Noname manuscript No.
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Proportional Bargaining Solutions, Strictly Comprehensive Sets and the Axiom of Continuity

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the date of receipt and acceptance should be inserted later

Abstract Continuity is an axiom of Kalai (1977)'s and Roth (1979b)'s characterizations of the proportional bargaining. In this paper we establish that, in the class of strictly comprehensive sets, continuity is a corollary of the other axioms. Consequently the proportional solution can be axiomatically defined in the class of strictly comprehensive sets without the axiom of continuity. We also show it is possible to tighten Kalai (1977)'s and Roth (1979b)'s axiomatization of the proportional bargaining solution, assuming a weaker version of the axiom of continuity.

Keywords axiomatic bargaining, proportional solution, continuity axiom.

The proportional solution is a longstanding solution for the axiomatic bargaining problem. It has been axiomatically characterized in two different classes of sets: in the class of comprehensive sets by Kalai (1977), and in the class of convex sets by Roth (1979b). Here we characterize it in a third bargaining class, the class of strictly comprehensive sets. While the class of convex sets describes the case where players can randomize over different bargaining agreements, the class of comprehensive sets adds to this the possibility of disposable utility. The class of strictly comprehensive sets is a subclass of comprehensive sets, and it represents the case where the marginal utility of the bargained item is strictly positive. Geometrically, the boundary of a strictly comprehensive set is never parallel to the axis. This is a natural class for bargaining to take place in, since it models the case where more of the good is better for all players. In this sense, at the Pareto frontier it cannot

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happen that some player diminishes his utility while all the others maintain theirs. That is, if someone is worst off then someone else must be better off.

In this work we first show that when the bargaining class has only strictly comprehensive sets, the axiom of continuity is a consequence of the other axioms (see Proposition 1). This result reveals that the axiom of continuity is needed only for a minor part of the comprehensive sets. Therefore, if we dismiss these sets, we obtain the proportional solution without assuming continuity. That is, the set of axioms that characterize the proportional solution to be the bargaining solution in the class of strictly comprehensive sets can omit the continuity axiom (see Theorem 1).

A second implication is that Kalai (1977)'s and Roth (1979b)'s results on proportional bargaining in convex and in comprehensive sets, respectively, can be slightly tightened. The continuity axioms in these characterization can be weakened to require that the solution preserves limits for each convergent sequence of problems in $\mathbb{X}$ whose limit is in $\mathbb{X} \setminus \mathbb{D}$.

Proportional bargaining is still a solution if we axiomatically impose continuity only on the sets that are comprehensive but not strictly comprehensive sets (see Corollary 1).

We end the paper with example 1 which describes a non-continuous bargaining solution for the class of comprehensive sets satisfying all the other axioms. This example shows that we cannot extend our result; that is, continuity is not a consequence of the other axioms in the class of comprehensive sets.

By a bargaining game we mean a pair $(S, \boldsymbol{d})$, where S is a set of outcome possibilities and $\boldsymbol{d}$ is a disagreement outcome. One outcome, a point in S or $\boldsymbol{d}$, is chosen as the bargaining solution of the game. The set S represents the admissible utility payoffs of the players. We assume that S is convex and compact; that there is an element $\boldsymbol{s} \in S$ with $\boldsymbol{s} > \boldsymbol{0}$; and[1] that $\boldsymbol{d} = \boldsymbol{0}$. The set of players is $N = \{1, 2, \dots, n\}$. A bargaining class is a family of subsets of $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq 0\}$. We use three bargaining classes: $\mathbb{B}$, the set of compact and convex sets in $\mathbb{R}^n_+$; $\mathbb{D}$, the set of compact, convex and comprehensive sets in $\mathbb{R}^n_+$ and $\overline{\mathbb{D}}$, the set of compact, convex and strictly comprehensive sets in $\mathbb{R}^n$. A convex set D is comprehensive, where $D \in \mathbb{D} \subset \mathbb{B}$, if the following property is verified

$$\boldsymbol{x} \in D, \text{ then } \boldsymbol{x'} \in D \text{ for any } \boldsymbol{0} \leq \boldsymbol{x'} \leq \boldsymbol{x}.$$

[1] By $\boldsymbol{x} < \boldsymbol{y}$ and $\boldsymbol{x} \leq \boldsymbol{y}$ it is meant, respectively, that $x_i < y_i$ or $x_i \leq y_i$ for all $i = 1, \dots, n$.

A comprehensive set $\overline{D}$ is strictly comprehensive, where $\overline{D} \in \overline{\mathbb{D}} \subset \mathbb{D}$, when

$$\boldsymbol{x}, \boldsymbol{x'} \in \overline{D} \text{ with } \boldsymbol{x} \leq \boldsymbol{x'} \text{ and } \boldsymbol{x} \neq \boldsymbol{x'}, \text{ there exists } \tilde{\boldsymbol{x}} \in \overline{D} \text{ such that } \boldsymbol{x} < \tilde{\boldsymbol{x}}.$$

For $a \in \mathbb{R}_+$ and S, let $aS = \{a\boldsymbol{s} : \boldsymbol{s} \in S\}$. We denote by δS the Pareto optimal frontier of the set S in $\mathbb{R}^n_+$; that is, $\delta S = \{\boldsymbol{x} \in S : \nexists \boldsymbol{y} \in S, \boldsymbol{y} > \boldsymbol{x}\}$. Our classes of sets are metric spaces under the Hausdorff distance, for any $A, B \in \mathbb{B}$, $d(A,B) = \max\big\{\max_{\boldsymbol{a}\in A}\min_{\boldsymbol{b}\in B} d(\boldsymbol{a},\boldsymbol{b}), \max_{\boldsymbol{b}\in B}\min_{\boldsymbol{a}\in A} d(\boldsymbol{a},\boldsymbol{b})\big\}$. In $\mathbb{R}_n^+$ we use the metric induced by the uniform norm, $d(\boldsymbol{a},\boldsymbol{b}) = \max_{i\in N}|a_i - b_i|$.A bargaining solution in the class $\mathbb{X} \in \mathcal{F} = \{\mathbb{B}, \mathbb{D}, \overline{\mathbb{D}}\}$ is a function $f : \mathbb{X} \to \mathbb{R}^n_+$ such that for all $S \in \mathbb{X}$ satisfies $f(S) \in S$. This function should respect several of the relations specified in the following axioms.

Axioms [2]

1. **Weakly Pareto Optimal.** For all $S \in \mathbb{X}$ there is no $\boldsymbol{s} \in S$ such that $\boldsymbol{s} > f(S)$.
2. **Homogeneity.** For all $S \in \mathbb{X}$ and all $a \in \mathbb{R}_+ \setminus \{0\}$, $f(aS) = af(S)$.
3. **Strong Individual Rationality.** For all $S \in \mathbb{X}$, $f(S) \geq \boldsymbol{0}$ and $f\big(\mathrm{cch}\{S\}\big) > \boldsymbol{0}$.
4. **Monotonicity.** For all $S, S' \in \mathbb{X}$ with $S \subseteq S'$, then $f(S) \leq f(S')$.

6a. **Independence of Irrelevant Alternatives.** For all $S, S' \in \mathbb{X}$ with $S \subseteq S'$ and $f(S') \in S$ then $f(S) = f(S')$.

6b. **Individual Monotonicity.** For all $S, S' \in \mathbb{X}$ with $S \subseteq S'$ and $\mathrm{cch}\,\{S\} \cap \{x \in \mathbb{R}^n : x_i = 0\} = \mathrm{cch}\,\big\{S'\big\} \cap \{x \in \mathbb{R}^n : x_i = 0\}$, for a given i, then $f_i(S) \leq f_i(S')$.

6c. **Continuity** If $S_k \in \mathbb{X}$, for $k \in \mathbb{N}$, converge to $S \in \mathbb{X}$, then $f(S_k) \to f(S)$.

A solution is proportional in $\mathbb{X} \in \mathcal{F}$ if there exists $\boldsymbol{p} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$ such that for any $S \in \mathbb{X}$, $f(S) = t^{\boldsymbol{p}}_S \boldsymbol{p}$, where

$$t^{\boldsymbol{p}}_S = \max\{t \in \mathbb{R}_+ : t\boldsymbol{p} \in S\}.$$

Kalai (1977) and Roth (1979b) both considered a bargaining solution that respected a subset of these axioms. Those theorems will be stated to simplify the comparison with the results in this paper. Kalai (1977) proved the following theorem[3] for the class $\mathbb{D}$:

Theorem (K). *A solution in $\mathbb{D}$ satisfies axioms* (2), (3), (6a), (6b) *and* (6c) *if and only if it is proportional.*

Roth (1979b) generalized this result to the larger class of convex sets $\mathbb{B}$ and provided the following theorem.

[2] When referring to the axioms it should be clear from the context to which bargaining class $\mathbb{X}$ each axiom applies. We leave out axiom number 5 to make this list comparable with the list of axioms in Roth (1979b).

[3] Theorem **K** is an improved version of the original theorem, attributed to Roth (1979b).

Theorem (R). *A solution in $\mathbb{B}$ that satisfies axioms* (2)*,* (3)*,* (6a)*,* (6b) *in $\mathbb{B}$, and* (6c) *in $\mathbb{D}$ must be proportional.*

Next, we will prove an alternative characterization of the proportional bargaining solution, this time in the class $\overline{\mathbb{D}}$. The central result is that the continuity of the solution in $\overline{\mathbb{D}}$ is a consequence of axioms (2), (3), (6a) and (6b).

