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Homomorphism to \mathbb{R} generated by abstract length functions: a dynamical construction

Gastão Bettencourt¹ and Sérgio Mendes²

¹Departamento de Matemática, Fac. Ciências, Universidade da Beira Interior, Covilhã, Portugal, ²Departamento de Matemática, ISCTE - Lisbon University Institute, Lisbon, Portugal

E-mail: gastao@ubi.pt, sergio.mendes@iscte.pt

Abstract. Erschler and Karlsson in [5] construct a homomorphism of a finitely generated group G to \mathbb{R} using a random walk approach. Central to their construction were the word length ℓ and a well behaved measure μ on G . In this note we define a class of abstract length functions and prove that Erschler and Karlsson construction can also be applied to this class.

1. Introduction

It was Pólya [11], in 1921, who started the study of random walks on \mathbb{Z}^n . It took almost 40 years, until Kesten [10], in 1959, started the study of random walks on infinite, finitely generated groups. After that, in 1972 Avez [1] introduced the entropy h of a random walk on a finitely generated group, arguably the most important numerical invariant associated to a group endowed with a probability measure, and connected entropy with the growth rate v of the group. In his subsequent work [2], Avez put in evidence the strong connection between entropy and the existence of bounded harmonic functions on the group. The drift ℓ is another number that has emerged in the study of random walks. These numbers are linked via the fundamental inequality popularized by Vershik [12], $h \leq v\ell$. For a comprehensive reference in a more general setting see Kaimanovich and Vershik [9].

The length function of a finitely generated group is a well established concept in geometric group theory. An abstract length function f , for a group G , associates to an element $g \in G$ a real number subject to certain axioms. In an attempt to understand cancelation properties for elements of free groups, Lyndon [7] defined integer valued length functions. Later, Harrison [6] used Lyndon axioms to define real abstract length functions on groups.

In this note we consider abstract length functions which are subject to the only basic axioms of being subadditive and inverse invariant. Hence, we consider a broader class of length functions which includes Lyndon's length functions. Our class of abstract length functions give rise to a pseudometric in G , that is, we admit some nonidentity elements to have (abstract) length zero. The set of the elements $\{g \in G : f(g) = 0\}$ actually has the structure of a subgroup of G which we conveniently call the kernel of f . Some of its properties are studied in section 3.

There are several ways to construct nontrivial homomorphisms, depending on the setting. For example, if G is amenable any homogeneous quasimorphism to \mathbb{R} is in fact a homomorphism, see [4]. Karlsson and Ledrappier in [8] proved that if a finite first moment random walk has zero entropy and positive drift, then there exists a non-trivial homomorphism to \mathbb{R} . Erschler and



Karlsson [5] used a dynamical construction to establish the existence of such homomorphism in the case where the drift can also be zero. In [3], the authors followed Erschler and Karlsson construction to define a homomorphism to \mathbb{R} for a certain class of semidirect products. The original construction starts by making a judicious choice of a sequence of functions T_n , and of a numerical sequence $\alpha(n)$ such that the product $\alpha(n)T_n$ will have a sublimit T which is precisely the desired homomorphism.

We give now a brief description of this note. In section 2 we define abstract length functions on a group G and give some examples. In section 3, we collect some elementary properties of abstract length functions. Finally, in section 4 we focus on a dynamical way of obtaining homomorphisms from a finitely generated group G to \mathbb{R} . Specifically, we show that Erschler and Karlsson construction still works for the broader class of abstract length functions. An interesting question relating this construction with the notion of the kernel of a subadditive inverse invariant functions f (see sections §2, 3) is to find under what conditions the kernel K_f equals the kernel of the homomorphism $T = \lim_k T_{n_k}^\alpha$, see Remark (4.7).

