Static Hedging with Repeated Richardson Extrapolation

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MSc in Mathematical Finance

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Resumo

Esta tese explora a repeated Richardson extrapolation technique quando aplicada a metodologias de replicação estáticas na avaliação de opções com barreira europeias utilizando o modelo constant elasticity of variance (CEV) e o modelo jump to default extended constant elasticity of variance (JDCEV).

A extrapolação de Richardson é uma ferramenta computacional usada para melhorar a eficiência de vários métodos numéricos. Nesta tese os seus benefícios vão ser explorados ao ser aplicada a métodos de replicação estática.

Abstract

This thesis explores the Repeated Richardson extrapolation technique when applied to static replication methodologies in the valuation of European-style barrier options under both the constant elasticity of variance (CEV) model and the jump to default extended constant elasticity of variance (JDCEV) model.

The Richardson extrapolation is a computational tool used throughout the literature in order to improve the efficiency of numerous numerical methods. In this thesis is going to be studied the benefits of its use when applied to static replication methods.

Keywords: Static replication; Richardson extrapolation; Option pricing; Barrier options; CEV; JDCEV

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CHAPTER 1

Introduction

This thesis studies the improvement in efficiency of different static hedging methodologies when applying the repeated Richardson extrapolation technique. It focuses exclusively on the pricing of European-style barrier options under the constant elasticity of variance (CEV) model proposed by Cox (1975) [**14**], and the jump to default extended constant elasticity of variance (JDCEV) model offered by Carr and Linetsky (2006) [**10**].

Barrier options are path-dependent exotic options characterized by a strike level and one or two barrier values. The payoff of a barrier option is similar to the one of a plain-vanilla option, as it is described by the strike price. However, it is conditional on the underlying asset price crossing the barrier. This barrier can be of knock-in type, and the contract is iniciated only if the underlying price hits this level before maturity, or it can be of knock-out type and the contract is terminated, instead of iniciated, if the underlying price ever hits the barrier during the life of the option. Barrier options can have a single barrier level, in case of a single barrier option, or two barrier levels, in case of a double barrier option. In this particular case they are both knock-out or both knock-in barriers and the contract is either terminated or iniciated, depending on which, if the underlying price hits either one of the barriers during the life of the option.

The valuation of a European-style barrier option is already studied under the geometric Brownian motion (GBM) assumption, see for example Rich (1994) [**29**] for single barrier options, or Buchen and Konstandatos (2009) [**6**] for double barrier options. Although the GBM diffusion process provides analytical formulas, the implied log-normal distributional assumption fails to encompass some observable empirical findings, seen for example in Jackwerth and Rubinstein (1996) [**23**]. The CEV diffusion model of Cox (1975) [**14**] can better accommodate both the leverage effect i.e., the existence of a negative correlation between the returns and volatility of a stock, observed for example in Bekaert and Wu (2000) [**1**], as well as the implied volatility skew, described by the inverse relation between the implied volatility and the strike price of an option, addressed for example by Dennis and Mayhew (2002) [**16**].

This thesis will focus on valuation methodologies, that are contained in the static hedging portfolio (SHP) approach which can be divided into two main distinct formulation ideas: the first, proposed by Carr and Chou (1997) [**8**] and Carr et al. (1998) [**9**], constructs a SHP with European-style plain-vanilla options with the same maturity as the target option but with different strike values, being known as the vertical replication strategy; the second proposed by Derman et al. (1995) [**17**], constructs the SHP using again standard European-style options but with different maturities and fixing the strike value to the barrier boundary to match the target option price, also called the horizontal replication strategy, hereafter DEK method.

The DEK method only matches the target option at n evenly spaced time points where the underlying price equals the barrier. This discrete process generates some errors, while increasing the time points will result in an increase of the accuracy of the approximation, increasing the number of time points will also increase the computing time necessary to replicate the target option price. The modified DEK method proposed by Chung et al. (2010) [**12**] aims to reduce these replication errors and provides a faster convergence to the price of the option. The DEK method and the modified DEK method are going to be the two methodolgies observed when pricing single barrier options.

The SHP approach, proposed by Dias et al. (2015) [**19**], is going to be used for the valuation of double barrier options. This methodology extends the modified DEK under the JDCEV model and subsequently the CEV model as well. This thesis studies the improvement in terms of efficiency of the Repeated Richardson Extrapolation Technique applied to these static hedging methods, under the CEV and the JDCEV model following the work in Dias et al. (2015) [**19**].

The repeated Richardson extrapolation technique is a numerical method already used throughout the literature to derive computationally efficient approximations for different kinds of pricing methods. For example, Broadie et al. (1996) [**5**] and Heston et al. (2000) [**22**] apply the Richardson extrapolation to binomial tree models in option pricing. Or, Bunch et al. (1992) [**7**] and Chung et al. (2007) [**11**] in American-style option pricing. In this thesis, the Richardson extrapolation is going to be used in order to try to improve the efficiency of static replication methods when pricing European-style barrier options.

The rest of this thesis is structured as follows. Section 2 provides a summary of the JDCEV model characteristics. Section 3 will describe the three SHP methods used and the analytical formulae for pricing European-style plain-vanilla options and European-style cash-or-northing binary options. Section 4 will outline the Richardson extrapolation technique that will be implemented and tested in the numerical results in section 5. Finally, section 6 contains some concluding remarks.

CHAPTER 2

JDCEV framework

The current section provides a summary of the JDCEV modelling architecture, first proposed by Carr and Linetsky (2006) [**10**] for the valuation of European-style standard options, and later used for pricing their American counterparts by Nunes (2009) [**27**] and Ruas et al. (2013) [**30**].

It is assumed throughout that the financial market is arbitrage-free and frictionless, and that trading takes place continuously on the time interval $\mathcal{T} := [t_0, T]$, for some fixed time $T > t_0$. The complete probability space $(\Omega, \mathcal{G}, \mathbb{Q})$ represents uncertainty, where $\mathbb{Q}$ is taken as given as the equivalent martingale measure associated to the numeráire 'money market account'.

The underlying asset price is represented by S_t for $t < \zeta$ or zero otherwise, where ζ represents a random time of default and assuming that the equity holders receive no recovery in the case of bankruptcy. The pre-default asset price is characterized by the following time-inhomogeneous diffusion process under the risk-neutral measure $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = [r_t - q_t + \lambda(S,t)]dt + \sigma(S,t)dW_t^{\mathbb{Q}}, \tag{1}$$

where $S_{t_0} > 0$, $r_t \in \mathbb{R}$ represents the time-t riskless and short-term interest rate, $q_t \in \mathbb{R}$ describes the time-t dividend yield, $\lambda(S,t) \in \mathbb{R}_+$ denotes the *hazard rate* that compensates stockholders for default, $\sigma(S,t) \in \mathbb{R}_+$ is the time-t instantaneous volatility of asset returns, and $W_t^{\mathbb{Q}} \in \mathbb{R}$ corresponds to a standard Wiener process defined under measure $\mathbb{Q}$ generating the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq t_0\}$.

The underlying asset price process, akin to Carr and Linetsky (2006) [**10**], can either diffuse or jump to default. In the case of diffusion, bankrupcy takes place at the first passage time of the stock price to zero:

$$\tau_0 := \inf\{t > t_0 : S_t = 0\}. \tag{2}$$

The stock price can also jump to a *cemetery state* if the integrated *hazard rate*

$$\Lambda_t := \begin{cases} \int_{t_0}^{t} \lambda(S,u)du & ,\tau_0 > t \\ \infty & ,\tau_0 \leq t \end{cases} \tag{3}$$

is greater or equal to the level drawn from an exponential random variable Θ independent of $\{W_t^{\mathbb{Q}}; t \in \tau\}$ at the first jump time

$$\hat{\zeta} := \inf\{t > t_0 : \Lambda_t \geq \Theta\}, \tag{4}$$

of a doubly stochastic Poisson process with intensity $\lambda(S,t)$. Therefore, the time of default is simple given by:

$$\zeta = \tau_0 \wedge \hat{\zeta}, \tag{5}$$

and $\mathbb{D} = \{\mathcal{D}_t : t \geq t_0\}$ is the filtration generated by the default indicator process $\mathcal{D}_t = \mathbb{1}_{\{t>\zeta\}}$. In summary, the default time ζ is decomposable in two parts: a predictable component, τ_0, whenever the asset price hits a zero barrier by diffusion; and, a totally inaccessible part, $\hat{\zeta}$, associated to an unpredictable jump to default.

