Do the SPX and VIX co-jump?

Claudio Alberto Salinas Tejerina

MSc in Mathematical Finance

Dissertation supervised by:

Professor Ph.D João Pedro Vidal Nunes, Full Professor,

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DEPARTMENT OF MATHEMATICS

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Finally, I would like to dedicate this work to my grandmother, who is no longer with us, but who lives in my heart and gives me strength beyond measure.

Resumo

O objetivo desta tese é buscar evidências em saltos e volatilidade estocástica em condições do mercado em baixa, nomeadamente quebras e correções de mercado de 1990 a 2020. Usamos os índices SPX e VIX para analisar os saltos nos retornos e a volatilidade estocástica no mercado, respectivamente. Para detectar saltos, aplicamos o teste não paramétrico de Lee and Mykland (2008) para identificar os tempos de chegada dos saltos e os tamanhos dos saltos realizados. Primeiro, avaliamos o desempenho do teste usando dados de baixa e de alta frequência, comparando os saltos detectados com um banco de dados de referência para o período de estudo. Em seguida, avaliamos os resultados dos testes com o objetivo de responder às seguintes questões: A volatilidade aumenta exatamente quando o mercado está em baixa? Os índices SPX e VIX saltam simultaneamente quando o mercado está em baixa? Os índices SPX e VIX saltam em direções opostas quando o mercado está em baixa? Historicamente, qual dos índices salta com mais frequência? Os resultados revelam várias descobertas sobre a relação entre os retornos do SPX e as mudanças no VIX. Quando há um evento relacionado ao mercado em baixa, os índices saltam em direções opostas. O VIX, em nosso estudo de alta frequência, salta em todas as datas de referência, o que nos leva a concluir que a volatilidade aumenta exatamente quando há uma quebra ou correção de mercado.

Palavras-chave: Teste de salto não paramétrico, Salto simultâneo, Dinâmica de salto, SPX & VIX

Abstract

The objective of this thesis is to look for evidence in jumps in returns and stochastic volatility in bear market conditions, namely market crashes and market corrections from 1990 to 2020. We use the SPX and VIX indexes to analyze the jumps in returns and stochastic volatility in the market, respectively. To detect jumps, we apply the nonparametric test of Lee and Mykland (2008) to identify jump arrival times and realized jump sizes. First, we assess the performance of the test using low and high-frequency data, comparing the detected jumps with a benchmark database for the study period. Then, we assess the outcomes of the tests aiming to answer the following questions: Does volatility spikes exactly when there is a stock market crash or correction? Do the SPX and VIX indexes co-jump when there is a stock market crash or correction? Do the SPX and VIX indexes jump in opposite directions when there is a stock market crash or correction? Historically, which of the indexes jump more frequently? The results reveal several findings regarding the relationship between the SPX returns and the changes in the VIX. In a nutshell, we conclude that the SPX tends to jump more frequently than the VIX. When there is an event related to a market crash or market correction, the indices co-jump in opposite directions. The VIX, in our high-frequency study, jumped in all benchmark dates, which leads us to conclude that volatility spikes exactly when there is a market crash or correction.

Keywords: Nonparametric jump test, Co-jump, Jump dynamics, SPX & VIX

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1. Introduction

1.1. Motivation

Since 1900 many scientists have been studying the theory of option pricing. Bachelier (1900) was the first to deduce an option pricing formula based on the assumption that stock prices follow a Brownian motion with zero drift. A breakthrough in option pricing theory appeared in Black and Scholes (1973). This classic article presents an analysis based only on observable variables. The formulas do not require investors' tastes or beliefs on the expected returns from the underlying stock prices. To derive the option pricing formula, Black and Scholes assumed ideal market conditions for the stock and option to avoid creating arbitrage possibilities. The ideal market conditions described in Black and Scholes (1973) are given below:

1. Frictionless market;
2. The short-term interest rate is known and constant through time;
3. The stock pays no dividends during the life of the option;
4. The option can only be exercised at the expiration date (European-style option);
5. The stock price follows a geometric Brownian motion through time, producing a log-normal distribution for the stock price between any two points in time.

Merton (1973) demonstrates that the analysis proposed by Black and Scholes is obtained even when the interest rate is stochastic, the share pays dividends, and the option can be exercised before maturity. In addition, Merton shows that, as long as the share price dynamics can be described by a continuous diffusion process, whose sample path is continuous with probability one, such arbitrage technique is still valid. It would be exaggerated to say that the Black-Scholes analysis is invalid because continuous trading is impossible and no empirical time series has a continuous sample path. In fact, Merton and Samuelson (1974) prove that the continuous-trading solution will be a valid asymptotic approximation to the discrete-trading reality, as long as the dynamics have continuous sample-paths. In essence, the validity of the Black-Scholes formula depends on whether the stock price changes or not by a small amount.

Since that time, there has been a great effort in modeling option prices, where each new model attempts to explore some of the restrictive premises of the Black-Scholes model. Efforts include (i) the stochastic-interest-rate option models; (ii) the jump-diffusion/pure jump models; (iii) the local volatility models; (iv) the stochastic-volatility models; (v) the stochastic-volatility and stochastic-interest-rates models; and (vi) the stochastic-volatility jump-diffusion models. For examples, please see the work of Bakshi et al. (1997).

Pricing an option relies on complex mathematical formulas. Still, the common ingredients of each of the models are i) the underlying price process, ii) the interest-rate process, and iii) the variance of the asset returns process. The underlying price may follow a process in either continuous or discrete time. Examples of continuous-time processes are the pure diffusion process, a Poisson jump process, a combination of jump and diffusion components with or without stochastic volatility, and with or without random jumps, among many other possibilities. For the interest-rate and variance processes, the choices are similar.

Merton (1976) was the first to explore the continuous-time stock price motion as a discontinuous (so-called jumps) stochastic process defined in continuous time and derived an option pricing formula where the underlying stock returns are generated by a combination of both diffusion and jump processes. In a nutshell, this process allows a stock price change of significant magnitude, no matter how small the time interval between two observations can be. Since the work of Merton, several empirical and theoretical studies proved the existence of these discontinuities and pointed out that if models fail to incorporate jump characteristics, the result will be mispricing —see, for example, the work of Jorion (1988), Bakshi et al. (1997) and Das and Sundaram (1999). As a result, authors started to incorporate jump processes in stock and/or option market analysis —see, for instance, the work of Chan and Maheu (2002) and Eraker et al. (2003)— and studied the impact on option pricing, bond pricing, risk management, and hedging —see Bakshi et al. (1997) and Lee and Mykland (2008).

More recently, several studies have focused on looking for evidence for jumps in returns and stochastic volatility. Since the stochastic volatility is not directly observable, the observable VIX index is used instead and its jump characteristics are compared to the returns of the SPX index. Wagner and Szimayer (2004) investigated the implied volatility index's jump characteristics, concluding that evidence of significant positive jumps exists. Dotsis et al. (2007) explored continuous-time diffusion and jump-diffusion processes to capture the dynamics of implied volatility indices over time; the authors found that the addition of jumps is necessary to capture the evolution of the implied volatility index. Becker et al. (2009) also discuss jumps in the VIX. Their findings indicate that the VIX reflects past jump activity in the SPX index. Lin and Lee (2010) analyzed the discontinuous jump and the time-varying correlated jump intensity for the changes in the VIX and the SPX returns. Their results provide evidence of a strong negative relationship between the SPX returns and the changes in the VIX volatility index.

One of the common findings among recent studies is that jumps are empirically difficult to detect since only discrete data are available for models that assume continuous-time trading. Correspondingly, a variety of jump tests have been developed in recent decades —for a brief revision, see Jiang and Oomen (2008). An approach that has recently gained momentum is the study of nonparametric tests for the presence of price jumps using high-frequency data —see, for example, the work of Barndorff-Nielsen and Shephard (2006), Lee and Mykland (2008), Jiang and Oomen (2008), Aït-Sahalia and Jacod (2009), Jacod and Todorov (2009) and Dumitru and Urga (2012). Such methods are straightforward to apply, as they require high-frequency transaction prices only. Furthermore, they are developed in a model-free framework, incorporating different classes of stochastic volatility models.

As seen in the brief discussion above, great effort has been put into enhancing pricing models in the stock and option markets. The inclusion and detection of price jumps have become a necessity and are no longer a model choice. In this sense, this work aims to further look for evidence for jumps in returns and stochastic volatility, focusing on historically bear market conditions, namely market crashes and corrections. We will use a nonparametric methodology to assess the jump dynamics of the SPX and VIX indexes from 1990 to 2020. Most of the aforementioned studies focused on detecting jumps in specific short periods; hence our contribution is to study a broader time span.

1.2. Objective and methodology

This thesis aims to look for evidence for jumps in returns and stochastic volatility in bear market conditions, namely market crashes and market corrections from 1990 to 2020. We use the SPX and VIX indexes to analyze the jumps in returns and stochastic volatility in the market, respectively. In order to detect jumps, we apply the nonparametric test of Lee and Mykland (2008), hereafter the "LM test", to detect jump arrival times and realized jump sizes. There is evidence that this test outperforms other nonparametric tests —see Lee and Mykland (2008) and Dumitru and Urga (2012)— such as the linear test of Barndorff-Nielsen and Shephard (2006) and the difference test of Jiang and Oomen (2008). Moreover, Todorov (2010) used the LM test to prove the persistent effect of jumps on the variance risk-premium, finding evidence that the SPX and VIX indexes co-jumped between 1990 and 2002.

First, we assess the performance of the LM test using low-frequency (one-day data) and high-frequency intraday data (one-hour, 30-minute, and 5-minute data), comparing the detected jumps with a benchmark database where market crashes and corrections dates are listed for the time span of the study. Even though nonparametric tests perform better when high-frequency data is used, we decided to include low-frequency data as it is free and easily obtained.

Then, we assess the outcomes of the tests aiming to answer the following questions:

- Does volatility spikes exactly when there is a stock market crash or correction?

- Do the SPX and VIX indexes co-jump when there is a stock market crash or correction?

- Do the SPX and VIX indexes jump in opposite directions when there is a stock market crash or correction?

- Historically, which of the indexes jump more frequently?

The thesis is structured as follows. Section 2 derives the LM test, setting up the theoretical framework and the test description. Section 3 discusses the data and the implementation of the LM test on the SPX and VIX indexes. Section 4 discusses the results obtained. Finally, the conclusions and recommendations for future work are given in Section 5.

2. The Lee and Mykland (2008) nonparametric test

The LM test takes a nonparametric approach aiming for the robustness of the results concerning the model specifications and the nonstationarity of the price process. The authors identify two main motivations for their study. First, to enhance the mapping of stochastic features of jump arrivals and their association to market information. Secondly, the improvement of derivative hedging as the presence of jumps can create incomplete markets, and the degree of market incompleteness build upon the jump structure — i.e., the size and intensity of jumps — is related to the magnitude of derivative hedging inaccuracy. The test has been built in a way that applies to any financial time series, including equity returns and volatility, interest rates, and exchange rates as long as high-frequency data are used. The test outputs the direction and size of the detected jumps making possible the characterization of the jump size distribution as well as the stochastic jump intensity.

2.1. The stochastic differential equations (SDE)

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, where $\mathbb{P}$ denotes the real world (or physical) probability measure. The continuously compounded return equals $d\ln S_t$ for $t \geq 0$, where S_t is the underlying asset price at time t. The underlying price is governed by the following SDE **when there are no jumps in the market**:

$$d\ln S_t = \mu(t)\,dt + \sigma(t)\,dW_t^{\mathbb{P}}, \tag{2.1}$$

where $\left\{W_t^{\mathbb{P}}(u) : 0 \leq u \leq t\right\}$ is an $\mathcal{F}_t$-adapted standard Brownian motion defined under $\mathbb{P}$. The drift $\mu(t)$ and spot volatility $\sigma(t)$ are $\mathcal{F}_t$-adapted processes, such that the underlying process is an Itô's process with continuous sample paths. On the other hand, **when there are jumps**, the underlying price is given by the following SDE:

$$d\ln S_t = \mu(t)\,dt + \sigma(t)\,dW_t^{\mathbb{P}} + Y(t)\,dJ_t^{\mathbb{P}}, \tag{2.2}$$

where $Y(t)$ is the random jump size and $J_t^{\mathbb{P}}$ is a counting predictable process defined under $\mathbb{P}$. The model assumes that the random components $W(t)$, $Y(t)$ and $J(t)$ are independent and the jump sizes $Y(t)$ are independent and identically distributed.

