Research Article

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Vector bundles E on $\mathbb{P}^3$ with homological dimension 2 and $\chi(\mathrm{End}\,E) = 1$

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Abstract: We find the complete integer solutions of the equation $X^2 + Y^2 + Z^2 - 4XY - 4YZ + 10XZ = 1$. As an application, we prove that, for each solution (a, b, c) such that $a > 0$, $b - 2a > 0$ and $(b - 2a)^2 \geq 4a$, there is a vector bundle E on $\mathbb{P}^3$ defined by a minimal linear resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0$. In particular, E satisfies $\chi(\mathrm{End}\,E) = 1$.

Keywords: Homological dimension, linear resolutions, Diophantine equations, ternary quadratic forms

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1 Introduction

Let E be a vector bundle on $\mathbb{P}^n$, with $n \geq 2$, of homological dimension 1 and defined by a minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^a \to \mathcal{O}_{\mathbb{P}^n}^b \to E \to 0. \tag{1.1}$$

These bundles were first introduced by Dolgachev and Kapranov [5], and they are usually called Steiner bundles. Their study is extensive and has been addressed from very different points of view. We refer in particular to [2, 3, 9–11].

As it can be easily deduced, e.g. from the formula given in [11, Proposition 3.4], the Euler characteristic of $\mathrm{End}(E)$ is $\chi(\mathrm{End}(E)) = a^2 + b^2 - (n + 1)ab$. It is worth recalling that often one is interested in studying vector bundles satisfying $\chi(\mathrm{End}\,E) = 1$, for they provide good candidates for exceptional bundles, important objects when studying stability and derived categories. We know that, for $n \geq 2$, all positive integer solutions of the quadratic form

$$X^2 + Y^2 - (n + 1)XY = 1 \tag{1.2}$$

are pairs $(X, Y) = (u_k, u_{k+1})$, with $k \geq 1$, where $\{u_k\}_{k\geq 0}$ is the sequence defined recursively by

$$\begin{cases} u_0 = 0, \\ u_1 = 1, \\ u_{k+2} = (n + 1)u_{k+1} - u_k. \end{cases}$$

So, if E is a vector bundle as in (1.1) and $\chi(\mathrm{End}(E)) = 1$, then $a = u_k$ and $b = u_{k+1}$ for some $k \geq 1$. Conversely, for each solution pair (u_k, u_{k+1}) of equation (1.2), we know that there exists a vector bundle E defined by an

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exact sequence of type (1.1), with $a = u_k$ and $b = u_{k+1}$. In fact, in this case, we know how to construct all such vector bundles (see [13, Lemma 2.2.5 and Theorem 2.2.7]).

In the present work, we address the case of homological dimension 2. More precisely, we consider a non-splitting vector bundle E defined by a minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-2)^a \to \mathcal{O}_{\mathbb{P}^n}(-1)^b \to \mathcal{O}_{\mathbb{P}^n}^c \to E \to 0, \tag{1.3}$$

with $n \geq 3$. From [11, Proposition 3.4], the Euler characteristic of End E is the ternary quadratic form

$$\chi(\mathrm{End}(E)) = a^2 + b^2 + c^2 - (n+1)ab - (n+1)bc + \binom{n+2}{2}ac.$$

If $\chi(\mathrm{End}(E)) = 1$, then a, b and c are positive integer solutions of the Diophantine equation

$$X^2 + Y^2 + Z^2 - (n+1)XY - (n+1)YZ + \binom{n+2}{2}XZ = 1. \tag{1.4}$$

We want to know if, given an arbitrary positive integer solution (a, b, c) of the above equation, there exists a vector bundle E with a linear resolution of type (1.3) and exponents (also called the Betti numbers) a, b and c.

The first obstacle we undergo is to find all positive integer solutions of (1.4). In Section 2, we delve into the theory of ternary quadratic forms, following [4]. We begin by recalling some generalities and proceed with the classification of the family of ternary quadratic forms (1.4), for n up to 20, using Legendre's theorem. Describing all positive solutions is a much harder problem than the corresponding one in the case of homological dimension 1, so we will need to restrict our study to $n = 3$ for which the above equation becomes

$$X^2 + Y^2 + Z^2 - 4XY - 4YZ + 10XZ = 1. \tag{1.5}$$

We end Section 2 with the complete integers solutions of (1.5) (Proposition 2.6).

In Section 3, we treat the existence problem and prove our main result, Theorem 3.3. For each solution (a, b, c) of equation (1.5) such that $a > 0$, $b - 2a > 0$ and $(b - 2a)^2 \geq 4a$, we construct a vector bundle E on $\mathbb{P}^3$ defined by a minimal linear resolution of length 2, thus satisfying $\chi(\mathrm{End}(E)) = 1$. The main result behind is the existence of a curve C on $\mathbb{P}^3$ whose ideal sheaf $\mathcal{I}_C$ is defined by a linear resolution of length 2 and such that $\mathrm{Ext}^1(\mathcal{I}_C(b-2a), \mathcal{O}_{\mathbb{P}^3}) \neq 0$ (see Proposition 3.2). Then we can take a nontrivial extension E of $\mathcal{O}_{\mathbb{P}^3}$ by $\mathcal{I}_C(b - 2a)$ and prove that E has the required resolution.