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2. A class of subadditive functions

Let $f : G \rightarrow \mathbb{R}$ be a real function on a group G which is subadditive and inverse invariant, i.e.

$$f(gh) \leq f(g) + f(h) \tag{1}$$

$$f(g^{-1}) = f(g). \tag{2}$$

These properties are a subset of the properties that characterize the usual length function on groups, $\ell(g)$. We denote the set of functions verifying (1) and (2) simply by *SII* (subadditive inverse invariant).

Denoting by e the identity of G , we have

$$f(e) = f(gg^{-1}) \leq f(g) + f(g^{-1}) = 2f(g)$$

Taking $g = e$ in the above inequality we obtain $f(e) \geq 0$. Hence,

$$f(g) \geq 0$$

for every $g \in G$.

Now, suppose there is some $g_o \in G$ such that $f(g_o) = 0$. Define the following subset of G

$$K = \{g \in G : f(g) = 0\}. \tag{3}$$

Since $0 \leq f(e) \leq 2f(g_o)$ then $f(e) = 0$ and $e \in K$. Also if $g \in K$ then $g^{-1} \in K$ by (2). On the other hand, if $g, h \in K$ it follows that

$$0 \leq f(gh) \leq f(g) + f(h) = 0$$

and $gh \in K$. So we conclude the following

Proposition 2.1. *Let $f \in SII$ and suppose K is not empty. Then K is a subgroup of G .*

This result motivates the following definition.

Definition 2.2. *Let $f \in SII$. $K = \{g \in G : f(g) = 0\}$ will be called the kernel of f .*

We introduce some simple examples and remarks to elucidate these concepts.

Example 2.3. Let $G = (\mathbb{R}^+, \cdot)$ be the multiplicative group of positive real numbers and let $f : G \rightarrow \mathbb{R}$ be the characteristic function of the set of positive irrational numbers

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q}^+ \\ 1 & x \in \mathbb{R}^+ \setminus \mathbb{Q}^+ \end{cases}$$

Since $x \in \mathbb{Q}^+$ if and only if $x^{-1} \in \mathbb{Q}^+$ it follows that f is inverse invariant. It is also easy to check subadditivity. We have $K = \mathbb{Q}^+$.

Example 2.4. More general, let $G = (\mathbb{R}^+, \cdot)$ and let $f : G \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \alpha & x \in \mathbb{Q}^+ \\ \beta & x \in \mathbb{R}^+ \setminus \mathbb{Q}^+ \end{cases}$$

If $0 \leq \alpha \leq 2\beta$ then $f \in SII$, which generalizes last example. Regarding the kernel, we have

- $\beta = 0$ then $f = 0$ and $K = \mathbb{R}^+$
- $\alpha = 0$ and $\beta > 0$ then $K = \mathbb{Q}^+$
- $\alpha > 0$ then $K = \emptyset$.

Here is a nonabelian example.

Example 2.5. Let $G = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$ denote the discrete Heisenberg group and let $f : G \rightarrow \mathbb{R}$ be defined by

$$f(g) = |x + z|.$$

We see that $f \in SII$ and the kernel is

$$K = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{Z} \right\}$$

We note that K is a normal subgroup of G .

Remark 2.6. It is interesting to note that if we consider a subadditive function which instead of (2) verifies $f(g^{-1}) = f(g)$ only for $g \in K$ we still have that K is a subgroup. Other properties still remain valid, e.g. see (3.1), (3.2).

Some ways to generate new inverse invariant subadditive functions are given by the next three results, whose proof is straightforward.

Proposition 2.7. Let $f_1, f_2 \in SII$, $\alpha > 0$ and $\varphi : H \rightarrow G$ be a group homomorphism. We have