2.2. Test assumptions

Definition 1 *Following Protter (2004), for random vectors* $\{X_n\}_{n>0}$ *and non-negative random variables* $\{d_n\}_{n>0}$*, write* $X_n = O_p(d_n)$ *to mean: for each* $\delta > 0$*, there exists a finite constant* M_δ *such that* $\mathbb{P}(|X_n| > M_\delta d_n) < \delta$ *eventually.*

Assumption 1 *For any* $\epsilon > 0$ *and discrete times* $0 = t_0 < t_1 < \ldots < t_n = T$*,*

$$\sup_i \sup_{t_i \le u \le t_{i+1}} |\mu(u) - \mu(t_i)| = O_p\left(\Delta t^{\frac{1}{2}-\epsilon}\right), \tag{2.3}$$

where $\Delta t = t_i - t_{i-1}$ *for* $i = 1, \ldots, n$*.*

Assumption 2 *For any* $\epsilon > 0$ *and discrete times* $0 = t_0 < t_1 < \ldots < t_n = T$*,*

$$\sup_i \sup_{t_i \le u \le t_{i+1}} |\sigma(u) - \sigma(t_i)| = O_p\left(\Delta t^{\frac{1}{2}-\epsilon}\right), \tag{2.4}$$

where $\Delta t = t_i - t_{i-1}$ *for* $i = 1, \ldots, n$*.*

Note that the assumptions above consider equally spaced discrete times $\Delta t = t_i - t_{i-1}$. Notwithstanding, this simplified structure can be easily generalized to non-equidistant cases by letting $\max(t_i - t_{i-1}) \to 0$. Assumptions 1 and 2 can be interpreted as the drift and diffusion coefficients maintaining a stable variation over a short time period, respectively. Moreover, the assumptions allow the drift and diffusion to depend on the process itself.

2.3. Intuition for the jump test

The main question that the test seeks to answer is how to distinguish market jumps from the diffusive part of the pricing models. Suppose that a jump occurs at some time t_i; it is expected that the absolute realized return be considerably greater than the usual continuous change. However, it can also happen that the spot volatility at t_i is also high, even if there is no jump and, therefore, as we can only observe prices at discrete times, the realized return may be as high as the return expected by an actual jump in the market. In order to discriminate between these two scenarios, a common practice is to standardize the return by a measure that explains the local variation that corresponds to the continuous (diffusive) part of the process. In essence, the ratio of realized return to estimated volatility creates the test statistic for jumps. A commonly used nonparametric estimator for variance is the realized power quadratic variation:

Definition 2 *Consider a semimartingale* $(X_t)_{t \geq 0}$*, and let* $0 = t_0 < t_1 < \ldots < t_n = T$ *be any sequence of partitions of the time-interval* $[0,T]$*. The* ***power quadratic variation*** *of the stochastic process* $(X_t)_{t \geq 0}$ *until time* T *is equal to:*

$$\langle X, X \rangle (T) := \operatorname*{plim}_{n \to \infty} \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}} \right)^2, \tag{2.5}$$

as long as $sup_i \{t_i - t_{i-1}\} \longrightarrow 0$ *for* $n \longrightarrow \infty$*.*

However, this estimator is inconsistent as it fails to separate the contributions of the diffusion and jump components. Indeed, let r_{t_i} be the continuously compounded rate of return in the time interval $[t_{i-1}, t_i]$, this is:

$$r_{t_i} := \ln \frac{S_{t_i}}{S_{t_{i-1}}}. \tag{2.6}$$

Then, following Barndorff-Nielsen and Shephard (2004), the power quadratic variation is given by:

$$\langle r, r \rangle (t) = \int_0^t \sigma^2 (u) \, du + \sum_{i=1}^{J_t^{\mathbb{P}}} Y^2 (t_i). \tag{2.7}$$

A slightly modified alternative version, called the realized bipower variation has been proposed by Barndorff-Nielsen and Shephard (2004):

Definition 3 *The bipower variation process is given by:*

$$\{r, r\}^{[r,s]} (t) = \operatorname*{plim}_{\delta \to 0} \delta^{1-(r+s)/2} \sum_{i=2}^{[t/\delta]-1} |r_{t_i}| \left| r_{t_{i-1}} \right|, \quad r, s \geq 0, \tag{2.8}$$

with $r = s = 1$*, i.e.*

$$\{r, r\}^{[1,1]} (t) := \operatorname*{plim}_{n \to \infty} \sum_{i=2}^{n} |r_{t_i}| \left| r_{t_{i-1}} \right|. \tag{2.9}$$

The bipower quadratic variation process (2.9) is a consistent estimator of the (integrated) variance process:

Proposition 1 (Barndorff-Nielsen and Shephard (2004)) *For* $t > 0$*,*

$$\{r, r\}^{[1,1]} (t) = c^2 \int_0^t \sigma^2 (u) \, du, \tag{2.10}$$

with

$$c := \sqrt{\frac{2}{\pi}}. \tag{2.11}$$

Proof. (Barndorff-Nielsen and Shephard, 2004, Page 9 and Remark 3) show that

$$\{r,r\}^{[r,s]}(t) = \mu_r \mu_s \int_0^t \sigma^{r+s}(u)\, du, \tag{2.12}$$

with

$$\mu_r = 2^{r/2} \frac{\Gamma\left(\frac{1}{2}(r+1)\right)}{\Gamma\left(\frac{1}{2}\right)}, \tag{2.13}$$

and where $\Gamma(\cdot)$ represents the Euler gamma function as given in (Olver et al., 2010, Equation 5.2.1).

Making $r = s = 1$, it follows that

$$\{r,r\}^{[1,1]}(t) = \mu_1 \mu_1 \int_0^t \sigma^2(u)\, du, \tag{2.14}$$

with

$$\mu_1 := \sqrt{2} \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}, \tag{2.15}$$

Since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\Gamma(1) = 1$, then

$$\mu_1 = \sqrt{\frac{2}{\pi}} = c, \tag{2.16}$$

and equation (2.10) follows from equations (2.14) and (2.16). ■

Therefore, the realized bipower variation is a consistent estimator for the integrated volatility no matter how jumps are mixed with the diffusive part of pricing models.

2.4. The LM test statistic

Definition 4 *The statistic that tests, at time t_i, whether there was a jump from t_{i-1} to t_i is defined as*

$$L_\mu(t_i) := \frac{r_{t_i} - \hat{m}_i}{\hat{\sigma}_i}, \tag{2.17}$$

with

$$\hat{m}_i := \frac{1}{K-1} \sum_{j=i-K+1}^{i-1} r_{t_j}, \tag{2.18}$$

and

$$\hat{\sigma}_i^2 := \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left|r_{t_j}\right| \left|r_{t_{j-1}}\right|, \tag{2.19}$$

and where K is the estimation window size, $\hat{m}_i$ is the average rate of return (drift) in the window, and $\hat{\sigma}_i^2$ is the average bipower (quadratic) variation realized over the window.

The window size should be large enough to dilute the effect of jumps on estimating the instantaneous variance but small enough so that the variance can be assumed approximately constant over the window. Later, precise bounds will be given to define K.

2.5. Preliminary results

This subsection aims to introduce several propositions that will be used to study the asymptotic behaviour of the jump detection statistic.

Proposition 2 *For a window size equal to* $K = O_p(\Delta t^{\alpha})$, *with* $\alpha \in]-1,-0.5[$,

$$\sup_{i,t\leq t_i}\left|\int_{t_{i-K}}^{t}\left[\mu(u)-\mu(t_{i-K})\right]du\right| = O_p\left(\Delta t^{\frac{3}{2}+\alpha-\epsilon}\right). \tag{2.20}$$

Proof. Since $t_i - t_{i-K} = K\times\Delta t$, then

$$\int_{t_{i-K}}^{t_i}\left[\mu(u)-\mu(t_{i-K})\right]du = \int_{t_{i-K}}^{t_i}\mu(u)\,du - \mu(t_{i-K})\times K\times\Delta t, \tag{2.21}$$

for $t_{i-K}\leq t_i$. Given Assumption 1,

$$\int_{t_{i-K}}^{t_i}\left[\mu(u)-\mu(t_{i-K})\right]du = O_p\left(\Delta t^{\frac{1}{2}+-\epsilon+1}\right) - O_p\left(\Delta t^{\frac{1}{2}-\epsilon+\alpha+1}\right) = O_p\left(\Delta t^{\frac{3}{2}+\alpha-\epsilon}\right) \tag{2.22}$$

uniformly in all i. This implies equation (2.20). ■

Proposition 3 *For a window size equal to* $K = O_p(\Delta t^{\alpha})$, *with* $\alpha \in]-1,-0.5[$,

$$\sup_{i,t\leq t_i}\left|\int_{t_{i-K}}^{t}\left[\sigma(u)-\sigma(t_{i-K})\right]dW_u^{\mathbb{P}}\right| = O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right), \tag{2.23}$$

where $\delta \in \left]0,\frac{3}{2}+\alpha\right[$.

Proof. Let $\int_{t_{i-K}}^{t_i}\left[\sigma(u)-\sigma(t_{i-K})\right]dW_u^{\mathbb{P}} := X$, $\langle X,X\rangle_T$ be its quadratic variation and X_T^* be the maximum process, where T is a finite time.

Since X is an Itô's integral, the quadratic variations is given by:

$$\langle X,X\rangle_{t_i} = \int_{t_{i-K}}^{t_i}\left[\sigma(u)-\sigma(t_{i-K})\right]^2 du. \tag{2.24}$$

Moreover, the maximum process is given by:

$$X_{t_i}^* = \sup_{i,t\leq t_i}\left|\int_{t_{i-K}}^{t}\left[\sigma(u)-\sigma(t_{i-K})\right]dW_u^{\mathbb{P}}\right|. \tag{2.25}$$

Since X is a local martingale, we can apply the Burkholder inequality —see Protter (2004):

$$\mathbb{E}_{\mathbb{P}}\left\{(X_T^*)^p\right\} \leq C_p\times\mathbb{E}_{\mathbb{P}}\left\{\langle X,X\rangle_T^{\frac{p}{2}}\right\}, \tag{2.26}$$

where $C_p = \left\{q^p\left(\frac{p(p-1)}{2}\right)\right\}^{\frac{p}{2}}$, with $\frac{1}{p}+\frac{1}{q}=1$. Therefore, applying equation (2.26) with $p=2$ and $T=t_i$:

$$\mathbb{E}_{\mathbb{P}}\left\{\left(X^*_{t_i}\right)^2\right\} \leq 4\times\mathbb{E}_{\mathbb{P}}\left\{\langle X,X\rangle_{t_i}\right\}, \tag{2.27}$$

and, hence,

$$\left\{\mathbb{E}_{\mathbb{P}}\left(X^*_{t_i}\right)\right\}^2 \leq \mathbb{E}_{\mathbb{P}}\left\{\left(X^*_{t_i}\right)^2\right\} \leq 4\times\mathbb{E}_{\mathbb{P}}\left\{\langle X,X\rangle_{t_i}\right\}, \tag{2.28}$$

using equations (2.24) and (2.25),

$$\left\{\mathbb{E}_{\mathbb{P}}\left[\sup_{i,t\leq t_i}\left|\int_{t_{i-K}}^{t}\left[\sigma(u)-\sigma(t_{i-K})\right]dW_u^{\mathbb{P}}\right|\right]\right\}^2 \leq 4\times\mathbb{E}_{\mathbb{P}}\left\{\int_{t_{i-K}}^{t_i}\left[\sigma(u)-\sigma(t_{i-K})\right]^2du\right\}. \tag{2.29}$$

Assumption 2 yields,

$$\int_{t_{i-K}}^{t_i}\sigma^2(u)\,du-2\sigma(t_{i-K})\int_{t_{i-K}}^{t_i}\sigma(u)\,du+\sigma^2(t_{i-K})\times K\times\Delta t = O_p\left(\Delta t^{2\left(\frac{1}{2}-\epsilon\right)+\alpha+1}\right), \tag{2.30}$$

and, hence,

$$\mathbb{E}_{\mathbb{P}}\left\{\int_{t_{i-K}}^{t_i}\left[\sigma(u)-\sigma(t_{i-K})\right]^2du\right\} = O_p\left(\Delta t^{2\left(\frac{1}{2}-\epsilon\right)+2\alpha+2}\right). \tag{2.31}$$

