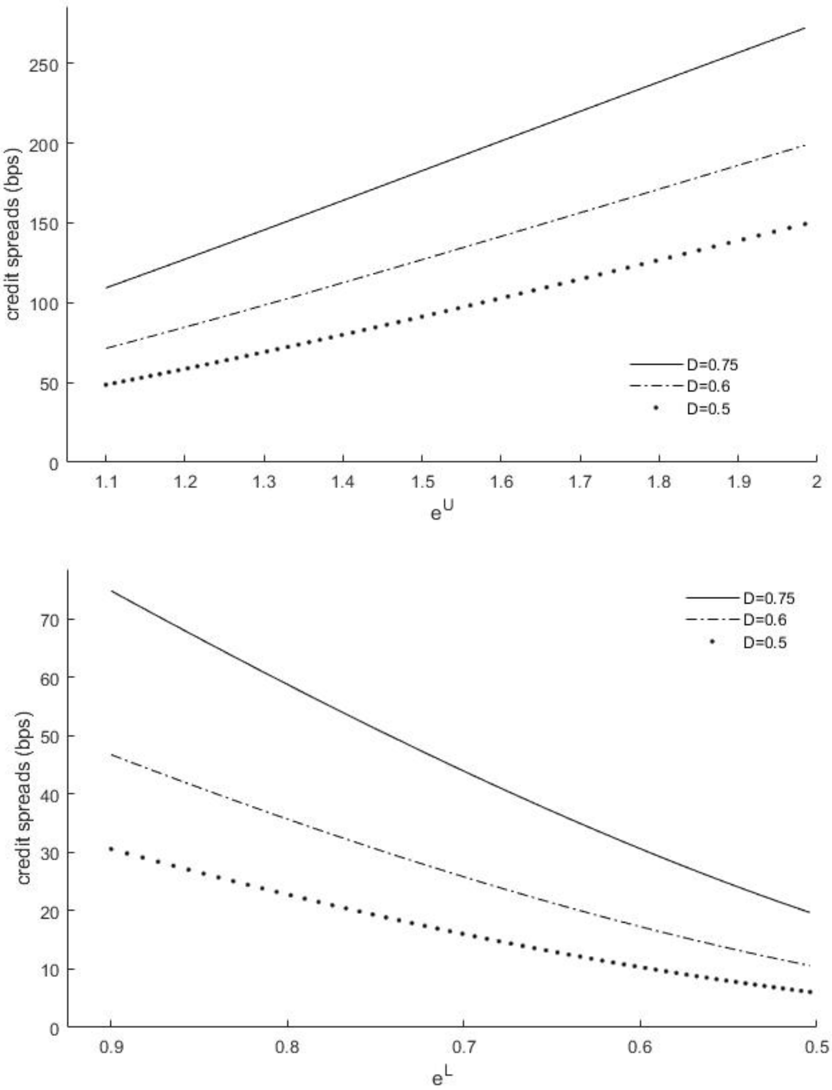

Figure 9: Credit spreads on the possibility of debt increase and on the possibility of debt decrease. The standard parameters with $T = 15$ are used while $D = \{0.5, 0.6, 0.75\}$. U spans between 1.1 and 2, that is, between an increase of 10% and 100% and L spans between 0.9 and 0.5, that is, a decrease of 10% and 50%.

the greater possibility of default and thus reducing the spreads. This is confirmed by the figure, as the spreads for the $\rho_{IV} = 0$ cases is always higher. The impact is greater in the short term than in the long term and when the initial leverage is higher than when it is lower.

The last two figures study the subordinated debt case under the possibility of a debt increase. Following what is presented in equation (1) of Gorton and Santomero (1990), where the senior debt value is calculated as regular debt, the values for the senior debt are calculated as in the B^U case.