- $\alpha f_1 \in SII$
- $f_1 + f_2 \in SII$
- $f_1 \circ \varphi \in SII$

Proposition 2.8. Let $f \in SII$ and $\varphi : G \rightarrow G$ a group homomorphism. Then we have

$$\frac{1}{n} \sum_{k=0}^{n-1} f \circ \varphi^k \in SII$$

Proposition 2.9. Let $f_k \in SII$ for all $k \in \mathbb{N}$ and $\varphi : G \rightarrow G$ a group homomorphism. We have

$$\frac{1}{n} \sum_{k=0}^{n-1} f_k \circ \varphi \in SII$$

We collect now some random remarks. The first connects elements of *SII* with the familiar concept of distance, the second puts in evidence that subadditivity is the fundamental concept in the defining equations (1) and (2), the third remark justify why the class *SII* is somehow unusual to a reader familiar with group theory and homomorphisms, and finally, the last one connects simple groups, a subclass of functions of *SII* and bi-invariant metrics. The first time we saw this result was in the work of Polterovich.

Remark 2.10. *If it happens that $f(e) = 0$, a left invariant pseudo metric d on G is defined as $d(g, h) = f(g^{-1}h)$. If the kernel K of f is nonempty and trivial, then f defines a metric.*

Remark 2.11. *We also note that if we have a subadditive function f that is not inverse invariant we can always construct an inverse invariant subadditive function by making*

$$\tilde{f}(g) = \frac{f(g) + f(g^{-1})}{2}.$$

Remark 2.12. *In the context of homomorphisms the concept of inverse invariance is void, that is, if $\psi : G \rightarrow \mathbb{R}$ is a homomorphism to \mathbb{R} which is inverse invariant, then ψ is trivial. In fact,*

$$0 = \psi(e) = \psi(gg^{-1}) = 2\psi(g)$$

so, $\psi(g) = 0$ for every $g \in G$.

Remark 2.13. *Assume now that the group G is simple and suppose that $f \in SII$ is also conjugacy invariant, in the sense $f(sgs^{-1}) = f(g)$, for all $s, g \in G$. In particular for $h \in K$, $g \in G$ we have $f(ghg^{-1}) = f(h) = 0$, and $ghg^{-1} \in K$, that is, K is normal on G . This means that either $K = G$, which amounts to say that f is trivial, or $K = \{e\}$. Regarding the first remark, keeping in mind that a right invariant metric is generated by a conjugacy invariant function of *SII*, we can say that a bi-invariant non trivial pseudo metric on a simple group is indeed a metric.*

3. Some properties of the kernel

We recall that when $\phi : G \rightarrow H$ is a group homomorphism, and h is an element of the kernel then, for every $g \in G$ we have

$$\phi(gh) = \phi(g)\phi(h) = \phi(g).$$

In our setting we still have the same result. For the remaining part of this section, G is an arbitrary group and $f \in SII$. Suppose that K is nonempty and let $h \in K$. We have, for every $g \in G$,

$$f(g) = f(ghh^{-1}) \leq f(gh) + f(h^{-1}) \leq f(gh).$$

Since

$$f(gh) \leq f(g) + f(h) = f(g)$$

it follows that $f(gh) = f(g)$.

Conversely, if $f(gh) = f(g)$, for every g , then

$$f(eh) = f(e)$$

and since K is nonempty, $f(e) = 0$, and $h \in K$. We have established the following criteria for the elements of K

Proposition 3.1. *Let K be nonempty. $h \in K$ if, and only if, $f(gh) = f(g)$ for every $g \in G$.*

Evidently we also have the same statement regarding $f(hg) = f(g)$. A condition of normality of K is available:

Proposition 3.2. *Let K be nonempty. K is normal on G if, and only if, for every $g, \bar{g} \in G$, $h \in K$ we have $f(gh\bar{g}) = f(g\bar{g})$*

Proof. If K is normal on G we have $ghg^{-1} \in K$ for $g \in G, h \in K$. So $gh = h'g$ and $f(gh\bar{g}) = f(h'g\bar{g}) = f(g\bar{g})$ where in the last identity we used proposition (3.1). On the other hand, if the condition is fulfilled then for $\bar{g} = g^{-1}$, we have $f(ghg^{-1}) = f(gg^{-1}) = f(e)$. But $f(e) = 0$ since K is nonempty. In this way we see that $ghg^{-1} \in K$ and K is normal on G . \square