As in the classical CEV model of Cox (1975) [**14**], Carr and Linetsky (2006) [**10**] accomodate the leverage effect and the implied volatility skew by specifying the instantaneous stock volatility as a power function:

$$\sigma(S,t) = a_t S_t^{\bar{\beta}}, \tag{6}$$

where $\bar{\beta} < 0$ is the volatility elasticity parameter and $a_t > 0$, $\forall t$, is a deterministic volatility scale function. Furthermore, and to include a positive correlation between default probabilities and equity volatility, Carr and Linetsky (2006) [**10**] also assume that the default intensity is an increasing affine function of the instantaneous stock variance:

$$\lambda(S,t) = b_t + c\sigma^2(S,t), \tag{7}$$

where $c \geq 0$, and $b_t \geq 0$, $\forall t$, is a deterministic function of time.

The proposed modelling framework described by equations (1) − (5) encompasses several well-known option pricing models as special cases, namely the standard GBM and the CEV model, allowing the valuation framework to be compared with alternative methods available in the literature for pricing barrier options under these two modelling assumptions.

CHAPTER 3

The static hedging approach

This section describes the static replication approach proposed by Derman et al. (1995) [**17**], DEK method, for valuing and hedging European barrier options, the modified DEK method offered by Chung et al. (2010) [**12**], and the SHP approach provided by Dias et al. (2015) [**19**] for efficiently pricing and hedging European double barrier knock-out and knock-in options. Before introducing these methods, the analytical formulae for pricing European standard options and European cash-or-nothing binary options under the JDCEV modelling architecture will be presented.

3.1. European-style standard options

For the time-homogeneous case, and assuming that $\zeta > t_0$, Carr and Linetsky (2006, Proposition 5.5) [**10**] show that the time-t_0 price of a standard call ($\phi = -1$) or put ($\phi = 1$) option with strike price K and expiry date $T(\geq t_0)$ condition on no default is given by:

$$\begin{aligned} v_{t_0}^0(S_{t_0}, K, T; \phi) = &-\phi e^{-q\tau} S_{t_0} \Phi_{+1}\left(0, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)\right) \\ &+\phi e^{-(r+b)\tau} K \tilde{x}(S_{t_0}, t_0, T)^{\frac{1}{2|\bar{\beta}|}} \Phi_{+1}\left(-\frac{1}{2|\bar{\beta}|}, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)\right), \end{aligned} \tag{8}$$

where

$$\tau = T - t_0, \tag{9}$$

$$\delta_+ := \frac{2c+1}{|\bar{\beta}|} + 2, \tag{10}$$

$$\tilde{x}(S_{t_0}, t_0, T) := \frac{\left(\frac{1}{|\bar{\beta}|} S_{t_0}^{|\bar{\beta}|}\right)^2}{\theta}, \tag{11}$$

$$\tilde{y}(K, t_0, T) := \frac{\left(\frac{1}{|\bar{\beta}|} K^{|\bar{\beta}|} e^{-|\bar{\beta}|(r-q+b)\tau}\right)^2}{\theta}, \tag{12}$$

and

$$\theta \equiv \theta(t_0, T) := \begin{cases} a^2(T - t_0) & , r - q + b = 0 \\ \frac{a^2}{2|\bar{\beta}|(r-q+b)}\left(1 - e^{-2|\bar{\beta}|(r-q+b)(T-t_0)}\right) & , r - q + b \neq 0 \end{cases} . \tag{13}$$

The functions $\Phi_\xi(p, y; v, \lambda) := \mathbb{E}^{\chi^2(v,\lambda)}(X^p \mathbb{1}_{\xi X \geq \xi y})$ as defined in Carr and Linetsky (2006, Equations 5.11 and 5.12) [**10**] represent, for $\xi \in \{-1, 1\}$, the truncated p-th moments of a noncentral chi-square random variable X with v degrees of freedom and noncentrality parameter λ.

The value of the recovery part of the put option at t_0 (note that the recovery part of the call is zero) is given by:

$$v_{t_0}^D(S_{t_0}, R, T; 1, T) = Re^{-r(T-t_0)}[1 - SP(S_{t_0}, t_0; T)], \tag{14}$$

where

$$SP(S_{t_0}, t_0; T) = e^{-b(T-t_0)} \left(\frac{\chi^2(S_{t_0})}{\theta}\right)^{\frac{1}{2|\bar{\beta}|}} \mathcal{M}\left(-\frac{1}{2|\bar{\beta}|}; \delta_+, \frac{\chi^2(S_{t_0})}{\theta}\right) \tag{15}$$

is understood as the risk-neutral probability of surviving beyond time $T > t_0$, with $\mathcal{M}(p; v, \lambda)$ representing the p-th raw moment of a noncentral chi-square random variable X with v degrees of freedom and noncentrality parameter λ. In the numerical analysis, the algorithm presented in Dias and Nunes (2018) [**18**] is used, for valuing the truncated p-th moments $\Phi_\xi(p, y; v, \lambda)$, with $\xi \in \{-1, 1\}$, and the raw moments $\mathcal{M}(p; v, \lambda)$ using the identity shown in Carr and Linetsky (2006, Equation 5.13) [**10**].

The theta of the standard call ($\phi = -1$) or put ($\phi = 1$) option conditional on no default at t_0 is given by:

$$\begin{aligned}
\Theta_{t_0}^{v^0}(S_{t_0}, K, T; \phi) :=& -\frac{\partial v_{t_0}^0(S_{t_0}, K, T; \phi)}{\partial \tau} \\
=&\phi K e^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times \Phi_{-\phi}(p, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)) \\
&\times \left[(r + b) - \frac{\theta'(\tau)}{\theta(\tau)}\left(p + \frac{\tilde{x}(S_{t_0}, t_0, T)}{2}\right)\right] \\
&+\phi K e^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times \frac{\theta'(\tau)}{\theta(\tau)}\tilde{\Phi}_{-\phi}(p, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)) \\
&+K e^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times 2^p e^{-\frac{\tilde{y}(K,t_0,T)}{2} - \frac{\tilde{x}(S_{t_0},t_0,T)}{2}} \left(\frac{\tilde{y}(K, t_0, T)}{2}\right)^{\frac{\delta_+}{2}+p} \\
&\times \left[2|\bar{\beta}|(r - q + b) + \frac{\theta'(\tau)}{\theta(\tau)}\right] H(\tilde{x}(S_{t_0}, t_0, T), \tilde{y}(K, t_0, T), \delta_+) - \phi S_{t_0} q e^{-q\tau} \\
&\times \Phi_{-\phi}(0, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)) \\
&-S_{t_0} e^{-q\tau}\Big\{\tilde{y}(K, t_0, T) p(\tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)) \left[2|\bar{\beta}|(r - q + b) + \frac{\theta'(\tau)}{\theta(\tau)}\right] \\
&-\tilde{x}(S_{t_0}, t_0, T) p(\tilde{y}(K, t_0, T); \delta_+ + 2, \tilde{x}(S_{t_0}, t_0, T)) \frac{\theta'(\tau)}{\theta(\tau)}\Big\},
\end{aligned} \tag{16}$$

where $\tilde{\Phi}_\xi(p, w; v, \lambda)$, for $\xi \in \{-1, 1\}$, is described in Ruas et al. (2013, Equations 37 and 35) [**30**], $p(w; v, \lambda)$ represents the probability density function of a noncentral chi-square distribution with v degrees of freedom and noncentrality parameter λ,

$$p := -\frac{1}{2|\bar{\beta}|}, \tag{17}$$

and

$$H(x, y, z) := \sum_{i=0}^{\infty} \frac{\left(\frac{xy}{4}\right)^i}{i!\Gamma(\frac{z}{2} + i)}. \tag{18}$$

The theta of the recovery part of the put option is given by:

$$\begin{aligned} \Theta_{t_0}^{v^D}(S_{t_0}, R, T; 1) &:= -\frac{\partial v_{t_0}^D(S_{t_0}, R, T; 1)}{\partial \tau} \\ &= -Re^{-r\tau}\left\{S\,P(S_{t_0}, t_0; T) \times \left[(r+b) - \frac{\theta'(\tau)}{\theta(\tau)}\left(p + \frac{\tilde{x}(S_{t_0}, t_0, T)}{2}\right)\right] - r\right\} \\ &\quad -Re^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p}\frac{\theta'(\tau)}{\theta(\tau)} \times \tilde{\mathfrak{M}}(p, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)), \end{aligned} \tag{19}$$

with

$$\tilde{\mathfrak{M}}(p, w; v, \lambda) := \tilde{\Phi}_{-1}(p, w; v, \lambda) + \tilde{\Phi}_{+1}(p, w; v, \lambda). \tag{20}$$