Proposition 1 *If a solution in $\overline{\mathbb{D}}$ satisfies axioms* (2)*,* (3)*,* (6a) *and* (6b)*, then it satisfies* (6c) *in $\overline{\mathbb{D}}$.*

The proofs are in the appendix. For the above proposition we now introduce the main ideas and give a sketch of the argument for the case of $N = \{1, 2, 3\}$. We leave for the appendix the general case, that is, for any number of players $n \in \mathbb{N}$, because it lacks the simplicity and geometric appeal of the other cases.

We argue by contradiction, assuming there is a sequence of sets S_k converging to S, with the sequence of solutions $f(S_k) = \boldsymbol{s}_k$ not converging to $f(S) = \boldsymbol{s}$. We start by noting that, it can be assumed, without loss of generality, that $f(S) \in S_k$ for all $k \in \mathbb{N}$, (see Lemma 3). We then find two sets, A and B, with $B \subseteq A$, satisfying the conditions of axiom (6b). Thus $f_i(B) \leq f_i(A)$, for any $i \in N$. However, those sets are such that $f(A), f(B) \in \delta S$ and $f(A) \neq f(B)$. Therefore, since S is strictly comprehensive, there must exist a $j \in N$ such that $f_j(A) < f_j(B)$. Hence, we obtain a contradiction.

To create these sets ideally we would use only $\boldsymbol{s_k}$ and $\boldsymbol{s}$. However, this is not possible. For example, the set $cch\{\boldsymbol{s_k}, \boldsymbol{s}\}$ is not strictly comprehensive. Thus, we need to use some other points to generate A and B. To create these points we must introduce some new concepts. For a subset $M \subseteq N = \{1, 2, 3\}$ we define $\pi_M(\boldsymbol{x})$, the projection in M of $\boldsymbol{x} = (x_1, x_2, x_3)$. The projection $\pi_M(\boldsymbol{x})$ is the vector whose coordinates in M are equal to those of $\boldsymbol{x}$ and those not in M are zero. For example, with $M = \{1, 3\}$, $\pi_M(\boldsymbol{x}) = (x_1, 0, x_3)$. Notice that

$$cch\{\boldsymbol{s_k}, \boldsymbol{s}\} = cch\Big\{\{\pi_M(\boldsymbol{s_k}), \pi_M(\boldsymbol{s})\}_{\{M \in 2^N \setminus \{\emptyset\}\}}\Big\}.$$

As already mentioned, this set does not belong to $\overline{\mathbb{D}}$. Therefore, to create a set that does belong to $\overline{\mathbb{D}}$ we use points that are close to the projections $\pi_M(\boldsymbol{s_k})$ and $\pi_M(\boldsymbol{s})$. For each subset $M \subseteq N$ we have two points, $\boldsymbol{x_M}$ and $\boldsymbol{y_M}$. Let $\boldsymbol{x_M} = \pi_M(\boldsymbol{s}) + (n - \#M)\epsilon\pi_M(1, 1, 1)$. For instance, $\boldsymbol{x_{\{2\}}} = (0, s_2 + 2\epsilon, 0)$ and $\boldsymbol{x_{\{1,3\}}} = (s_1 + \epsilon, 0, s_3 + \epsilon)$. The vector $\boldsymbol{y_M}$ is created in the same way but with $\pi_M(\boldsymbol{s})$ replaced by $\rho_k\pi_M(\boldsymbol{s_k})$; that is, $\boldsymbol{y_M} = \rho_k\pi_M(\boldsymbol{s_k}) + (n - \#M)\epsilon\pi_M(1, 1, 1)$, where $\rho_k = \max\{\rho \in \mathbb{R}_+ : \rho\boldsymbol{s_k} \in S\}$.

The sets A and B are as follows:

$$A = cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M\in 2^N\setminus\{\emptyset\}\}}\Big\} \qquad B = cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M\in 2^N\setminus\{\emptyset,N\}\}}, \rho_k\boldsymbol{s}, \rho_k\boldsymbol{s_k}\Big\}.$$

The set of generating points is almost the same, the difference being that $\boldsymbol{s}$ is used as a generator of A and $\rho_k\boldsymbol{s}$ as a generator of B. Thus, the sets A and B are very similar. In particular, $A \cap \{\boldsymbol{x} \in \mathbb{R}^3_+ : x_i = 0\} = B \cap \{\boldsymbol{x} \in \mathbb{R}^3_+ : x_i = 0\}$, for any $i \in \{1, 2, 3\}$ (*cf.* axiom 6b).

Since $\rho_k < 1$, we have that $\rho_k\boldsymbol{s} \in A$ and $B \subset A$. With a judicious choice of ϵ we can ensure that all generators of A are in S and all generators of B are in $\rho_k S_k$. Therefore, using axiom (6a), we conclude that $f(A) = \boldsymbol{s}$ and $f(B) = \rho_k\boldsymbol{s_k}$. Both these choices belong to δS, the boundary of S, and since S is a strictly comprehensive set, for some $j \in N$, we have $f_j(A) < f_j(B)$. On the other hand we can apply axiom (6b), since

$$A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_j = 0\} = B \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_j = 0\} \text{ for all } j \in N.$$

This is a consequence of the generating vectors of A and B, which have at least one coordinate equal to zero, being the same. From $A, B \in \overline{\mathbb{D}}$, $B \subseteq A$ and the previous equality, axiom (6b) implies that $f(B) \leq f(A)$. And we reach a contradiction with $f_j(B) > f_j(A)$ for some $j \in N$. As referred, the full proof is in the appendix.

Proposition 1 enables us to find a slightly tighter version of Theorems **K** and **R**. Indeed, instead of assuming that the solution is continuous in $\mathbb{D}$, we only need to assume that the solution is continuous in the sets in $\mathbb{D} \setminus \overline{\mathbb{D}}$. Then, using this hypothesis and the previous proposition, we can conclude that the solution is continuous in $\mathbb{D}$. Hence, the solution is proportional. We first need to define continuity in $\mathbb{D} \setminus \overline{\mathbb{D}}$, since this involves a subtle point. In particular we are using that the solution f is continuous in $\mathbb{D} \setminus \overline{\mathbb{D}}$ when the domain of the solution is $\mathbb{D}$. That is, although the limiting set S must be in $\mathbb{D} \setminus \overline{\mathbb{D}}$, the elements S_k of the convergent sequence might be in $\overline{\mathbb{D}}$.

Axiom 6c'. For any sequence $S_k \in \mathbb{D}$ converging to $S \in \mathbb{D} \setminus \overline{\mathbb{D}}$, $f(S_k) \to f(S)$.

We can now present the new versions of Theorems **K** and **R**.

Corollary 1 *A solution in* $\mathbb{D}$ *satisfies* (2), (3), (6a), (6b) *and* (6c') *if and only if it is proportional. A solution in* $\mathbb{B}$ *satisfies* (2), (3), (6a), (6b) *and* (6c') *if and only if it is proportional.*

Our next result proves that the solution is proportional in the bargaining class of strictly comprehensive sets $\overline{\mathbb{D}}$. In comparison with **K** and **R** there is no need for the axiom of continuity, since it follows from the other axioms. By shrinking the class of bargaining sets from $\mathbb{D}$ to $\overline{\mathbb{D}}$, one of the axioms becomes unnecessary.

Theorem 1 *A solution in* $\overline{\mathbb{D}}$ *satisfies* (2)*,* (3)*,* (6a)*, and* (6b) *if and only if it is proportional.*

We now show that Theorem 1 cannot be generalized for the larger class of comprehensive sets $\mathbb{D}$. We provide an example of a non-continuous solution in $\mathbb{D}$ which respects all the axioms assumed in Theorem 1. In this example the solution is proportional to a given $\boldsymbol{q} \in \mathbb{R}^2$ in the majority of the sets, and it is proportional to $\boldsymbol{p} \in \mathbb{R}^2$ in the remaining sets. This family of sets, call it $\mathcal{B}$, where the solution is proportional to $\boldsymbol{p}$, is characterized by part of the set's boundary being parallel to the ordinates. In particular, the solution is $\boldsymbol{p}S^1$ whenever $\boldsymbol{p}S^1 \in S$. Since the family of sets $\mathbb{D} \setminus \mathcal{B}$, where the solution is proportional to $\boldsymbol{q}$, is not closed, the solution is non-continuous.

Example 1

$$g(S) = \begin{cases} \boldsymbol{p}S^1 & \text{if } \boldsymbol{p}S^1 \in S \\ \boldsymbol{q}t_S^{\boldsymbol{q}} & \text{if } \boldsymbol{p}S^1 \notin S \end{cases}$$

where $\boldsymbol{p} = (1, p_2), \boldsymbol{q} = (1, q_2) \in \mathbb{R}^2$, $0 < q_2 < p_2 < 1$ and $S^1 = max\{x : (x, y) \in S\}$.

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1 Appendix

To prove Proposition 1 we need two auxiliary lemmas. The first shows that two sets with different bargaining choices cannot simultaneously contain the other's set choice. The second lemma shows that when a sequence of convex and comprehensive sets converges to S, then the boundary of that sequence converges to the boundary of S.

