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A family of vector bundles on $\mathbb{P}^3$ of homological dimension 2 and $\chi\,(\mathrm{End} E) = 1$

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Abstract

For any odd integer $\gamma \leq -5$, we construct a family of rank 3 vector bundles $\{E_\gamma\}$ on $\mathbb{P}^3$ with minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\frac{3\gamma^2+8\gamma-3}{8}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\frac{3\gamma^2+2\gamma-17}{4}} \to \mathcal{O}_{\mathbb{P}^n}^{\frac{3\gamma^2-4\gamma-7}{8}} \to E_\gamma \to 0.$$

and satisfying $\chi\,(\mathrm{End}\, E_\gamma) = 1$.

1 Introduction

Notation 1.1. *If E is a vector bundle on $\mathbb{P}^n$ we denote*

$$\mathrm{h}^q(E) = \dim \mathrm{H}^q(E) = \dim \mathrm{H}^q(\mathbb{P}^n, E).$$

The Euler characteristic of E is defined by the integer

$$\chi(E) = \textstyle\sum_{i=0}^{n}(-1)^i \mathrm{h}^i(E).$$

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Recall furthermore that we have an isomorphism $\operatorname{End} E \cong E^{\vee} \otimes E$.

If E is a vector bundle on $\mathbb{P}^n$, with $n \geq 2$, of homological dimension 1 and defined by a minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^a \to \mathcal{O}_{\mathbb{P}^n}^b \to E \to 0 \tag{1}$$

then $\chi(\operatorname{End} E) = a^2 + b^2 - (n+1)ab$ is the Euler characteristic of $\operatorname{End} E$. It is well-known (see e.g. [6], Lemma 2.2.3) that $\chi(\operatorname{End} E) = 1$ if and only if $(a, b) = (u_k, u_{k+1})$, with $k \geq 1$, where $\{u_k\}_{k\geq 0}$ is the sequence defined recursively by

$$\begin{cases} u_0 = 0, \\ u_1 = 1, \\ u_{k+1} = (n+1)u_k - u_{k-1}. \end{cases} \tag{2}$$

Moreover, for each pair of the form $(a, b) = (u_k, u_{k+1})$, where $\{u_k\}_{k\geq 0}$ is as above, the existence of a vector bundle E defined by a resolution of type (1) and thus satisfying $\chi(\operatorname{End} E) = 1$ is guaranteed, for all $n \geq 2$ ([6], Theorem 2.2.7). It is worth recalling that one of the reasons for studying vector bundles satisfying $\chi(\operatorname{End} E) = 1$ is that they provide good candidates for exceptional bundles, important objects when studying stability.

If we look at the case of homological dimension 2 or, more precisely, to vector bundles E defined by a minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2)^a \to \mathcal{O}_{\mathbb{P}^n}(-1)^b \to \mathcal{O}_{\mathbb{P}^n}^c \to E \to 0,$$

with $n \geq 3$, then the Euler characteristic of $\operatorname{End} E$ is now given by the ternary quadratic form

$$\chi(\operatorname{End} E) = a^2 + b^2 + c^2 - (n+1)ab - (n+1)bc + \binom{n+2}{2}ac.$$

Finding a general formula of all integer and positive solutions of the equation $\chi(\operatorname{End} E) = 1$ is much harder, so we will restrict to the case when $n = 3$ and for which this equation becomes

$$a^2 + b^2 + c^2 - 4ab - 4bc + 10ac = 1. \tag{3}$$

Besides, we would also like to know which solutions (a, b, c) of (3) give rise to a vector bundle E of homological dimension 2 and linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^a \to \mathcal{O}_{\mathbb{P}^3}(-2)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0. \tag{4}$$

In [5], Proposition 4.3, the authors already proved the existence of a certain family of vector bundles defined this way and satisfying $\chi(\mathrm{End}\,E) = 1$. We can restate the referred proposition as follows.

Proposition 1.2. *There exists a vector bundle E of homological dimension 2 and linear resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0,$$

with $(a, b, c) = (u_s, u_{s+1}, 4u_{s+1} - 10u_s)$, for each $s \geq 1$, where $\{u_s\}_{s\geq 1}$ is the sequence (2).

In particular, $\chi(\mathrm{End}\,E) = 1$.

In this note we construct a new family of vector bundles with the required properties corresponding to 3-uples (a, b, c) different from the one provided by the above proposition, that is coming from triples of the form $(a, b, c) = (u_s, u_{s+1}, 4u_{s+1} - 10u_s)$.

This construction is described in the next section and inspired the authors for a more general result in a work in progress in [4].