In [11], two of the authors approached the problem of characterizing exceptional vector bundles of arbitrary homological dimension. They were able to explicitly construct a family of exceptional vector bundles of homological dimension 2 and linear resolution on the projective space $\mathbb{P}^n$ (see [11, Proposition 4.3]). When $n = 3$, this family is a subset of the set of vector bundles constructed herein.

Notation 1.1. If E is a vector bundle on $\mathbb{P}^n$, we denote $\mathrm{h}^q(E) = \dim \mathrm{H}^q(E) = \dim \mathrm{H}^q(\mathbb{P}^n, E)$. The Euler characteristic of E is defined by the integer

$$\chi(E) = \sum_{i=0}^{n} (-1)^i \mathrm{h}^i(E).$$

Recall furthermore that we have an isomorphism $\mathrm{End}\, E \cong E^\vee \otimes E$. The minimal length of a resolution of E, i.e. the homological dimension of E, is denoted by $\mathrm{hd}(E)$.

2 The solutions of the Diophantine equation $\chi(\mathrm{End}\, E) = 1$

Before we specify to the case $n = 3$, let us consider the more general family of Diophantine equations (1.4), arising from the algebraic geometric problem described in the introduction

$$X^2 + Y^2 + Z^2 - (n+1)XY - (n+1)YZ + \binom{n+2}{2}XZ = 1, \quad n \geq 3,$$

where $\binom{n}{k}$ denote the binomial coefficients.

The solution of this Diophantine equation is related to the classical problem of representing integers by ternary quadratic forms. A well-known example, solved by Gauss [12, p. 79], is to determine which integers can be written in the form $x^2 + y^2 + z^2$.

The first step is to write the above equation in a reduced form. Then, using Legendre's theorem [8], we classify each reduced form for a given $n \geq 3$ as being either isotropic or anisotropic. Each one of these two cases has a different technique. As mentioned before, we shall concentrate in the isotropic case $n = 3$ using the methods described in [4]. As we will see, although the method is explained in Cassel's book, explicit formulae of the integral solutions for a given n are not easy to obtain. In Cassel's own words [4, p. 303]: "The numerical details could be tedious in any given case."

2.1 The reduced equations

Equation (1.4) can be written as

$$\left(\frac{n+1}{2}X - Y + \frac{n+1}{2}Z\right)^2 - \frac{n^2+3n-2}{8}(X-Z)^2 - \frac{n^2+n-4}{8}(X+Z)^2 = 1$$

and also

$$((n+1)X - 2Y + (n+1)Z)^2 - \frac{n^2+n-4}{2}(X+Z)^2 - \frac{n^2+3n-2}{2}(X-Z)^2 = 4.$$

Write

$$a = a_n = \frac{n^2+n-4}{2} \quad \text{and} \quad b = b_n = \frac{n^2+3n-2}{2}. \tag{2.1}$$

Observe that a_n and b_n are positive integers for $n \geq 2$ (recall that we are concerned only with $n \geq 3$). Moreover, a_n and b_n are related by $b_n = a_n + n + 1$ for all n. Now, we make a first change of variables

$$u = (n+1)X - 2Y + (n+1)Z, \quad v = X + Z, \quad w = X - Z.$$

Equation (1.4) becomes

$$u^2 = 4 + \frac{n^2+n-4}{2}v^2 + \frac{n^2+3n-2}{2}w^2,$$

or with the notation introduced by (2.1), simply $u^2 = 4 + av^2 + bw^2$. Rewriting this last equation as

$$\left(\frac{u}{2}\right)^2 - a\left(\frac{v}{2}\right)^2 - b\left(\frac{w}{2}\right)^2 = 1$$

and performing a second change of variables

$$x = \frac{u}{2}, \quad y = \frac{v}{2}, \quad z = \frac{w}{2},$$

we obtain

$$x^2 - ay^2 - bz^2 = 1. \tag{2.2}$$

If a and b are not perfect squares, then $f = x^2 - ay^2 - bz^2$ is the reduced ternary form associated with equation (1.4). Otherwise, we need to change variables one last time to obtain the reduced ternary form.

2.2 Notation and terminology

For the sake of completeness, we recall some basic terminology about rational quadratic forms (see [4] for a detailed account). As usual, $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{Z}$ the ring of integers. Let $\mathcal{I}$ be a ring in $\mathbb{Q}$. We allow the possibility $\mathcal{I} = \mathbb{Q}$. We say that a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{i,j} a_{ij}x_ix_j,$$

with $a_{ij} = a_{ji} \in \mathbb{Q}$ in n variables $x_1, \dots, x_n$, represents an element $c \in \mathbb{Q}$ over $\mathcal{I}$ if there is some

$$\mathbf{b} = (b_1, \dots, b_n) \in \mathcal{I}^n \quad \text{such that} \quad f(\mathbf{b}) = c.$$

In particular, we shall speak of rational representations if $\mathcal{I} = \mathbb{Q}$ and of integral representations if $\mathcal{I} = \mathbb{Z}$. In this note, we are only concerned with integral representations. The quadratic form is integer-valued if $f(\mathbf{b}) \in \mathcal{I} = \mathbb{Z}$ for all $\mathbf{b} \in \mathcal{I}^n$. In terms of coefficients, it means that $a_{ii} \in \mathcal{I}$ and $2a_{ij} \in \mathcal{I}$ $(1 \le i \le j \le n)$. We say that f is classically integral if $a_{ij} \in \mathcal{I}$ $(1 \le i \le n, 1 \le j \le n)$, that is if the elements of the Gram matrix are in $\mathcal{I}$. When f is classically integral, we say that it is primitive if $\mathrm{g.c.d.}(a_{ij}) = 1$, where g.c.d. denotes the greatest common divisor.