Therefore, and similarly to Lemma 1 in Mykland and Zhang (2006), there must exist a $\delta \in \left]0,\frac{3}{2}+\alpha\right[$ such that:

$$\sup_{i,t\leq t_i}\left|\int_{t_{i-K}}^{t}\left[\sigma(u)-\sigma(t_{i-K})\right]dW_u^{\mathbb{P}}\right| = O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \tag{2.32}$$

■

Proposition 4 *As $n\to\infty$ and for $\theta>0$,*

$$\frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}+O_p\left(\Delta t^{\theta}\right)\right|\left|r_{t_{i-1}}+O_p\left(\Delta t^{\theta}\right)\right| = \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}\right|\left|r_{t_{i-1}}\right|+O_p\left(\Delta t^{\theta}\right). \tag{2.33}$$

Proof. Since $\left|r_{t_i}+O_p\left(\Delta t^{\theta}\right)\right| \leq |r_{t_i}|+O_p\left(\Delta t^{\theta}\right)$ and $O_p\left(\Delta t^{\theta}\right)\geq 0$, then

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}+O_p\left(\Delta t^{\theta}\right)\right|\left|r_{t_{i-1}}+O_p\left(\Delta t^{\theta}\right)\right| \\
\leq\ &\frac{1}{n}\sum_{i=1}^{n}\left[|r_{t_i}|+O_p\left(\Delta t^{\theta}\right)\right]\left[\left|r_{t_{i-1}}\right|+O_p\left(\Delta t^{\theta}\right)\right] \\
=\ &\frac{1}{n}\sum_{i=1}^{n}\left[|r_{t_i}|\left|r_{t_{i-1}}\right|+O_p\left(\Delta t^{\theta}\right)\left(|r_{t_i}|+\left|r_{t_{i-1}}\right|\right)+\left(O_p\left(\Delta t^{\theta}\right)\right)^2\right].
\end{aligned} \tag{2.34}$$

Moreover,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|r_{t_i}| = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left|r_{t_{i-1}}\right| = \mathbb{E}\left[|r_t|\right] = O_p(1), \tag{2.35}$$

because the expectation is a constant, and, therefore, inequality (2.34) yields

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}+O_p\left(\Delta t^{\theta}\right)\right|\left|r_{t_{i-1}}+O_p\left(\Delta t^{\theta}\right)\right| \\
\leq\ & \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}\right|\left|r_{t_{i-1}}\right|+2O_p\left(\Delta t^{\theta}\right)\times O_p(1)+\left(O_p\left(\Delta t^{\theta}\right)\right)^2 \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}\right|\left|r_{t_{i-1}}\right|+O_p\left(\Delta t^{\theta}\right). \qquad (2.36)
\end{aligned}$$

Alternatively, and since $\left|r_{t_i}+O_p\left(\Delta t^{\theta}\right)\right| \geq \left|r_{t_i}\right|-O_p\left(\Delta t^{\theta}\right)$ and $O_p\left(\Delta t^{\theta}\right)\geq 0$, then

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}+O_p\left(\Delta t^{\theta}\right)\right|\left|r_{t_{i-1}}+O_p\left(\Delta t^{\theta}\right)\right| \\
\geq\ & \frac{1}{n}\sum_{i=1}^{n}\left[\left|r_{t_i}\right|-O_p\left(\Delta t^{\theta}\right)\right]\left[\left|r_{t_{i-1}}\right|-O_p\left(\Delta t^{\theta}\right)\right] \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left[\left|r_{t_i}\right|\left|r_{t_{i-1}}\right|-O_p\left(\Delta t^{\theta}\right)\left(\left|r_{t_i}\right|+\left|r_{t_{i-1}}\right|\right)+\left(O_p\left(\Delta t^{\theta}\right)\right)^2\right] \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}\right|\left|r_{t_{i-1}}\right|-2O_p\left(\Delta t^{\theta}\right)\times O_p(1)+\left(O_p\left(\Delta t^{\theta}\right)\right)^2 \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left|r_{t_i}\right|\left|r_{t_{i-1}}\right|+O_p\left(\Delta t^{\theta}\right). \qquad (2.37)
\end{aligned}$$

Combining inequalities (2.36) and (2.37), equation (2.33) arises. ■

Proposition 5 *For a sequence of independent and identically distributed (iid) random variables* $\{U_i\}_{i=1}^{n}$ *with a standard normal probability law,*

$$\frac{1}{n}\sum_{i=1}^{n}|U_i|\,|U_{i-1}| = c^2, \qquad (2.38)$$

where $c=\mathbb{E}_{\mathbb{P}}(|U_i|)$*, for all* $i\in\{1,2,\ldots,n\}$*, is defined in equation (2.11).*

Proof. Since

$$(|U_i|-c)(|U_{i-1}|-c) = |U_i|\,|U_{i-1}|-c(|U_i|+|U_{i-1}|)+c^2, \qquad (2.39)$$

and because $COV_{\mathbb{P}}(|U_i|,|U_{i-1}|)=0$, given that $|U_i|$ and $|U_{i-1}|$ are independent, then

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}|U_i|\,|U_{i-1}| &= \frac{1}{n}\sum_{i=1}^{n}(|U_i|-c)(|U_{i-1}|-c)+c\left(\frac{1}{n}\sum_{i=1}^{n}|U_i|+\frac{1}{n}\sum_{i=1}^{n}|U_{i-1}|\right)-c^2 \\
&= COV_{\mathbb{P}}(|U_i|,|U_{i-1}|)+c(\mathbb{E}_{\mathbb{P}}(|U_i|)+\mathbb{E}_{\mathbb{P}}(|U_i|))-c^2 \\
&= 0+2c^2-c^2=c^2. \qquad (2.40)
\end{aligned}$$

■

Proposition 6 *Let* $X = O_p\left(\Delta t^{\theta}\right)$ *and* $Y = O_p\left(\Delta t^{\beta}\right)$, *for* $\theta, \beta \in \mathbb{R}$. *Then,*

$$\frac{X + O_p(\Delta t^{\gamma})}{\sqrt{Y + O_p(\Delta t^{\gamma})}} = \frac{X}{\sqrt{Y}} + O_p\left(\Delta t^{\min\left(\theta+\gamma-\frac{3\beta}{2}, \gamma-\frac{\beta}{2}, 2\gamma-\frac{3\beta}{2}\right)}\right), \tag{2.41}$$

for $\gamma \in \mathbb{R}$ *and* $\theta, \beta < \gamma$.

Proof. Using the difference of squares formula,

$$\left\{[Y + O_p(\Delta t^{\gamma})]^{-\frac{1}{2}} - Y^{-\frac{1}{2}}\right\}\left\{[Y + O_p(\Delta t^{\gamma})]^{-\frac{1}{2}} + Y^{-\frac{1}{2}}\right\} = [Y + O_p(\Delta t^{\gamma})]^{-1} - Y^{-1},$$

$$[Y + O_p(\Delta t^{\gamma})]^{-1} - Y^{-1} = \frac{Y - Y - O_p(\Delta t^{\gamma})}{[Y + O_p(\Delta t^{\gamma})]\,Y}, \tag{2.42}$$

and, hence,

$$\begin{aligned}
[Y + O_p(\Delta t^{\gamma})]^{-\frac{1}{2}} &= Y^{-\frac{1}{2}} - \frac{O_p(\Delta t^{\gamma})}{[Y + O_p(\Delta t^{\gamma})]\,Y\left\{[Y + O_p(\Delta t^{\gamma})]^{-\frac{1}{2}} + Y^{-\frac{1}{2}}\right\}} \\
&= Y^{-\frac{1}{2}} \\
&\quad - \frac{O_p(\Delta t^{\gamma})}{O_p\left(\Delta t^{\min(\gamma,\beta)}\right) O_p\left(\Delta t^{\beta}\right)\left\{\left[O_p\left(\Delta t^{\min(\gamma,\beta)}\right)\right]^{-\frac{1}{2}} + O_p\left(\Delta t^{-\frac{\beta}{2}}\right)\right\}} \\
&= Y^{-\frac{1}{2}} - \frac{O_p(\Delta t^{\gamma})}{O_p\left(\Delta t^{2\beta}\right) O_p\left(\Delta t^{-\frac{\beta}{2}}\right)} \\
&= Y^{-\frac{1}{2}} - \frac{O_p(\Delta t^{\gamma})}{O_p\left(\Delta t^{\frac{3\beta}{2}}\right)}.
\end{aligned} \tag{2.43}$$

Using equation (2.43), then

$$\begin{aligned}
\frac{X + O_p(\Delta t^{\gamma})}{\sqrt{Y + O_p(\Delta t^{\gamma})}} &= [X + O_p(\Delta t^{\gamma})]\,[Y + O_p(\Delta t^{\gamma})]^{-\frac{1}{2}} \\
&= [X + O_p(\Delta t^{\gamma})]\left[Y^{-\frac{1}{2}} - O_p\left(\Delta t^{\gamma-\frac{3\beta}{2}}\right)\right] \\
&= \frac{X}{\sqrt{Y}} - XO_p\left(\Delta t^{\gamma-\frac{3\beta}{2}}\right) + O_p(\Delta t^{\gamma})\left[Y^{-\frac{1}{2}} - O_p\left(\Delta t^{\gamma-\frac{3\beta}{2}}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
\frac{X+O_p\left(\Delta t^{\gamma}\right)}{\sqrt{Y+O_p\left(\Delta t^{\gamma}\right)}} &= \frac{X}{\sqrt{Y}}-O_p\left(\Delta t^{\theta}\right)O_p\left(\Delta t^{\gamma-\frac{3\beta}{2}}\right) \\
&\quad +O_p\left(\Delta t^{\gamma}\right)\left[O_p\left(\Delta t^{-\frac{\beta}{2}}\right)-O_p\left(\Delta t^{\gamma-\frac{3\beta}{2}}\right)\right] \\
&= \frac{X}{\sqrt{Y}}-O_p\left(\Delta t^{\theta+\gamma-\frac{3\beta}{2}}\right)+O_p\left(\Delta t^{\gamma}\right)O_p\left(\Delta t^{\min\left(\gamma-\frac{\beta}{2},2\gamma-\frac{3\beta}{2}\right)}\right) \\
&= \frac{X}{\sqrt{Y}}+O_p\left(\Delta t^{\min\left(\theta+\gamma-\frac{3\beta}{2},\gamma-\frac{\beta}{2},2\gamma-\frac{3\beta}{2}\right)}\right). \qquad (2.44)
\end{aligned}
$$

■

2.6. Asymptotic behavior of the test statistic

2.6.1. Under the absence of jumps

The scope of this subsection is to describe the asymptotic behavior of the jump detection statistic, $L_\mu(t_i)$, when there is no jump at time t_i. The realized return from time t_{i-1} to t_i follows the diffusion part of equation (2.1) or equation (2.2).