2 A family of rank 3 bundles on $\mathbb{P}^3$ of homological dimension 2 and $\chi(\mathrm{End}\,E) = 1$

Let E be a vector bundle on $\mathbb{P}^3$ of homological dimension 2 and linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0, \tag{5}$$

satisfying $\chi(\mathrm{End}\,E) = 1$, that is, such that a, b and c satisfy

$$a^2 + b^2 + c^2 - 4ab - 4bc + 10ac = 1.$$

So a, b and c are integer positive solutions of the Diophantine equation

$$X^2 + Y^2 + Z^2 - 4XY - 4YZ + 10XZ = 1. \tag{6}$$

We would like to find for which solutions (a, b, c) of (6) there is a vector bundle E defined by a linear resolution of the form (5) with Betti numbers a, b and c (so that E will satisfy $\chi(\operatorname{End} E) = 1$).

According to [4], any solution (a, b, c) of (6) is a triple of integers of the form

$$\begin{aligned} a &= \frac{\alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 + 2\alpha\beta + 4\gamma\delta}{8}, \\ b &= \frac{\alpha^2 + 2\gamma^2 - 6\beta^2 - 12\delta^2}{4}, \\ c &= \frac{\alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 - 2\alpha\beta - 4\gamma\delta}{8}, \end{aligned} \tag{7}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, with α even and $\alpha\delta - \beta\gamma = \pm 1$. In order to these integers to be the Betti numbers of E they furthermore must satisfy

$$\begin{cases} a > 0 \quad \Leftrightarrow \quad (\alpha + \beta)^2 + 2(\gamma + \delta)^2 > 3\left(\beta^2 + 2\delta^2\right), \\ a \leq b \quad \Leftrightarrow \quad (\alpha - \beta)^2 + 2(\gamma - \delta)^2 \geq 11\left(\beta^2 + 2\delta^2\right). \end{cases} \tag{8}$$

Note also that under this parameterisation we have

$$\operatorname{rk} E = a - b + c = \beta^2 + 2\delta^2. \tag{9}$$

Recall (Proposition 4.3 in [5]) that the existence of such a family of vector bundles, $\{E_s\}_{s \geq 1}$, is proved in the case when (a, b, c) is a triple of the form $(u_s, u_{s+1}, 4u_{s+1} - 10u_s)$, for $s \geq 1$, where $\{u_s\}_{s \geq 1}$ is the sequence (2). When $s = 1$ we get $(a, b, c) = (1, 4, 6)$ and E is the rank 3 vector bundle on $\mathbb{P}^3$ with minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-1)^4 \to \mathcal{O}_{\mathbb{P}^3}^6 \to E \to 0.$$

In fact, this is the only rank 3 case among the set of triples of this form. Using (7)–(9) we can easily deduce that this triple $(1, 4, 6)$ corresponds to

4 solutions of type $(\alpha, \beta, \gamma, \delta)$: $(-4, 1, -3, 1)$, $(-4, 1, 3, -1)$, $(4, -1, -3, 1)$ and $(4, -1, 3, -1)$.

We will next see that the triple $(1, 4, 6)$ does not exhaust all rank 3 possible solutions. We will construct a family $\{E_\gamma\}_{\{\gamma\in\mathbb{Z}\,:\,\gamma \text{ odd},\ \gamma\le -5\}}$ of rank 3 vector bundles with minimal linear resolution of type (5) and such that $\chi(\operatorname{End} E) = 1$.

Suppose first that a vector bundle E of rank 3 exists defined by a linear resolution of type (5). Since E is globally generated with $\mathrm{h}^0(E) = c$ and $3 = \operatorname{rk} E \le c$, we can take 2 general global sections of E, $\sigma_1, \sigma_2 \in \mathrm{H}^0(E)$, and consider the curve $C = (\sigma_1 \wedge \sigma_2)_0$ defined as the degeneracy locus in $\mathbb{P}^3$ where σ_1 and σ_2 are linearly dependent. We thus have a short exact sequence

$$0 \to \mathcal{O}^2_{\mathbb{P}^3} \to E \to \mathcal{I}_C(c_1(E)) \to 0,$$

where $c_1(E) = b - 2a$ denotes the first Chern class of E. Taking cohomology of this sequence we get

$$0 \to \mathrm{H}^0\left(\mathcal{O}^2_{\mathbb{P}^3}\right) \cong \mathbb{C}^2 \to \mathrm{H}^0(E) \cong \mathbb{C}^c \to \mathrm{H}^0\left(\mathcal{I}_C(c_1(E))\right) \to 0,$$

and so $\mathrm{H}^0\left(\mathcal{I}_C(c_1(E))\right) \neq 0$. Hence C is contained in a surface of degree $c_1(E)$ and therefore, $c_1(E) = b - 2a > 0$. We have just proved the following.

Proposition 2.1. *If E is a rank 3 vector bundle on $\mathbb{P}^3$ of homological dimension 2 and minimal linear resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}^c_{\mathbb{P}^3} \to E \to 0,$$

then it has positive first Chern class, i.e. $b - 2a > 0$.