Lemma 1 *Let f satisfy* (6a). *Let $S, S' \in \mathbb{X} \in \mathcal{F}$ and $f(S) \neq f(S')$. If $f(S) \in S'$, then $f(S') \notin S$.*

Proof Since $f(S) \in S'$, $f(S) \in S \cap S'$. By axiom (6a), $f(S \cap S') = f(S)$. Then $f(S \cap S') \neq f(S')$ and, since $S \cap S' \subseteq S'$, axiom (6a) implies that $f(S') \notin S \cap S'$. Therefore, $f(S') \notin S$. □

Lemma 2 *If $S_k \to S$, with $S_k, S \in \mathbb{D}$, then $\delta S_k \to \delta S$.*

Proof Throughout this proof we use the well-know result that the proportional solution is continuous for any $\boldsymbol{p} \in \mathbb{R}^n_+ \setminus \{\boldsymbol{0}\}$; that is $t^{\boldsymbol{p}}_{S_k} \to t^{\boldsymbol{p}}_S$, whenever $S_k \to S$ and $S_k, S \in \mathbb{D}$. We first assume that $d(\delta S_k, \delta S)$ does not converge to zero. In this case, there is an $\epsilon > 0$ such that at least one of the next two conditions holds:

$$\text{There exists an } \boldsymbol{y_k} \in \delta S, \text{with } d(\delta S_k, \boldsymbol{y_k}) > \epsilon, \text{ for an infinite number of } k\text{'s.} \quad (1)$$

$$\text{There exists an } \boldsymbol{x_k} \in \delta S_k, \text{with } d(\boldsymbol{x_k}, \delta S) > \epsilon, \text{ for an infinite number of } k\text{'s.} \quad (2)$$

In the argument, to avoid double and triple subscripts, we use the original notation of a sequence when taking a subsequence. If (1) holds, then, there exists an $\boldsymbol{y_k} \in \delta S$, such that, for any $\boldsymbol{x_k} \in \delta S_k$, $d(\boldsymbol{y_k}, \boldsymbol{x_k}) > \epsilon$. Let $\boldsymbol{y} \in \delta S$ be the limit of a convergent subsequence of $\boldsymbol{y_k}$. Let $K \in \mathbb{N}$ be such that $d(\boldsymbol{y_k}, \boldsymbol{y}) < \epsilon/2$, for $k \geq K$. Then $d(\boldsymbol{x_k}, \boldsymbol{y}) > d(\boldsymbol{x_k}, \boldsymbol{y_k}) - d(\boldsymbol{y_k}, \boldsymbol{y}) > \epsilon/2$, for any $k \geq K$ and $\boldsymbol{x_k} \in \delta S_k$. With $\boldsymbol{x_k} = t^{\boldsymbol{y}}_{S_k} \boldsymbol{y}$, we conclude that $d(t^{\boldsymbol{y}}_{S_k} \boldsymbol{y}, \boldsymbol{y}) > \epsilon/2$, for any $k \geq K$. Thus, $\lim t^{\boldsymbol{y}}_{S_k} \boldsymbol{y} \neq \boldsymbol{y}$ and so $\lim t^{\boldsymbol{y}}_{S_k} \neq 1 = t^{\boldsymbol{y}}_S$. Therefore, we get a contradiction with the continuity of the proportional solution.

If (2) holds, then, for each k, there exists an $\boldsymbol{x_k} \in \delta S_k$ such that, for any $\boldsymbol{y} \in \delta S$, $d(\boldsymbol{x_k}, \boldsymbol{y}) > \epsilon$. Assume that $\boldsymbol{x}$ is the limit of a convergent subsequence of $\boldsymbol{x}_k$. This subsequence exists once we can enclose the converging sequence of sets S_k in a larger compact set. Take $K \in \mathbb{N}$ such that $d(\boldsymbol{x}_k, \boldsymbol{x}) < \epsilon/2$, for $k \geq K$. Then, $d(\boldsymbol{x}, \boldsymbol{y}) > d(\boldsymbol{x}_k, \boldsymbol{y}) - d(\boldsymbol{x}, \boldsymbol{x}_k) > \epsilon/2$. If $\boldsymbol{x}_k - t^{\boldsymbol{x}}_{S_k} \boldsymbol{x} \to 0$ we obtain a contradiction, since

$$\epsilon/2 < d(\boldsymbol{x}, t^{\boldsymbol{x}}_S \boldsymbol{x}) \leq d(t^{\boldsymbol{x}}_S \boldsymbol{x}, t^{\boldsymbol{x}}_{S_k} \boldsymbol{x}) + d(t^{\boldsymbol{x}}_{S_k} \boldsymbol{x}, \boldsymbol{x}_k) + d(\boldsymbol{x}_k, \boldsymbol{x}) \to 0.$$

Here we used the facts that $\epsilon/2 < d(\boldsymbol{x}, \boldsymbol{y})$ with $\boldsymbol{y} = t^{\boldsymbol{x}}_S \boldsymbol{x} \in S$ and that the proportional solution is continuous. In particular, $d(t^{\boldsymbol{x}}_{S_k} \boldsymbol{x}, t^{\boldsymbol{x}_k}_S) \to 0$.

To prove that $\boldsymbol{x}_k - t^{\boldsymbol{x}}_{S_k} \boldsymbol{x} \to 0$ we first consider the case where $\boldsymbol{x} > \boldsymbol{0}$. Let

$$\alpha_k = \max\{\alpha \in \mathbb{R}_+ : \alpha \boldsymbol{x} \leq \boldsymbol{x}_k\}, \qquad \text{and} \qquad \beta_k = \min\{\beta \in \mathbb{R}_+ : \beta \boldsymbol{x} \geq \boldsymbol{x}_k\}.$$

Since $\boldsymbol{x}_k \to \boldsymbol{x}$, $\alpha_k \to 1$ and $\beta_k \to 1$. The set S_k is a comprehensive set and $\boldsymbol{x}_k \in \delta S_k$, thus $\alpha_k \le t^{\boldsymbol{x}}_{S_k} \le \beta_k$ and $t^{\boldsymbol{x}}_{S_k} \to 1$ and we have $t^{\boldsymbol{x}}_{S_k}\boldsymbol{x} - \boldsymbol{x}_k \to \mathbf{0}$.

I then consider the case where $\boldsymbol{x} \ge \mathbf{0}$ but not $\boldsymbol{x} > \mathbf{0}$. Let $\tilde{\boldsymbol{x}}_k$ be such that $\tilde{x}_{k,i} = x_{k,i}$ when $x_i > 0$ and $\tilde{x}_{k,i} = 0$ when $x_i = 0$. It is easy to see that $\tilde{\boldsymbol{x}}_k \to \boldsymbol{x}$. Again, using

$$\tilde{\alpha}_k = \max\{\alpha \in \mathbb{R}_+ : \alpha\boldsymbol{x} \le \tilde{\boldsymbol{x}}_k\} \qquad \text{and} \qquad \beta_k = \min\{\beta \in \mathbb{R}_+ : \beta\boldsymbol{x} \ge \tilde{\boldsymbol{x}}_k\},$$

we can conclude that $t^{\boldsymbol{x}}_{S_k}\boldsymbol{x} - \tilde{\boldsymbol{x}}_k \to \mathbf{0}$ and, since $\tilde{\boldsymbol{x}}_k - \boldsymbol{x}_k \to \mathbf{0}$, that $t^{\boldsymbol{x}}_{S_k}\boldsymbol{x} - \boldsymbol{x}_k \to \mathbf{0}$. □

We will prove in Proposition 1 that in $\overline{\mathbb{D}}$ axioms (2), (3), (6a) and (6b) imply axiom (6c). The proof is done by contradiction and we start by assuming the existence of a sequence of sets, $S_k \in \overline{\mathbb{D}}$ convergent to $S \in \overline{\mathbb{D}}$ whose solution $f(S_k) = \boldsymbol{s_k}$ does not converge to the solution of S. The next lemma shows that when such sequence $\{S_k\}_k$ exists we can assume in addition that $f(S) \in S_k$ for all k.

Lemma 3 *Let the solution f in $\overline{\mathbb{D}}$ satisfy axioms* (2), (3), (6a) *and* (6b). *Let $\{S_k\}_k \subseteq \overline{\mathbb{D}}$ be a convergent sequence to S and $f(S_k) \to \boldsymbol{s}'$. Then, there is a sequence $\{S'_k\}_{k\in\mathbb{N}} \subseteq \overline{\mathbb{D}}$, such that S'_k converges to S, $f(S'_k) \to \boldsymbol{s}'$ and $f(S) \in S'_k$.*

Proof The solution, according to Roth (1979b)[proposition 2][4], is weakly Pareto optimal, so $\boldsymbol{s} = f(S) \in \delta S$ and $t^{\boldsymbol{s}}_S = 1$. Since $\beta\boldsymbol{s} \in S_k$ for all $0 \le \beta \le t^{\boldsymbol{s}}_{S_k}$, we have that $\boldsymbol{s} \in \alpha S_k$ for all $\alpha \ge \frac{1}{t^{\boldsymbol{s}}_{S_k}} = \alpha_k$. Since the proportional solution is continuous, $\alpha_k \to 1$.

We now know that: 1) $\boldsymbol{s} \in \alpha_k S_k$; 2) $\alpha_k S_k \to S$, because $\alpha_k \to 1$ and $S_k \to S$; and 3) $f(\alpha_k S_k) = \alpha_k f(S_k) = \alpha_k \boldsymbol{s_k} \to \boldsymbol{s'}$. We proved the statement of the lemma with $S'_k = \alpha_k S_k$. □

Proposition 1. *If a solution in $\overline{\mathbb{D}}$ satisfies axioms* (2), (3), (6a) *and* (6b), *then it satisfies* (6c) *in $\overline{\mathbb{D}}$.*

Proof We will argue by contradiction. If f is not continuous in $\overline{\mathbb{D}}$, there exists a sequence of converging sets $S_k \to S$, with $S_k, S \in \overline{\mathbb{D}}$, such that the sequence $f(S_k) = \boldsymbol{s_k}$ does not converge to $f(S) = \boldsymbol{s}$. We initially assume that $\boldsymbol{s_k} \to \boldsymbol{s'} \ne \boldsymbol{s}$, i.e. we assume that the sequence $\boldsymbol{s_k}$ is convergent. According to Lemma 3 it can be assumed, without loss of generality, that $\boldsymbol{s} \in S_k$ for all $k \in \mathbb{N}$.