3.2. European-style cash-or-nothing options

Assuming that $\zeta > t_0$, and under the financial model described by equations (1) – (5), the time-t_0 value of a cash-or-nothing call ($\phi = -1$) or put ($\phi = 1$) option with strike price K, expiry date $T(\geq t_0)$ and fixed compensation amount X, conditional on no default is given by:

$$Bin_{t_0}^0(S_{t_0}, K, T, X; \phi) := Xe^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times \Phi_{-\phi}(p, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)). \tag{21}$$

Regarding the recovery part of the put option, its value is achieved as:

$$Bin_{t_0}^D(S_{t_0}, K, T; 1) = v_{t_0}^D(S_{t_0}, K, T; 1). \tag{22}$$

The theta of the cash-or-nothing call ($\phi = -1$) or put ($\phi = 1$) conditional on no default at t_0 is given by:

$$
\begin{aligned}
\Theta_{t_0}^{Bin^0}(S_{t_0}, K, T, X; \phi) &:= -\frac{\partial Bin_{t_0}^0(S_{t_0}, K, T, X; \phi)}{\partial \tau} \\
&= Xe^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times \Phi_{-\phi}(p, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)) \\
&\times \left[(r+b) - \frac{\theta'(\tau)}{\theta(\tau)} \left(p + \frac{\tilde{x}(S_{t_0}, t_0, T)}{2} \right) \right] \\
&+ Xe^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times \frac{\theta'(\tau)}{\theta(\tau)} \tilde{\Phi}_{-\phi}(p, \tilde{y}(K, t_0, T); \delta_+, \tilde{x}(S_{t_0}, t_0, T)) \\
&+ \phi Xe^{-(r+b)\tau}[\tilde{x}(S_{t_0}, t_0, T)]^{-p} \times 2^p e^{-\frac{\tilde{y}(K,t_0,T)}{2} - \frac{\tilde{x}(S_{t_0},t_0,T)}{2}} \left(\frac{\tilde{y}(K, t_0, T)}{2} \right)^{\frac{\delta_+}{2}+p} \\
&\times \left[2|\bar{\beta}|(r-q+b) + \frac{\theta'(\tau)}{\theta(\tau)} \right] \times H(\tilde{x}(S_{t_0}, t_0, T), \tilde{y}(K, t_0, T), \delta_+).
\end{aligned}
\tag{23}
$$

The theta of the recovery part of the cash-or-nothing put option, $\Theta_{t_0}^{Bin^D}(S_{t_0}, X, T; 1)$, is given by equation (19) with $R = X$.

3.3. DEK method

The static replication approach proposed by Derman et al. (1995) [**17**] aims to formulate a static hedge portfolio of standard European options with different maturities to match the terminal and boundary conditions of a target barrier option. To correspond the value of the portfolio to the target option at the barrier before maturity at n evenly spaced time points, $t_0 = 0, t_1, ..., t_{n-1}$, DEK adds W_i units of a standard option, maturing at time t_{i+1} and with a strike price equal to the barrier, to the portfolio. The weight is then computed using a *value-matching* condition. By working its way backwards, the DEK method can establish the amount of standard options used and, therefore, it replicates the payoffs of the target barrier option.

3.3.1. Single barrier options

The replication method begins by analysing the expiration date, composing a starting portfolio which value matches the value of the target option when assuming the barrier is never hit. Then, working backwards, new options are added to the replicating portfolio in order to match both values along the barrier at each time-step.

Let us first consider a knock-out single barrier (SB) option, that consists of a standard option if the barrier is never hit, or zero otherwise. The replication method begins with a portfolio consisting of a European-style call or put option, depending on the target option being a SB call or put, with the same contract characteristics as the target option.

In the case of a knock-in SB option, when assuming the barrier is never hit until maturity its value will be zero, or that of a standard option if touched. Hence, the starting portfolio will be empty, again, by assuming the barrier is never touched until maturity.

Next, moving back one time-step $T - h$ (with $h = \Delta t$), we can add to the starting portfolio a position in an ordinary call option with expiration at T and strike equal to the barrier, in case of an up barrier. If it is a down barrier, the position added is an ordinary put. The weight of the new position must be such that when multiplied to the theoretical value of the new option and added to the theoretical value of the portfolio, both at the barrier and at time $T - h$, will yield the value of the target SB (0 in case of a knock-out, or the value of a standard option with the same characteristics as the target option in the case of a knock-in).

Following up by going one time-step further ($T - 2h$) and, in the same way, adding another position to the replicating portfolio, this time with expiration $T - h$, to ensure that it matches the target SB value at the boundary at $T - 2h$. Continuing this backward process until t_0 completes the static replicating portfolio in such manner that it has the same value as the target SB at all time-steps. The new positions expire out of the money if the barrier is never touched (since their strike is equal to the barrier) and therefore they do not alter the cash position of the portfolio that already replicate the target option at that specific point in time.

Considering an up-and-out call (UOC) as the target option, the starting portfolio would contain a single call option with a contract with the same characteristics as the UOC. The value of the replicating portfolio at the barrier (B) and at time t_i will be equal to $C(B, K, T - t_i) + \sum_{j=i}^{n-1} w_j C(B, B, t_{j+1} - t_i)$, for $i = n - 1, n - 2, ..., 0$. So, the value matching condition at each time-step is:

$$C(B, K, T - t_i) + \sum_{j=i}^{n-1} w_j C(B, B, t_{j+1} - t_i) = 0. \quad (24)$$

When computing the DEK method, we solve the above equation at each time-step to compute the weight values, as the only unknown variable is W_i. To list another example, consider a down-and-in put as the target option. The starting portfolio will be empty, at each time-step a new put is added and the value of the portfolio at the barrier and at time t_i is equal to $\sum_{j=i}^{n-1} W_j P(B, B, t_{j+1} - t_i)$. At the barrier, the target option will be equivalent to a standard put option with the same contract specifications, so, the value matching condition in this case is given by:

$$\sum_{j=i}^{n-1} w_j P(B, B, t_{j+1} - t_i) = P(B, K, T - t_i). \quad (25)$$

In order to increase the accuracy of the DEK method, the number n of evenly spaced discrete points can be increased, which will result in a speed-accuracy trade-off that is analyzed in section 5.

3.3.2. Double barrier options

Implementing the DEK method to replicate double barrier options is similar to the process described above. Using two, instead of just one value matching condition, one for each barrier, the portfolio is constituted by standard European options to match the value of the target option on both the upper (U) and lower (L) barriers. To hedge the upper barrier we add standard call options with strike price at the upper barrier, while at the lower barrier we hedge by adding standard put options with the lower barrier as the strike price.

Under the JDCEV framework the possibility of jump to default has to be accounted for. Taking a double barrier knock-out option, assuming the barrier is never touched before the maturity date implies that there is no jump to default. Furthermore, the jump to default would trigger the knock-out event and bring the value of the target option to zero. So, the replication is made by adding standard options conditional on no default.

In case of a double barrier knock-in option, its value is equal to a standard option at both barriers. Assuming the barriers are never reached until the maturity date, indicates again that there was no jump to default and the value of the target option is zero. If a jump to default occurs during the life of the target option, in case of a put, its value is going to be equal to the recovery component with the same contract characteristics as the DB option, and for that reason the replicating porfolio will start with one unit of the recovery component, and the hedging is continued by adding standard options conditional on no default.

Considering a double barrier knock-out call option, the starting portfolio is constituted by one unit of standard call option with the same contract characteristics as the target option. Next, to ensure a zero value at both barriers, we use the two following value matching conditions at each time step:

$$C(U, K, T - t_i) + \sum_{j=i}^{n-1} w_j C(U, U, t_{j+1} - t_i) + \sum_{j=i}^{n-1} g_j P(U, L, t_{j+1} - t_i) = 0, \tag{26}$$

$$C(L, K, T - t_i) + \sum_{j=i}^{n-1} w_j C(L, U, t_{j+1} - t_i) + \sum_{j=i}^{n-1} g_j P(L, L, t_{j+1} - t_i) = 0. \tag{27}$$

The first condition matches the value of the replicating portfolio to the value of the target option at time t_i at the upper barrier, while the second matches the values at the lower barrier. There are only two unknown variables, w_i and g_i which correspond to the weights attached to the new call and put options added to the portfolio and are calculated by solving both equations.

3.4. Modified DEK method

The hedging performance of the DEK method, although theoretically attractive, when applied discretely might not be so accurate. Due to this discrete implementation, the value of the replicating portfolio does not match the value of the target option at the barrier except at those discrete time points, being this the major source for replication errors. Chung et al. (2010) [**12**] perceived that the DEK portfolio may be sensitive to the change of time, i.e. theta different from zero, which result also in replicating errors. To address this issue of large theta values on the barrier, Chung et al. (2010) [**12**] propose a modified static hedge portfolio.