Theorem 7 ((Lee and Mykland, 2008, Theorem 1.1)) *Under Assumptions 1 and 2, for a window size equal to $K=O_p(\Delta t^{\alpha})$, with $\alpha\in\,]-1,-0.5[$, and as $\Delta t\to 0$,*

$$\sup_{i\in A_n}\left|L_\mu(t_i)-\hat{L}_\mu(t_i)\right|=O_p\left(\Delta t^{\frac{1}{2}-\delta+\alpha-\epsilon}\right), \qquad (2.45)$$

where $\delta\in\left]0,\frac{3}{2}+\alpha\right[$, A_n is the subset of $i\in\{1,2,\ldots,n\}$ where there is no jump at time t_i,

$$\hat{L}_\mu(t_i)=\frac{U_i-\bar{U}_{i-1}}{c} \qquad (2.46)$$

yields the probability law of the test statistic (2.17) under the diffusion process (2.1), U_i is a standard normal random variable, $c=\mathbb{E}_{\mathbb{P}}(|U_i|)$, for all $i\in\{1,2,\ldots,n\}$, is defined in equation (2.11), and

$$\bar{U}_{i-1}:=\frac{1}{K-1}\sum_{j=i-K+1}^{i-1}U_j. \qquad (2.47)$$

Proof. Propositions 2 and 3 allow the diffusion process (2.1) to be approximated by the Itô process

$$d\ln S_t^i=\mu(t_{i-K})\,dt+\sigma(t_{i-K})\,dW_t^{\mathbb{P}}, \qquad (2.48)$$

for $t \in [t_{i-K}, t_i]$ because

$$\begin{aligned}
& \left| \ln \frac{S_t}{S_{t_{i-K}}} - \ln \frac{S_t^i}{S_{t_{i-K}}^i} \right| \\
= & \left| \int_{t_{i-K}}^{t} \mu(u)\, du + \int_{t_{i-K}}^{t} \sigma(u)\, dW_u^{\mathbb{P}} - \int_{t_{i-K}}^{t} \mu(t_{i-K})\, du - \int_{t_{i-K}}^{t} \sigma(t_{i-K})\, dW_u^{\mathbb{P}} \right| \\
= & \left| \int_{t_{i-K}}^{t} [\mu(u) - \mu(t_{i-K})]\, du + \int_{t_{i-K}}^{t} [\sigma(u) - \sigma(t_{i-K})]\, dW_u^{\mathbb{P}} \right| \\
= & \left| \int_{t_{i-K}}^{t} [\mu(u) - \mu(t_{i-K})]\, du + \int_{t_{i-K}}^{t} [\sigma(u) - \sigma(t_{i-K})]\, dW_u^{\mathbb{P}} \right| \\
= & \left| O_p\left(\Delta t^{\frac{3}{2}+\alpha-\epsilon}\right) + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \right| = O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right).
\end{aligned} \tag{2.49}$$

Starting with the numerator of the test statistic (2.17), and using definitions (2.6) and (2.18), as well as equation (2.49), for all i and j such that $t_j \in [t_{i-K}, t_i]$,

$$\begin{aligned}
r_{t_j} - \hat{m}_j &= \ln \frac{S_{t_j}}{S_{t_{j-1}}} - \frac{1}{K-1} \sum_{l=j-K+1}^{j-1} \ln \frac{S_{t_l}}{S_{t_{l-1}}} \\
&= \ln \frac{S_{t_j}^i}{S_{t_{j-1}}^i} - \frac{1}{K-1} \sum_{l=j-K+1}^{j-1} \ln \frac{S_{t_l}^i}{S_{t_{l-1}}^i} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right),
\end{aligned}$$

and using equation (2.48),

$$\begin{aligned}
r_{t_j} - \hat{m}_j &= \int_{t_{j-1}}^{t_j} \mu(t_{i-K})\, du + \int_{t_{j-1}}^{t_j} \sigma(t_{i-K})\, dW_u^{\mathbb{P}} \\
&\quad - \frac{\sum_{l=j-K+1}^{j-1} \left[\int_{t_{l-1}}^{t_l} \mu(t_{i-K})\, du + \int_{t_{l-1}}^{t_l} \sigma(t_{i-K})\, dW_u^{\mathbb{P}}\right]}{K-1} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\
&= \mu(t_{i-K}) \times \Delta t + \sigma(t_{i-K}) \times \left(W_{t_j}^{\mathbb{P}} - W_{t_{j-1}}^{\mathbb{P}}\right) \\
&\quad - \frac{\sum_{l=j-K+1}^{j-1} \left[\mu(t_{i-K}) \times \Delta t + \sigma(t_{i-K}) \times \left(W_{t_l}^{\mathbb{P}} - W_{t_{l-1}}^{\mathbb{P}}\right)\right]}{K-1} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\
&= \mu(t_{i-K}) \times \Delta t + \sigma(t_{i-K}) \times \left(W_{t_j}^{\mathbb{P}} - W_{t_{j-1}}^{\mathbb{P}}\right) - \mu(t_{i-K}) \times \Delta t \\
&\quad - \frac{\sigma(t_{i-K}) \times \sum_{l=j-K+1}^{j-1} \left(W_{t_l}^{\mathbb{P}} - W_{t_{l-1}}^{\mathbb{P}}\right)}{K-1} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)
\end{aligned} \tag{2.50}$$

Since

$$U_i := \frac{W_{t_i}^{\mathbb{P}} - W_{t_{i-1}}^{\mathbb{P}}}{\sqrt{\Delta t}} \tag{2.51}$$

possesses a standard normal probability law, and using definition (2.47), then equation (2.50) can be restated as

$$\begin{aligned} r_{t_j} - \hat{m}_j &= \sqrt{\Delta t}\sigma\left(t_{i-K}\right)U_j - \frac{\sigma\left(t_{i-K}\right) \times \sum_{l=j-K+1}^{j-1} \sqrt{\Delta t} U_l}{K-1} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\ &= \sqrt{\Delta t}\sigma\left(t_{i-K}\right) \times \left(U_j - \frac{\sum_{l=j-K+1}^{j-1} U_l}{K-1}\right) + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\ &= \sqrt{\Delta t}\sigma\left(t_{i-K}\right) \times \left(U_j - \bar{U}_{j-1}\right) + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right). \end{aligned} \tag{2.52}$$

Concerning the denominator of the test statistic (2.17), and to cope with the drift of the diffusion processes (2.1) and (2.48), a measure change can be made. Girsanov's theorem implies that the stochastic differential equation (2.1) can be rewritten as

$$d \ln S_t = \sigma\left(t\right) dW_t^{\bar{\mathbb{P}}} \tag{2.53}$$

where

$$dW_t^{\bar{\mathbb{P}}} = \frac{\mu\left(t\right)}{\sigma\left(t\right)} dt + dW_t^{\mathbb{P}} \tag{2.54}$$

is a Brownian motion increment but under a new equivalent probability measure $\bar{\mathbb{P}}$, and equation (2.53) can be approximated by the Itô process

$$d \ln S_t^i = \sigma\left(t_{i-K}\right) dW_t^{\bar{\mathbb{P}}} \tag{2.55}$$

for $t \in [t_{i-K}, t_i]$. Therefore, equations (2.49) and (2.53) imply that

$$\begin{aligned} \hat{\sigma}_i^2 &= \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left|r_{t_j}\right| \left|r_{t_{j-1}}\right| \\ &= \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left|\ln \frac{S_{t_j}}{S_{t_{j-1}}}\right| \left|\ln \frac{S_{t_{j-1}}}{S_{t_{j-2}}}\right| \\ &= \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} \left|\ln \frac{S_{t_j}^i}{S_{t_{j-1}}^i} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\right| \\ &\quad \times \left|\ln \frac{S_{t_{j-1}}^i}{S_{t_{j-2}}^i} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\right|. \end{aligned} \tag{2.56}$$

Proposition 4 allows equation (2.56) to be restated as

$$\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\ln\frac{S_{t_j}^i}{S_{t_{j-1}}^i}\right|\left|\ln\frac{S_{t_{j-1}}^i}{S_{t_{j-2}}^i}\right| + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\\
&= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\int_{t_{j-1}}^{t_j}\sigma\left(t_{i-K}\right)dW_u^{\bar{\mathbb{P}}}\right|\left|\int_{t_{j-2}}^{t_{j-1}}\sigma\left(t_{i-K}\right)dW_u^{\bar{\mathbb{P}}}\right| + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\\
&= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\sigma\left(t_{i-K}\right)\times\left(W_{t_j}^{\bar{\mathbb{P}}}-W_{t_{j-1}}^{\bar{\mathbb{P}}}\right)\right|\left|\sigma\left(t_{i-K}\right)\times\left(W_{t_{j-1}}^{\bar{\mathbb{P}}}-W_{t_{j-2}}^{\bar{\mathbb{P}}}\right)\right|\\
&\quad + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\\
&= \frac{\sigma^2\left(t_{i-K}\right)}{K-2}\times\sum_{j=i-K+2}^{i-1}\left|\sqrt{\Delta t}U_j\right|\left|\sqrt{\Delta t}U_{j-1}\right| + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right),
\end{aligned} \tag{2.57}$$

where the last equality follows from equation (2.51). Finally, Proposition 5 allows equation (2.57) to be rewritten as

$$\begin{aligned}
\hat{\sigma}_i^2 &= \frac{\sigma^2\left(t_{i-K}\right)}{K-2}\times\Delta t\times\sum_{j=i-K+2}^{i-1}\left|U_j\right|\left|U_{j+1}\right| + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\\
&= \frac{\sigma^2\left(t_{i-K}\right)}{K-2}\times\Delta t\times\left[\left(K-2\right)c^2\right] + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\\
&= \sigma^2\left(t_{i-K}\right)\times\Delta t\times c^2 + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right).
\end{aligned} \tag{2.58}$$

Combining equations (2.17), (2.52) and (2.58), then

$$L_\mu\left(t_i\right) = \frac{\sqrt{\Delta t}\sigma\left(t_{i-K}\right)\times\left(U_i-\bar{U}_{i-1}\right) + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)}{\sqrt{\sigma^2\left(t_{i-K}\right)\times\Delta t\times c^2 + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)}}. \tag{2.59}$$

Since $\sigma\left(t_{i-K}\right)$ is a constant ,$\sigma\left(t_{i-K}\right) = O_p\left(1\right)$ (at time t_i) and defining $X := \sqrt{\Delta t}\sigma\left(t_{i-K}\right)\times\left(U_i-\bar{U}_{i-1}\right) = O_p\left(\Delta t^{\frac{1}{2}}\right)$ and $Y := \sigma^2\left(t_{i-K}\right)\times\Delta t\times c^2 = O_p\left(\Delta t\right)$, we can apply equation (2.41) to equation (2.59):

$$L_\mu\left(t_i\right) = \frac{\sqrt{\Delta t}\sigma\left(t_{i-K}\right)\times\left(U_i-\bar{U}_{i-1}\right)}{\sqrt{\sigma^2\left(t_{i-K}\right)\times\Delta t\times c^2}} + O_p\left(\Delta t^{\min\left(\theta+\gamma-\frac{3\beta}{2},\gamma-\frac{\beta}{2},2\gamma-\frac{3\beta}{2}\right)}\right), \tag{2.60}$$

where $\theta = \frac{1}{2}$, $\beta = 2\theta = 1$ and $\gamma = \frac{3}{2} - \delta + \alpha - \epsilon$. Hence,

$$\begin{aligned}
\min\left(\theta + \gamma - \frac{3\beta}{2}, \gamma - \frac{\beta}{2}, 2\gamma - \frac{3\beta}{2}\right) &= \min\left(\gamma - 2\theta, \gamma - \theta, 2\gamma - 3\theta\right) \\
&= \min\left(\gamma - 1, \gamma - \frac{1}{2}, 2\gamma - \frac{3}{2}\right) \\
&= \min\left(\gamma - 1, 2\gamma - \frac{3}{2}\right) \\
&= \gamma - 1 + \min\left(0, \gamma - \frac{1}{2}\right). \qquad (2.61)
\end{aligned}$$

Note that since $\delta \in \left]0, \frac{3}{2} + \alpha\right[$, then $(-\delta + \alpha) \in \left]-\frac{3}{2}, \alpha\right[$ and $\gamma \in \left]-\epsilon, \frac{3}{2} + \alpha - \epsilon\right[$. However, since $\alpha \in]-1, -0.5[$, then $\gamma > \frac{1}{2}$, and $\min\left(0, \gamma - \frac{1}{2}\right) = 0$. Consequently,

$$\min\left(\theta + \gamma - \frac{3\beta}{2}, \gamma - \frac{\beta}{2}, 2\gamma - \frac{3\beta}{2}\right) = \gamma - 1. \qquad (2.62)$$

Therefore,

$$L_{\mu}(t_i) = \frac{\left(U_i - \bar{U}_{i-1}\right)}{c} + O_p\left(\Delta t^{\frac{1}{2} - \delta + \alpha - \epsilon}\right), \qquad (2.63)$$

and equation (2.45) follows. ■

Remark 1 *Lee and Mykland (2008) find that* $\sup_{i \in A_n} \left| L_{\mu}(t_i) - \hat{L}_{\mu}(t_i) \right| = O_p\left(\Delta t^{\frac{3}{2} - \delta + \alpha - \epsilon}\right)$ *as long as* $\Delta t \to 0$, *whilst our finding is* $\sup_{i \in A_n} \left| L_{\mu}(t_i) - \hat{L}_{\mu}(t_i) \right| = O_p\left(\Delta t^{\frac{1}{2} - \delta + \alpha - \epsilon}\right)$ *as long as* $\Delta t \to 0$. *The difference in the findings appears when dealing with Equation (2.59).*

Theorem 7 states that the jump detection statistic $L_{\mu}(t_i)$ follows approximately the same distribution as $\hat{L}_{\mu}(t_i)$. In turn, $\hat{L}_{\mu}(t_i) \sim N\left(0, \frac{1}{c^2}\right)$ since U_i is a standard normal random variable.