Now suppose we have two elements g, h such that $f(gh) = 0$. What can be said about $f(g)$ or $f(h)$? To answer this question, just notice that

$$f(g) = f(ghh^{-1}) \leq f(gh) + f(h) = f(h)$$

$$f(h) = f(g^{-1}gh) \leq f(g) + f(gh) = f(g)$$

Hence, we have the following

Proposition 3.3. *If $gh \in K$ then $f(g) = f(h)$.*

This result is obvious for $g, h \in K$. The interesting point is that it works for all $g, h \in G$, and in particular for all $g, h \in G \setminus K$. For concreteness let $a, b \in G$ such that $ab \in K$ and neither a and b belongs to K . Let $A = \{a, b, a^{-1}, b^{-1}\}$. The previous result can be reformulated as

Proposition 3.4. *Let $a, b \in G \setminus K$ such that $ab \in K$, and A as above. We have $f(A) = \{f(a)\}$*

In the same vein of reasoning we can ask what are the possible images of $f(xy)$ for $x, y \in A$. We have

Proposition 3.5. *Let $a, b \in G \setminus K$ such that $ab \in K$, and A as above. We have*

$$f(A^2) \subset \{0, f(a^2), f(b^2), f(ba)\}.$$

Proof. Enumerating all possible cases we have

Case 1, ($x = a$):

$$\{f(ay) : y \in A\} = \{f(aa), f(aa^{-1}), f(ab), f(ab^{-1})\} = \{f(a^2), f(e), 0, f(ab^{-1})\}$$

Using proposition (3.1)

$$f(ab^{-1}) = f(abb^{-1}b^{-1}) = f(b^{-2}) = f(b^2)$$

and we have

$$\{f(ay) : y \in A\} = \{0, f(a^2), f(b^2)\}$$

Case 2, ($x = a^{-1}$):

$$\{f(a^{-1}y) : y \in A\} = \{f(e), f(a^{-1}a^{-1}), f(a^{-1}b), f(a^{-1}b^{-1})\}$$

Again, by proposition (3.1)

$$f(a^{-1}b) = f(a^{-1}a^{-1}ab) = f(a^2) \text{ and } f(a^{-1}b^{-1}) = f(ba).$$

In this case we have

$$\{f(a^{-1}y) : y \in A\} = \{0, f(a^2), f(ba)\}$$

Case 3, ($x = b$):

$$\{f(by) : y \in A\} = \{f(ba), f(ba^{-1}), f(b^2), 0\} = \{f(ba), f(ab^{-1}), f(b^2), 0\} = \{f(ba), f(b^2), 0\}$$

Case 4, ($x = b^{-1}$):

$$\{f(b^{-1}y) : y \in A\} = \{f(b^{-1}a), f(b^{-1}a^{-1}), 0, f(b^{-2})\}.$$

Since $f(b^{-1}a) = f(a^{-1}b) = f(a^2)$, we have

$$\{f(b^{-1}y) : y \in A\} = \{f(a^2), f(ab), f(b^2), 0\}.$$

Summing up we have the result. □

If a and b are as in the former proposition but in addition, $f(ba) = 0$, the values that $f(xy)$ can attain are even more restricted. In this case we have

$$f(b^2) = f(bb) = f(ba^{-1}ab) = f(ba^{-1}) = f(baa^{-1}a^{-1}) = f(a^2).$$

This is resumed as

Corollary 3.6. *Let $a, b \in G \setminus K$ such that $ab \in K$, and A as above. Assume $ba \in K$. We have*

$$f(A^2) \subset \{0, f(a^2)\}.$$

We note that if we have $a, b \in G \setminus K$ such that $ab \in K$, and if a and b commute we have $f(ab) = f(ba)$. Another situation where this can also happen is when K is a normal subgroup of G . Just note that $ab \in K \triangleleft G$ means that for all $g \in G$ $gabg^{-1} = h' \in K$. Let $g = a^{-1}$. We have $f(a^{-1}aba) = f(h') = 0$, that is $f(ba) = 0$.