We find that there exist sets A and B such that: 1) $A, B \in \overline{\mathbb{D}}$; 2) $B \subset A$; 3) $f(A), f(B) \in \delta S$; 4) $f(A) \ne f(B)$; and 5) $A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\} = B \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\}$ for all $i \in N$. With these sets we get a contradiction: according to axiom (6b),

[4] This proposition was proved for the class $\mathbb{D}$, but the exact same proof is valid for the class $\overline{\mathbb{D}}$.

$f(A) \geq f(B)$. Since $f(A) \neq f(B)$, $f(A), f(B) \in \delta S$ and S is strictly comprehensive, there exists a vector $\boldsymbol{z} \in S$ such that $\boldsymbol{z} > f(B)$. Therefore, $f(B) \notin \delta S$, which contradicts 3).

The proof of these claims requires new concepts, which we now introduce. For any vector $\boldsymbol{a} \in \mathbb{R}^n_+$ let $\pi_M(\boldsymbol{a})$ be the projection of $\boldsymbol{a}$ over the coordinates $M \subseteq N$. The vector $\boldsymbol{y} = \pi_M(\boldsymbol{a})$ is such that $y_i = a_i$ for $i \in M$ and $y_i = 0$ for $i \in \overline{M} = N - M$. Let $\boldsymbol{e} \in \mathbb{R}^n$ be the vector with all entries equal to 1. Thus $\pi_M(\boldsymbol{e})$ is such that $\pi_M(\boldsymbol{e})_i = 1$ for $i \in M$ and $\pi_M(\boldsymbol{e})_i = 0$ for $i \in \overline{M}$.

We also need to define an order relation for each set of coordinates $M \subseteq N$. Let $\boldsymbol{x} <_M \boldsymbol{y}$ if $x_i < y_i$ for all $i \in M$. Likewise, we define $\boldsymbol{x} \leq_M \boldsymbol{y}$ and $\boldsymbol{x} =_M \boldsymbol{y}$ accordingly.

Since S is a comprehensive set and there is an $\tilde{\boldsymbol{s}} > \boldsymbol{0}$ such that $\tilde{\boldsymbol{s}} \in S$, the set $\{\rho \in \mathbb{R}_+ : \rho \boldsymbol{s_k} \in S\}$ is non-empty. $S \subseteq \mathbb{R}^n$ is also compact; thus, $t_S^{\boldsymbol{s_k}} = \max\{\rho \in \mathbb{R}_+ : \rho \boldsymbol{s_k} \in S\}$ is well defined. To simplify notation we will, from now on, use $\rho_k = t_S^{\boldsymbol{s_k}}$. Since $\boldsymbol{s} \in S_k$, Lemma 1 implies that $\boldsymbol{s_k} \notin S$. Thus, $\rho_k < 1$. The bargaining solution $\boldsymbol{s_k}$ is weakly Pareto optimal in S_k; that is, $\boldsymbol{s_k} \in \delta S_k$, and, using Lemma 2, we conclude that the limit of $\boldsymbol{s_k}$ belongs to the boundary of S, i.e., $\boldsymbol{s_k} \to \boldsymbol{s'} \in \delta S$. Taking, if necessary, subsequences from ρ_k we see that $\rho_k \boldsymbol{s_k} \in \delta S$ and that $\boldsymbol{s_k} \to \boldsymbol{s'} \in \delta S$, and since S is strictly comprehensive we conclude that $\rho_k \to 1$.

Axiom (3) implies that $f(S) = \boldsymbol{s} > \boldsymbol{0}$. Then, for any $\emptyset \neq M \subset N$, $\pi_M(\boldsymbol{s}) \leq \boldsymbol{s}$ and $\pi_M(\boldsymbol{s}) \neq \boldsymbol{s}$. Since S is strictly comprehensive, there exists a $\boldsymbol{z_M} \in S$ with $\pi_M(\boldsymbol{s}) < \boldsymbol{z_M}$. Since $\rho_k S_k \to S$ there exists, for large k, a value $\boldsymbol{w_M} \in \rho_k S_k$ close to $\boldsymbol{z_M}$ and then $\pi_M(\boldsymbol{s}) < \boldsymbol{w_M}$. The same is true for $\rho_k \boldsymbol{s_k} > \boldsymbol{0}$, that is, there exists a point $\tilde{\boldsymbol{z}}_{\boldsymbol{M}} \in S$ and $\tilde{\boldsymbol{w}}_{\boldsymbol{M}} \in \rho_k S_k$, for each $M \subset N$, such that $\rho_k \pi_M(\boldsymbol{s_k}) < \tilde{\boldsymbol{z}}_{\boldsymbol{M}}$ and $\rho_k \pi_M(\boldsymbol{s_k}) < \tilde{\boldsymbol{w}}_{\boldsymbol{M}}$. So, there is an $\epsilon > 0$, that satisfies the following conditions, for all $M \in 2^N \setminus \{\emptyset, N\}$:

$$(*) \qquad \begin{array}{ll} \pi_M(\boldsymbol{s}) + (n\epsilon)\pi_M(\boldsymbol{e}) < \boldsymbol{z_M} & \rho_k \pi_M(\boldsymbol{s_k}) + (n\epsilon)\pi_M(\boldsymbol{e}) < \tilde{\boldsymbol{z}}_{\boldsymbol{M}} \\ \pi_M(\boldsymbol{s}) + (n\epsilon)\pi_M(\boldsymbol{e}) < \boldsymbol{w_M} & \rho_k \pi_M(\boldsymbol{s_k}) + (n\epsilon)\pi_M(\boldsymbol{e}) < \tilde{\boldsymbol{w}}_{\boldsymbol{M}}. \end{array}$$

For each $M \in 2^N \setminus \{\emptyset\}$, let

$$\boldsymbol{x_M} = \pi_M(\boldsymbol{s}) + \big(m(M)\epsilon\big)\pi_M(\boldsymbol{e}) \qquad \boldsymbol{y_M} = \rho_k \pi_M(\boldsymbol{s_k}) + \big(m(M)\epsilon\big)\pi_M(\boldsymbol{e}).$$

where $m(M) = n - \#M$. Define

$$A = cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M \in 2^N \setminus \{\emptyset\}\}}\Big\} \qquad B = cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M \in 2^N \setminus \{\emptyset, N\}\}}, \rho_k \boldsymbol{s}, \rho_k \boldsymbol{s_k}\Big\}.$$

Note that the only difference between the elements that generate A and those that generate B is that $\boldsymbol{x_N} = \boldsymbol{s}$ is replaced by $\rho_k \boldsymbol{s}$ ($\boldsymbol{y_N} = \rho_k \boldsymbol{s}_k$ because $m(N) = 0$).

Claim $A, B \in \overline{\mathbb{D}}$.

To prove that A is strictly comprehensive we need to show that for any $\tilde{\boldsymbol{z}}, \boldsymbol{z} \in A$ with $\tilde{\boldsymbol{z}} \leq \boldsymbol{z}$ and $\tilde{\boldsymbol{z}} \neq \boldsymbol{z}$, there exists $\boldsymbol{z'} \in A$ such that $\tilde{\boldsymbol{z}} < \boldsymbol{z'}$. In other words, when the set $M_{\boldsymbol{z}} = \{i \in N : \tilde{z}_i < z_i\} \neq N$ we will find a vector $\boldsymbol{z'} \in A$ such that $M_{\boldsymbol{z'}} = \{i \in N : \tilde{z}_i < z'_i\} = N$.

We then start with a vector $\boldsymbol{z}$ such that $M_{\boldsymbol{z}} \neq N$, and find a new vector $\boldsymbol{z'}$ such that $M_{\boldsymbol{z'}} = N$. Let $\boldsymbol{z} = \sum_{M \in 2^N \setminus \{\emptyset\}} (\alpha_M \boldsymbol{x_M} + \beta_M \boldsymbol{y_M})$ (all parameters are non-negative). The new vector $\boldsymbol{z'}$ is similar to $\boldsymbol{z}$ but with one parameter reduced and some other with a corresponding increase. These parameters' changes will be done in such a way that the coordinates in $M_{\boldsymbol{z}}$ are still in $M_{\boldsymbol{z'}}$ and some more are added, i.e., $M_{\boldsymbol{z}} \subset M_{\boldsymbol{z'}}$.