Figure 11 presents three curves, the spread for the junior debt with the possibility

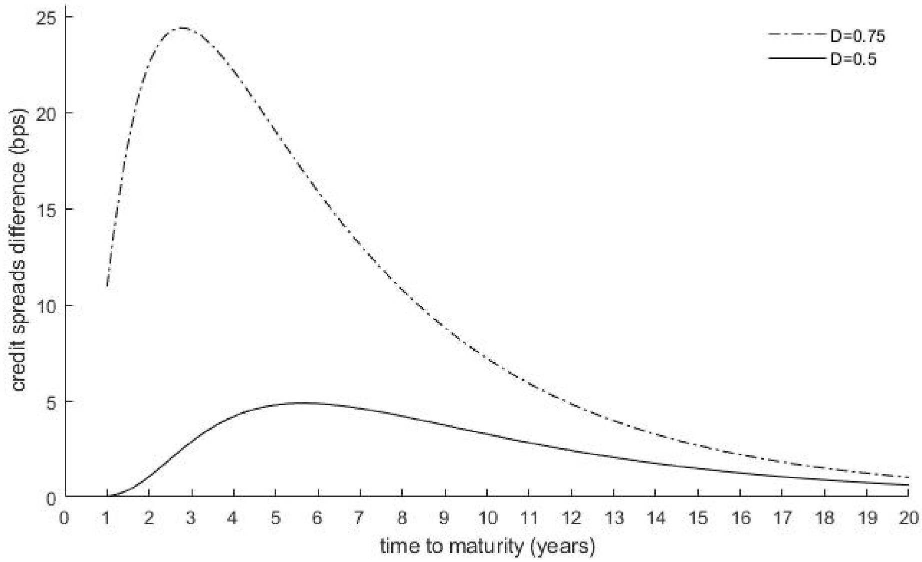

Figure 10: Difference between spreads on the possibility of debt decrease. Standard parameters with are assumed with $\rho_{IV} = \{0, -1\}$, with the zero correlation taking the plus sign in the difference and the correlation of minus one with the minus sign.

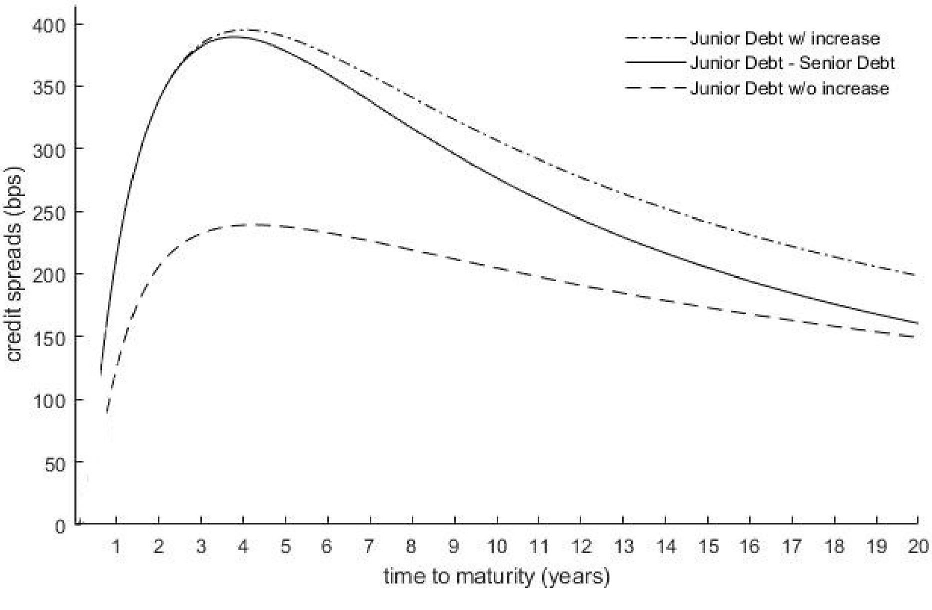

Figure 11: Junior debt spreads and its difference to the spread on the senior debt with $D^S = D^J = 0.25$. All the remaining parameters are the standard values presented in this section

of a senior debt increase, the junior debt spread without the possibility of increasing the debt and the difference between the junior debt and the senior debt spreads. As it can be noted, the possibility of the debt ratchet increases the spreads with more intensity over the lower maturities, reinforcing the hump-shape of the spread curve. The curve that represents the spreads difference shows that the spread on the senior debt with the possibility of being increased is significantly smaller than the one on the junior debt. Therefore, it is observed that the impact of the possibility of a debt increase of the senior debt is greater on the junior debt.