Corollary 3.7. *Let $a, b \in G \setminus K$ such that $ab \in K$, A as above and $K \triangleleft G$. We have*

$$f(A^2) \subset \{0, f(a^2)\}$$

These type of results are generalizable. One typical result is given by the next reasoning. Assume that $a, b, c \in G \setminus K$ are such that $f(abc) = 0$. We have

$$f(a) = f(abcc^{-1}b^{-1}) = f(c^{-1}b^{-1}) \leq f(b) + f(c)$$

that is

$$f(a) - f(c) \leq f(b).$$

We also have

$$f(c) = f(b^{-1}a^{-1}abc) = f(b^{-1}a^{-1}) \leq f(a) + f(b),$$

which implies

$$f(c) - f(a) \leq f(b).$$

Hence, we have proved the following:

Proposition 3.8. *Let $a, b, c \in G \setminus K$ such that $abc \in K$. We have*

$$|f(a) - f(c)| \leq f(b).$$

4. A dynamical construction

From now on we restrict to infinite, finitely generated groups G , endowed with a probability measure μ . Denote by e the identity of G . Let $S = \{s_1, \dots, s_p\} \subset G$ be a finite set of generators. The word length of g is defined to be

$$l(g) = \min\{n \in \mathbb{N} : g = s_1 \cdots s_n, s_i \in S\}.$$

The set of generators is called symmetric if $X^{-1} = X$. In that case the word length is generated in the usual way by the distance associated with the Cayley graph of G and satisfy the usual properties:

$$l(gh) \leq l(g) + l(h). \tag{4}$$

$$l(g^{-1}) = l(g). \tag{5}$$

Definition 4.1. We say that μ has finite first moment if

$$\sum_{g \in G} l(g)\mu(g) < +\infty$$

Given probability measures μ and ν on G , define the convolution measure as

$$\begin{aligned} \mu * \nu(g) &= \sum_{h \in G} \mu(h)h\nu(g) \\ &= \sum_{h \in G} \mu(h)\nu(h^{-1}g) \end{aligned}$$

where $h\nu(g) = \nu(h^{-1}g)$. Denote by μ^{*n} the n -fold convolution.

Using the word length define a sequence L_n as

$$L_n = \sum_{h \in G} l(h)\mu^{*n}(h). \tag{6}$$

The sequence L_n is subadditive, $L_{n+m} \leq L_n + L_m$. Hence, the classical Fekete lemma ensures that the limit

$$\ell_\mu = \lim_{n \rightarrow +\infty} \frac{L_n}{n}$$

exists in $\mathbb{R} \cup \{-\infty, +\infty\}$. Since $l \geq 0$ and L_n is subadditive we have $L_n \leq nL_1$ and since μ has finite first moment it follows that $\ell_\mu < +\infty$.

It will be useful to define the following map

$$K_g^n(h) = g\mu^{*n}(h) - \mu^{*n}(h) \tag{7}$$

which play the role of a kernel and verifies

$$\sum_{g \in G} K_g^n(h)\mu(g) = \mu^{*(n+1)}(h) - \mu^{*n}(h). \tag{8}$$

Now, given an abstract length function f , we may still define in analogy with (6)

$$F_n = \sum_{h \in G} f(h)\mu^{*n}(h). \tag{9}$$

We note that F_n verifies $F_{n+m} \leq F_n + F_m$, hence we have $F_n \leq nF_1$. In view of this, for an abstract length function f in G , we consider a measure μ such that

$$\sum_{g \in G} f(g)\mu(g) < +\infty$$

Therefore, we have $F_1 < +\infty$.