Therefore, our restrictions (8) above can be improved:

$$\begin{cases} a > 0 & \Leftrightarrow \quad (\alpha+\beta)^2 + 2(\gamma+\delta)^2 > 3\left(\beta^2 + 2\delta^2\right), \\ b - 2a > 0 & \Leftrightarrow \quad -\alpha\beta - 2\gamma\delta > 2\left(\beta^2 + 2\delta^2\right). \end{cases} \tag{10}$$

Furthermore, from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}^2 \to E \to \mathcal{I}_C\left(b-2a\right) \to 0,$$

where $C = \left(\sigma_1 \wedge \sigma_2\right)_0$, we get a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^3}\left(-2\right)^a & & \mathcal{O}_{\mathbb{P}^3}\left(-2\right)^a & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^3}\left(-1\right)^b & & \mathcal{O}_{\mathbb{P}^3}\left(-1\right)^b & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^3}^2 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^c & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c-2} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^2 & \longrightarrow & E & \longrightarrow & \mathcal{I}_C\left(b-2a\right) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

So, in order to reach our goal we will first prove the existence of a curve C in $\mathbb{P}^3$ with a resolution of type

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^{c-2} \to \mathcal{I}_C(b-2a) \to 0.$$

We will then show that this implies the existence of a vector bundle E defined by (5) and satisfying $\chi\left(\operatorname{End} E\right) = 1$.

Given any positive integer p, denote

$$D_p = \frac{p(p+1)}{2}$$

and

$$d_p = \begin{cases} \frac{p(p+2)}{4}, & \text{if } p \text{ is even,} \\ \frac{(p+1)^2}{4}, & \text{if } p \text{ is odd.} \end{cases}$$

Proposition 2.2. *Given an odd integer $\gamma \leq -5$, let*

$$p = -\frac{3\gamma + 7}{2} \quad \text{and} \quad s = \frac{3\gamma^2 + 8\gamma - 3}{8}.$$

Then, there is a smooth connected curve $C_\gamma \subset \mathbb{P}^3$ of degree

$$\deg(C_\gamma) = \binom{p+1}{2} - s,$$

arithmetic genus

$$p_a = \frac{(2p+3)(p-1)(p-2)}{6} - s(p-1),$$

of maximal rank and with a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-p)^s \to \mathcal{O}_{\mathbb{P}^3}(-1-p)^{2s+p} \to \mathcal{O}_{\mathbb{P}^3}(-p)^{s+p+1} \to \mathcal{I}_{C_\gamma} \to 0.$$

Proof. Let $d = \binom{p+1}{2} - s$ and let us first prove that $d_p \leq d \leq D_p$. In fact, on the one hand

$$d = \binom{p+1}{2} - s = \frac{(p+1)\,p}{2} - s \leq \frac{(p+1)\,p}{2} = D_p.$$

On the other hand, if p is even, we have

$$\begin{aligned} & d_p = \frac{p(p+2)}{4} \leq \binom{p+1}{2} - s = d \\ \Leftrightarrow \quad & \frac{p(p+2)}{4} \leq \frac{(p+1)\,p}{2} - s \\ \Leftrightarrow \quad & 4s \leq p^2 \Leftrightarrow \gamma \leq -5 \end{aligned}$$

If p is odd, we similarly obtain

$$d_p = \frac{(p+1)^2}{4} \leq \binom{p+1}{2} - s \Leftrightarrow 4s \leq p^2 - 1 \Leftrightarrow \gamma \leq -7,$$

and observe that when $\gamma = -5$, we get $p = 4$ even. From our hypotheses on γ and p we may conclude that it also holds $d_p \leq d$.

Therefore, by Proposition 2.2 in [1], there exists a smooth connected curve C_γ of degree $d = \binom{p+1}{2} - s$ such that

$$\mathrm{H}^0\left(\mathbb{P}^3, \mathcal{I}_{C_\gamma}(p-1)\right) = \mathrm{H}^1\left(\mathbb{P}^3, \mathcal{I}_{C_\gamma}(p-1)\right) = \mathrm{H}^2\left(\mathbb{P}^3, \mathcal{I}_{C_\gamma}(p-1)\right) = 0.$$

In addition, by Remark 2.6 and Corollary 2.4 in [1], this curve has maximal rank and arithmetic genus

$$p_a = (p-1)d + 1 - \binom{p+2}{3} = \frac{(2p+3)(p-1)(p-2)}{6} - s(p-1).$$

Now, we may apply Corollary 5.1 in [2] to conclude that this curve C_γ has a p-linear resolution, that is, the ideal sheaf $\mathcal{I}_C$ has a p-linear resolution, and that this resolution is of the following type:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-p)^s \to \mathcal{O}_{\mathbb{P}^3}(-1-p)^{2s+p} \to \mathcal{O}_{\mathbb{P}^3}(-p)^{s+p+1} \to \mathcal{I}_{C_\gamma} \to 0.$$

□

From now on we will concentrate on those solutions (a, b, c) of the diophantine equation (6) such that $\beta = \delta = 1$ and $\alpha = \gamma + 1$, with α an even integer (or equivalently, γ an odd integer).