2.6.2. Under the presence of jumps

This subsection describes the behavior of $L_{\mu}(t_i)$ when a jump arrives. The realized return from t_{i-1} to t_i follows the jump-diffusion process (2.2). Next theorem shows that as $\Delta t \to 0$, the test statistic becomes so large that the jump arrival at time t_i can be detected.

Theorem 8 *Let* $L_{\mu}(t_i)$ *be as defined in equation (2.17) and the window size be* $K = O_p(\Delta t^{\alpha})$, *with* $\alpha \in]-1, -0.5[$, *and as* $\Delta t \to 0$. *Suppose the price process follows equation (2.2) and that Assumptions 1 and 2 are satisfied. Moreover, suppose there is a jump at any time* $\tau \in (t_{i-1}, t_i]$. *Then,*

$$L_{\mu}(t_i) \simeq \frac{U_i - \bar{U}_{i-1}}{c} + \frac{Y(\tau)}{c\sigma\sqrt{\Delta t}} \mathbb{1}_{\tau \in (t_{i-1}, t_i)}, \qquad (2.64)$$

where $Y(\tau)$ is the actual jump size at the jump time τ. Therefore, $L_\mu(t_i) \to \infty$, as $\Delta t \to 0$. If there is no jump at any time $\tau \in (t_{i-1}, t_i]$, $L_\mu(t_i)$ has the asymptotic behavior described in Theorem 7.

Proof. Propositions 2 and 3 allow the price process (2.2) to be approximated by the Itô process

$$d \ln S_t^i = \mu(t_{i-K})\,dt + \sigma(t_{i-K})\,dW_t^{\mathbb{P}} + Y(t)\,dJ_t^{\mathbb{P}}, \tag{2.65}$$

for $t \in [t_{i-K}, t_i]$ because

$$\begin{aligned}
&\left|\ln \frac{S_t}{S_{t_{i-K}}} - \ln \frac{S_t^i}{S_{t_{i-K}}^i}\right| \\
= & \left|\int_{t_{i-K}}^t \mu(u)\,du + \int_{t_{i-K}}^t \sigma(u)\,dW_u^{\mathbb{P}} + \int_{t_{i-K}}^t Y(u)\,dJ_u^{\mathbb{P}}\right. \\
& \left. - \int_{t_{i-K}}^t \mu(t_{i-K})\,du - \int_{t_{i-K}}^t \sigma(t_{i-K})\,dW_u^{\mathbb{P}} - \int_{t_{i-K}}^t Y(u)\,dJ_u^{\mathbb{P}}\right| \\
= & \left|\int_{t_{i-K}}^t [\mu(u) - \mu(t_{i-K})]\,du + \int_{t_{i-K}}^t [\sigma(u) - \sigma(t_{i-K})]\,dW_u^{\mathbb{P}}\right| \\
= & \left|O_p\left(\Delta t^{\frac{3}{2}+\alpha-\epsilon}\right) + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\right| \\
= & O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right).
\end{aligned} \tag{2.66}$$

Starting with the numerator of the test statistic (2.17), and using definitions (2.6) and (2.18), as well as equation (2.66), for all i and j such that $t_j \in [t_{i-K}, t_i]$,

$$\begin{aligned}
r_{t_j} - \hat{m}_j &= \ln \frac{S_{t_j}}{S_{t_{j-1}}} - \frac{1}{K-1} \sum_{l=j-K+1}^{j-1} \ln \frac{S_{t_l}}{S_{t_{l-1}}} \\
&= \ln \frac{S_{t_j}^i}{S_{t_{j-1}}^i} - \frac{1}{K-1} \sum_{l=j-K+1}^{j-1} \ln \frac{S_{t_l}^i}{S_{t_{l-1}}^i} \\
&\quad + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right),
\end{aligned} \tag{2.67}$$

and using equation (2.65),

$$\begin{aligned}
r_{t_j} - \hat{m}_j = & \int_{t_{j-1}}^{t_j} \mu(t_{i-K})\,du + \int_{t_{j-1}}^{t_j} \sigma(t_{i-K})\,dW_u^{\mathbb{P}} + \int_{t_{i-1}}^t Y(u)\,dJ_u^{\mathbb{P}} \\
& - \frac{\sum_{l=j-K+1}^{j-1} \left[\int_{t_{l-1}}^{t_l} \mu(t_{i-K})\,du + \int_{t_{l-1}}^{t_l} \sigma(t_{i-K})\,dW_u^{\mathbb{P}} + \int_{t_{i-1}}^t Y(u)\,dJ_u^{\mathbb{P}}\right]}{K-1} \\
& + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right),
\end{aligned} \tag{2.68}$$

i.e.,

$$
\begin{aligned}
r_{t_j} - \hat{m}_j &= \mu(t_{i-K}) \times \Delta t + \sigma(t_{i-K}) \times \left(W^{\mathbb{P}}_{t_j} - W^{\mathbb{P}}_{t_{j-1}}\right) \\
&+ \sum_{m=1}^{J^{\mathbb{P}}_{t_j} - J^{\mathbb{P}}_{t_{j-1}}} Y(\tau_m) \times \mathbb{1}_{\{\tau_m \in]t_{j-1}, t_j]\}} \\
&- \frac{\sum_{l=j-K+1}^{j-1} \left[\begin{array}{c} \mu(t_{i-K}) \times \Delta t + \sigma(t_{i-K}) \times \left(W^{\mathbb{P}}_{t_l} - W^{\mathbb{P}}_{t_{l-1}}\right) \\ + \sum_{m=1}^{J^{\mathbb{P}}_{t_l} - J^{\mathbb{P}}_{t_{l-1}}} Y(\tau_m) \times \mathbb{1}_{\{\tau_m \in]t_{l-1}, t_l]\}} \end{array} \right]}{K-1} \\
&+ O_p\left(\Delta t^{\frac{3}{2} - \delta + \alpha - \epsilon}\right),
\end{aligned} \tag{2.69}
$$

or

$$
\begin{aligned}
r_{t_j} - \hat{m}_j &= \mu(t_{i-K}) \times \Delta t + \sigma(t_{i-K}) \times \left(W^{\mathbb{P}}_{t_j} - W^{\mathbb{P}}_{t_{j-1}}\right) \\
&+ \sum_{m=1}^{J^{\mathbb{P}}_{t_j} - J^{\mathbb{P}}_{t_{j-1}}} Y(\tau_m) \times \mathbb{1}_{\{\tau_m \in]t_{j-1}, t_j]\}} \\
&- \mu(t_{i-K}) \times \Delta t - \frac{\sigma(t_{i-K}) \times \sum_{l=j-K+1}^{j-1} \left(W^{\mathbb{P}}_{t_l} - W^{\mathbb{P}}_{t_{l-1}}\right)}{K-1} \\
&- \frac{\sum_{l=j-K+1}^{j-1} \sum_{m=1}^{J^{\mathbb{P}}_{t_l} - J^{\mathbb{P}}_{t_{l-1}}} Y(\tau_m) \times \mathbb{1}_{\{\tau_m \in]t_{l-1}, t_l]\}}}{K-1} \\
&+ O_p\left(\Delta t^{\frac{3}{2} - \delta + \alpha - \epsilon}\right).
\end{aligned} \tag{2.70}
$$

Using equations (2.47) and (2.51), and since $K \to \infty$ when $\Delta t \to 0$, equation (2.70) can be restated as

$$
r_{t_j} - \hat{m}_j \simeq \sqrt{\Delta t}\sigma(t_{i-K}) \times \left(U_j - \bar{U}_{j-1}\right) + Y(\tau) \mathbb{1}_{\tau \in (t_{i-1}, t_i)}. \tag{2.71}
$$

Concerning the denominator of the test statistic (2.17), and to cope with the drift of the diffusion processes (2.2) and (2.65), a measure change can be made. Girsanov's theorem implies that the stochastic differential equation (2.2) can be rewritten as

$$
d \ln S_t = \sigma(t) dW^{\bar{\mathbb{P}}}_t + \bar{Y}(t) dJ^{\bar{\mathbb{P}}}_t, \tag{2.72}
$$

which is still a jump-diffusion process but under a new equivalent probability measure $\bar{\mathbb{P}}$, and equation (2.72) can be approximated by the Itô process

$$
d \ln S^i_t = \sigma(t_{i-K}) dW^{\bar{\mathbb{P}}}_t + \bar{Y}(t) dJ^{\bar{\mathbb{P}}}_t \tag{2.73}
$$

for $t \in [t_{i-K}, t_i]$. Therefore, equation (2.66) implies that

$$
\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|r_{t_j}\right|\left|r_{t_{j-1}}\right| \\
&= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\ln\frac{S_{t_j}}{S_{t_{j-1}}}\right|\left|\ln\frac{S_{t_{j-1}}}{S_{t_{j-2}}}\right| \\
&= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\ln\frac{S^i_{t_j}}{S^i_{t_{j-1}}} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\right| \\
&\quad \times\left|\ln\frac{S^i_{t_{j-1}}}{S^i_{t_{j-2}}} + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right)\right|, \qquad (2.74)
\end{aligned}
$$

and Proposition 4 allows equation (2.74) to be restated as

$$
\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\ln\frac{S^i_{t_j}}{S^i_{t_{j-1}}}\right|\left|\ln\frac{S^i_{t_{j-1}}}{S^i_{t_{j-2}}}\right| + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\
&= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\int_{t_{j-1}}^{t_j}\sigma\left(t_{i-K}\right)dW_u^{\bar{\mathbb{P}}} + \int_{t_{j-1}}^{t_j}\bar{Y}\left(u\right)dJ_u^{\bar{\mathbb{P}}}\right| \\
&\quad \times\left|\int_{t_{j-2}}^{t_{j-1}}\sigma\left(t_{i-K}\right)dW_u^{\bar{\mathbb{P}}} + \int_{t_{j-2}}^{t_{j-1}}\bar{Y}\left(u\right)dJ_u^{\bar{\mathbb{P}}}\right| \\
&\quad + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right), \qquad (2.75)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}\left|\sigma\left(t_{i-K}\right)\times\left(W_{t_j}^{\bar{\mathbb{P}}} - W_{t_{j-1}}^{\bar{\mathbb{P}}}\right) + \bar{Y}\left(\tau_j\right)\times\mathbb{1}_{\tau_j\in\left(t_{j-1},t_j\right)}\right| \\
&\quad \times\left|\sigma\left(t_{i-K}\right)\times\left(W_{t_{j-1}}^{\bar{\mathbb{P}}} - W_{t_{j-2}}^{\bar{\mathbb{P}}}\right) + \bar{Y}\left(\tau_{j-1}\right)\times\mathbb{1}_{\tau_{j-1}\in\left(t_{j-2},t_{j-1}\right)}\right| \\
&\quad + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right), \qquad (2.76)
\end{aligned}
$$

or

$$
\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1}|\sigma\left(t_{i-K}\right)^2\times U_jU_{j+1}\Delta t \\
&\quad + \sigma\left(t_{i-K}\right)\times\bar{Y}\left(\tau_{j-1}\right)\times\mathbb{1}_{\tau_{j-1}\in\left(t_{j-2},t_{j-1}\right)}\times\left(W_{t_j}^{\bar{\mathbb{P}}} - W_{t_{j-1}}^{\bar{\mathbb{P}}}\right) \\
&\quad + \sigma\left(t_{i-K}\right)\times\bar{Y}\left(\tau_j\right)\times\mathbb{1}_{\tau_j\in\left(t_{j-1},t_j\right)}\times\left(W_{t_{j-1}}^{\bar{\mathbb{P}}} - W_{t_{j-2}}^{\bar{\mathbb{P}}}\right) \\
&\quad + \bar{Y}\left(\tau_j\right)\times\bar{Y}\left(\tau_{j-1}\right)\times\mathbb{1}_{\tau_j\in\left(t_{j-1},t_j\right),\tau_{j-1}\in\left(t_{j-2},t_{j-1}\right)}| \\
&\quad + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right). \qquad (2.77)
\end{aligned}
$$