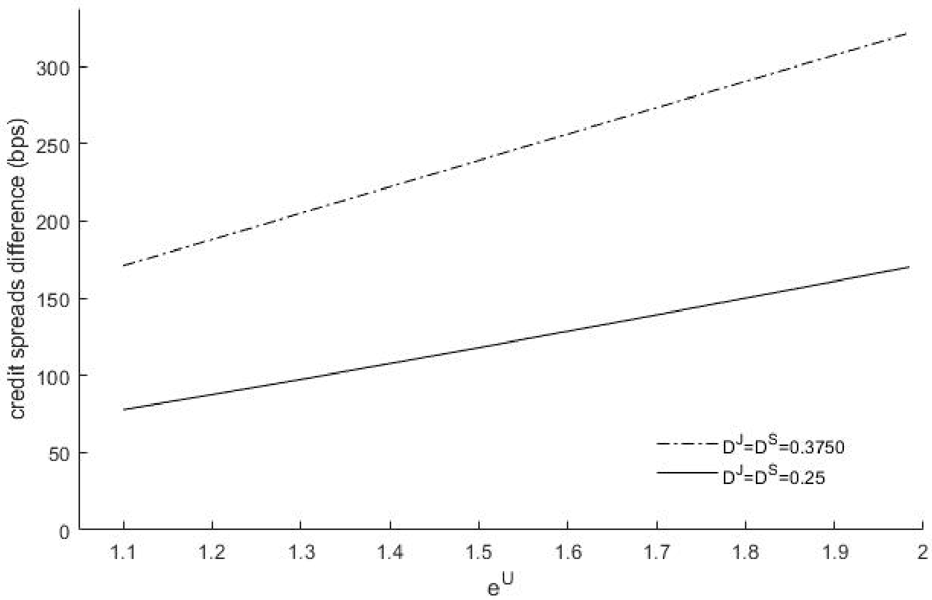

Figure 12: Difference between the spreads on the junior debt and the senior debt with $D^S = D^J = \{0.25, 0.375\}$, $T = 15$, U represents and increase of the debt between 10% and 100%, while the remaining parameters are the standard values presented in this section

.

Finally, Figure 12 explores the difference on the junior debt and the senior debt spreads, with the latter under possibility of being increased. As it is observed, as the size of the debt increase is augmented, the spread difference becomes greater, and the impact is greater in the higher leverage case.

19 Conclusions

In this part of the thesis, a model where the concept from Das and Kim (2015) — debt changing in discrete jumps — was explored in a different fashion. Instead of relying on the firm value crossing certain barriers, the changes in debt have their own process, although it can still be linked to the firm value through correlation of the processes.

Through changes of measure, we obtain the formulae for various cases that depart from the baseline Merton model: for the possibility of an increase, the possibility of a decrease and a combination of both. We observe that the possibility of increasing magnifies the credit spreads, while the possibility of decreasing reduces them. The case where debt can increase and or decrease is ambiguous, with the spreads being increased over the short term, reduced on the medium term and close to equal over the long term. We also find that the inclusion of correlations influences the impact of the debt changes.

Finally, given the flexibility of this dynamic debt framework, we extend it to the subordinated debt case of Gorton and Santomero (1990), and observe the impact of increasing the senior over itself and the junior debt. We notice that the impact over the junior debt is greater.

Part V

Conclusions

Part I gives a literature overview that motives the topics studied in this thesis. The remaining three parts of this thesis explore dynamic credit risk models, with a special emphasis on those based on barrier options as the in Das and Kim (2015) but also going beyond that setting, exploring the dynamic debt idea but through intensity process instead of first passage times.

In Part II, we expanded the two barrier and three barrier first-then-options presented by Jun and Ku (2012) to the JDCEV model of Carr and Linetsky (2006). We obtained formulae using the stopping time (ST) approach that was first developed for barrier options in Dias et al. (2015) and in Dias et al. (2021). The results were obtained for the various contracts, including the puts' recovery values. With the results, one is able to obtain the option's prices with the variance being a function of the underlying asset value and the possibility of a jump to default ruled by an affine function. These results also highlight the impact of the increased amount of barriers, by in general, reducing the value of the options, as more barriers must be crossed in order to activate the options' payoffs.

In part III, several of the barrier option formulae from the first part are used to obtain the Das and Kim (2015) dynamic debt model. This is done in the well known CEV model, which is nested in the JDCEV model where many of the formulae were obtained. The debt discounts formulae were obtained, and the corresponding spreads were computed. This allowed to observe how the dynamic debt model cases behave, when the CEV volatility for cases where it increases for decreasing underlying asset values.