A quadratic form f is said to be isotropic if it represents 0 nontrivially; otherwise, it is anisotropic.

Two forms f and g are $\mathbb{Z}$-equivalent if there is a square integral matrix T such that $g(\mathbf{X}) = f(T\mathbf{X})$ and $\det(T) \in \mathcal{U}(\mathbb{Z}) = \{\pm 1\}$, where $\mathcal{U}(\mathbb{Z})$ is the group of units of $\mathbb{Z}$. The equivalence is proper if $\det(T) = +1$ and improper if $\det(T) = -1$. Clearly, proper equivalence is an equivalence relation. An equivalence of a form f with itself is an integral automorph. Denote by $\mathcal{O}(f)$ the set of automorphs of f and by $\mathcal{O}^+(f)$ the set of proper automorphs. The set $\mathcal{O}^+(f)$ forms a group under composition, the group of (proper) automorphs of f, also called the orthogonal group of f.

We also need some terminology on binary quadratic forms. Let $f = ax^2 + bxy + cy^2$ be an integral binary quadratic form in the non-classical definition. The discriminant of f is given by $\Delta = b^2 - 4ac$, and it is related with the determinant $d(f)$ of f, i.e. of the associated Gram matrix, by $\Delta(b) = -4d(f)$.

2.3 Classification of ternary forms

In order to decide if an indefinite ternary quadratic form is isotropic or anisotropic, we may use the beautiful theorem of Legendre (see [14, Chapter II, Section XIV, Chapter IV and Appendix I] for a historical perspective). The version bellow can be found in [8, Proposition 17.3.2] and is more suitable for our purposes. Let $\mathcal{R}$ denote the following binary relation over the set of nonzero integers: given $m, n \in \mathbb{Z}\backslash\{0\}$,

$$m\mathcal{R}n \iff x^2 = m \pmod{n}$$

for some integer x.

Theorem 2.1 (Legendre's theorem). *Let a, b be positive squarefree integers. Then $x^2 = ay^2 + bz^2$ has a nontrivial solution, that is the form $f = x^2 - ay^2 - bz^2$ is isotropic if, and only if, the following conditions are satisfied:*
(i) $a\mathcal{R}b$;
(ii) $b\mathcal{R}a$;
(iii) $-\frac{ab}{d^2}\mathcal{R}d$, *where* $d = \mathrm{g.c.d.}(a, b)$.

The following remark is useful for explicit computations.

Remark 2.2. Note that if b is odd, then

$$a\mathcal{R}b \implies \left(\frac{a}{b}\right) = 1,$$

where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacobi symbol. We recall that, when $b = p$ is prime, the Legendre symbol is defined to be

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a nonzero solution,} \\ -1 & \text{if } x^2 \equiv a \pmod{p} \text{ has no solution,} \end{cases}$$

and $\left(\frac{a}{p}\right) = 0$ when p divides a. The Jacobi symbol generalizes the Legendre symbol as follows. Given an arbitrary odd integer b, if $b = p_1^{e_1} \dots p_k^{e_k}$, where p_i are odd primes, set

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)^{e_1} \cdots \left(\frac{a}{p_k}\right)^{e_k}.$$

Example 2.3. For $n = 4$, the (reduced) ternary form (2.2) is $x^2 = 8y^2 + 13z^2 = 2(2y)^2 + 13z^2$. With the obvious change of variables, we may represent it by $x^2 = 2y^2 + 13z^2$. Now, $13\mathcal{R}2$ since $9 = 13 - 2 \times 2$. However,

$$\left(\frac{2}{13}\right) = -\left(\frac{1}{13}\right) = -1,$$

and so, by the above remark, the form is anisotropic.

We now present a list of reduced ternary quadratic forms associated with equation (2.2) for n from 3 up to 20. The classification of each form $f = x^2 - ay^2 - bz^2$ is obtained using Legendre's theorem and some basic computations with the Jacobi symbol.

n	$f = x^2 - ay^2 - bz^2$	Classification
3	$x^2 - 4y^2 - 8z^2$	isotropic
4	$x^2 - 8y^2 - 13z^2$	anisotropic
5	$x^2 - 13y^2 - 19z^2$	anisotropic
6	$x^2 - 19y^2 - 29z^2$	anisotropic
7	$x^2 - 26y^2 - 34z^2$	anisotropic
8	$x^2 - 34y^2 - 43z^2$	anisotropic
9	$x^2 - 43y^2 - 53z^2$	isotropic
10	$x^2 - 54y^2 - 64z^2$	isotropic
11	$x^2 - 64y^2 - 76z^2$	isotropic
12	$x^2 - 76y^2 - 89z^2$	anisotropic
13	$x^2 - 89y^2 - 103z^2$	anisotropic
14	$x^2 - 103y^2 - 118z^2$	anisotropic
15	$x^2 - 118y^2 - 134z^2$	anisotropic
16	$x^2 - 134y^2 - 151z^2$	anisotropic
17	$x^2 - 151y^2 - 169z^2$	isotropic
18	$x^2 - 169y^2 - 188z^2$	isotropic
19	$x^2 - 188y^2 - 208z^2$	anisotropic
20	$x^2 - 208y^2 - 229z^2$	anisotropic

Table 1

One case is particularly easy. In fact, the form is isotropic if either a or b is a square. For suppose a is a square. Then $(\sqrt{a}, 1, 0)$ is a nontrivial integral representation of 0 by f. This is the case when $n = 3, 10, 11, 17$ and 18; see Table 1.