Since $\sigma(t_{i-K}) \times \bar{Y}(\tau_{j-1}) \times \mathbb{1}_{\tau_{j-1}\in(t_{j-2},t_{j-1})} \times \left(W_{t_j}^{\bar{\mathbb{P}}} - W_{t_{j-1}}^{\bar{\mathbb{P}}}\right) = O_p(1) \times O_p(1) \times O_p(1) \times O_p\left(\Delta t^{\frac{1}{2}}\right) = O_p\left(\Delta t^{\frac{1}{2}}\right)$ and $\bar{Y}(\tau_j) \times \bar{Y}(\tau_{j-1}) \times \mathbb{1}_{\tau_j\in(t_{j-1},t_j),\tau_{j-1}\in(t_{j-2},t_{j-1})} = O_p(1) \times O_p(1) = O_p(1)$, equation (2.77) can be restated as

$$\begin{aligned}
\hat{\sigma}_i^2 &= \frac{1}{K-2}\sum_{j=i-K+2}^{i-1} \left|\sigma(t_{i-K})^2 \times U_j U_{j+1}\Delta t + O_p\left(\Delta t^{\frac{1}{2}}\right) + O_p\left(\Delta t^{\frac{1}{2}}\right) + O_p(1)\right| \\
&\quad + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\
&= \frac{\sigma^2(t_{i-K})}{K-2} \times \Delta t \times \sum_{j=i-K+2}^{i-1} |U_j||U_{j+1}| + O_p\left(\Delta t^{\frac{1}{2}}\right) + O_p\left(\Delta t^{\frac{3}{2}-\delta+\alpha-\epsilon}\right) \\
&= \frac{\sigma^2(t_{i-K})}{K-2} \times \Delta t \times \left[(K-2)c^2\right] + O_p\left(\Delta t^{\min\left(\frac{1}{2},\frac{3}{2}-\delta+\alpha-\epsilon\right)}\right) \\
&= \sigma^2(t_{i-K}) \times \Delta t \times c^2 + O_p\left(\Delta t^{\min\left(\frac{1}{2},\frac{3}{2}-\delta+\alpha-\epsilon\right)}\right).
\end{aligned} \tag{2.78}$$

Combining equations (2.17), (2.71) and (2.78), then

$$\begin{aligned}
L_\mu(t_i) &\simeq \frac{\sqrt{\Delta t}\sigma(t_{i-K}) \times \left(U_j - \bar{U}_{j-1}\right) + Y(\tau)\mathbb{1}_{\tau\in(t_{i-1},t_i)}}{\sqrt{\sigma^2(t_{i-K}) \times \Delta t \times c^2}} \\
&\simeq \frac{U_i - \bar{U}_{i-1}}{c} + \frac{Y(\tau)}{c\sigma\sqrt{\Delta t}}\mathbb{1}_{\tau\in(t_{i-1},t_i)}.
\end{aligned} \tag{2.79}$$

■

2.6.3. Rejection region of the test

As seen in Theorem 7 and Theorem 8, the test statistics present different limiting behavior depending on the existence of jumps at the testing times. If there is no jump at the testing time, the test follows approximately a normal distribution, whilst the test becomes very large if there is a jump. The main question is how large the test statistic can be when there is no jump? The answer to this question relies upon choosing a relevant threshold to distinguish the presence of jumps at a testing time. Therefore, it is reasonable to study the asymptotic distribution of maximums of the test statistics under the absence of jumps at any time in $(t_{-1}, t_i]$.

Lemma 9 *If the conditions for $L_\mu(t_i)$, K, c, and A_n are as in Theorem 7, then as $\Delta t \to 0$,*

$$\frac{\max_{i\in A_n} |L_\mu(t_i)| - C_n}{S_n} \to \xi, \tag{2.80}$$

where ξ has a cumulative distribution function $\mathbb{P}(\xi \le x) = \exp\left(-e^{-x}\right)$,

$$C_n = \frac{(2\log n)^{\frac{1}{2}}}{c} - \frac{\log 4\pi + \log(\log n)}{2c(2\log n)^{\frac{1}{2}}}, \tag{2.81}$$

and

$$S_n = \frac{1}{c(2\log n)^{\frac{1}{2}}}, \tag{2.82}$$

where n is the number of observations.

Proof. *Since $L_\mu(t_i)$ follows approximately the same distribution as $\hat{L}_\mu(t_i) \sim N\left(0, \frac{1}{c^2}\right)$ when there is no jump, the proof of this lemma is an extension of Section 2.3.2 of Galambros (1987) and follows after applying Theorem 2.1.3 of Galambros (1987) for a normal distribution with variance equals to $\frac{1}{c^2}$, $Z_n := \max_{i\in A_n} |L_\mu(t_i)|$, $a_n := C_n$ and $b_n := S_n$.*

Remark 2 *In the paper of Lee and Mykland (2008), there is a typo in the numerator of the second component of equation (2.81).*

■

In a nutshell, the idea of selecting a rejection region is that if the observed test statistics are not even within the usual region of maximums, it is unlikely that the realized return is from the diffusion part of the jump-diffusion model. For instance, setting a significance level of 1%, the threshold for $\frac{|L_\mu(t_i)|-C_n}{S_n}$ is ς, such that $\mathbb{P}(\xi \le \varsigma) = \exp\left(-e^{-\varsigma}\right) = 0.99$. Equivalently, $\varsigma = -\log(-\log(0.99)) = 4.6001$.Therefore, if $\frac{|L_\mu(t_i)|-C_n}{S_n} > 4.6001$, then the hypothesis of no jump at time t_i is rejected.

2.7. Misclassifications

In this subsection, we discuss the probability of misclassifications as a function of the frequency of observations. For a single testing time, t_i, there can be two kinds of misclassification:

1. Failure to detect the actual jump (FTD_i) at time t_i: There is a jump in the interval $(t_{i-1}, t_i]$, but the test fails to reveal its existence;

2. Spurious detection of the jump (SD_i) at time t_i: There is no jump in the interval $(t_{i-1}, t_i]$, but the test wrongly concludes there is one.

In the case that we do the test several times with time-series data, the global extension of these concepts is straightforward:

1. Global failure to detect actual jump ($GFTD$): There are some jumps over the whole interval $[0, T]$, but the test fails to detect any of them;

2. Global spurious detection of jump (GSD): There are some returns that are not due to jumps, but the test wrongly declares any of them as due to a jump.

Lee and Mykland (2008) show that $\mathbb{P}(GFTD|N) \to 0$ and $\mathbb{P}(GSD|N) \to 0$ as long as the significance level of the test $\alpha_n \to 0$ (alternatively, $\beta_n \to \infty$) and $\Delta t \to 0$, where N is the number of jumps in $[0, T]$ and β_n be the $(1-\alpha_n)$th percentile of the limiting distribution of ξ in Lemma 9. The authors examined the effectiveness of the test using Monte Carlo simulation over one thousand series of returns over one year at several different frequencies. For a series generation, the authors used the Euler-Maruyama stochastic differential equation discretization scheme. Figure 2.1 illustrates that increasing the frequency of observations reduces the probability of spurious detection of jumps for the constant and stochastic volatility cases. Moreover, including stochastic volatility in the model increases the probability of spurious detection of jumps by one order of magnitude compared to the constant volatility cases.

Figure 2.2 shows the probability of success in detecting an actual jump. The authors chose different jump sizes to illustrate that it is harder to detect smaller-sized jumps at low frequencies. Notwithstanding, as the frequency of observations increases, the test enhances its detection power even for very small-sized jumps.

In summary, the authors' study confirms that if the frequency of observation increases, the test improves its power of jump detection.

Probability of spurious detection $P(SD_i)$

freq	$\sigma = 0.3$	(SE)	$\sigma = 0.6$	(SE)	SV	(SE)
24-hour	1.3305e-03	(7.4050e-05)	1.3305e-03	(7.6239e-05)	3.9110e-03	(1.3319e-04)
12-hour	5.7380e-04	(3.4901e-05)	5.3222e-04	(3.3570e-05)	2.3306e-03	(7.7336e-05)
6-hour	2.0696e-04	(1.4460e-05)	2.1209e-04	(1.4790e-05)	1.3289e-03	(4.3670e-05)
2-hour	5.2879e-05	(4.3701e-06)	5.5911e-05	(4.2684e-06)	4.8131e-04	(1.6731e-05)
1-hour	2.1775e-05	(1.9032e-06)	2.5126e-05	(2.0353e-06)	2.7688e-04	(1.0062e-05)
30-minute	8.8436e-06	(8.3749e-07)	8.6768e-06	(8.3965e-07)	1.4467e-04	(7.2952e-06)
15-minute	3.4947e-06	(3.7430e-07)	4.1876e-06	(4.2736e-07)	8.9449e-05	(4.2818e-06)

The encompassing model is $d \log S(t) = \mu(t)dt + \sigma(t)dW(t)$. Constant volatility sets $\sigma(t) = \sigma$ at 30% and 60%. Stochastic volatility (SV) assumes the Affine model of Heston (1993), specified as $d\sigma^2(t) = (\theta_0 + \theta_1\sigma^2(t))\,dt + \omega\sigma(t)dB(t)$, where $B(t)$ denotes a Brownian motion. The table shows means and standard errors (in parentheses) of probability of spurious detection $P(SD_i)$ at time t_i. The significance level α is 5%. *freq* denotes the frequency of observations.

Figure 2.1.: Probability of spurious detection of jump $\mathbb{P}(SD_i)$ from Lee and Mykland (2008).

Jump Size	3σ	2σ	1σ	0.5σ	0.25σ	0.1σ
			Constant volatility σ at 30%			
freq = 24-hour	0.9920	0.9880	0.9810	0.9270	0.4690	0.0260
	(0.0028)	(0.0034)	(0.0043)	(0.0082)	(0.0158)	(0.0050)
freq = 6-hour	0.9860	0.9780	0.9820	0.9700	0.9050	0.1520
	(0.0037)	(0.0046)	(0.0042)	(0.0054)	(0.0093)	(0.0114)
freq = 1-hour	0.9950	0.9860	0.9890	0.9890	0.9770	0.8880
	(0.0022)	(0.0037)	(0.0033)	(0.0033)	(0.0047)	(0.0100)
freq = 15-minute	0.9980	0.9970	0.9960	0.9920	0.9970	0.9820
	(0.0014)	(0.0017)	(0.0020)	(0.0028)	(0.0017)	(0.0042)
Jump Size	$3\widetilde{\sigma(t)}$	$2\widetilde{\sigma(t)}$	$1\widetilde{\sigma(t)}$	$0.5\widetilde{\sigma(t)}$	$0.25\widetilde{\sigma(t)}$	$0.1\widetilde{\sigma(t)}$
			Stochastic volatility (SV)			
freq = 24-hour	0.9470	0.9330	0.8540	0.5720	0.2500	0.0320
	(0.0071)	(0.0079)	(0.0112)	(0.0157)	(0.0137)	(0.0056)
freq = 6-hour	0.9770	0.9690	0.9410	0.8480	0.5320	0.1400
	(0.0047)	(0.0055)	(0.0075)	(0.0114)	(0.0158)	(0.0110)
freq = 1-hour	0.9870	0.9860	0.9830	0.9610	0.8770	0.5260
	(0.0036)	(0.0037)	(0.0041)	(0.0061)	(0.0104)	(0.0158)
freq = 15-minute	0.9970	0.9990	0.9980	0.9920	0.9610	0.8100
	(0.0017)	(0.0010)	(0.0014)	(0.0028)	(0.0061)	(0.0130)

The encompassing model is $d \log S(t) = \mu(t)dt + \sigma(t)dW(t) + Y(t)dJ(t)$. Constant volatility sets $\sigma(t) = \sigma$ at 30%. Stochastic volatility (SV) assumes the Affine model of Heston (1993), specified as $d\sigma^2(t) = (\theta_0 + \theta_1\sigma^2(t))\,dt + \omega\sigma(t)dB(t)$, where $B(t)$ denotes a Brownian motion. The table contains means and standard errors (in parentheses) of probability of detecting actual jumps $[1 - P(FTD_i)]$ at time t_i. The significance level α is 5%. The jump sizes are set in comparison with volatility level: 3σ means jump sizes are set at three times of volatility level. For stochastic volatility, the jump size depends on the mean of volatility $\widetilde{\sigma(t)} = E[\sigma(t)]$. *freq* denotes the frequency of observations.