In part IV, the debt increasing and decreasing features of the dynamic debt model of Das and Kim (2015) are explored in a different style. Instead of setting that debt changes when certain barriers are crossed by the leverage value, the changes are ruled by hazard processes which can be correlated to the firm value. After obtained closed formulae through measure changes, the credit spreads are obtained, allowing to see how these are affected by the possibility of debt changes through an intensity process. The possibility of debt increases augments spreads, the possibility of debt decreases reduces spreads, and the possibility of both events has mixed effects over time. In addition, the framework is extended to the case of the presence of subordinated debt, as presented in Gorton and Santomero (1990). It is shown that the possibility of debt increases in the senior debt has a bigger impact on the spreads of the junior debt.

References

Andersen, L., Andreasen, J., 2000. Volatility skews and extensions of the Libor market model. Applied Mathematical Finance 7, 1–32.

Arora, N., Bohn, J., Zhu, F., 2005. Reduced form vs. structural models of credit risk: A case study of three models. J. Invest. Manag. 3.

Bekaert, G., Wu, G., 2000. Asymmetric volatility and risk in equity markets. Review of Financial Studies 13, 1–42.

Benton, D., Krishnamoorthy, K., 2003. Computing discrete mixtures of continuous distributions: Noncentral chisquare, noncentral t and the distribution of the square of the sample multiple correlation coefficient. Computational Statistics and Data Analysis 43, 249–267.

Black, F., 1976. The pricing of commodity contracts. Journal of Financial Economics 3, 167–179.

Black, F., Cox, J.C., 1976. Valuing corporate securities: Some effects of bond indenture provisions. Journal of Finance 31, 351–367.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–654.

Bollerslev, T., Chou, R.Y., Krone, K.F., 1992. Arch modeling in finance: A review of the theory and empirical evidence. Journal of Econometrics 52, 5–59.

Bowie, J., Carr, P., 1994. Static simplicity. Risk 7, 45–49.

Boyle, P.P., Tian, Y., 1999. Pricing lookback and barrier options under the CEV process. Journal of Financial and Quantitative Analysis 34, 241–264.

Buchen, P., Konstandatos, O., 2009. A new approach to pricing double-barrier options with arbitrary payoffs and exponential boundaries. Applied Mathematical Finance 16, 497–515.

Buonocore, A., Giorno, V., Nobile, A.G., Ricciardi, L.M., 1990. On the two-boundary first-crossing-time problem for diffusion processes. Journal of Applied Probability 27, 102–114.

Buraschi, A., Jackwerth, J., 2001. The price of a smile: Hedging and spanning in option markets. Review of Financial Studies 14, 495–527.

Campbell, J.Y., Taksler, G.B., 2003. Equity volatility and corporate bond yields. Journal of Finance 58, 2321–2349.

Campi, L., Polbennikov, S., Sbuelz, A., 2009. Systematic equity-based credit risk: A CEV model with jump to default. Journal of Economic Dynamics and Control 33, 93–108.

Carr, P., 1998. Randomization and the American put. Review of Financial Studies 11, 597–626.

Carr, P., Chou, A., 1997. Breaking barriers. Risk 10, 139–145.

Carr, P., Linetsky, V., 2006. A jump to default extended CEV model: An application of Bessel processes. Finance and Stochastics 10, 303–330.

Carr, P., Wu, L., 2010. Stock options and credit default swaps: A joint framework for valuation and estimation. Journal of Financial Econometrics 8, 409–449.

Chen, L., Collin-Dufresne, P., Goldstein, R.S., 2008. On the Relation Between the Credit Spread Puzzle and the Equity Premium Puzzle. The Review of Financial Studies 22, 3367–3409.

Chen, M., 2015. Equity volatility smile and skew under a cev-based structural leverage model. Applied Mathematics and Information Sciences 9, 1095–1104.

Christie, A.A., 1982. The stochastic behavior of common stock variances: Value, leverage and interest rate effects. Journal of Financial Economics 10, 407–432.

Chung, S.L., Shih, P.T., Tsai, W.C., 2010. A modified static hedging method for continuous barrier options. Journal of Futures Markets 30, 1150–1166.

Collin-Dufresne, P., Goldstein, R.S., 2001. Do credit spreads reflect stationary leverage ratios? Journal of Finance 56, 1929–1957.

Collin-Dufresne, P., Goldstein, R.S., Martin, J.S., 2001. The determinants of credit spread changes. Journal of Finance 56, 2177–2207.