Example 2.4. Consider $n = 9$. The form is $f = x^2 - 43y^2 - 53z^2$, and both 43 and 53 are squarefree (actually, prime numbers). We have $19^2 = 43 + 6 \times 53$, so $43\mathcal{R}53$, and also $15^2 = 53 + 4 \times 43$, that is $53\mathcal{R}43$. By Legendre's theorem, f is isotropic.

2.4 Isotropic ternary forms

It turns out that the study of the representation of integers by isotropic ternary forms reduces to the study of the form $f_0(x, y, z) = xz - y^2$.

We now follow [4, p. 301]. Let

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}^{\pm}(2, \mathbb{Z}),$$

where $\mathrm{SL}^{\pm}(2, \mathbb{Z})$ is the modular group of elements of $\mathrm{GL}(2, \mathbb{Z})$ with determinant ± 1. Then

$$T(S) = \tilde{S} = \begin{pmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\gamma & \delta^2 \end{pmatrix}$$

is an automorph of f_0.

According to [4, Lemma 5.2], the map $S \mapsto T(S)$ is a homomorphism of $\mathrm{SL}^{\pm}(2, \mathbb{Z})$ onto $\mathcal{O}^{+}(f_0)$ with kernel $\{\pm I\}$. In other words, we have the following description of the group of automorphs of f_0:

$$\mathcal{O}^{+}(f_0) \cong \mathrm{SL}^{\pm}(2, \mathbb{Z})/\{\pm I\}.$$

Given any other isotropic ternary form f, there exist an integer m and an integral matrix M such that

$$mf(\mathbf{X}) = f_0(M\mathbf{X});$$

see [4, p. 303]. Hence, any integral representation $\mathbf{b} \in \mathbb{Z}^3$ of a given integer $r \in \mathbb{Z}$ by f gives rise to the integral representation $M\mathbf{b}$ of mr by f_0. On the other hand, every representation of mr by f_0 is of the type $T(S)c$, where c runs through a finite set, say Φ_f, and $S \in \mathrm{SL}^{\pm}(2, \mathbb{Z})/\{\pm I\}$. Furthermore, $T(S)c$ comes from an integral representation of r by f whenever

$$M^{-1}T(S)c \in \mathbb{Z}^3. \tag{2.3}$$

We conclude that, in order to find all the integral solutions of the Diophantine equation (2.2),

$$x^2 - ay^2 - bz^2 = 1,$$

with $f = x^2 - ay^2 - bz^2$ isotropic, it amounts to the following:
(i) find M and m;
(ii) identify the finite set of $\Phi_f = \{c_1, \dots, c_k\}$;
(iii) solve a system of three modular equations (mod $\det(M)$).
If a and b in equation (2.2) are both squarefree, it can be difficult to determine explicitly M and m. These computations are not explained in Cassel's book. But the case where a or b is a square is fairly simple.

Proposition 2.5. *Let $f = x^2 - ay^2 - bz^2$ with a and b positive integers.*
(i) *If a is a square, then f is isotropic and we may take*

$$m = \tilde{b} \quad \text{and} \quad M = M_{\tilde{b}} = \begin{bmatrix} \tilde{b} & \tilde{b} & 0 \\ 0 & 0 & \tilde{b} \\ 1 & -1 & 0 \end{bmatrix},$$

where $\tilde{b} \mid b$ and $\tilde{b}$ has squarefree prime factors.
(ii) *If b is a square, then f is isotropic and we may take*

$$m = \tilde{a} \quad \text{and} \quad M = M_{\tilde{a}} = \begin{bmatrix} \tilde{a} & 0 & \tilde{a} \\ 0 & \tilde{a} & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

where $\tilde{a} \mid a$ and $\tilde{a}$ has squarefree prime factors.

Proof. (i) Since a is a square, we obtain a new ternary form, which is primitive,

$$\tilde{f} = x^2 - (\sqrt{a}y)^2 - \tilde{b}(\sqrt{c}z)^2,$$

with the obvious change of variables. Here, $b = \tilde{b}c$, $\tilde{b}$ is a squarefree factor and c is a square (with the possibility that $\sqrt{c} = 1$). In matrix form, $f_0(M_{\tilde{b}}\mathbf{X})$ with $M_{\tilde{b}}$ as above becomes:

$$M_{\tilde{b}}^t M_{f_0} M_{\tilde{b}} = \begin{bmatrix} \tilde{b} & 0 & 1 \\ \tilde{b} & 0 & -1 \\ 0 & \tilde{b} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{b} & \tilde{b} & 0 \\ 0 & 0 & \tilde{b} \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{b} & 0 & 0 \\ 0 & -\tilde{b} & 0 \\ 0 & 0 & -\tilde{b}^2 \end{bmatrix} = \tilde{b} M_{\tilde{f}}.$$

The case (ii) is similar. □

Now, we identify the set Φ_f. By [4, Lemma 5.3], the elements c_i correspond to representatives of proper equivalence classes of classically integral binary forms of determinant m. Surely, there are only finitely many such classes.