Suppose that $\alpha_K > 0$ or $\beta_K > 0$ for some $K \subseteq M_{\boldsymbol{z}}$. We assume that $\alpha_K > 0$ (the case where $\beta_K > 0$ follows analogously). In this case we can diminish slightly α_K to α'_K in a way that the induced reduction in the coordinates in K is smaller than the initial difference between $\boldsymbol{z}$ and $\tilde{\boldsymbol{z}}$ at all those coordinates, i.e.

$$\boldsymbol{0} <_K (\alpha_K - \alpha'_K)\boldsymbol{x_K} <_K \boldsymbol{z} - \tilde{\boldsymbol{z}}. \tag{3}$$

The decrease in α_K is compensated by an equal increase in $\alpha_{\overline{K}}$,

$$\alpha'_{\overline{K}} - \alpha_{\overline{K}} = \alpha'_K - \alpha_K,$$

and all other parameters are kept equal. Let the new vector be given by $\boldsymbol{z'} = \sum_{M \in 2^N \setminus \{\emptyset\}} (\alpha'_M \boldsymbol{x_M} + \beta'_M \boldsymbol{y_M})$. We then have that

$$\begin{aligned}
\boldsymbol{z'} = & \sum_{M \in 2^N \setminus \{\emptyset\}} (\alpha_M \boldsymbol{x_M} + \beta_M \boldsymbol{y_M}) + (\alpha'_K - \alpha_K)\boldsymbol{x_K} + (\alpha'_{\overline{K}} - \alpha_{\overline{K}})\boldsymbol{x_{\overline{K}}} \\
= & \boldsymbol{z} - (\alpha_K - \alpha'_K)\boldsymbol{x_K} + (\alpha_K - \alpha'_K)\boldsymbol{x_{\overline{K}}}.
\end{aligned}$$

We claim that $M_{\boldsymbol{z'}} = N$. To prove that $K \subseteq M_{\boldsymbol{z'}}$ we use (3),

$$\begin{aligned}
\pi_K(\boldsymbol{z'}) &= \pi_K(\boldsymbol{z}) - (\alpha_K - \alpha'_K)\boldsymbol{x_K} \\
&>_K \pi_K(\boldsymbol{z}) - \pi_K\Big(\boldsymbol{z} - \tilde{\boldsymbol{z}}\Big) = \pi_k(\tilde{\boldsymbol{z}}).
\end{aligned}$$

Now we prove the inclusion $\overline{K} \subseteq M_{\boldsymbol{z'}}$. From $\tilde{\boldsymbol{z}} \leq \boldsymbol{z}$ we have that $\tilde{\boldsymbol{z}} \leq_{\overline{K}} \boldsymbol{z}$. Note that $\pi_{\overline{K}}(\boldsymbol{z'}) = \pi_{\overline{K}}(\boldsymbol{z}) + (\alpha_K - \alpha'_K)\boldsymbol{x_{\overline{K}}}$. From (3) we know that $\alpha_K - \alpha'_K > 0$. Moreover, from the definition of $\boldsymbol{x_{\overline{K}}}$ we have that $\boldsymbol{x_{\overline{K}}} >_{\overline{K}} \boldsymbol{0}$. Therefore, $\boldsymbol{z} >_{\overline{K}} \tilde{\boldsymbol{z}}$.

Suppose now that $\alpha_K = 0$ and $\beta_K = 0$ for all $K \subseteq M_{\boldsymbol{z}}$. Then there is a $K \subset N$ with $\alpha_K^2 + \beta_K^2 > 0$ such that $K \cap M_{\boldsymbol{z}} \neq \emptyset$ and $K \cap \overline{M}_{\boldsymbol{z}} \neq \emptyset$. We assume that $\alpha_K > 0$ (the case where $\beta_K > 0$ follows analogously). We proceed in two steps; first we find a vector $\boldsymbol{z'}$ such that $M_{\boldsymbol{z'}}$ is strictly bigger than $M_{\boldsymbol{z}}$; second, we find $\boldsymbol{z''}$ with $M_{\boldsymbol{z''}} = N$. To obtain $\boldsymbol{z'}$ from $\boldsymbol{z}$ we start by diminishing α_K to α'_K in a way that the induced reduction in the coordinates in $K \cap M_{\boldsymbol{z}}$ is smaller then the initial difference in those coordinates, i.e.,

$$\mathbf{0} <_K (\alpha_K - \alpha'_K)\boldsymbol{x_K} <_{K \cap M_{\boldsymbol{z}}} \boldsymbol{z} - \tilde{\boldsymbol{z}}. \tag{4}$$

The decrease in α_K is compensated by an equal increase in $\alpha_{K\cap\overline{M}_{\boldsymbol{z}}}$. That is,

$$\alpha'_{K\cap\overline{M}_{\boldsymbol{z}}} - \alpha_{K\cap\overline{M}_{\boldsymbol{z}}} = \alpha_K - \alpha'_K.$$

All other parameters are kept equal. The vector $\boldsymbol{z'}$ is given by

$$\begin{aligned}
\boldsymbol{z'} &= \sum_{M\in 2^N\setminus\{\emptyset\}} (\alpha'_M \boldsymbol{x_M} + \beta'_M \boldsymbol{y_M}) \\
&= \sum_{M\in 2^N\setminus\{\emptyset\}} (\alpha_M \boldsymbol{x_M} + \beta_M \boldsymbol{y_M}) + (\alpha'_K - \alpha_K)\boldsymbol{x_K} + (\alpha'_{K\cap\overline{M}_{\boldsymbol{z}}} - \alpha_{K\cap\overline{M}_{\boldsymbol{z}}})\boldsymbol{x_{K\cap\overline{M}_z}} \\
&= \boldsymbol{z} + (\alpha_K - \alpha'_K)(\boldsymbol{x_{K\cap\overline{M}_z}} - \boldsymbol{x_K})
\end{aligned}$$

We now prove that $M_{\boldsymbol{z'}}$ is strictly bigger than $M_{\boldsymbol{z}}$, in particular we prove that $M_{\boldsymbol{z'}} = M_{\boldsymbol{z}} \cup K$. Using (4) we conclude that

$$\begin{aligned}
\pi_{M_{\boldsymbol{z}}\cap K}(\boldsymbol{z'}) &= \pi_{M_{\boldsymbol{z}}\cap K}(\boldsymbol{z}) - (\alpha_K - \alpha'_K)\pi_{M_{\boldsymbol{z}}\cap K}(\boldsymbol{x_K}) \\
&>_{M_{\boldsymbol{z}}\cap K} \pi_{M_{\boldsymbol{z}}\cap K}(\boldsymbol{z}) - \pi_{M_{\boldsymbol{z}}\cap K}\big(\boldsymbol{z} - \tilde{\boldsymbol{z}}\big) = \pi_{M_{\boldsymbol{z}}\cap K}\big(\tilde{\boldsymbol{z}}\big).
\end{aligned}$$

Thus $K \cap M_{\boldsymbol{z}} \subseteq M_{\boldsymbol{z'}}$. It is evident that $\overline{K} \cap M_{\boldsymbol{z}} \subseteq M_{\boldsymbol{z'}}$, since $\pi_{M_{\boldsymbol{z}}\cap\overline{K}}(\boldsymbol{z'}) = \pi_{M_{\boldsymbol{z}}\cap\overline{K}}(\boldsymbol{z}) >_{M_{\boldsymbol{z}}\cap\overline{K}} \tilde{\boldsymbol{z}}$.

To prove that $K\cap\overline{M_{\boldsymbol{z}}} \subset M_{\boldsymbol{z'}}$, we start by noting that $m(K\cap\overline{M}_{\boldsymbol{z}}) \geq m(K)$, and, by definition, we have that $\boldsymbol{x_K} <_{K\cap\overline{M}_{\boldsymbol{z}}} \boldsymbol{x_{K\cap\overline{M}_z}}$. Therefore, the reduction in the coordinates of $(K\cap\overline{M}_{\boldsymbol{z}})$ created by the reduction in α_K is more than compensated by the increase in the same coordinates due to the change in $\alpha_{K\cap\overline{M}_{\boldsymbol{z}}}$. To be more precise, we have that

$$\pi_{K\cap\overline{M}_{\boldsymbol{z}}}(\boldsymbol{z'}) = \pi_{K\cap\overline{M}_{\boldsymbol{z}}}(\boldsymbol{z}) + (\alpha_K - \alpha'_K)\pi_{K\cap\overline{M}_{\boldsymbol{z}}}(\boldsymbol{x_{K\cap\overline{M}_z}} - \boldsymbol{x_K}),$$

it then follows, using $(\alpha_k - \alpha'_k)(\boldsymbol{x_{K\cap \overline{M}_z}} - \boldsymbol{x_K}) >_{K\cap \overline{M}_{\boldsymbol{z}}} \mathbf{0}$, that

$$\pi_{K\cap \overline{M}_{\boldsymbol{z}}}(\boldsymbol{z'}) >_{K\cap \overline{M}_{\boldsymbol{z}}} \pi_{K\cap \overline{M}_{\boldsymbol{z}}}(\boldsymbol{z}) \geq_{K\cap \overline{M}_{\boldsymbol{z}}} \tilde{\boldsymbol{z}}.$$

Thus, $K \cap \overline{M}_{\boldsymbol{z}} \subseteq M_{\boldsymbol{z'}}$.

We have then created a vector $\boldsymbol{z'}$ in such a way that $M_{\boldsymbol{z'}}$ is strictly bigger than $M_{\boldsymbol{z}}$, $M_{\boldsymbol{z}} \subset M_{\boldsymbol{z'}} = M_{\boldsymbol{z}} \cup (K \cap \overline{M}_{\boldsymbol{z}}) = M_{\boldsymbol{z}} \cup K$. The vector $\boldsymbol{z'}$ is such that the parameter $\alpha'_{K\cap \overline{M}_{\boldsymbol{z}}} > 0$, and we also know that $K \cap \overline{M}_{\boldsymbol{z}} \subseteq M_{\boldsymbol{z'}}$. Thus we can use the reasoning of the previous case to create a new vector $\boldsymbol{z''}$ with $M_{\boldsymbol{z''}} = N$.

Claim $B \subset A$.

The set of generators of A and B are the same, except that $\boldsymbol{s}$ is a generator of A and not of B, and $\rho_k\boldsymbol{s}$ of B and not of A. We know that $\rho_k < 1$. Hence, $\rho_k\boldsymbol{s} < \boldsymbol{s}$ and, since A is strictly comprehensive, $\rho_k\boldsymbol{s} \in A$. Therefore $B \subset A$.

Claim $f(A), f(B) \in \delta S$.