The intention of the modified DEK method is to construct a portfolio which has not merely the same value but also the same theta on the barrier as the target option at the discrete time points. To meet both goals, positions in cash-or-nothing binary options are also added to the portfolio along with standard options.

The process is very similar to the DEK method: the starting portfolio is constructed in the same manner, but when calculating the weights of the new options in addition to the single value matching condition the modified DEK method also has a theta matching condition. At each time-step the weights of the new standard option and binary option are computed by solving both matching conditions.

To match the value and theta conditions on the barrier before maturity at n evenly spaced time points, $t_0 = 0, t_1, ..., t_{n-1}$, w_i units of a standard option and $\hat{w}_i$ units of a binary option maturing at t_{i+1} with a strike price equalling to the barrier are added to the portfolio. The modified DEK method, just as the DEK method, continues this backward process to complete the replication in the same way.

Considering an up-and-out put (UOP) as the target option, the starting portfolio would contain a single put option with a contract with the same characteristics as the UOP. The value of the replicating portfolio at the barrier (B) and at time t_i will be equal to $P(B, K, T - t_i) + \sum_{j=i}^{n-1} w_j C(B, B, t_{j+1} - t_i)$, for $i = n-1, n-2, ..., 0$. So, the two matching conditions at each time-step are given by:

$$P(B, K, T - t_i) + \sum_{j=i}^{n-1} w_j C(B, B, t_{j+1} - t_i) + \sum_{j=i}^{n-1} \hat{w}_j Bin_c(B, B, t_{j+1} - t_i) = 0, \tag{28}$$

$$\Theta_p(B, K, T - t_i) + \sum_{j=i}^{n-1} w_j \Theta_c(B, B, t_{j+1} - t_i) + \sum_{j=i}^{n-1} \hat{w}_j \Theta_{Bin_c}(B, B, t_{j+1} - t_i) = 0. \tag{29}$$

The first condition matches the value of the replicating portfolio to the value of the target option at time t_i at the barrier, while the second matches their corresponding thetas. There are only two unknown variables, w_i and $\hat{w}_i$ which correspond to the weights attached to the new standard call and binary call options added to the portfolio and are calculated by solving both equations.

In the case of double barrier options the SHP approach offered by Dias et al. (section 4.2, 2015) [**19**] describes the same process outlined above, but, following Chung et al. (2013) [**13**] also adds cash-or-nothing options to the replicating portfolio just before the maturity of the target option. Hence, the four matching conditions are defined as shown in Dias et al. (equations 47-50, 2015) [**19**].

CHAPTER 4

Repeated Richardson extrapolation

Consider the problem of calculating an unknown value, a_0, which can be approximated by a calculable function $f(h)$, depending on a step-size $h > 0$ and provided by some numerical scheme, such that $f(0) = \lim_{h\to 0} f(h) = a_0$. In order to approximate the unknown quantity a_0, the Richardson extrapolation technique can be implemented provided that the complete expansion of the truncation error about $f(h)$ is known. Based on the previous statement, it is assumed that $f(h)$ can be written as:

$$f(h) = a_0 + a_1 h^{p_1} + a_2 h^{p_2} + a_3 h^{p_3} + \cdots + a_k h^{p_k} + O(h^{p_{k+1}}), \tag{30}$$

with known exponents $0 < p_1 < p_2 < \cdots < p_{k+1}$, but unknown parameters $a_0, a_1, \cdots, a_k$, where $O(h^{p_{k+1}})$ denotes a quantity whose size is proportional to $h^{p_{k+1}}$, or possibly smaller, and $h \in [0, H]$ for some basic step $H > 0$.

The intention behind the Richardson extrapolation technique is to improve the accuracy of the approximation via a linear combination of two different step sizes. the function $f(h)$ can be successively computed using each time smaller step sizes, $h_1 > h_2 > \cdots > 0$. To address the possible problem of non-uniform convergence, Omberg (1987) [**28**] and Chang et al. (2007) [**11**] suggest using geometric-spaced time points achieved by continuously doubling the number fo uniformly-spaced exercise dates for compound and American options. The Romberg sequence is going to be applied when implementing the repeated Richardson extrapolation technique in response to this issue. Therefore, we carry out the following progression of step-sizes: $h_1 = T/8$ and $h_{i+1} = h_i/2$ for $\{i = 1, 2, 3, \cdots, n-1\}$, where T is the time to maturity of the target barrier option. By using the Romberg sequence, the possible non-uniform convergence is overcame as it allows the nesting of the previous replication set, meaning that the time steps observed in $f(h_2)$ include all the time steps in $f(h_1)$ and so forth, ensuring that every new approximation is at least as good as the last one. According to Schmidt (1968) [**31**], the following algorithm when $p_m = p \times m, m = 1, \cdots, k$ can be formulated.

Algorithm:

For $i \geq 2$ and $j = 1, 2, 3, \cdots, i-1$, the Richardson extrapolation technique and its repeated version are defined by

$$f_{i,j} = f_{i,j-1} + \frac{f_{i,j-1} - f_{i-1,j-1}}{\left(\frac{h_{i-j}}{h_i}\right)^p - 1}, \tag{31}$$

where $f_{i,0} = f(h_i)$ represents the price of a barrier option computed by either the DEK or the modified DEK method with a step size of h_i, and $f_{i,j}$ its approximated value obtained from the repeated Richardson extrapolation computed j times using step sizes of $h_i, h_{i+1}, \cdots, h_{i+j}$. We set $p = 1$, which is one of the commonly used extrapolation schemes. The extrapolation, stopped at the n-th order, can be represented in the following arrangement:

$$\begin{array}{rcccccc}
f(h_1) = f_{1,0} & & & & & & \\
f(h_2) = f_{2,0} & f_{2,1} & & & & & \\
f(h_3) = f_{3,0} & f_{3,1} & f_{3,2} & & & & \\
\vdots & & & \ddots & & & \\
f(h_{n-1}) = f_{n-1,0} & f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & \cdots & f_{n-1,n-2} & \\
f(h_n) = f_{n,0} & f_{n,1} & f_{n,2} & f_{n,3} & \cdots & f_{n,n-2} & f_{n,n-1}
\end{array}$$

The first columnm, terms $f_{i,0}$, are calculated by one of the methods that will be presented in the next section, while the following columns, terms $f_{i,j}$, are computed through the recursion shown in equation 31. The two-point Richardson extrapolation technique is then repeated to achieve a fast numerical scheme that can provide an increase to the accuracy observed in two different ways: while going up the i axis we get a decrease in the step-size value, while going up the j axis we get a better aproximation as a result of the repetion of the Richardson extrapolation technique, implying that the quantity $f_{n,n-1}$ is the most accurate approximation.

CHAPTER 5

Numerical results

This section compares the static replication performance of the DEK, the modified DEK and the repeated Richardson extrapolation applied to both techniques, under the CEV model and the general JDCEV framework. Accuracy is measured by the mean absolute percentage error (MAPE) relative to a benchmark value and efficiency is obtained by the total CPU time (in seconds) necessary to replicate the target option.

First, the performance of each method is going to be tested under the CEV model in both single barrier and double barrier type options. Secondly, only the performance of the replication of double barrier options is going to be tested using the general JDCEV modelling framework. To compare the replication of double barrier options in both modelling frameworks the paper of Dias et al. (2015) [**19**] is going to be followed.

5.1. CEV model

The CEV model of Cox (1975) [**14**], as previously shown by Carr and Linetsky (2006) [**10**], can be nested into the general framework described by equations (1)-(5) as a result of the following conditions: $r_t = r$, $q_t = q$, $\lambda(S,t) = 0$, and $\sigma(S,t) = \delta S_t^{\beta/2-1}$, with $\delta \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$. Thus, $\bar{\beta} = \beta/2 - 1$.

The analytical formulas for computing the standard options value and respective thetas needed for the modified DEK method are shown in Larguinho et al. (2013) [**24**]. To achieve closed-form solutions for the cash-or-nothing options under the CEV model would be straightforward following the same line of thought. The computation of the complementary distribution function of a noncentral chi-square random variable $Q_{\chi^2(\nu,\lambda)}(x)$ with ν degrees of freedom and noncentrality parameter λ is made with the algorithm proposed by Benton and Krishnamoorthy (2003) [**2**].

Considering a European up-and-out call option, the following table compares the DEK method, proposed by Derman et al. (1995) [**17**], and the repeated Richardson extrapolation employed in the DEK method.