2.5 The case $n = 3$

Now, we specify to the case $n = 3$. The quadratic form $f = x^2 - 4y^2 - 8z^2$ is clearly isotropic, for $(\pm 2, \pm 1, 0)$ are nontrivial representations of 0 by f. We want to find all the integral solutions of

$$f = 1 \iff x^2 - (2y)^2 - 2(2z)^2 = 1.$$

With the change variables $X_1 = x$, $X_2 = 2y$, $X_3 = 2z$, we have

$$f(X_1, X_2, X_3) = X_1^2 - X_2^2 - 2X_3^2 = 1. \tag{2.4}$$

Equation (2.4) is the reduced, primitive form of equation (1.5). From the above discussion, there exist $m \in \mathbb{Z}$ and an integral 3×3 matrix M such that $mf(\mathbf{X}) = f_0(M\mathbf{X})$, with $\mathbf{X} = (X_1, X_2, X_3) \in \mathbb{Z}^3$. By Proposition 2.5, we have

$$m = 2 \quad \text{and} \quad M = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Then

$$f(\mathbf{X}) = 1 \implies f_0(M\mathbf{X}) = 2 \times 1 = 2.$$

There is only one proper equivalence class of classically integral binary forms of determinant 2; it corresponds to the reduced binary form $(1, 0, 2)$. It has, of course, discriminant $\Delta = -8$.

Now, from (2.3), the solutions of equation (2.4) are given by

$$\begin{aligned} M^{-1}T(S)c = \frac{1}{8}\begin{bmatrix} 2 & 0 & 4 \\ 2 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} T(S)c &= \frac{1}{4}\begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} \alpha^2 & 2\alpha\gamma & \gamma^2 \\ \alpha\beta & \alpha\delta + \beta\gamma & \gamma\delta \\ \beta^2 & 2\beta\gamma & \delta^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} \alpha^2 + 2\gamma^2 + 2\beta^2 + 4\delta^2 \\ \alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 \\ 2\alpha\beta + 4\gamma\delta \end{bmatrix}. \end{aligned} \tag{2.5}$$

Therefore, we obtain a system with three modular equations,

$$\begin{cases} \alpha^2 + 2\gamma^2 + 2\beta^2 + 4\delta^2 \equiv 0 \pmod 4, \\ \alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 \equiv 0 \pmod 4, \\ 2\alpha\beta + 4\gamma\delta \equiv 0 \pmod 4 \end{cases}$$

$$\iff \begin{cases} \alpha^2 + 2\gamma^2 + 2\beta^2 \equiv 0 \pmod 4, \\ 2\alpha\beta \equiv 0 \pmod 4 \end{cases} \implies \begin{cases} \alpha^2 + 2\gamma^2 + 2\beta^2 \equiv 0 \pmod 2, \\ 2\alpha\beta \equiv 0 \pmod 2 \end{cases}$$

We conclude that $\alpha^2 \equiv 0 \pmod 2$. In other words, the integral solutions of the Diophantine equation (2.4) are parameterized by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}^{\pm}(2, \mathbb{Z}),$$

with α even.

From the above, we may now obtain the integer solutions of equation (1.5) applying formulae (2.5) and then going backwards doing the proper change of variables. Hence, we have the following proposition.

Proposition 2.6. *The integer solutions of equation* (1.5) *are given by*

$$\begin{aligned} X &= \frac{\alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 + 2\alpha\beta + 4\gamma\delta}{8}, \\ Y &= \frac{\alpha^2 + 2\gamma^2 - 6\beta^2 - 12\delta^2}{4}, \\ Z &= \frac{\alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 - 2\alpha\beta - 4\gamma\delta}{8}, \end{aligned} \tag{2.6}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, α even and $\alpha\delta - \beta\gamma = \pm 1$.

3 Vector bundles on $\mathbb{P}^3$ of homological dimension 2 and $\chi(\operatorname{End} E) = 1$

Let E be a vector bundle on $\mathbb{P}^3$ of homological dimension 2 defined by a minimal linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0 \tag{3.1}$$

and such that $\chi(\operatorname{End} E) = 1$. Then, from [11, Proposition 3.4], this is equivalent to the fact that the Betti numbers a, b and c satisfy $a^2 + b^2 + c^2 - 4ab - 4bc + 10ac = 1$. The 3-tuple (a, b, c) is thus a solution of the Diophantine equation (1.5) and, as proved in Proposition 2.6, a, b and c are (positive) integers of form (2.6) for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, α even and $\alpha\delta - \beta\gamma = \pm 1$. Observe that, according to this parameterization, we easily deduce that $\operatorname{rk} E = a - b + c = \beta^2 + 2\delta^2$.