Following the construction of Erschler-Karlssoon [5], we will define the function $T_n(g)$ as follows:

$$\begin{aligned} T_n(g) &= \sum_{h \in G} (f(gh) - f(h))\mu^{*n}(h) \\ &= \sum_{h \in G} f(h)K_g^n(h). \end{aligned} \tag{10}$$

The function T_n verifies the following properties

Lemma 4.2. *Let $T_n(g)$ be defined as above. Then*

- (i) $T_n(e) = 0$
- (ii) $|T_n(g)| \leq f(g)$
- (iii) $\sum_{g \in G} T_n(g)\mu(g) = F_{n+1} - F_n$.

Proof. The proof is a simple calculation. □

$T_n(g)$ is not a homomorphism, but it will be fundamental to understand how far it is from being one. The next result bounds this difference. For the sake of readability we will sometimes use the following notation

$$\text{Def}_{T_n}(g_1, g_2) = T_n(g_1g_2) - T_n(g_1) - T_n(g_2). \tag{11}$$

Lemma 4.3. *We have*

$$|\text{Def}_{T_n}(g_1, g_2)| \leq f(g_1) \sum_{h \in G} |K_{g_2}^n(h)|. \tag{12}$$

Note that, since

$$\sum_{h \in G} |K_g^n(h)| \leq 2$$

we can consider a sequence $\beta(n)$ such that for every $g \in G$ there exists $C(g)$ with

$$\sum_{h \in G} |K_g^n(h)| \leq C(g)\beta(n).$$

In fact, the above constant $C(g)$ can be made more explicit.

Lemma 4.4. *If $\sum_{h \in G} |K_g^n(h)| \leq C(g)\beta(n)$ then there exists a constant C_o such that*

$$\sum_{h \in G} |K_g^n(h)| \leq l(g)C_o\beta(n).$$

See [3] for the proof of lemmas (4.3) and (4.4). It is convenient to introduce the following notation:

$$\gamma(n) = \max_{s \in S} |T_n(s)|.$$

We have the following

Proposition 4.5. *Let $T_n(g)$ be defined as above, μ a probability measure on G and $g \in \langle \text{supp } \mu \rangle$. We have*

$$|T_n(g)| \leq C_1\beta(n)l^2(g) + l(g)\gamma(n). \tag{13}$$

Proof. Let $g \in \langle \text{supp } \mu \rangle \subset G$. Since G is generated by the finite set $S = \{s_1, \dots, s_p\}$ we can write $g = s_1 \cdots s_m$ where $m = l(g)$. By the previous lemmas we have

$$\begin{aligned} |\text{Def}_{T_n}(s_1 \cdots s_{m-1}, s_m)| &\leq f(s_1 \cdots s_{m-1}) \sum_{h \in G} |K_{s_m}^n(h)| \\ &\leq \left(\sum_{i=1}^{m-1} f(s_i) \right) \sum_{h \in G} |K_{s_m}^n(h)| \leq \left(\sum_{i=1}^{m-1} f(s_i) \right) C_o\beta(n). \end{aligned}$$

Writing $f_o = \max_{s \in S} f(s)$, we get

$$|\text{Def}_{T_n}(s_1 \cdots s_{m-1}, s_m)| \leq (m-1)f_oC_o\beta(n)$$

Therefore,

$$\begin{aligned} |T_n(g)| &= |T_n(s_1 \cdots s_m)| \\ &= |T_n(s_1 \cdots s_m) - T_n(s_1 \cdots s_{m-1}) - T_n(s_m) + \\ &\quad + T_n(s_1 \cdots s_{m-1}) - T_n(s_1 \cdots s_{m-2}) - T_n(s_{m-1}) + \cdots \\ &\quad \cdots + T_n(s_1 s_2) - T_n(s_1) - T_n(s_2) + \sum_{i=1}^m T_n(s_i)| \\ &\leq f_oC_o\beta(n)((m-1) + (m-2) + \cdots + 1) + \sum_{i=1}^m |T_n(s_i)| \\ &\leq f_oC_o\beta(n)((m-1) + (m-2) + \cdots + 1) + m\gamma(n) \\ &\leq f_oC_o\beta(n)m^2 + m\gamma(n). \end{aligned}$$

□

Denote $\Delta_n L = L_{n+1} - L_n$ and $\Delta_n F = F_{n+1} - F_n$. We may now state and prove the main result of this section.