The first column indicates the guideline i which represents the step sizes ($\Delta t = h_i = T/2^{i+2}$) used in the same row. The second column displays the values obtained by employing the regular DEK method. The third column reports the computed values obtained by the implementation of the repeated Richardson extrapolation technique only once ($j = 1$), the subsequent colomuns follow by employing the Richardson extrapolation twice, three times, until j times. The values in parentheses represent the CPU computing time to replicate the target option in seconds.

Table 1. Repeated Richardson Extraplotion employed in the DEK method under the CEV model

$f_{i,j}$	DEK	Repeated Richardson Extrapolation of DEK				
i	$j=0$	1	2	3	4	5
1	1.2218 (0.0156)					
2	1.0413 (0.0156)	0.8607 (0.0192)				
3	0.9546 (0.0469)	0.8679 (0.0477)	0.8704 (0.0481)			
4	0.9123 (0.1081)	0.8700 (0.1095)	0.8707 (0.1103)	0.8708 (0.1122)		
5	0.8915 (0.2272)	0.8706 (0.2274)	0.8708 (0.2281)	0.8708 (0.2336)	0.8708 (0.2351)	
6	0.8811 (0.4083)	0.8708 (0.4106)	0.8708 (0.4127)	0.8708 (0.4160)	0.8708 (0.4209)	0.8708 (0.4233)

Note: This table employs the repeated Richardson extrapolation technique in the DEK method in the static replication of a European up-and-out call option with the following contract characteristics: $S_{t_0} = 100$, $K = 100$, barrier value of 120, $T - t_0 = 1$ year, $\beta = 0$, $\delta S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $q = 0$. The benchmark value of the target option is computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and is equal to 0.8708.

As observed, the use of the repeated Richarson extrapolation in the DEK method generates superior results. While we achieve the benchmark value of 0.8708 with the Richardson extrapolation in $f_{4,3}$, with the DEK method the best approximation in 0.8811 in $f_{6,0}$ needing more than double the amount of CPU computing time.

To investigate this result even further, we can look at the reduction in the replication error. We have $f_{3,1} = f_{3,0} + (f_{3,0} - f_{2,0})/(2-1) = 0.8679$, after employing the Richardson extrapolation once, that represents a reduction of $(|f_{3,0} - a_0| - |f_{3,1} - a_0|)/|f_{3,0} - a_0| = 96.54\%$. By employing it twice, we get $f_{3,2} = 0.8704$ and a reduction of 99.52% in the replication error when comparing to $f_{3,0}$.

In the succeeding table we make a similar comparison by computing the static replication of the same European up-and-out call option employing the modified DEK method, proposed by Chung et al. (2010) [**12**] and Tsai (2014) [**32**], and the repeated Richardson extrapolation applied to the modified DEK method.

Similar to Table 1, the first column indicates the number of step sizes ($\Delta t = h_i = T/2^{i+2}$) used within the same row. The second column displays the values obtained by the modified DEK method. The third column reports the computed values by the implementation of the repeated Richardson extrapolation technique only once ($j = 1$), and the subsequent collumns follow by employing the Richardson extrapolation twice, three times, until j times. The values in parentheses represent the CPU computing time to replicate the target option in seconds.

Table 2. Repeated Richardson Extrapolation employed in the modified DEK method under the CEV model

$f_{i,j}$	Mod. DEK	Repeated Richardson Extrapolation of Mod. DEK				
i	$j = 0$	1	2	3	4	5
1	0.9023 (0.0275)					
2	0.8827 (0.0555)	0.8631 (0.0607)				
3	0.8752 (0.0836)	0.8678 (0.1231)	0.8693 (0.1266)			
4	0.8725 (0.1681)	0.8697 (0.2443)	0.8703 (0.2492)	0.8704 (0.2516)		
5	0.8714 (0.3427)	0.8704 (0.5045)	0.8707 (0.5072)	0.8707 (0.5080)	0.8707 (0.5100)	
6	0.8711 (0.7195)	0.8707 (0.9848)	0.8708 (0.9850)	0.8708 (0.9889)	0.8708 (0.9911)	0.8708 (0.9955)

Note: This table employs the repeated Richardson extrapolation technique in the modified DEK method in the static replication of an European up-and-out call option with the following contract characteristics: $S_{t_0} = 100$, $K = 100$, barrier value of 120, $T - t_0 = 1$ year, $\beta = 0$, $\delta S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $q = 0$. The benchmark value of the target option is computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and is equal to 0.8708.

Firstly, we observe that the modified DEK method exceeds in performance relatively to the DEK method: it produces a far more accurate estimation in $f_{3,0} = 0.8752$, than the $f_{6,0} = 0.8811$ observed in Table 1, while requiring less computing time.

Applying the Richardson extrapolation in the modified DEK method, just like in Table 1, also improves its efficiency. Looking at $f_{3,0} = 0.8752$, we get a reduction in the replication error of 31.82% after employing the Richardson extrapolation once, with $f_{3,1} = 0.8678$. Employing it twice, $f_{3,2} = 0.8693$, results in a reduction of 65.91% relatively to $f_{3,0} = 0.8752$.

Although employing the Richardson extrapolation in the modified DEK method improves its results, it can be preferable its application to the DEK method, which has greater benefits. For example, $f_{3,2} = 0.8704$ of Table 1 is a more accurate estimation than $f_{3,2} = 0.8693$ of Table 2, requiring only 0.0481 seconds to the 0.1266 seconds required when using the Richardson extrapolation in the modified DEK method.

The next four tables value an up-and-in call and a down-and-in put in order to understand if the results obtained are consistent for other types of single barrier options. These tables are modelled after the previous two, i.e. the first column indicates the guideline i which represents the step sizes ($\Delta t = h_i = T/2^{i+2}$) used within in the same row. The second column presents the values obtained by either the modified DEK or the DEK method. The last columns show the values computed by the Richardson extrapolation employed in the method considered before, and follow as already explained above. The values in parentheses represent the CPU computing time to replicate the target option in seconds.

Table 3. Numerical results of up-and-in call prices under the CEV model based on the DEK method

$f_{i,j}$	DEK	Repeated Richardson Extrapolation of DEK				
i	$j = 0$	1	2	3	4	5
1	13.7804 (0.0115)					
2	13.9609 (0.0240)	14.1415 (0.0253)				
3	14.0476 (0.0510)	14.1342 (0.0537)	14.1318 (0.0540)			
4	14.0899 (0.1141)	14.1322 (0.1147)	14.1315 (0.1151)	14.1314 (0.1153)		
5	14.1107 (0.2287)	14.1316 (0.2456)	14.1314 (0.2471)	14.1314 (0.2472)	14.1314 (0.2490)	
6	14.1211 (0.4723)	14.1314 (0.4975)	14.1314 (0.4977)	14.1314 (0.5058)	14.1314 (0.5061)	14.1314 (0.5069)

Note: This table employs the repeated Richardson extrapolation technique in the DEK method in the static replication of an European up-and-in call option with the following contract characteristics: $S_{t_0} = 100$, $K = 100$, barrier value of 120, $T - t_0 = 1$ year, $\beta = 0$, $\delta S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $q = 0$. The benchmark value of the target option is computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and is equal to 14.1314.

Table 3 values a European-style up-and-in call option using the DEK method. As seen before the performance of the method is enhanced when employing the Richardson extrapolation, which achieves the benchmark value at the position $f_{4,3}$. Taking more than double the computing time, the standard DEK method did not produce the correct value.

Table 4. Numerical results of up-and-in call prices under the CEV model based on the modified DEK method

$f_{i,j}$	Mod. DEK	Repeated Richardson Extrapolation of Mod. DEK				
i	$j = 0$	1	2	3	4	5
1	14.0999 (0.0525)					
2	14.1195 (0.1053)	14.1391 (0.1098)				
3	14.1270 (0.2166)	14.1344 (0.2270)	14.1329 (0.2281)			
4	14.1297 (0.4453)	14.1325 (0.4639)	14.1319 (0.4641)	14.1318 (0.4753)		
5	14.1308 (0.8646)	14.1318 (0.9859)	14.1315 (0.9862)	14.1315 (0.9864)	14.1315 (0.9886)	
6	14.1311 (1.9153)	14.1315 (2.0722)	14.1314 (2.0733)	14.1314 (2.0750)	14.1314 (2.0751)	14.1314 (2.0772)

Note: This table employs the repeated Richardson extrapolation technique in the modified DEK method in the static replication of an European up-and-in call option with the following contract characteristics: $S_{t_0} = 100$, $K = 100$, barrier value of 120, $T - t_0 = 1$ year, $\beta = 0$, $\delta S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $q = 0$. The benchmark value of the target option is computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and is equal to 14.1314.