Now, we are interested in finding those integer solutions (2.6) which correspond to such a vector bundle E. It is obvious that there are infinitely many that do not give rise to any such bundle. So, for instance, if $\alpha = 0$, then $\beta = \pm 1$ and $\gamma = \pm 1$, and we get

$$a = \frac{-4\delta^2 \pm 4\delta}{8} = -\frac{1}{2}\delta^2 \pm \frac{1}{2}\delta \leq 0$$

(note that we do not have a complete description of all positive solutions of (1.5) as in the case of homological dimension 1). Also, the set of solutions obtained include cases we are excluding; in the case when $\delta = 0$, and so $\beta = \pm 1$, the rank of E is $\operatorname{rk} E = \beta^2 = 1$, contradicting the fact that $\operatorname{rk} E \geq 2$ (provided E is a non-splitting vector bundle; see [2, Corollary 1.7]).

We next prove the following general fact concerning the first Chern class of any vector bundle defined by (3.1).

Proposition 3.1. *If E is a vector bundle on $\mathbb{P}^3$ of homological dimension 2 and minimal linear resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0,$$

then it has positive first Chern class, i.e. $b - 2a > 0$.

Proof. Since E is globally generated with $\mathrm{h}^0(E) = c$ and $r := \operatorname{rk} E \leq c$, we can take $r - 1$ global sections of E, $\sigma_1, \dots, \sigma_{r-1} \in \mathrm{H}^0(E)$, and consider the curve $C = (\sigma_1 \wedge \cdots \wedge \sigma_{r-1})_0$, defined as the degeneracy locus in $\mathbb{P}^3$, where $\sigma_1, \dots, \sigma_{r-1}$ are linearly dependent. We thus have a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}^{r-1} \to E \to \mathcal{I}_C(c_1(E)) \to 0,$$

where $c_1(E) = b - 2a$ denotes the first Chern class of E. Taking cohomology of this sequence, we get

$$0 \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}^{r-1}) \cong \mathbb{C}^{r-1} \to \mathrm{H}^0(E) \cong \mathbb{C}^c \to \mathrm{H}^0(\mathcal{I}_C(c_1(E))) \to 0,$$

and so $\mathrm{H}^0(\mathcal{I}_C(c_1(E))) \neq 0$. Hence C is contained in a surface of degree $c_1(E)$, and therefore, $c_1(E) > 0$. □

In order to prove our main result, we will first prove the existence of curves in $\mathbb{P}^3$ whose ideal sheaf is defined by a minimal linear resolution of length 2.

For any positive integer p, denote

$$D_p := \frac{p(p+1)}{2} \quad \text{and} \quad d_p := \begin{cases} \frac{p(p+2)}{4} & \text{if } p \text{ is even}, \\ \frac{(p+1)^2}{4} & \text{if } p \text{ is odd}. \end{cases}$$

Proposition 3.2. *Suppose $p, s > 0$ are positive integers that satisfy*

$$\begin{cases} p^2 \geq 4s & \text{if } p \text{ is even}, \\ p^2 - 1 \geq 4s & \text{if } p \text{ is odd}. \end{cases} \tag{3.2}$$

Then there exists a smooth connected curve $C \subset \mathbb{P}^3$ with minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-p)^s \to \mathcal{O}_{\mathbb{P}^3}(-1-p)^{2s+p} \to \mathcal{O}_{\mathbb{P}^3}(-p)^{s+p+1} \to \mathcal{I}_C \to 0,$$

maximal rank, degree

$$\deg(C) = \binom{p+1}{2} - s$$

and arithmetic genus

$$p_a = \frac{(2p+3)(p-1)(p-2)}{6} - s(p-1).$$

Furthermore, $\mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(p-4)) \neq 0$.

Proof. Let $d = \binom{p+1}{2} - s$, and let us first prove that $d_p \le d \le D_p$. In fact, on the one hand,

$$d = \binom{p+1}{2} - s = \frac{(p+1)p}{2} - s \le \frac{(p+1)p}{2} = D_p.$$

On the other hand, if p is even, we have

$$d_p = \frac{p(p+2)}{4} \le \binom{p+1}{2} - s = d \iff \frac{p(p+2)}{4} \le \frac{(p+1)p}{2} - s \iff 4s \le p^2.$$

If p is odd, we similarly obtain

$$d_p = \frac{(p+1)^2}{4} \le \binom{p+1}{2} - s \iff 4s \le p^2 - 1.$$

Because of our hypotheses (3.2), we also conclude that $d_p \le d$.

Therefore, $d_p \le d \le D_p$, and by [1, Proposition 2.2], there exists a smooth connected curve C of degree $d = \binom{p+1}{2} - s$ such that

$$\mathrm{H}^0(\mathbb{P}^3, \mathcal{I}_C(p-1)) = \mathrm{H}^1(\mathbb{P}^3, \mathcal{I}_C(p-1)) = \mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(p-1)) = 0.$$

In addition, by [1, Corollary 2.4 and Remark 2.6], this curve is not contained in any surface of degree less than p, it has maximal rank and arithmetic genus

$$p_a = (p-1)d + 1 - \binom{p+2}{3} = \frac{(2p+3)(p-1)(p-2)}{6} - s(p-1).$$

Now, we may apply [6, Corollary 5.1] to conclude that this curve C has a p-linear resolution, that is the ideal sheaf $\mathcal{I}_C$ has a p-linear resolution, and that this resolution is of the following type:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-p)^s \to \mathcal{O}_{\mathbb{P}^3}(-1-p)^{2s+p} \to \mathcal{O}_{\mathbb{P}^3}(-p)^{s+p+1} \to \mathcal{I}_C \to 0.$$