We will first prove that $A \subset S$ and $B \subseteq \rho_k S_k$ We know that $m(M) = n - \#M \leq n$, for any $M \subseteq N$. Using $(*)$, when $M \neq N$ we have that

$$\boldsymbol{x_M} = \pi_M(\boldsymbol{s}) + \big(m(M)\epsilon\big)\pi_M(\boldsymbol{e}) \leq \pi_M(\boldsymbol{s}) + n\epsilon\pi_M(\boldsymbol{e}) < \boldsymbol{z_M}.$$

S is a comprehensive set and $\boldsymbol{z_M} \in S$ thus $\boldsymbol{x_M} \in S$. An analogous argument proves that $\boldsymbol{y_M} \in S$. In the case that $M = N$, we have that $\boldsymbol{x_N} = \boldsymbol{s} \in S$. Since all the elements that generate A are in S, then $A \subset S$. Since $f(S) = \boldsymbol{s} \in A \subseteq S$ axiom $(6a)$ implies that $f(A) = \boldsymbol{s}$. The solution $f(\cdot)$ is weakly Pareto optimal (Roth (1979b)[proposition 2]), and so $\boldsymbol{s} \in \delta S$. A similar reasoning proves that $B \subseteq \rho_k S_k$. Axiom $(6a)$ implies that $f(B) = \rho_k \boldsymbol{s}_k$. From the definition of $\rho_k = \max\{\rho \in \mathbb{R}_+ : \rho\boldsymbol{s_k} \in S\}$ we conclude that $f(B) = \rho_k\boldsymbol{s}_k \in \delta S$. In claim 3 we proved that $f(A) = \boldsymbol{s}$ and that $f(B) = \rho_k\boldsymbol{s_k}$. By hypothesis $\rho_k\boldsymbol{s_k} \to \boldsymbol{s'} \neq \boldsymbol{s}$. Thus, with k sufficiently large, $f(B) = \rho_k\boldsymbol{s_k} \neq \boldsymbol{s} = f(A)$.

Claim $A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\} = cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M\in 2^N_{-i}\}}\Big\} = B \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\}$ for all $i \in N$, with $2^N_{-i} = \{M \subset N : i \notin M \text{ and } M \neq \emptyset\}$.

The inclusion $A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\} \supseteq cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M\in 2^N_{-i}\}}\Big\} = A_{-i}$ is obvious, once we know that $A \supseteq \{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M\in 2^N_{-i}\}}$ and, by definition, the vectors in $\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M\in 2^N_{-i}\}}$ have the i^{th} coordinate equal to zero, that is $\boldsymbol{x}_{\boldsymbol{M},i} = 0 = \boldsymbol{y}_{\boldsymbol{M},i}$, for any $M \in 2^N_{-i}$.

To prove the reverse inclusion, we will observe that for any $\boldsymbol{z} \in A$ there is a $\boldsymbol{z'} \in A_{-i}$ such that $\boldsymbol{z} \leq_{N-i} \boldsymbol{z'}$. Then, in particular, for any $\boldsymbol{z} \in A \cap \{\boldsymbol{x} \in$

$\mathbb{R}^n : x_i = 0\}$ there is a $\boldsymbol{z'} \in A_{-i}$ such that $\boldsymbol{z} \leq_{N-i} \boldsymbol{z'}$. The i^{th} coordinate of any vector $\boldsymbol{z} \in A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\}$ and of any $\boldsymbol{z'} \in A_{-i}$ is equal to zero. Thus $\boldsymbol{z} \leq \boldsymbol{z'}$. Since A_{-i} is comprehensive, $\boldsymbol{z} \in A_{-i}$. We will initially assume that $\boldsymbol{z} = \sum_{M \in 2^N \setminus \{\emptyset\}} (\alpha_M \boldsymbol{x_M} + \beta_M \boldsymbol{y_M})$ where $\sum \alpha_M + \beta_M = 1$ and $\alpha_M, \beta_M \geq 0$. Let $\boldsymbol{z'} = \sum_{M \in 2^N \setminus \{\emptyset\}} (\alpha'_M \boldsymbol{x_M} + \beta'_M \boldsymbol{y_M})$, with

$$\begin{aligned} \alpha'_M &= \alpha_M + \alpha_{M \cup i}, & \beta'_M &= \beta_M + \beta_{M \cup i} & &\text{if } M \in 2^N_{-i}, \\ \alpha'_M &= 0, & \beta'_M &= 0 & &\text{if } M \notin 2^N_{-i}. \end{aligned}$$

For any $M \in 2^N_{-i}$, we have that $\boldsymbol{x_M} >_{N-i} \boldsymbol{x_{M \cup i}}$. Then

$$\begin{aligned} (\alpha_M + \alpha_{M \cup i}) \boldsymbol{x_M} &\geq_{N-i} \alpha_M \boldsymbol{x_M} + \alpha_{M \cup i} \boldsymbol{x_{M \cup i}}, \\ (\beta_M + \beta_{M \cup i}) \boldsymbol{x_M} &\geq_{N-i} \beta_M \boldsymbol{x_M} + \beta_{M \cup i} \boldsymbol{x_{M \cup i}}. \end{aligned}$$

From these inequalities we conclude that $\boldsymbol{z'} = \pi_{N-i}(\boldsymbol{z'}) \geq_{N-i} \pi_{N-i}(\boldsymbol{z})$. And we found a vector $\boldsymbol{z'} \in A_{-i}$ such that $\boldsymbol{z'} \leq_{N-i} \boldsymbol{z}$.

When $\boldsymbol{z}$ does not belong to the boundary of A, there is a vector $\tilde{\boldsymbol{z}}$ in the boundary such that $\boldsymbol{z} \leq \tilde{\boldsymbol{z}}$. To find a vector in the referred conditions apply the previous reasoning to the vector $\tilde{\boldsymbol{z}}$. We then have $\boldsymbol{z} \leq \tilde{\boldsymbol{z}} \leq_{N-i} \boldsymbol{z'}$. To prove that $cch\Big\{\{\boldsymbol{x_M}, \boldsymbol{y_M}\}_{\{M \in 2^N_{-i}\}}\Big\} = B \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\}$ we follow an analogous argument, noting that $\rho_k \boldsymbol{s} < \boldsymbol{s} <_{N-i} \boldsymbol{y_{N-i}}$ and that $\rho_k \boldsymbol{s_k} <_{N-i} \boldsymbol{y_{N-i}}$, for any $i \in N$. We have hence proved that there exist sets A and B satisfying conditions 1)-5). Therefore, it can not exist a convergent sequence $f(S_k)$ which does not converge to $f(S)$, when $S_k \to S$. However, it might happen that $f(S_k)$ is not convergent. In this case, since $S_k \to S$, for large K there is a compact set $\overline{S} \subseteq \mathbb{R}^n$ such that $S_k \subset \overline{S}$, for all $k > K$. A sequence in a compact set is not convergent if there are (at least) two subsequences converging to different values. But, as we saw, any converging subsequence $\boldsymbol{s_{k_i}}$ must converge to $\boldsymbol{s}$. Since it is impossible to have two subsequences converging to a different value, the sequence $f(S_k)$ must be convergent and f continuous. □

Corollary 1. *A solution in* $\mathbb{D}$ *satisfies* (2)*,* (3)*,* (6a)*,* (6b) *and* (6c') *if and only if it is proportional. A solution in* $\mathbb{B}$ *satisfies* (2)*,* (3)*,* (6a)*,* (6b) *and* (6c') *if and only if it is proportional.*

Proof If f is continuous in $\mathbb{D}$ both results follow from Theorems **K** and **R**. We then prove that for any $S \in \mathbb{D}$ and any sequence $S_k \in \mathbb{D}$, where $k \in \mathbb{N}$, convergent to S, we have that $f(S_k) \to f(S)$.

Assume that $S \in \mathbb{D} \setminus \overline{\mathbb{D}}$. If the sequence $S_k \in \mathbb{D}$, $k \in \mathbb{N}$, is such that $S_k \to S$, axiom (6c') implies that $f(S_k) \to f(S)$.

Assume now that $S \in \overline{\mathbb{D}}$. We need to analyze two (non-mutually exclusive) cases: when there are infinite $S_k \in \overline{\mathbb{D}}$ and when there are infinite $S_k \in \mathbb{D} \setminus \overline{\mathbb{D}}$. In the first case, let S_{k_p} be the subsequence formed by the elements of $\{S_k\}$ that are in $\overline{\mathbb{D}}$. Clearly such subsequence is convergent, $S_{k_p} \to S$, and, by Proposition 1, we have that $f(S_{k_p}) \to f(S)$.

It remains to be proved that $f(S_{k_p}) \to f(S)$, when $S \in \overline{\mathbb{D}}$ and S_{k_p} is a subsequence formed by the elements of $\{S_k\}$ that are in $\mathbb{D} \setminus \overline{\mathbb{D}}$. When the limit exists, let $\boldsymbol{s'} = \lim f(S_k) = \boldsymbol{s_k}$. We conclude that $\boldsymbol{s'} > \mathbf{0}$. An argument similar to that used in Lemma (3) is valid, and we can assume, without loss of generality that $\boldsymbol{s} \in S_k$, for any $k \in \mathbb{N}$. Let $A_k = \operatorname{cch}\{\boldsymbol{s_k}, \boldsymbol{s}\}$. Axiom ($6a$) implies that $f(A_k) = \boldsymbol{s_k}$, since $\boldsymbol{s_k}, \boldsymbol{s} \in S_k$. The sequence $\{A_k\}_k$ is convergent, $A_k \to A = \operatorname{cch}\{\boldsymbol{s'}, \boldsymbol{s}\}$. Clearly $A \in \mathbb{D} \setminus \overline{\mathbb{D}}$, and by axiom ($6c'$), $f(A) = \lim f(A_k) = \boldsymbol{s}'$. On the other hand, axiom (2) implies that $f(A) > \mathbf{0}$. Hence, $\boldsymbol{s'} > \mathbf{0}$.