Table 4 values the same up-and-in call option but using the modified DEK method. Using the Richardson extrapolation the benchmark value is obtained in about 2 seconds, in the position $f_{6,2}$. Compared to the up-and-out call seen before, both attained the benchmark at the same position, although this example takes 1 second more.

The repeated Richardson extrapolation technique, when employed to the DEK method, clearly not only improves the speed-accuracy trade-off of the replication but also produces better results than when employed in the modified DEK method.

TABLE 5. Numerical results of down-and-in put prices under the CEV model based on the DEK method

$f_{i,j}$	DEK	Repeated Richardson Extrapolation of DEK				
i	$j = 0$	1	2	3	4	5
1	5.3633 (0.0095)					
2	5.3975 (0.0200)	5.4317 (0.0203)				
3	5.4141 (0.0420)	5.4307 (0.0443)	5.4303 (0.0445)			
4	5.4222 (0.0876)	5.4303 (0.0913)	5.4302 (0.0919)	5.4302 (0.0923)		
5	5.4262 (0.1521)	5.4302 (0.1919)	5.4302 (0.1924)	5.4302 (0.1931)	5.4302 (0.1936)	
6	5.4282 (0.2216)	5.4302 (0.4164)	5.4302 (0.4168)	5.4302 (0.4177)	5.4302 (0.4181)	5.4302 (0.4188)

Note: This table employs the repeated Richardson extrapolation technique in the DEK method in the static replication of an European down-and-in put option with the following contract characteristics: $S_{t_0} = 100$, $K = 100$, barrier value of 90, $T - t_0 = 1$ year, $\beta = 0$, $\delta S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $q = 0$. The benchmark value of the target option is computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and is equal to 5.4302.

Table 5 values a European-style down-and-in put option using the DEK method. The results are consistent to what was shown before: the Richardson extrapolation applied to the DEK method highly improves its performance. The benchmark value is obtained first at the position $f_{4,2}$ with 0.09 seconds of computing time while the best estimation of the DEK method, with the same computing time, is 5.4222 at position $f_{4,0}$.

Table 6. Numerical results of down-and-in put prices under the CEV model based on the modified DEK method

$f_{i,j}$	Mod. DEK	Repeated Richardson Extrapolation of Mod. DEK				
i	$j = 0$	1	2	3	4	5
1	5.4225 (0.0460)					
2	5.4270 (0.0931)	5.4315 (0.0983)				
3	5.4290 (0.1891)	5.4311 (0.2017)	5.4309 (0.2023)			
4	5.4298 (0.3913)	5.4305 (0.4268)	5.4303 (0.4277)	5.4302 (0.4279)		
5	5.4300 (0.7785)	5.4303 (0.8723)	5.4302 (0.8725)	5.4302 (0.8733)	5.4302 (0.8738)	
6	5.4301 (0.7915)	5.4302 (1.1648)	5.4302 (1.1888)	5.4302 (1.2088)	5.4302 (1.2108)	5.4302 (1.2148)

Note: This table employs the repeated Richardson extrapolation technique in the modified DEK method in the static replication of an European down-and-in put option with the following contract characteristics: $S_{t_0} = 100$, $K = 100$, barrier value of 90, $T - t_0 = 1$ year, $\beta = 0$, $\delta S_0^{\beta/2-1} = 0.25$, $r = 0.10$, and $q = 0$. The benchmark value of the target option is computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and is equal to 5.4302.

Table 6 uses the modified DEK method to replicate the same target option as Table 5. In this example too the Richardson extrapolation converges faster to the benchark when employed to the DEK method. Using the modified DEK method it takes about 0.43 seconds contrasting with the 0.09 of the DEK.

Next, the application of the Richardson extrapolation technique is used on the replication of European-style double barrier options in order to understand its impact on the speed-accuracy trade-off of the DEK method and of the SHP approach formulated in Dias et al. (2015) [**19**] to extend the modified DEK method to double barrier options.

Figure 1 compares the speed-accuracy trade-off of three methods: the SHP approach, the Richardson extrapolation employed to the DEK method, and the Richardson extrapolation employed to the SHP approach. The standard DEK method is not used in this figure since to achieve the same levels of accuracy it would take a much higher number of time steps and therefore a substantial amount of computing time.

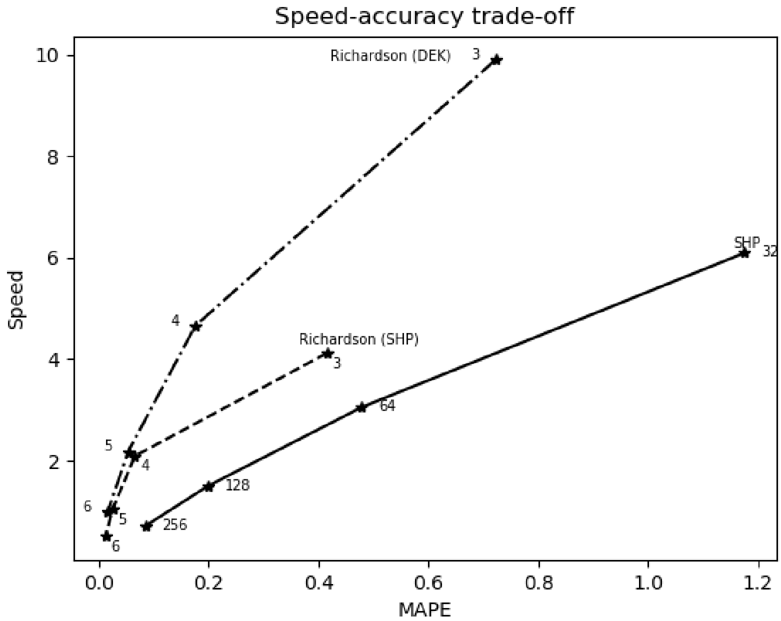

FIGURE 1. The figure compares the speed and accuracy of the SHP approach, the Richardson extrapolation employed in the DEK method and in the SHP approach when valuing European-style double knock-out calls under the CEV model.

Figure 1 is computed with a subset (1696) of a random sample of 2000 calls (those with a value bellow 0.5 were excluded) whose parameters were generated from a uniform distribution and within the following ranges: strike price values between 70 and 130, instantaneous volatility between 10% and 60%, $\beta < 2$, time to maturity within 0.1 and 1 years with a probability of 75% and within 1 and 3 years otherwise, risk free rate ranging from 0% to 10%, dividend yield also between 0% and 10% but only with a probability of 80% or equal to 0%, the upper barrier value between 110 and 140, the lower barrier value ranging from 60 to 90. The inicial spot value is fixed with a value of 100.

The MAPE is computed against a benchmark value, obtained by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps, and are presented multiplied by a factor of 10^3. The speed corresponds to the number of option prices calculated per second.

The SHP approach is calculated with 32, 64, 128 and 256 time steps. The Richardson extrapolation employed in the DEK method and in the SHP approach are computed using a starting column of 3, 4, 5 and 6 elements (numbers shown in the figure), which are used to apply the Richardson extrapolation repeated two, three, four and five times respectively.

Figure 1 reinforces what was shown before and clearly highlights the fact that the Richardson extrapolation when employed to the DEK method is able to achieve high levels of accuracy faster, as its curve is always to the left of the other two, meaning that for the same level of accuracy, this method has a higher speed.

Altough the improvement the Richardson extrapolation provides to the SPH approach is not as significative as the one seen when employed to the DEK method, it is observable that its curve is always above and to the left of the SHP approach without it.

Table 7 is an extension of Table 1 in Dias et al. (2015) [**19**], comparing the efficiency of the SHP approach against the implementation of the repeated Richardson extrapolation technique employed in the DEK method, based on the previous results only the DEK method is going to be considered with the Richardson extrapolation. It values knock-out double barrier call options, following the same model parameters as in Davydov and Linetsky (2001, Table 1) [**15**].