Consigli, G., 2004. Credit default swaps and equity volatility: Theoretical modeling and market evidence. Departement of Applied Mathematics, University Ca'Foscari .

Cox, J.C., 1975. Notes on option pricing I: Constant elasticity of variance diffusions. Working Paper, Stanford University. Reprinted in *Journal of Portfolio Management*, 23 (1996), 15-17.

Cox, J.C., Ingersoll, Jr., J.E., Ross, S.A., 1985. A theory of the term structure of interest rates. Econometrica 53, 385–408.

Cremers, M., Driessen, J., Maenhout, P., Weinbaum, D., 2008. Individual stock-option prices and credit spreads. Journal of Banking and Finance 32, 2706–2715.

Das, S.R., Kim, S., 2015. Credit spreads with dynamic debt. Journal of Banking and Finance 50, 121–140.

Davydov, D., Linetsky, V., 2001. Pricing and hedging path-dependent options under the CEV process. Management Science 47, 949–965.

Davydov, D., Linetsky, V., 2003. Pricing options on scalar diffusions: An eigenfunction expansion approach. Operations Research 51, 185–209.

Delianedis, G., Geske, R., 1998. Credit risk and risk neutral default probabilities: Information about rating migrations and defaults. Working Paper, Available in SSRN-id424301.

Dennis, P., Mayhew, S., 2002. Risk-neutral skewness: Evidence from stock options. Journal of Financial and Quantitative Analysis 37, 471–493.

Dennis, P., Mayhew, S., Stivers, C., 2004. Stock returns, implied volatility innovations and the asymmetric volatility phenomenon. Journal of Financial and Quantitative Analysis 41, 381–406.

Derman, E., Ergener, D., Kani, I., 1995. Static options replication. Journal of Derivatives 2, 78–95.

Dias, J., Nunes, J., Cruz, A., 2020. A note on options and bubbles under the cev model:

implications for pricing and hedging. Review of Derivatives Research 23, 1–24.

Dias, J.C., Nunes, J.P., 2011. Pricing real options under the constant elasticity of variance diffusion. Journal of Futures Markets 31, 230–250.

Dias, J.C., Nunes, J.P., Ruas, J.P., 2015. Pricing and static hedging of European-style double barrier options under the jump to default extended CEV model. Quantitative Finance 15, 1995–2010.

Dias, J.C., Nunes, J.P., Ruas, J.P., 2021. Almost everything you need to know about barrier options under a markovian diffusion process with killing. - working paper, ISCTE Business School .

Driessen, J., 2005. Is default event risk priced in corporate bonds? Review of Financial Studies 18, 165–195.

Du, D., Elkamhi, R., Ericsson, J., 2019. Time-varying asset volatility and the credit spread puzzle. The Journal of Finance 74, 1841–1885.

Duffee, G.R., 1999. Estimating the price of default risk. Review of Financial Studies 12, 197–226.

Duffie, D., Gârleanu, N., 2001. Risk and valuation of collateralized debt obligations. Financial Analysts Journal 57, 41–59.

Duffie, D., Pedersen, L.H., Singleton, K.J., 2003. Modeling sovereign yield spreads: a case study of russian debt. Journal of Finance 58, 119–160.

Duffie, D., Singleton, K.J., 1997. An econometric model of the term structure of interest-rate swap yields. Journal of Finance 52, 1287–1321.

Eisenthal-Berkovitz, Y., Feldhütter, P., Vig, V., 2020. Leveraged buyouts and bond credit spreads. Journal of Financial Economics 135, 577–601.

Emanuel, D.C., MacBeth, J.D., 1982. Further results on the constant elasticity of variance call option pricing model. Journal of Financial and Quantitative Analysis 17, 533–554.

Eom, Y.H., Helvege, J., Huang, J.Z., 2004. Structural models of corporate bond pricing: An empirical analysis. Review of Financial Studies 17, 499–544.

Fard, F.A., 2015. Analytical pricing of vulnerable options under a generalized jump-diffusion model. Insurance: Mathematics and Economics 60, 19–28.

Flannery, M.J., Nikolova, S., Öztekin, O., 2012. Leverage expectations and bond credit spreads. Journal of Financial and Quantitative Analysis 47, 689–714.