After applying the required substitutions in (7), we get

$$a = \frac{3\gamma^2 + 8\gamma - 3}{8}, \quad b = \frac{3\gamma^2 + 2\gamma - 17}{4}, \quad c = \frac{3\gamma^2 - 4\gamma - 7}{8}.$$

Moreover, the inequalities (10) become

$$\begin{cases} a > 0 & \Leftrightarrow \quad \frac{3\gamma^2+8\gamma-3}{8} > 0, \\ b - 2a > 0 & \Leftrightarrow \quad \frac{-3\gamma-7}{2} > 0 \end{cases}$$

and we get that γ is an odd integer less than or equal to -5 ($\gamma \leq -5$).

Observing that $b - 2a = \frac{-3\gamma-7}{2}$ and $a = \frac{3\gamma^2+8\gamma-3}{8}$ are the integers p and s of Proposition 2.2, we know that there is a curve C_γ whose ideal sheaf has a linear resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-p)^s \to \mathcal{O}_{\mathbb{P}^3}(-1-p)^{2s+p} \to \mathcal{O}_{\mathbb{P}^3}(-p)^{s+p+1} \to \mathcal{I}_{C_\gamma} \to 0,$$

or equivalently, of the form

$$0\to\mathcal{O}_{\mathbb{P}^3}\left(-2\right)^a\to\mathcal{O}_{\mathbb{P}^3}\left(-1\right)^b\to\mathcal{O}_{\mathbb{P}^3}^{b-a+1}\to\mathcal{I}_{C_\gamma}(b-2a)\to 0.$$

Now, Proposition 5.2 in [3] guarantees that we can take a non-trivial element

$$e_\gamma\in\mathrm{Ext}^1\left(\mathcal{I}_{C_\gamma}(b-2a),\mathcal{O}_{\mathbb{P}^3}\right)\cong\mathrm{H}^2\left(\mathbb{P}^3,\mathcal{I}_{C_\gamma}(b-2a-4)\right)$$

$$\cong\mathrm{H}^1\left(C,\mathcal{O}_{C_\gamma}(b-2a-4)\right)\cong\mathrm{H}^0\left(C,\omega_{C_\gamma}(4+2a-b)\right),$$

and so we have a diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{O}_{\mathbb{P}^3}\left(-2\right)^a & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathcal{O}_{\mathbb{P}^3}\left(-1\right)^b & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{b-a+1} & \longrightarrow & 0 \\
 & & & & & \overset{\phi}{\searrow} & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} & \longrightarrow & E_\gamma & \longrightarrow & \mathcal{I}_{C_\gamma}(b-2a) & \longrightarrow & 0. \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Applying the covariant functor $\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^3}^c,-\right)$ to the bottom row we get the short exact sequence

$$0\to\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^3}^c,\mathcal{O}_{\mathbb{P}^3}^{c-b+a-1}\right)\to\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^3}^c,E_\gamma\right)\to\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^3}^c,\mathcal{I}_{C_\gamma}(b-2a)\right)\to 0.$$

Taking the morphism $\phi\in\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^3}^c,\mathcal{I}_{C_\gamma}(b-2a)\right)$ we may conclude that there is a non-trivial morphism $\varphi\in\mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^3}^c,E_\gamma\right)$ that allows us to complete the above diagram to the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^3}(-2)^a & = & \mathcal{O}_{\mathbb{P}^3}(-2)^a & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{\mathbb{P}^3}(-1)^b & = & \mathcal{O}_{\mathbb{P}^3}(-1)^b & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c} & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{b-a+1} & \longrightarrow & 0 \\
 & & & & \downarrow \varphi & \searrow \phi & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} & \longrightarrow & E_\gamma & \longrightarrow & \mathcal{I}_{C_\gamma}(b-2a) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where the middle column is a resolution of E_γ. So, the following theorem is proved.

Theorem 2.3. *Let $\gamma \leq -5$ be any odd integer. Then, there is a rank 3 vector bundle E_γ of homological dimension 2 defined by a linear resolution of type*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\frac{3\gamma^2+8\gamma-3}{8}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\frac{3\gamma^2+2\gamma-17}{4}} \to \mathcal{O}_{\mathbb{P}^n}^{\frac{3\gamma^2-4\gamma-7}{8}} \to E_\gamma \to 0,$$

and such that $\chi(\operatorname{End} E_\gamma) = 1$.

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