Figure 2.2.: Probability of detecting actual jump $[1 - \mathbb{P}(FTD_i])$ from Lee and Mykland (2008).

3. Data and implementation of the LM test

In this section, we discuss the data used for the study and describe the implementation of the LM test.

3.1. The data

In order to assess the performance of the test as a function of the data frequency, we use low-frequency (daily data) and high-frequency intraday data (one-hour, 30-minute, and 5-minute data). The motivation for using these two sets is to assess whether the test can detect jumps at times of high volatility (as seen in market crashes or market corrections) where big jump sizes are expected, even when using low-frequency data since the probability of misclassification, as described in Subsection 2.7 is low. We use daily data from the Wall Street Journal (WSJ) and the Chicago Board Options Exchange (CBOE) databases for the SPX and VIX indexes, respectively. The time span is 30 years, from 2nd January 1990 to 31st December 2020. On the other hand, we use high-frequency data from the FirstRate Data LLC database for both indexes. The time span covers 14 years, from 27th April 2007 to 31st December 2020.

To check the consistency of the test, we compare the jumps detected from the test with a database created from the largest SPX daily percentage losses in each year of the study. Not surprisingly, these losses are associated with market crashes and market corrections, as can be seen in Table 3.1.

Table 3.1.: Benchmark dates - Largest SPX daily percentage losses and associated events.

Date	SPX Daily Change	Event	Source
6/8/1990	-3.1%	Iraq-Kuwait crises	The New York Times
15/11/1991	-3.7%	Concerns about credit card legislation	The New York Times
7/4/1992	-1.9%	Indian stock market scam	The New York Times
16/2/1993	-2.4%	Slower US economy forecast	Bloomberg
4/2/1994	-2.3%	Federal Reserve raised interest rates	The Washington Post
18/12/1995	-1.6%	Victory of the communist party in Russia	Bloomberg
8/3/1996	-3.1%	Cut of interest rates	Bloomberg
27/10/1997	-7.1%	Economic crisis in Asia	CNN Money
31/8/1998	-7.0%	Russian financial crisis	CNN Money
15/10/1999	-2.8%	Signs of inflation	CNN Money
14/4/2000	-6.0%	Dot-com bubble crash	CNN Money
17/9/2001	-5.0%	Aftermath of 9/11	CNN Money
3/9/2002	-4.2%	Weak economic data	CNN Money
24/3/2003	-3.6%	Iraq war concerns	CNN Money
5/8/2004	-1.6%	Increase in interest rates	CNN Money
15/4/2005	-1.7%	Weak economic data	CNN Money
20/1/2006	-1.8%	Disappointing results of big stocks	CNN Money
27/2/2007	-3.5%	Chinese stock bubble	CNN Money
15/10/2008	-9.5%	Global financial crises	CNN Money
20/1/2009	-5.4%	Inauguration of Obama term	CNN Money
20/5/2010	-4.0%	Flash crash in commodities	Butler Research
8/8/2011	-6.9%	USA credit-rating downgrade	CNN Money
1/6/2012	-2.5%	Fear of global slowdown	The Guardian
20/6/2013	-2.5%	Fear of global slowdown	The Guardian
3/2/2014	-2.3%	Weak economic data	CNN Business
24/8/2015	-4.0%	China's economic slowdown	CNN Business
24/6/2016	-3.7%	Brexit	CNN Business
17/5/2017	-1.8%	President Trump's allegations	CNN Business
5/2/2018	-4.2%	Increasing interest rates	CNBC
5/8/2019	-3.0%	Trump - China politics	CNBC
16/3/2020	-12.8%	COVID-19 crash	CNN Business

Summary of the biggest daily losses of SPX in the years under study. For each date, we look in different sources for an event that justifies the index's loss. As can be seen in the table, the major events related to market crashes and corrections are mapped, such as the dot-com bubble crash (2000), subprime crisis (2008), and more recently, the market crash related to COVID-19 (2020) to name a few. For our study period, there are a total of 31 dates on which a jump is expected to be detected.

Finally, in order to retain the benefit of bipower variation on the test statistic, the window size K must be large enough so that the effect of jumps on estimating instantaneous volatility disappears, yet it must be smaller than the number of observations n. The condition $K = O(\Delta t^{\alpha})$ with $\alpha \in]-1, -0.5[$ satisfies this requirement and since $\Delta t = \frac{1}{252 \times nobs}$ where $nobs$ is the number of observations per day, the integers between $\sqrt{252 \times nobs}$ and $252 \times nobs$ are candidates for K. The authors recommendation of window sizes for one-day, one-hour, 30-minute, and 5-minute are 16, 78, 110, and 270, respectively. Notwithstanding, we found that these candidates are quite small for our study. The test fails to detect jumps in key dates as the effect of high volatility impact the instantaneous volatility estimation. Consequently, we use different window sizes for our study. Table 3.2 illustrates the number of observations per day (assuming 6h30min of daily trading), the window size candidates (K), and the total number of observations (n).

Table 3.2.: Nobs, K, and n for different frequencies.

Frequency	nobs	K				n	
		Low	High	Average	Authors	SPX	VIX
One-day	1	16	252	134	16	7,812	7,808
One-hour	7	42	1,764	903	78	27,488	34,565
30-minute	13	57	3,276	1,667	110	48,233	63,400
5-minute	78	140	19,656	9,898	270	274,596	362,835

Summary of the number of observations per day (*nobs*), the window size candidates (K), and the total number of observations (n) for each of the data frequencies. As aforementioned, the window size candidates range between $\sqrt{252 \times nobs}$ and $252 \times nobs$; hence we have a low and high value for each frequency. Moreover, we also included the average between the extremes of the interval and the authors' recommendation for window size candidates. The total number of observations is associated with 30 and 14 years for one-day data and high-frequency data, respectively.

3.2. Implementation of the LM test

The implementation of the LM test is described as follows:

1. Calculate the index returns by taking the difference of log index close prices;
2. Calculate the sampling frequency given by $\Delta t = \frac{1}{252 \times nobs}$, where *nobs* is the number of observations per day;
3. Calculate the window size K. As K is given by $K = \Delta t^{\alpha}$ with $\alpha \in]-1, -0.5[$, the integers between $\sqrt{252 \times nobs}$ and $252 \times nobs$ are candidates for K;
4. Calculate the average rate of return (drift) and instantaneous volatility given in equation (2.18) and by taking the square root of equation (2.19), respectively;
5. Calculate the test statistic given in Equation (2.17). It is noteworthy that the authors recommend neglecting the drift term as its order, dt, is negligible compared to the diffusion (of order $\sqrt{dt}$) and jump components (of order 1). Nevertheless, we include the drift term in our study;
6. Calculate the selection region parameters C_n and S_n given in equations (2.81) and (2.82), respectively;
7. Calculate the threshold parameter given by $\mathbb{P}(\xi \leq \varsigma) = \exp\left(-e^{-\varsigma}\right) = 1 - \alpha = 0.95$, where α is the significance level (5% in our study);
8. Check the jump detection test: if $\frac{|L_\mu(t_i)| - C_n}{S_n} > \varsigma$, then the hypothesis of no jump at t_i is rejected;
9. Once a jump is detected, we assume that the jump size dominates the return;
10. The test outcomes are: jump arrival date and jump size (equal to the return).

The MATLAB code written to run the LM test can be found in the Appendix.

4. Discussion of the results

4.1. Low-frequency data

For the one-day data, our first approach was to assess the results using the window size suggested by the authors, i.e., $K = 16$. In this run, we found that the SPX jumped 31 times, 9 (29%) times upward, and 22 (71%) times downward, whereas the VIX jumped 46 times, 39 (85%) times upward, and 7 (15%) times downward. The indexes co-jumped 16 times (8 times in the benchmark dates) and in all cases in opposite directions. Indeed, every time that one of the indexes jumped, the other showed a return in the opposite direction. Compared to the benchmark dates, the test failed to detect jumps in dates when a jump was expected. For instance, the test failed to detect key events such as the 9/11 aftermath (2001), the subprime crises (2008), the COVID-19 crash (2020), among other moments where the market plumed. Table 4.1 summarizes the test performance compared to the benchmark dates:

Table 4.1.: SPX and VIX jumps for K=16.

Date	Event	Jump	
		SPX	**VIX**
6/8/1990	Iraq-Kuwait crises	No	No
15/11/1991	Concerns about credit card legislation	Yes	Yes
7/4/1992	Indian stock market scam	No	Yes
16/2/1993	Slower US economy forecast	Yes	Yes
4/2/1994	Federal Reserve raised interest rates	Yes	Yes
18/12/1995	Victory of the communist party in Russia	No	Yes
8/3/1996	Cut of interest rates	No	No
27/10/1997	Economic crisis in Asia	Yes	Yes
31/8/1998	Russian financial crisis	No	No
15/10/1999	Signs of inflation	No	No
14/4/2000	Dot-com bubble crash	Yes	No
17/9/2001	Aftermath of 9/11	No	No
3/9/2002	Weak economic data	No	No
24/3/2003	Iraq war concerns	No	No
5/8/2004	Increase in interest rates	No	No

(continued)

Table 4.1—*continued*

Date	Event	Jump	
		SPX	VIX
15/4/2005	Weak economic data	No	No
20/1/2006	Disappointing results of big stocks	No	Yes
27/2/2007	Chinese stock bubble	Yes	Yes
15/10/2008	Global financial crises	No	No
20/1/2009	Inauguration of Obama term	No	No
20/5/2010	Flash crash in commodities	No	No
8/8/2011	USA credit-rating downgrade	Yes	Yes
1/6/2012	Fear of global slowdown	No	No
20/6/2013	Fear of global slowdown	No	No
3/2/2014	Weak economic data	No	No
24/8/2015	China's economic slowdown	No	No
24/6/2016	Brexit	Yes	No
17/5/2017	President Trump's allegations	Yes	Yes
5/2/2018	Increasing interest rates	Yes	Yes
5/8/2019	Trump - China politics	No	No
16/3/2020	COVID-19 crash	No	No

Summary of the jumps detected by the test for $K = 16$. "Yes" means that the index jumped, whilst "No" means the index did not jump. The SPX and VIX jumped 10 and 11 times in the benchmark dates, respectively. The indexes co-jumped 8 times in the benchmark dates.

Since the test only managed to detect 10 (SPX) and 11 (VIX) jumps of the 31 expected, we decided to experiment with different window size candidates aiming to dilute the effect of high volatility on the bipower variation calculation when using low-frequency data. Tables 4.2 and 4.3 depict the findings.

Table 4.2.: One-day data - SPX and VIX jumps for different window sizes.