Geman, H., Yor, M., 1996. Pricing and hedging double-barrier options: A probabilistic approach. Mathematical Finance 6, 365–378.

Geske, R., 1977. The valuation of corporate liabilities as compound options. Journal of Financial and Quantitative Analysis 12, 541–552.

Giorno, V., Nobile, A.G., Ricciardi, L.M., Sato, S., 1989. On the evaluation of first-passage-time probability densities via non-singular integral equations. Advances in Applied Probability 21, 20–36.

Gorton, G., Santomero, A.M., 1990. Market discipline and bank subordinated debt. Journal of Money, Credit and Banking 22, 119–128.

Gutiérrez, R., Ricciardi, L.M., Román, P., Torres, F., 1997. First-passage-time densities for time-non-homogeneous diffusion processes. Journal of Applied Probability 34, 623–631.

Hansen, L.P., 1982. Large sample properties of generalized method of moments estimators. Econometrica 50, 1029–1054.

Haug, E.G., 2001. Closed form valuation of American barrier options. International Journal of Theoretical and Applied Finance 4, 355–359.

Haug, E.G., 2006. The Complete Guide to Option Pricing Formulas. McGraw-Hill, New York, NY. 2nd edition.

Heston, S.L., 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies 6, 327–343.

Hilscher, J., 2007. Is the corporate bond market forward looking? European Central Bank Working Paper Series 800, 2706–2715.

Huang, J.Z., Huang, M., 2012. How Much of the Corporate-Treasury Yield Spread Is Due to Credit Risk? The Review of Asset Pricing Studies 2, 153–202.

Huang, J.Z., Shi, Z., Zhou, H., 2019. Specification Analysis of Structural Credit Risk Models. Review of Finance 24, 45–98.

Hui, C.H., 1996. One-touch double barrier binary option values. Applied Financial Economics 6, 343–346.

Hull, J.C., Nelken, I., White, A.D., 2005. Merton's model, credit risk and volatility skews. Journal of Credit Risk 1, 3–27.

Hung, M.W., Liu, Y.H., 2004. Pricing vulnerable options in incomplete markets. The Journal of Future Markets 25, 135–170.

Jackwerth, J.C., Rubinstein, M., 1996. Recovering probability distributions from option prices. Journal of Finance 51, 1611–1631.

Jarrow, R.A., Turnbull, S.M., 1995. Pricing derivatives on financial securities subject to credit risk. Journal of Finance 50, 53–85.

Jeanblanc, M., Yor, M., Chesney, M., 2009. Mathematical Methods for Financial Markets. Springer, New York.

Johnson, H., Stulz, R., 1987. The pricing of options with default risk. The Journal of Finance 42, 267–280.

Jones, E.P., Mason, S.P., Rosenfeld, E., 1984. Contingent claims analysis of corporate capital structures: An empirical investigation. Journal of Finance 39, 611–625.

Jun, D., Ku, H., 2012. Cross a barrier to reach barrier options. Journal of Mathematical Analysis and Applications 389, 968–978.

Jun, D., Ku, H., 2013. Pricing chained options with curved barriers. Mathematical Finance 23, 763–776.

Klein, P., 1996. Pricing black-scholes options with correlated credit risk. Journal of Banking and Finance 20, 1211–1229.

Klein, P., Inglis, M., 2001. Pricing vulnerable european options when the option's payoff can increase the risk of financial distress. Journal of Banking and Finance 25, 993–1012.

Klein, P., Yang, J., 2010. Vulnerable american options. Managerial Finance 36, 414–430.

Koo, E., Kim, G., 2017. Explicit formula for the valuation of catastrophe put option with exponential jump and default risk. Chaos, Solitons and Fractals 101, 1–7.

Kuan, G.C.H., Webber, N., 2003. Pricing barrier options with one-factor interest rate models. Journal of Derivatives 10, 33–50.

Kunitomo, N., Ikeda, M., 1992. Pricing options with curved boundaries. Mathematical Finance 2, 275–298.

Lando, D., 1998. On cox processes and credit risky securities. Review of Derivatives Research 2, 99–120.

Leland, H.E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance 49, 1213–1252.

Linetsky, V., 2004. The spectral decomposition of the option value. International Journal of Theoretical and Applied Finance 7, 337–384.

Madan, D.B., Unal, H., 1998. Pricing the risks of default. Review of Derivatives Research 2, 121–160.