Table 7. European double barrier knock-out call prices under the CEV model

				SHP Approach			Richardson extrapolation		
K	β	δ	Boyle & Tian	100	200	1000	4	5	6
95	1	2.50E+00	1.8805	1.8802	1.8804	1.8805	1.8801	1.8804	1.8805
95	0	2.50E+01	2.0799	2.0797	2.0799	2.0800	2.0796	2.0799	2.0799
95	-2	2.50E+03	2.5528	2.5526	2.5527	2.5528	2.5525	2.5527	2.5528
95	-4	2.50E+05	3.1294	3.1292	3.1293	3.1294	3.1292	3.1294	3.1294
95	-6	2.50E+07	3.8088	3.8084	3.8086	3.8088	3.8086	3.8087	3.8088
100	1	2.50E+00	1.0957	1.0954	1.0956	1.0957	1.0956	1.0957	1.0957
100	0	2.50E+01	1.2383	1.2380	1.2382	1.2383	1.2381	1.2383	1.2383
100	-2	2.50E+03	1.5798	1.5795	1.5797	1.5798	1.5797	1.5798	1.5798
100	-4	2.50E+05	2.0021	2.0018	2.0020	2.0021	2.0020	2.0021	2.0021
100	-6	2.50E+07	2.5059	2.5055	2.5057	2.5059	2.5058	2.5059	2.5059
105	1	2.50E+00	0.5125	0.5122	0.5124	0.5125	0.5125	0.5125	0.5125
105	0	2.50E+01	0.5944	0.5941	0.5943	0.5944	0.5944	0.5944	0.5945
105	-2	2.50E+03	0.7960	0.7956	0.7958	0.7960	0.7959	0.7960	0.7960
105	-4	2.50E+05	1.0535	1.0531	1.0533	1.0535	1.0534	1.0535	1.0535
105	-6	2.50E+07	1.3696	1.3693	1.3695	1.3696	1.3696	1.3697	1.3697
CPU time (seconds)			218.53	0.34	0.68	3.93	0.23	0.46	0.93
MAPE ($\times 10^3$)				0.2899	0.1010	0.0062	0.1054	0.0207	0.0046

Note: This table values European knock-out double barrier call options under the CEV model, using the same model parameters as in Davydov and Linetsky (2001, table 1) [**15**], i.e. $S_{t_0} = 100$, lower barrier price of 90, upper barrier price of 120, $T - t_0 = 0.5$ years, $r = 10\%$ and $q = 0\%$ but with $\beta \in \{1, 0, -2, -4, -6\}$.

Table 7 extends the cases for $\beta < 2$ of Dias et al. (2015, Table 1) [**19**]. The benchmark values are computed by the transformed trinomial tree of Boyle and Tian (1999) [**4**] with 100,000 time steps and are reported in the forth column. The next three columns represent the prices computed by the SHP approach described in Dias et al. (2015, section 4.2) [**19**], using 100, 200 and 1000 time steps respectively.

The results of the Richardson extrapolation applied in the DEK method are presented in the last three columns. The first represents the prices computed with the Richardson extrapolation repeated three times over a starting column of four elements (prices computed by the DEK method, with 8, 16, 32, and 64 time steps respectively). The prices shown in the eighth column are equivalent to the position $f_{4,3}$ in Table 1, the ninth column are equivalent to the position $f_{5,4}$, and the tenth to the $f_{6,5}$ position.

The last two rows present the CPU time (in seconds) needed to compute the target option prices and the MAPE respectively, which is computed against the benchmark value and is shown multiplied by a factor of 10^3.

Table 7 shows that in terms of accuracy the Richardson extrapolation applied to the DEK method can, at least, match the SHP approach. When taking the speed into account, it can be observed that the Richardson extrapolation applied to the DEK method can outperform the SHP approach. The first column of the Richardson extrapolation can be compared to the second column of the SHP approach in terms of error but takes half the computing time. The second column of the Richardson extrapolation technique while not matching the error of the SHP approach with 1000 time steps, it is much smaller than the one observed in the SHP approach when the computing times are equal. Both methods are comparable in terms of accuracy in the respective last columns, but the Richardson extrapolation needing less computing time.

Comparing the last row as well as the values between both methods, it can be observed that the Richardson extrapolation outperform the SHP approach, matching the same accuracy needing less time.

There are a few cases where the price achieved with the Richardson extrapolation does not completely match the benchmark value, this fact could raise some concerns. When making an analysis with the exact values computed, it is observable that the MAPE of the Richardson extrapolation matches the one of the SHP approach.

5.2. JDCEV model

This subsection compares the performance of the SHP approach, proposed in Dias et al. (2015) [**19**], and the Richardson extrapolation employed in the DEK method for valuing double barrier options under the time-homogeneous JDCEV model.

Table 8 is an extension of Table 2 in Dias et al. (2015) [**19**], comparing the efficiency of the SHP approach against the implementation of the repeated Richardson extrapolation technique employed in the DEK method. It compiles the prices of 45 double barrier put options with different contract specifications.

The table is divided in three panels, the first corresponding to the nested CEV model with $b = c = 0$, the second with $b = 0$ but $c = 1$ and the last where the prices are computed with the parameter b equalling to 0.02 and parameter c equalling to 0.5. The fourth and fifth columns present the different components of a put option price computed by the equations 8 and 14 respectively, and the sixth column shows their sum.

Columns seven, eight and nine implement the SHP approach, discussed in Dias et al. (2015) [**19**], for pricing knock-out double barrier put options using 100, 200 and 1000 time steps respectively. The tenth column reports the prices of a knock-in double barrier put options computed by the same SHP approach with 1000 time steps.

The last four columns present prices calculated by the Richardson extrapolation applied in the DEK method. The first three, represent the prices computed with the Richardson extrapolation repeated three, four and five times over a starting column of four, five and six elements, respectively. These columns are respective to the prices of knock-out double barrier put options while the last, presents the prices of knock-in double barrier put options using the Richardson extrapolation repeated five times over a starting column six elements.

Table 8: European double barrier knock-out and knock-in put prices under the JDCEV model

			Standard European put			SHP approach				Richardson extrapolation			
K	$\bar{\beta}$	a	$v_{t_0}^0$	$v_{t_0}^D$	v_{t_0}	100	200	1000	KI1000	4	5	6	KI6
Panel A: $b = c = 0$													
95	-0.5	2.50E+00	3.0297	0.0000	3.0297	0.0199	0.0200	0.0201	3.0096	0.0201	0.0201	0.0201	3.0096
95	-1	2.50E+01	3.1094	0.0000	3.1094	0.0177	0.0179	0.0179	3.0915	0.0180	0.0180	0.0180	3.0915
95	-2	2.50E+03	3.1999	0.0866	3.2865	0.0141	0.0142	0.0142	3.2723	0.0143	0.0143	0.0143	3.2723
95	-3	2.50E+05	2.6303	0.8679	3.4982	0.0111	0.0112	0.0112	3.4870	0.0113	0.0112	0.0112	3.4870
95	-4	2.50E+07	1.7565	2.0051	3.7616	0.0087	0.0087	0.0087	3.7529	0.0089	0.0088	0.0088	3.7528
100	-0.5	2.50E+00	4.7075	0.0000	4.7075	0.1552	0.1554	0.1554	4.5521	0.1554	0.1554	0.1554	4.5521
100	-1	2.50E+01	4.7145	0.0000	4.7145	0.1426	0.1427	0.1428	4.5717	0.1428	0.1428	0.1428	4.5717
100	-2	2.50E+03	4.6524	0.0912	4.7436	0.1199	0.1200	0.1200	4.6235	0.1201	0.1200	0.1200	4.6235
100	-3	2.50E+05	3.8841	0.9136	4.7977	0.1000	0.1000	0.1001	4.6976	0.1002	0.1001	0.1001	4.6976
100	-4	2.50E+07	2.7761	2.1106	4.8867	0.0827	0.0828	0.0828	4.8040	0.0830	0.0828	0.0828	4.8039
105	-0.5	2.50E+00	6.8961	0.0000	6.8961	0.4922	0.4922	0.4923	6.4038	0.4921	0.4922	0.4923	6.4038
105	-1	2.50E+01	6.8194	0.0000	6.8194	0.4654	0.4654	0.4655	6.3539	0.4653	0.4654	0.4654	6.3539
105	-2	2.50E+03	6.5869	0.0957	6.6826	0.4148	0.4149	0.4149	6.2677	0.4149	0.4149	0.4149	6.2677
105	-3	2.50E+05	5.6089	0.9592	6.5681	0.3675	0.3675	0.3675	6.2006	0.3677	0.3676	0.3676	6.2006
105	-4	2.50E+07	4.2628	2.2161	6.4789	0.3234	0.3234	0.3235	6.1555	0.3238	0.3236	0.3235	6.1554
Panel B: $b = 0$ and $c = 1$													
95	-0.5	2.50E+00	2.1700	2.7125	4.8825	0.0170	0.0171	0.0171	4.8654	0.0172	0.0172	0.0172	4.8654
95	-1	2.50E+01	2.1599	2.6874	4.8473	0.0151	0.0153	0.0153	4.8319	0.0153	0.0153	0.0153	4.8319
95	-2	2.50E+03	2.0156	2.8246	4.8402	0.0121	0.0121	0.0122	4.8280	0.0123	0.0122	0.0122	4.828
95	-3	2.50E+05	1.5470	3.3567	4.9037	0.0095	0.0096	0.0096	4.8941	0.0097	0.0096	0.0096	4.8941
95	-4	2.50E+07	1.0134	4.0114	5.0248	0.0074	0.0075	0.0075	5.0172	0.0076	0.0075	0.0075	5.0172
100	-0.5	2.50E+00	3.5004	2.8553	6.3557	0.1360	0.1361	0.1361	6.2196	0.1360	0.1361	0.1361	6.2196
100	-1	2.50E+01	3.4185	2.8288	6.2473	0.1251	0.1252	0.1252	6.1221	0.1252	0.1252	0.1252	6.1221
100	-2	2.50E+03	3.1210	2.9733	6.0943	0.1054	0.1054	0.1055	5.9888	0.1056	0.1055	0.1055	5.9888
100	-3	2.50E+05	2.4738	3.5333	6.0071	0.0881	0.0881	0.0881	5.9190	0.0883	0.0882	0.0882	5.9190
100	-4	2.50E+07	1.7503	4.2225	5.9728	0.0730	0.0731	0.0731	5.8997	0.0733	0.0732	0.0731	5.8996
105	-0.5	2.50E+00	5.3018	2.9981	8.2999	0.4415	0.4415	0.4415	7.8584	0.4413	0.4414	0.4415	7.8584
105	-1	2.50E+01	5.1380	2.9702	8.1082	0.4183	0.4183	0.4183	7.6900	0.4182	0.4183	0.4183	7.6900
105	-2	2.50E+03	4.6704	3.1219	7.7923	0.3744	0.3744	0.3744	7.4180	0.3745	0.3744	0.3744	7.4179
105	-3	2.50E+05	3.8323	3.7100	7.5423	0.3330	0.3330	0.3330	7.2093	0.3333	0.3331	0.3330	7.2093
105	-4	2.50E+07	2.9088	4.4336	7.3424	0.2941	0.2942	0.2942	7.0481	0.2946	0.2944	0.2943	7.0481