We now prove that there is a large $K \in \mathbb{N}$ such that

$$\pi_{N\setminus i}(\boldsymbol{s_k}) = \big(s_{k,1}, \dots, s_{k,i-1}, 0, s_{k,i-1}, \dots, s_{k,n}\big) \in S, \text{ for any } i \in N \text{ and } k \geq K. \tag{5}$$

We know, by Lemma 2, that $\boldsymbol{s'} \in \delta S$. Since S is (strictly) comprehensive, $\pi_{N\setminus i}(\boldsymbol{s'}) \in S$. The vector $\boldsymbol{s'} > \mathbf{0}$, thus $\pi_{N\setminus i}(\boldsymbol{s'}) \neq \boldsymbol{s'}$. The set S is strictly comprehensive, thus there exists $\boldsymbol{\gamma} \in S$ such that $\gamma > \pi_{N\setminus i}(\boldsymbol{s'})$. Continuity of the projection $\pi_{N\setminus i}(\cdot)$ implies that there is a $K_i \in \mathbb{N}$ such that for $k \geq K_i$, $\gamma > \pi_{N\setminus i}(\boldsymbol{s_k})$. Therefore, $\pi_{N\setminus i}(\boldsymbol{s_k}) \in S$. This argument holds for any $i \in N$; hence, we can find $K \in \mathbb{N}$ such that (5) holds.

For $k \geq K$, let

$$A = \operatorname{cch}\{\boldsymbol{s_k}, \boldsymbol{s}\} \text{ and } B = \operatorname{cch}\{\pi_{N\setminus 1}(\boldsymbol{s_k}), \dots, \pi_{N\setminus n}(\boldsymbol{s_k}), \boldsymbol{s}\}.$$

Since $\boldsymbol{s_k} \geq \pi_{N\setminus i}(\boldsymbol{s_k})$, for any $i \in N$, we have $B \subset A$. We now prove, using axiom ($6b$), that $f(B) \leq f(A)$. For that we need to verify that for any $i \in N$,

$$A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\} = B \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\}.$$

We prove this equality by proving the double inclusion. One inclusion is obvious from $B \subset A$. For the other inclusion, notice that

$$A \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\} = \big\{\boldsymbol{z} \in \mathbb{R}^n_+ : \boldsymbol{z} \leq \pi_{N\setminus i}(\boldsymbol{y}) \text{ for some } \boldsymbol{y} \in \operatorname{ch}\{\boldsymbol{s_k}, \boldsymbol{s}\}\big\},$$

since A is comprehensive. Let $\boldsymbol{y} = \alpha \boldsymbol{s} + (1-\alpha)\boldsymbol{s_k}$, with $\alpha \in [0,1]$, then

$$\pi_{N\setminus i}(\boldsymbol{y}) = \alpha \pi_{N\setminus i}(\boldsymbol{s}) + (1-\alpha)\pi_{N\setminus i}(\boldsymbol{s_k}) \leq \alpha \boldsymbol{s} + (1-\alpha)\pi_{N\setminus i}(\boldsymbol{s_k}) \in B.$$

Since B is comprehensive, for any $\boldsymbol{z} \leq \pi_{N\setminus i}(\boldsymbol{y})$ we have that $\boldsymbol{z} \in B$. Thus. $A \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\} \subseteq B \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0\}$. We can apply axiom (6$b$) for any $i \in N$, therefore $f(B) \leq f(A)$. Since $f(A) = \boldsymbol{s_k}$ and $f(B) = \boldsymbol{s}$, by taking limits we get that $\boldsymbol{s'} \leq \boldsymbol{s}$. Since S is strictly comprehensive, $\boldsymbol{s'}, \boldsymbol{s} \in \delta S$ and $\boldsymbol{s'} \leq \boldsymbol{s}$, then $\boldsymbol{s'} = \boldsymbol{s}$. With an argument identical to the one at the end of Proposition 1 we conclude that the sequence $\boldsymbol{s_k}$ must be convergent. Then, we have $f(S_k) \to f(S)$ whenever $S_k, S \in \mathbb{D}$ and $S_k \to S$. That is, f is continuous in $\mathbb{D}$. □

Theorem 1. *The bargaining solution in* $\overline{\mathbb{D}}$ *that satisfies* (2), (3), (6a) *and* (6b) *is proportional.*

Proof We follow closely theorem *12* in Roth (1979a), the only adjustment being the set X used by Roth (1979a) once it does not belong to the bargaining class $\overline{\mathbb{D}}$ that we are now working with[5]. Proposition 1 implies that f is continuous in $\overline{\mathbb{D}}$; that is, f satisfies axiom (6c) in $\overline{\mathbb{D}}$. Define the sets $\varDelta_1 = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \sum_j x_j \leq n\}$ and for positive and close to zero α, $\varDelta_\alpha = \bigcap_{i=1}^n \{\boldsymbol{x} \in \mathbb{R}^n_+ : \alpha x_i + \sum_{j\neq i} x_j \leq n\}$. Set $\boldsymbol{p} = f(\varDelta_1)$. Axiom (3) implies that $\boldsymbol{p} > \boldsymbol{0}$, thus $\boldsymbol{p} \in \text{int}(\varDelta_\alpha)$. Clearly if $A, B \in \overline{\mathbb{D}}$, then $A \cap B \in \overline{\mathbb{D}}$; thus $S_1 = S \cap t_S^{\boldsymbol{p}} \varDelta_1$ and $S_\alpha = S \cap t_S^{\boldsymbol{p}} \varDelta_\alpha$ both belong to $\overline{\mathbb{D}}$. We claim that $f(S_1) = f(S_\alpha)$. For this result we need to use axiom (6b), so first we observe that $S_\alpha \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\} = S_1 \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\}$. That the first set contains the second is an obvious consequence of $\alpha < 1$ and $S_\alpha \supseteq S_1$. If $\boldsymbol{y} \in S_\alpha \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\}$, then $y_i = 0$ and $\boldsymbol{y} \in S_\alpha = S \cap t_S^{\boldsymbol{p}} \varDelta_\alpha$; thus $\tilde{\boldsymbol{y}} = \boldsymbol{y}/t_S^{\boldsymbol{p}} \in \varDelta_\alpha$. Which implies that $\alpha \tilde{y}_k + \sum_{j\neq k} \tilde{y}_j \leq n$ for any $k \in N$. Then, using that $\tilde{y}_i = 0$ and taking $k = i$ in the previous inequality, we obtain $\alpha \tilde{y}_i + \sum_{j\neq i} \tilde{y}_j = \sum_j \tilde{y}_j \leq n$. Hence, $\tilde{\boldsymbol{y}} = \boldsymbol{y}/t_S^{\boldsymbol{p}} \in \varDelta_1$, and we may conclude that $\boldsymbol{y} \in S_1 \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\}$. So $S_\alpha \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\} \subseteq S_1 \cap \{\boldsymbol{x} \in \mathbb{R}^n_+ : x_i = 0\}$. Axiom (2) implies that $f(t_S^{\boldsymbol{p}} \varDelta_1) = t_S^{\boldsymbol{p}} \boldsymbol{p}$, axiom (6$a$) that $f(S_1) = t_S^{\boldsymbol{p}} \boldsymbol{p}$ and axiom (6b) that $f(S_\alpha) \geq f(S_1)$. But $S_\alpha \subseteq S$ and $f(S_1)$ is strictly Pareto optimal in S, thus $f(S_\alpha) = f(S_1) = t_S^{\boldsymbol{p}} \boldsymbol{p}$. Define the sets $T_a \in \overline{\mathbb{D}}$ for $a \in [0, 1]$ satisfying:

(1) $T_\beta \subseteq T_\gamma$, if $\beta \leq \gamma$ (2) $T_{\beta_k} \to T_\beta$ if $\beta_k \to \beta$ (3) $T_0 = S_\alpha$ and $T_1 = S$

We claim that

$$B = \{b \in [0,1] : f(T_b) \in t_S^{\boldsymbol{p}} \varDelta_\alpha\} = \{b \in [0,1] : f(T_b) = f(S_\alpha)\}.$$

If b is such that $f(T_b) \in t_S^{\boldsymbol{p}} \varDelta_\alpha$, since $T_b \subseteq S$, then $f(T_b) \in S \cap t_S^{\boldsymbol{p}} \varDelta_\alpha = S_\alpha$. In this case, axiom (6$a$) implies that $f(T_b) = f(S_\alpha)$, thus $\{b \in [0,1] : f(T_b) \in t_S^{\boldsymbol{p}} \varDelta_\alpha\} \subseteq \{b \in$

[5] Apart from this minor change all else is kept equal, all partial results are still true, the intermediate arguments keep their validity and we only provide this proof for the sake of completeness.

$[0,1] : f(T_b) = f(S_\alpha)\}$. On the other hand if $f(T_b) = f(S_\alpha) = t_S^{\boldsymbol{p}}\boldsymbol{p}$, since $\boldsymbol{p} \in \Delta_\alpha$, then $t_S^{\boldsymbol{p}}\boldsymbol{p} \in t_S^{\boldsymbol{p}}\Delta_\alpha$. The equality is proved.

We now prove that $B = [0,1]$. Observe that when $\beta \in B$ and $0 \le \beta' \le \beta$, then $\beta' \in B$. Since $S_\alpha \subseteq T_{\beta'} \subseteq T_\beta$ and $f(T_\beta) = f(S_\alpha) \in T_{\beta'}$, axiom (6$a$) implies that $f(T_{\beta'}) = f(T_\beta) = f(S_\alpha)$; thus, $\beta' \in B$. Then the set B is either $B = \{0\}$ or an interval. Since B is non-empty, $\bar{b} = \sup\{b : b \in B\}$ is well defined. We claim that $\bar{b} = 1$.