Merton, R.C., 1973. Theory of rational option pricing. Bell Journal of Economics and Management Science 4, 141–183.

Merton, R.C., 1974. On the pricing of corporate debt: The risk structure of interest rates. Journal of Finance 29, 449–470.

Merton, R.C., 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics 3, 125–144.

Mijatović, A., Pistorius, M., 2013. Continuously monitored barrier options under Markov processes. Mathematical Finance 23, 1–38.

Modigliani, F., Miller, M.H., 1958. The cost of capital, corporation finance and the theory of investment. American Economic Review 48, 261–297.

Musiela, M., Rutkowski, M., 2005. Martingale Methods in Financial Modelling. Springer-Verlag, Berlin. 2nd edition.

Nelson, D.B., 1991. Conditional heteroskedasticity in asset returns: A new approach. Econometrica 59, 347–370.

Nini, G., Smith, D.C., Sufi, A., 2012. Creditor control rights, corporate governance, and firm value. Review of Financial Studies 25, 1713–1761.

Nobile, A.G., Ricciardi, L.M., Sacerdote, L., 1985. A note on first-passage time and some related problems. Journal of Applied Probability 22, 346–359.

Nunes, J.P., 2009. Pricing American options under the constant elasticity of variance model and subject to bankruptcy. Journal of Financial and Quantitative Analysis 44, 1231–1263. Correction: Nunes, J. P., 2012, American-style Puts under the JDCEV Model: A Correction. JFQA web site at http://depts.washington.edu/jfqa/ in Unpublished Appendixes, Notes, Comments, and Corrections.

Nunes, J.P., Ruas, J.P., Dias, J.C., 2015. Pricing and static hedging of American-style knock-in options on defaultable stocks. Journal of Banking & Finance 58, 343–360.

Park, C., Schuurmann, F., 1976. Evaluations of barrier-crossing probabilities of Wiener paths. Journal of Applied Probability 13, 267–275.

Pelsser, A., 2000. Pricing double barrier options using Laplace transforms. Finance and Stochastics 4, 95–104.

Protter, P., 1992. Stochastic Integration and Differential Equation. Springer-Verlag, Berlin, Heidelberg. second edition.

Revuz, D., Yor, M., 1999. Continuous Martingales and Brownian Motion. Springer-Verlag, Berlin. 3rd edition.

Rich, D.R., 1994. The mathematical foundations of barrier option-pricing theory. Advances in Futures and Options Research 7, 267–311.

Roberts, M.R., Sufi, A., 2009. Renegotiation of financial contracts: Evidence from private credit agreements. Journal of Financial Economics 93, 159–184.

Ruas, J.P., Dias, J.C., Nunes, J.P., 2013. Pricing and static hedging of American options under the jump to default extended CEV model. Journal of Banking and Finance 37, 4059–4072.

Rubinstein, M., 1994. Implied binomial trees. Journal of Finance 49, 771–818.

Rubinstein, M., Reiner, E., 1991. Breaking down the barriers. Risk 4, 28–35.

Schaefer, S.M., Strebulaev, I.A., 2008. Structural models of credit risk are useful: Evidence from hedge rations on corporate bonds. Journal of Financial Economics 90, 1–19.

Schroder, M., 1989. Computing the constant elasticity of variance option pricing formula. Journal of Finance 44, 211–219.

Schröder, M., 2000. On the valuation of double-barrier options: Computational aspects. Journal of Computational Finance 3, 5–33.

Schwert, W., 1989. Why does stock market volatility change over time? Journal of Finance 44, 1115–1153.

Shi, Z., 2019. Time-varying ambiguity, credit spreads, and the levered equity premium. Journal of Financial Economics 134, 617–646.

Sidenius, J., 1998. Double barrier options: Valuation by path counting. Journal of Computational Finance 1, 63–79.

Vasicek, O., 1977. An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177–188.

Yang, S.J., Min-KuLee, Kim, J.H., 2014. Pricing vulnerable options under a stochastic volatility model. Applied Mathematics Letters 34, 7–12.

Zhang, B.Y., Zhou, H., Zhu, H., 2009. Explaining credit default swap spreads with the equity volatility and jump risks of individual firms. Review of Financial Studies 22, 5099–5131.

Zhou, C., 2001. The term structure of credit spreads with jump risk. Journal of Banking and Finance 25, 2015–2040.