continues on next page

Table 8: (continued)

K	$\bar{\beta}$	a	Standard European put			SHP approach				Richardson extrapolation			
			$v^0_{t_0}$	$v^D_{t_0}$	v_{t_0}	100	200	1000	KI1000	4	5	6	KI6
Panel C: $b = 0.02$ and $c = 0.5$													
95	-0.5	2.50E+00	2.3287	2.2560	4.5847	0.0174	0.0175	0.0176	4.5671	0.0176	0.0176	0.0176	4.5671
95	-1	2.50E+01	2.3568	2.2481	4.6049	0.0155	0.0156	0.0157	4.5892	0.0157	0.0157	0.0157	4.5892
95	-2	2.50E+03	2.3101	2.3748	4.6849	0.0122	0.0123	0.0123	4.6725	0.0124	0.0123	0.0123	4.6725
95	-3	2.50E+05	1.8363	2.9803	4.8166	0.0095	0.0096	0.0096	4.807	0.0097	0.0096	0.0096	4.8069
95	-4	2.50E+07	1.2126	3.7893	5.0019	0.0074	0.0074	0.0074	4.9944	0.0075	0.0074	0.0074	4.9944
100	-0.5	2.50E+00	3.7204	2.3747	6.0951	0.1389	0.1390	0.1390	5.9562	0.1389	0.1390	0.1390	5.9562
100	-1	2.50E+01	3.6791	2.3664	6.0455	0.1270	0.1271	0.1272	5.9183	0.1271	0.1272	0.1272	5.9183
100	-2	2.50E+03	3.4851	2.4998	5.9849	0.1058	0.1059	0.1059	5.8789	0.1060	0.1059	0.1059	5.8789
100	-3	2.50E+05	2.8322	3.1372	5.9694	0.0874	0.0874	0.0874	5.8819	0.0876	0.0875	0.0874	5.8819
100	-4	2.50E+07	2.0092	3.9887	5.9979	0.0716	0.0716	0.0716	5.9263	0.0718	0.0716	0.0716	5.9263
105	-0.5	2.50E+00	5.5889	2.4935	8.0824	0.4488	0.4488	0.4488	7.6336	0.4486	0.4487	0.4488	7.6336
105	-1	2.50E+01	5.4656	2.4847	7.9503	0.4227	0.4227	0.4227	7.5276	0.4225	0.4226	0.4227	7.5277
105	-2	2.50E+03	5.1031	2.6248	7.7279	0.3736	0.3736	0.3736	7.3542	0.3737	0.3737	0.3736	7.3542
105	-3	2.50E+05	4.2559	3.2940	7.5499	0.3280	0.3280	0.3280	7.2219	0.3282	0.3281	0.3281	7.2219
105	-4	2.50E+07	3.2225	4.1881	7.4106	0.2858	0.2859	0.2859	7.1247	0.2862	0.2860	0.2859	7.1247
CPU time (seconds)						0.43	0.86	4.82	5.11	0.35	0.72	1.52	1.42

Note: This table values European knock-out and knock-in double barrier put options under the JDCEV model, for various parameter configurations, fixing only $S_0 = 100$, *lower barrier price of 90, upper barrier price of 120,* $T - t_0 = 0.5$ *years,* $r = 10\%$ *and* $q = 0\%$.

Table 8 illustrates the efficiency of both methods for valuing double barrier options under the JDCEV model, and more clearly the improvement that the repeated Richardson extrapolation technique can bring to the DEK method. With the Richardson extrapolation the DEK method is able to achieve the same accuracy levels while improving the time needed to compute the double barrier option prices, surpacing in performance the SHP approach.

CHAPTER 6

Conclusions

In this thesis, the repeated Richardson extrapolation technique was employed in static hedging methodologies in order to study its improvement in efficiency. This study is focused solely on the valuation of European-style single and double barrier options under the CEV model, Cox (1975) [**14**], and the more general JDCEV model, Carr and Linetsky (2006) [**10**].

First, under the CEV model, the Richardson extrapolation was employed in the DEK and the modified DEK models. With the valuation of an up-and-out call, an up-and-in call and a down-and-in put it is clear that the Richardson extrapolation has the potential to improve the performance of the two static hedging methods. The modified DEK method is an improvement of the plain DEK method, achieving this by not only matching the value of the target option but also its theta along the barrier, while this provides a faster and more accurate method it also implies more calculations needed. In the results this extra calculations injure the performance of the method when used with the Richardson extrapolation, making the DEK method preferred in comparison.

The three examples of different types of single barrier options show that the repeated Richardson extrapolation increases the speed-accuracy trade-off of both methods tested. In order to understand if the results are also valid for double barrier options this thesis followed the work of Dias et al. (2015) [**19**] using the SHP approach proposed, instead of the modfied DEK.

Figure 1 presents 1696 different double knock-out call contracts computed with the SHP approach, the Richardson extrapolation employed in SHP approach and in the DEK method. The plain DEK method was also computed but to obtain the same levels of accuracy it needed a substantial amount of extra computing time, for that reason it is not shown. In addition to Figure 1, the Table 1 of Dias et al. (2015) [**19**] was reproduced to accomodate the Richardson extrapolation. The results were coherent in the valuation of both single and double barrier options, although the repeated Richardson improved both methods it provides a more significant outcome when employed in the DEK method.

This thesis also extends its scope by testing the valuation of double barrier options under the JDCEV model. Table 8 adds to Table 2 of Dias et al. (2015) [**19**] the Richardson extrapolation employed in the DEK method in contrast with the SHP approach. Looking at the table presented the repeated Richardson technique does not show significant levels of improvement, in lower orders of accuracy, but as the accuracy increases so does the Richardson extrapolation efficiency, as it can be seen in the last two columns.

Given the previous results in the CEV model, only the Richardson extrapolation employed in the DEK was considered under the JDCEV framework, to further explore the Richardson could also be applied to the SHP approach and compare its improvement in relation with the one obtained under the CEV model. Aside from double barrier options, there single barrier counterparts could also be tested.

Overall, the Richardson can improve the efficiency of the static hedging methodologies considered it proved to have better results when pared with the DEK method, the extra computations of theta and cash-or-nothing options seem to take some of that performance away. Another aspect worth noting is that the code used for the Richardson extrapolation could be better optimized, leading to even better results in the future.

The Repeated Richardson extrapolation technique is a valuable computational tool and proved to enhance the performance of the methodologies observed. Another point where it can be useful is through the estimation of the error in the approximation. Chang et al. (2007) [**11**] uses Schmidt's (1968) [**31**] inequality to specify the accuracy of the Richardson extrapolation approximation, and Farasi et al. (2010) [**20**] also provide step-size control by evaluating the approximation error.

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