Date	**Event**	**Jump (K=16)**		**Jump (K=32)**		**Jump (K=64)**		**Jump (K=128)**		**Jump (K=256)**	
		SPX	**VIX**	**SPX**	**VIX**	**SPX**	**VIX**	**SPX**	**VIX**	**SPX**	**VIX**
6/8/1990	Iraq-Kuwait crises	No	No	No	No	No	No	No	No	No	No
15/11/1991	Concerns about credit card legislation	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
7/4/1992	Indian stock market scam	No	Yes	No	No	No	No	No	No	No	No
16/2/1993	Slower US economy forecast	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes
4/2/1994	Federal Reserve raised interest rates	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
18/12/1995	Victory of the communist party in Russia	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes
8/3/1996	Cut of interest rates	No	No	No	Yes	No	No	Yes	No	Yes	Yes
27/10/1997	Economic crisis in Asia	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
31/8/1998	Russian financial crisis	No	No	Yes	No	Yes	No	Yes	No	Yes	No
15/10/1999	Signs of inflation	No	No	No	No	No	No	No	No	No	No
14/4/2000	Dotcom bubble crash	Yes	No	No	No	No	No	Yes	No	Yes	No
17/9/2001	Aftermath of 9/11	No	No	Yes	No	Yes	Yes	No	Yes	No	Yes
3/9/2002	Weak economic data	No	No	No	No	No	No	No	No	No	No
24/3/2003	Iraq war concerns	No	No	No	No	No	No	No	No	No	No
5/8/2004	Increase in interest rates	No	No	No	No	No	No	No	No	No	No
15/4/2005	Weak economic data	No	No	No	No	No	No	No	Yes	No	No
20/1/2006	Disappointing results of big stocks	No	Yes	No	Yes	No	No	No	No	No	No
27/2/2007	Chinese stock bubble	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
15/10/2008	Global financial crises	No	No	No	No	No	No	Yes	No	Yes	No
20/1/2009	Inauguration of Obama term	No	No	No	No	No	No	No	No	No	No
20/5/2010	Flash crash in commodities	No	No	No	No	No	No	No	No	No	No
8/8/2011	USA credit-rating downgrade	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
1/6/2012	Fear of global slowdown	No	No	No	No	No	No	No	No	No	No
20/6/2013	Fear of global slowdown	No	No	No	No	No	No	No	No	No	No
3/2/2014	Weak economic data	No	No	No	No	No	No	No	No	No	No
24/8/2015	China's economic slowdown	No	No	No	No	Yes	No	Yes	Yes	Yes	Yes
24/6/2016	Brexit	Yes	No	Yes	Yes	Yes	Yes	No	Yes	No	Yes
17/5/2017	President Trump's allegations	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes
5/2/2018	Increasing interest rates	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
5/8/2019	Trump - China politics	No	No	Yes	Yes	No	No	Yes	Yes	No	Yes
16/3/2020	COVID-19 crash	No	No	No	No	Yes	No	Yes	No	Yes	No

Summary of the jumps detected by the indexes for the window sizes: 16, 32, 64, 128, and 256. "Yes" means that the index jumped, whilst "No" means the index did not jump.

Table 4.3.: One-day data - Summary of jump dynamics for different window sizes.

K	Index	# Jumps	Up (%)	Down (%)	# Jumps in benchmark dates	# Co-jumps	# Co-jumps in benchmark dates
16	SPX	31	9 (29%)	22 (71%)	10	16	8
	VIX	46	39 (85%)	7 (15%)	11		
32	SPX	31	5 (16%)	26 (84%)	12	21	10
	VIX	48	42 (88%)	6 (12%)	13		
64	SPX	33	6 (18%)	27 (82%)	13	17	10
	VIX	38	32 (84%)	6 (16%)	11		
128	SPX	38	10 (26%)	28 (74%)	15	18	10
	VIX	41	36 (88%)	5 (12%)	14		
256	SPX	38	13 (34%)	25 (66%)	12	12	8
	VIX	33	27 (82%)	6 (18%)	14		

Summary of the jump dynamics for different window sizes: 16, 32, 64, 128, and 256. The number of jumps, jump direction, and the number of jumps in benchmark dates are depicted for each index. Moreover, the total number of co-jumps and the number of co-jumps in benchmark dates are also depicted.

On average, the SPX jumped 34 times, 25% of the times upward, and 75% of the times downward, whereas the VIX jumped 41 times, 85% of the times upward, and 15% of the times downward. The indexes co-jumped in the opposite direction 100% of the time, regardless of the window size. Moreover, every time one of the indexes jumped, the other showed a return in the opposite direction in almost 100% of the cases. The run with the most number of jumps in the benchmark dates uses a window size of 128 (very close to the average of the window size candidate interval). From the 31 expected jumps, the test identified 19 jumps where at least one of the indexes jumped, including key events such as the 9/11 aftermath (2001), the subprime crises (2008), and the COVID-19 crash (2020).

Generally, there are no clear trends of enhancement by increasing the window size. Indeed, using different window sizes resulted in different jump dates. Nevertheless, it is evident that there is a tendency of opposite jumping directions of the indexes, and if a jump is detected on a benchmark date, there is evidence that the indexes tend to co-jump. The SPX tend to jump more frequently downward, whilst the VIX upward. Moreover, the VIX tend to jump more than the SPX. In the next subsection, we further explore these pieces of evidence for the most robust case using high-frequency data.

4.2. High-frequency data

The data covers only 14 years for the high-frequency case, from 27th April 2007 to 31st December 2020. In this case, we have 13 benchmark dates from 2008 to 2020. For the window sizes, we use the average of the extremes of the window size candidate interval as depicted in Table 3.2 and following our findings in the low-frequency case. Tables 4.4 and 4.5 summarize the findings.

Table 4.4.: SPX and VIX jumps for different frequencies.

Date	Event	Jump (One-day)K=128		Jump (One-hour)K=903		Jump (30-minute)K=1,667		Jump (5-minute)K=9,898	
		SPX	VIX	SPX	VIX	SPX	VIX	SPX	VIX
15/10/2008	Global financial crises	Yes	No	Yes	No	Yes	No	Yes	Yes
20/1/2009	Inauguration of Obama term	No	No	No	No	No	Yes	No	Yes
20/5/2010	Flash crash in commodities	No	No	Yes	Yes	Yes	Yes	Yes	Yes
8/8/2011	USA credit-rating downgrade	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
1/6/2012	Fear of global slowdown	No	No	Yes	No	Yes	No	Yes	Yes
20/6/2013	Fear of global slowdown	No	No	Yes	No	Yes	Yes	Yes	Yes
3/2/2014	Weak economic data	No	No	No	No	Yes	Yes	Yes	Yes
24/8/2015	China's economic slowdown	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
24/6/2016	Brexit	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes
17/5/2017	President Trump's allegations	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
5/2/2018	Increasing interest rates	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
5/8/2019	Trump - China politics	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
16/3/2020	COVID-19 crash	Yes	No	Yes	Yes	Yes	Yes	Yes	Yes

Summary of the SPX and VIX jumps for different high frequencies. "Yes" means that the index jumped, whilst "No" means the index did not jump.

Table 4.5.: Summary of jump dynamics for different frequencies.

Frequency	Index	# Jumps	Up (%)	Down (%)	# Jumps in benchmark dates	# Co-jumps	# Co-jumps in benchmark dates
One-hour	SPX	365	166 (45%)	199 (55%)	11	121	8
	VIX	271	184 (68%)	87 (32%)	8		
30-minute	SPX	605	297 (49%)	308 (51%)	12	200	10
	VIX	515	334 (65%)	181 (35%)	11		
5-minute	SPX	2,162	1,170 (54%)	992 (46%)	12	904	12
	VIX	1,884	1,390 (57%)	1,063 (43%)	13		

Summary of the jump dynamics for different high frequencies. The number of jumps, jump direction, and the number of jumps in benchmark dates are depicted for each index. Moreover, the total number of co-jumps and the number of co-jumps in benchmark dates are also depicted.

The results from the high-frequency runs are quite interesting and successfully answer the queries of this work. First, from Table 4.5, we can conclude that in this case, the SPX jumped more than the VIX as opposed to the finding on the low-frequency study. On average, the SPX tends to jump slightly more often downward (51% of the cases), whereas the VIX tends upward (63% of the cases) in agreement with the low-frequency study, although in different proportions. Moreover, the runs show that every time one of the indexes jumped, the other showed a return in the opposite direction in 88% of the cases. In all co-jump dates, the indexes jumped in opposite directions in 99% of the cases, regardless of the intraday frequency. Naturally, as we increased the frequency, the test enhanced its performance, as shown in Table 4.4. For the 5-minute case (the best performer run), at least one of the indexes jumped in all the benchmark dates. Indeed, in 12 out of 13 dates, the indexes co-jumped in opposite directions, and since the VIX jumped in all 13 dates, this shows that volatility spikes when a market crash or correction event happen. Finally, another key finding is that the indexes tend to co-jump in ~45% of the cases when one of the indexes jump.

5. Conclusions

5.1. Remarks

This thesis utilizes the LM nonparametric test to look for evidence on jumps returns and stochastic volatility, in bear market conditions, using low and high-frequency data. We use the SPX and VIX indexes to analyze the jumps in returns and stochastic volatility, respectively. To check the consistency of the test, we compare the jumps detected from the test with a database created from the largest SPX daily percentage losses in each year of the study.

The results reveal several findings regarding the relationship between the SPX returns and the changes in the VIX. For the low-frequency data, our first approach was to use the window size suggested by the authors, but the results were not satisfactory as the test fails to detect jumps in dates where historical events suggest that a jump must have happened. Thus, we studied the impact of increasing the window size on the test's potential to detect jumps. In fact, there is an improvement trend up to the window size equal to 128 (very close to the average of the extremes of the window size candidates) when comparing the jumps associated with the benchmark dates. Notwithstanding, even with this improved run, the test fails to detect almost 50% of the expected jumps. On average, we find that the VIX jumps more frequently than the SPX. In addition, there is a clear jumping trend associated with negative events. In fact, the SPX jumps down 75% of the time and the VIX jumps up 85% of the time. The indexes co-jumped in the opposite direction 100% of the time in the benchmark dates, regardless of the window size. Moreover, every time one of the indexes jumped, the other showed a return in the opposite direction in almost 100% of the cases.

For the high-frequency data, we used for our runs the average of the extremes of the window size candidates, following our findings in the low-frequency data case. Indeed, our first approach was to use the window sizes recommended by the authors but the results were not satisfactory when compared to the average window size cases. The findings in this case are quite interesting and successfully answer the queries of this work. First, we can conclude that the SPX jumped more often than the VIX as opposed to the finding from the low-frequency study. On average, the SPX tends to jump slightly more often downward (51% of the cases), whereas the VIX tends upward (63% of the cases) in agreement with the low-frequency study and corroborating our hypothesis that the indexes' jumps are more often associated to negative events in the market. Furthermore, the runs show that every time one of the indexes jumped, the other showed a return in the opposite direction in 88% of the cases. In all co-jump dates, the indexes jumped in opposite directions in 99% of the cases, regardless of the intraday frequency. Naturally, as we increased the frequency, the test enhanced its performance. For the 5-minute case (the best performer run), at least one of the indexes jumped in all the benchmark dates. Indeed, in 12 out of 13 dates, the indexes co-jumped in opposite directions. The VIX jumped in all 13 dates, and this shows that volatility spikes when a market crash or correction event happens. Finally, another key finding is that the indexes tend to co-jump in ~45% of the cases when one of the indexes jumps.

Finally, we can conclude that the LM test is a powerful tool to detect jumps but only when high-frequency data is used. Returning to the questions raised in the objective of the thesis, we can conclude that the SPX tends to jump more frequently than the VIX. When there is an event related to a market crash or market correction, the indices co-jump in opposite directions. The VIX, in our high-frequency study, jumped in all benchmark dates, which leads us to conclude that, in fact, volatility spikes exactly when there is a market crash or correction.

5.2. Recommendations for future work

Our main recommendation for future work is to explore the existing improvements in the literature on bipower variation as it is the fundamental component of the test. In fact, there is evidence in the literature that the joint use of bipower variation and threshold estimation enhances the power of jump detection of the test. This enhancement can be applied to both low-frequency and high-frequency data.

Another recommendation for further work is to use other nonparametric tests and to check if, with the same data, other tests can outperform the LM test.

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A. Appendix - Matlab® Code

```
%Lee, Suzanne S., and Per A. Mykland (2007)
    %Nomparametric Test
    function[Summary_Table] = LM_test_drift(S,K,alfa)
    % S is the asset price
    % K is the window size
    % alfa is the significance level
    % Calculating the asset returns
    r = [NaN;diff(log(S))];
    % Calculating the realized bipower variation (bpv)
    bpv = abs(r(1:end)).*abs([NaN;r(1:end-1)]);
    bpv = [NaN;bpv(1:end-1)];
    % Calculating the average rate of return (drift)
    mi=movmean(r,[K-2 0]);
    % Estimating the instantaneous volatily
    sigma = sqrt(movmean(bpv,[K-3 0]));
    % Calculating the L statistic
    L = (r-mi)./sigma;
    % Selection of rejection region
    % Estimation of parameters
    n = numel(r)-K; % Number of observations
    c = (2/pi)^0.5; % Constant c=E|Ui|
```