Finally, to prove that $\mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(p-4)) \neq 0$, consider the short exact sequence

$$0 \to \mathcal{I}_C(p-4) \to \mathcal{O}_{\mathbb{P}^3}(p-4) \to \mathcal{O}_C(p-4) \to 0,$$

from which we see that

$$\mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(p-4)) \cong \mathrm{H}^1(C, \mathcal{O}_C(p-4)) \cong \mathrm{H}^0(C, \omega_C(4-p)).$$

Since C is smooth, [7, Proposition 5.2] guarantees the non-vanishing of $\mathrm{H}^0(C, \omega_C(3-p))$. Then we also have $\mathrm{H}^0(C, \omega_C(4-p)) \cong \mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(p-4)) \neq 0$, thus completing the proof. □

We are now ready to construct, for each solution of (1.5), a vector bundle on the 3-dimensional projective space with the required properties.

Suppose (a, b, c) is a solution of (1.5) and hence, as stated in Proposition 2.6, of form (2.6). Let

$$p = b - 2a = -\alpha\beta - 2\gamma\delta - 2(\beta^2 + 2\delta^2),$$
$$s = a = \frac{\alpha^2 + 2\gamma^2 - 2\beta^2 - 4\delta^2 + 2\alpha\beta + 4\gamma\delta}{8}.$$

Observe that p is even for a is even. If p and s are positive integers and furthermore $p^2 \geq 4s$, we know from Proposition 3.2 that there is a smooth curve C whose ideal sheaf has a linear resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2-p)^s \to \mathcal{O}_{\mathbb{P}^3}(-1-p)^{2s+p} \to \mathcal{O}_{\mathbb{P}^3}(-p)^{s+p+1} \to \mathcal{I}_C \to 0.$$

Twisting this sequence by $\mathcal{O}_{\mathbb{P}^3}(p)$ and rewriting it in terms of a, b and c, we get

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^{b-a+1} \to \mathcal{I}_C(b-2a) \to 0.$$

Moreover, also from Proposition 3.2, we know that $\mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(b-2a-4)) \neq 0$. Since

$$\mathrm{H}^2(\mathbb{P}^3, \mathcal{I}_C(b-2a-4)) \cong \operatorname{Ext}^1(\mathcal{I}_C(b-2a), \mathcal{O}_{\mathbb{P}^3}),$$

we can take a nontrivial element $e \in \operatorname{Ext}^1(\mathcal{I}_C(b-2a), \mathcal{O}_{\mathbb{P}^3})$. Then we have a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathcal{O}_{\mathbb{P}^3}(-2)^a \arrow[d] & \\
 & & & \mathcal{O}_{\mathbb{P}^3}(-1)^b \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{c} \arrow[r] \arrow[dr, dashed, "\phi"] & \mathcal{O}_{\mathbb{P}^3}^{b-a+1} \arrow[r] \arrow[d] & 0 \\
e\colon 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} \arrow[r] & E \arrow[r] & \mathcal{I}_C(b-2a) \arrow[r] \arrow[d] & 0 \\
 & & & 0, &
\end{tikzcd}$$

where E is a representative of the corresponding equivalence class of e. Applying the covariant functor $\operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^c, -)$ to the bottom row, we get the short exact sequence

$$0 \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^c, \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1}) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^c, E) \to \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^c, \mathcal{I}_C(b-2a)) \to 0.$$

Then, taking the morphism $\phi \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^c, \mathcal{I}_C(b-2a))$, there is a nontrivial morphism $\varphi \in \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^3}^c, E)$ that allows us to complete the above diagram to the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{O}_{\mathbb{P}^3}(-2)^a \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-2)^a \arrow[d] & \\
 & & \mathcal{O}_{\mathbb{P}^3}(-1)^b \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1)^b \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{c} \arrow[r] \arrow[d, "\varphi"] \arrow[dr, dashed, "\phi"] & \mathcal{O}_{\mathbb{P}^3}^{b-a+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}^{c-b+a-1} \arrow[r] & E \arrow[r] \arrow[d] & \mathcal{I}_C(b-2a) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

The left column is a resolution of E, and hence $\operatorname{hd}(E) \leq 2$. To check that the resolution is minimal, that is the homological dimension of E is exactly 2, observe that one can easily deduce from the diagram that $\mathrm{H}^1(E(-2)) \neq 0$. But this would contradict [11, Lemma 3.1] if we had $\operatorname{hd}(E) \leq 1$.

We have just proved our main theorem.

Theorem 3.3. *Suppose (a, b, c) is a solution of (1.5). If $a > 0$, $b - 2a > 0$ and $(b - 2a)^2 \geq 4a$, then there is a vector bundle E of homological dimension 2 and linear resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^a \to \mathcal{O}_{\mathbb{P}^3}(-1)^b \to \mathcal{O}_{\mathbb{P}^3}^c \to E \to 0.$$

In particular, E satisfies $\chi(\operatorname{End} E) = 1$.