Assume that $\bar{b} < 1$. In this case $\bar{b} \in B$, otherwise there exists a sequence of elements $b_k \in B = [0,\bar{b})$, such that $b_k \to \bar{b}$ and $f(T_{b_k}) \in t_S^{\boldsymbol{p}}\Delta_\alpha$ but $f(T_{\bar{b}}) \notin t_S^{\boldsymbol{p}}\Delta_\alpha$. Since $t_S^{\boldsymbol{p}}\Delta_\alpha$ is compact, when it exists, $\lim f(T_{b_k}) \in t_S^{\boldsymbol{p}}\Delta_\alpha$; thus $f(T_{b_k}) \not\to f(T_{\bar{b}})$ and the continuity of f is contradicted. However, $\bar{b} < 1$ and $\bar{b} \in B$ is also impossible. Since, if possible, there exists a sequence of elements $b_k \notin B$, $b_k \in (\bar{b},1]$, such that $b_k \to \bar{b}$. Moreover $f(T_{\bar{b}}) = f(S_\alpha) = t_S^{\boldsymbol{p}}\boldsymbol{p} \in \operatorname{int}(t_S^{\boldsymbol{p}}\Delta_\alpha)$, but $f(T_{b_k}) \notin t_S^{\boldsymbol{p}}\Delta_\alpha$, which again contradicts the continuity of f. In conclusion, $\bar{b} = 1$, and $f(T_b) = f(S_\alpha)$ for any $b \in [0,1)$, continuity of f in $\overline{\mathbb{D}}$ implies that $f(S) = \lim_{b\to 1} f(T_b) = t_S^{\boldsymbol{p}}\boldsymbol{p}$, so $B = [0,1]$. The solution f is proportional. □

Example 1 (a non-continuous solution for $\mathbb{D}$)**.**

$$g(S) = \begin{cases} \boldsymbol{p}S^1 & \textit{if } \boldsymbol{p}S^1 \in S \\ \boldsymbol{q}t_S^{\boldsymbol{q}} & \textit{if } \boldsymbol{p}S^1 \notin S \end{cases}$$

With $\boldsymbol{p} = (1,p_2), \boldsymbol{q} = (1,q_2) \in \mathbb{R}^2$, $0 < q_2 < p_2 < 2$ *and* $S^1 = max\{x : (x,y) \in S\}$.

Proof For $\boldsymbol{p}$ and $\boldsymbol{q}$ with $0 < q_2 < p_2 < 2$ the solution is clearly not continuous. Let $\Delta_\alpha = \bigcap_{i=1}^2 \{\boldsymbol{x} \in \mathbb{R}_+^2 : \alpha x_i + x_{-i} \le 2\}$. As $\Delta_0^1 = 2$ and $2\boldsymbol{p} \in \Delta_0$, then $g(\Delta_0) = 2\boldsymbol{p}$. For any $0 < \alpha < 1$, $\Delta_\alpha^1 = 2$ and clearly $\Delta_\alpha^1\boldsymbol{p} \notin \Delta_\alpha$. Thus, the solution is $g(\Delta_\alpha) = \boldsymbol{q}t_{\Delta_\alpha}^{\boldsymbol{q}}$. Therefore, when $\alpha \to 0$, $g(\Delta_\alpha) \to \boldsymbol{q}t_{\Delta_0}^{\boldsymbol{q}} = 2\boldsymbol{q}$, different from $2\boldsymbol{p} = g(\Delta_0)$. Since $\Delta_\alpha \to \Delta_0$, when $\alpha \to 0$, g is not continuous in $\mathbb{D}$.

We show that the axioms (2), (3), (6a) and (6b) are fulfilled by the solution g. The proof for axiom (2) is straightforward. If $S^1\boldsymbol{p} \in S$, then $cS^1\boldsymbol{p} \in cS$ and $g(cS) = cg(S)$. The same is true when $S^1\boldsymbol{p} \notin S$, $g(cS) = c\boldsymbol{q}t_S^{\boldsymbol{q}} = cg(S)$ and we proved that g satisfies axiom (2). Now we prove that g satisfies axiom (3). By hypothesis for any $S \in \mathbb{D}$ there exists an $\boldsymbol{s} \in S$ with $\boldsymbol{s} > \boldsymbol{0}$. Then $S^1 > 0$ and $t_S^{\boldsymbol{q}} > 0$, and we have that $g(S) > 0$. Axiom (3) is satisfied.

To prove g observes (6a), we assume that $S \subseteq T$ and that $g(T) \in S$. We need to consider two cases. The first is when $g(T) = \boldsymbol{p}T^1$. In this case, since $g(T) = (T^1, p_2T^1) \in S$ we have that $S^1 \ge T^1$. From $S \subseteq T$ we have that $T^1 \ge S^1$. In conclusion, $\boldsymbol{p}S^1 = \boldsymbol{p}T^1 = g(T) \in S$ and the solutions are equal in both sets

$g(S) = g(T)$. The second case is when $g(T) = \boldsymbol{q}t_T^{\boldsymbol{q}}$. If $S^1\boldsymbol{p} \notin S$ we have that $g(S) = t_S^{\boldsymbol{q}}\boldsymbol{q}$. And clearly $g(S)$ must be equal to $g(T) = t_T^{\boldsymbol{q}}\boldsymbol{q}$. In the case where $S^1\boldsymbol{p} \in S$, we find a contradiction, and that means that $S^1\boldsymbol{p} \notin S$. The solution of T is $g(T) = t_T^{\boldsymbol{q}}\boldsymbol{q}$, when $T^1\boldsymbol{p} \notin T$. We are assuming that $S^1\boldsymbol{p} \in S \subseteq T$; then $t_S^{\boldsymbol{p}} \geq S^1$ $T^1 > S^1$. Since $t_T^{\boldsymbol{q}}\boldsymbol{q} = (t_T^{\boldsymbol{q}}, q_2 t_T^{\boldsymbol{q}}) \in S$, it holds that

$$S^1 \geq t_T^{\boldsymbol{q}} \geq t_S^{\boldsymbol{q}} \geq t_S^{\boldsymbol{p}} \geq S^1,$$

(the third inequality results from $\boldsymbol{p} \geq \boldsymbol{q}$ and S being a comprehensive set). We conclude that $t_T^{\boldsymbol{q}} = S^1$. The vectors $S^1\boldsymbol{p} = (S^1, S^1 p_2) \in S$ and $(T^1, 0)$ belong to the convex set T. Then the convex combination $\boldsymbol{a}_\alpha = \alpha S^1\boldsymbol{p} + (1-\alpha)(T^1, 0)$ also belongs to T, for any $0 \leq \alpha \leq 1$. We know that $T^1 > S^1$ and $S^1 p_2 > S^1 q_2$, then, for values of α close to 1, $\boldsymbol{a}_\alpha > (S^1, q_2 S^1) = S^1\boldsymbol{q}$. A contradiction with $t_T^{\boldsymbol{q}} = S^1$ is found.

The proof of Individual Monotonicity ($6b$) is divided in 4 cases, one for each player and for each condition of the solution g. Let $S, T \in \mathbb{D}$ be such that $S \subseteq T$:

- Let $S \cap \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\} = T \cap \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$. For the case where $T^1\boldsymbol{p} \in T$. Since $g(T) = T^1\boldsymbol{p}$, we have that

  $$g_2(T) = p_2 T^1 > q_2 T^1 \geq q_2 t_T^{\boldsymbol{q}} \geq q_2 t_S^{\boldsymbol{q}}.$$

  Then g satisfies the axiom, since $g_2(T) \geq \max\{p_2 S^1, q_2 t_S^{\boldsymbol{q}}\} \geq g_2(S)$. For the case where $T^1\boldsymbol{p} \notin T$. The set S is comprehensive and $(S^1, 0) \in S$, then $S \cap \{x \in \mathbb{R}^2 : x_2 = 0\} = [0, S^1] \times \{0\}$. Likewise, for the set T, $T \cap \{x \in \mathbb{R}^2 : x_2 = 0\} = [0, T^1] \times \{0\}$. Both these sets are equal, therefore, $S^1 = T^1$. Since $T^1\boldsymbol{p} \notin T$, we have that $T^1\boldsymbol{p} = S^1\boldsymbol{p} \notin S$ and the solution at the set S is given by $g(S) = t_S^{\boldsymbol{q}}\boldsymbol{q}$. The second player is not worst off at T than at S; that is, $g_2(S) = q_2 t_S^{\boldsymbol{q}} \leq q_2 t_T^{\boldsymbol{q}} = g_2(T)$.
- Let $S \cap \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\} = T \cap \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$. For the case where $T^1\boldsymbol{p} \in T$. The solution of the first player satisfies $g_1(T) = T^1 \geq S^1 \geq t_S^{\boldsymbol{q}}$. Thus g respects axiom $6b$, since $g_1(T) \geq \max\{S^1, t_S^{\boldsymbol{q}}\} \geq g_1(S)$. For the case where $T^1\boldsymbol{p} \notin T$. Clearly $t_T^{\boldsymbol{q}} \geq t_S^{\boldsymbol{q}}$ so we only need to check the case where $g(S) = S^1\boldsymbol{p}$. In this case $S^1\boldsymbol{p} \in S \subseteq T$, T is comprehensive and $\boldsymbol{q} \leq \boldsymbol{p}$ so $S^1\boldsymbol{q} \in T$, and $g_1(T) = t_T^{\boldsymbol{q}} \geq S^1 = g_1(S)$.

□