Theorem 4.6. *Suppose that μ is non-degenerate, has finite second moment and for some sequence n_k it holds that $\Delta_{n_k} F \geq \Delta_{n_k} L > 0$ and*

$$\lim_{k \rightarrow +\infty} \frac{\beta(n_k)}{\Delta_{n_k} L} = 0.$$

Then G admits a non-trivial homomorphism to \mathbb{R} .

Proof. By proposition (4.5) we have $|T_n(g)| \leq C_1\beta(n)l^2(g) + l(g)\gamma(n)$ so

$$\begin{aligned} \sum_{g \in G} |T_n(g)|\mu(g) &\leq C_1\beta(n) \sum_{g \in G} l^2(g)\mu(g) + \gamma(n) \sum_{g \in G} l(g)\mu(g) \\ &= C_2\beta(n) + \gamma(n)C_3. \end{aligned}$$

Using the hypotheses of this proposition and lemma (4.2) we have

$$0 < \Delta_{n_k} L \leq \Delta_{n_k} F = \sum_{g \in G} T_{n_k}(g)\mu(g) = \left| \sum_{g \in G} T_{n_k}(g)\mu(g) \right|$$

$$\leq \sum_{g \in G} |T_{n_k}(g)| \mu(g) \leq C_2 \beta(n_k) + \gamma(n_k) C_3$$

Again using the hypotheses we have that for any $\varepsilon > 0$ there exists n_k such that $\beta(n_k) \leq \varepsilon \Delta_{n_k} L$ and we have

$$0 < \Delta_{n_k} L \leq C_2 \varepsilon \Delta_{n_k} L + \gamma(n_k) C_3.$$

Choosing ε such that $C_2 \varepsilon < 1/2$ we know that for n_k big enough we have

$$0 < (1/2) \Delta_{n_k} L \leq (1 - C_2 \varepsilon) \Delta_{n_k} L \leq \gamma(n_k) C_3. \tag{14}$$

From this inequality we can draw two conclusions. First we see that for n_k sufficiently big we have

$$0 < \gamma(n_k)$$

which allows us to define the coefficient

$$\alpha(n_k) = \frac{1}{\gamma(n_k)}$$

and the second conclusion is

$$0 < \alpha(n_k) \Delta_{n_k} L \leq 2C_3.$$

So we see that

$$0 \leq \alpha(n_k) \beta(n_k) = \alpha(n_k) \Delta_{n_k} L \frac{\beta(n_k)}{\Delta_{n_k} L} \leq 2C_3 \frac{\beta(n_k)}{\Delta_{n_k} L}$$

and using our convergence hypothesis we conclude that

$$\alpha(n_k) \beta(n_k) \rightarrow 0$$

Let us define

$$T_n^\alpha(g) = \alpha(n) T_n(g), \quad \text{and} \quad T(g) = \lim_{k \rightarrow +\infty} T_{n_k}^\alpha(g).$$

Using lemma (4.3) we can say that

$$|\text{Def}_{T_{n_k}^\alpha}(g_1, g_2)| \leq f(g_1) C(g_2) \beta(n_k) \alpha(n_k)$$

and

$$\lim_{k \rightarrow +\infty} \text{Def}_{T_{n_k}^\alpha}(g_1, g_2) = 0$$

which amounts to say that

$$T(g_1 g_2) = T(g_1) + T(g_2).$$

□

Remark 4.7. Let $h \in K_f$, i.e, $f(h) = 0$. By proposition 3.1, $f(gh) = f(g)$ for every $g \in G$ and so $T_n(g) = 0$, for every n . Hence, $K_f \subset \text{Ker} T$. It would be interesting to know under what conditions we have that $K_f = \text{Ker} T$.

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