We can apply Theorem 3.3 to get infinitely many families of such vector bundles. For example, consider the solutions of (1.5) with $\beta = \delta = 1$ and $\alpha = \gamma - 1$ even. In this case, we have

$$\begin{aligned}
a &= \frac{(\gamma-1)^2 + 2\gamma^2 - 6 + 2(\gamma-1) + 4\gamma}{8} = \frac{3\gamma^2 + 4\gamma - 7}{8},\\
b &= \frac{(\gamma-1)^2 + 2\gamma^2 - 18}{4} = \frac{3\gamma^2 - 2\gamma - 17}{4},\\
c &= \frac{(\gamma-1)^2 + 2\gamma^2 - 6 - 2(\gamma-1) - 4\gamma}{8} = \frac{3\gamma^2 - 8\gamma - 3}{8},
\end{aligned}$$

where γ is an odd integer. Moreover, the conditions

$$\begin{cases}
a > 0 & \Longleftrightarrow \quad 3\gamma^2 + 4\gamma - 7 > 0,\\
b > 2a & \Longleftrightarrow \quad \gamma < -5/3,\\
(b-2a)^2 \geq 4a & \Longleftrightarrow \quad 3\gamma^2 + 22\gamma + 39 \geq 0
\end{cases}$$

imply γ is an odd integer less than or equal to -5. Hence, by Theorem 3.3, for each odd integer $\gamma \leq -5$, there is a vector bundle E_γ of rank 3 and homological dimension 2 defined by

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\frac{3\gamma^2+8\gamma-3}{8}} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\frac{3\gamma^2+2\gamma-17}{4}} \to \mathcal{O}_{\mathbb{P}^3}^{\frac{3\gamma^2-4\gamma-7}{8}} \to E_\gamma \to 0.$$

For instance, if $\gamma = -5$, we get the following rank-3 vector bundle of homological dimension 2 with resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^4 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{12} \to \mathcal{O}_{\mathbb{P}^3}^{11} \to E_{-5} \to 0$$

satisfying $\chi(\operatorname{End} E_{-5}) = 1$.

More generally, we can state the existence of the following family of rank-3 vector bundles on $\mathbb{P}^3$.

Corollary 3.4. *For any odd integer $\gamma \leq -5$, there is a rank-3 vector bundle E_γ defined by a minimal linear resolution of type*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\frac{3\gamma^2+4\gamma-7}{8}} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\frac{3\gamma^2-2\gamma-17}{4}} \to \mathcal{O}_{\mathbb{P}^3}^{\frac{3\gamma^2-8\gamma-3}{8}} \to E_\gamma \to 0.$$

In particular, E satisfies $\chi(\operatorname{End} E) = 1$.

In [11, Proposition 4.3], the authors had proved the existence of vector bundles of homological dimension 2 and linear resolution

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^{u_s} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{u_{s+1}} \to \mathcal{O}_{\mathbb{P}^3}^{4u_{s+1}-10u_s} \to E \to 0,$$

and satisfying $\chi(\operatorname{End} E) = 1$, where $\{u_s\}_{s\geq 0}$ is the sequence

$$\begin{cases}
u_0 = 0,\\
u_1 = 1,\\
u_{s+2} = 4u_{s+1} - u_s.
\end{cases}$$

Hence, the 3-tuples of the form $(a, b, c) = (u_s, u_{s+1}, 4u_{s+1} - 10u_s)$ are solutions of the Diophantine equation (1.5). It is obvious that these solutions form a subset of the set of general solutions obtained in this work. So a natural question is how to determine when a general solution (2.6) comes from a solution of this form.

We can also ask whether Theorem 3.3 gives us all vector bundles, up to isomorphism, of this type (in the case of homological dimension 1, we have a description of all vector bundles with linear resolution of length 1 and $\chi(\operatorname{End} E) = 1$).

In addition, note that the hypotheses $a > 0$ and $b - 2a > 0$ in the theorem are necessary restrictions on a and b so that E exists. Obviously, if E exists, then one must have $a > 0$. In general, among all solutions of (1.5), the only feasible ones are those 3-tuples (a, b, c) such that

$$\begin{aligned} a > 0 \quad &\Longleftrightarrow \quad (\alpha+\beta)^2 + 2(\gamma+\delta)^2 > 3(\beta^2 + 2\delta^2),\\ b > 0 \quad &\Longleftrightarrow \quad \alpha^2 + 2\delta^2 > 6(\beta^2 + 2\delta^2),\\ c > 0 \quad &\Longleftrightarrow \quad (\alpha-\beta)^2 + 2(\gamma-\delta)^2 > 3(\beta^2 + 2\delta^2),\\ a \le b \quad &\Longleftrightarrow \quad (\alpha-\beta)^2 + 2(\gamma-\delta)^2 \ge 11(\beta^2 + 2\delta^2). \end{aligned}$$

But we immediately observe that the fourth inequality implies the third one. And it is also easy to check that if the first and the last conditions hold, then the second one also holds. So we need only to consider the first and the fourth inequalities, $a > 0$ and $a \le b$. Proposition 3.1 implies that the feasible solutions can be further restricted to the two conditions in the theorem, $a > 0$ and $b - 2a > 0$.

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