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Livschitz Theorem in Suspension Flows and Markov Systems: Approach in Cohomology of Systems

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Abstract: It is presented and proved a version of Livschitz Theorem for hyperbolic flows pragmatically oriented to the cohomological context. Previously, it is introduced the concept of cocycle and a natural notion of symmetry for cocycles. It is discussed the fundamental relationship between the existence of solutions of cohomological equations and the behavior of the cocycles along periodic orbits. The generalization of this theorem to a class of suspension flows is also discussed and proved. This generalization allows giving a different proof of the Livschitz Theorem for flows based on the construction of Markov systems for hyperbolic flows.

Keywords: cocycles; cohomological equations; Anosov Closing Lemma; hyperbolic flows; Livschitz Theorem; Markov systems; suspension flows

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1. Introduction

This paper presents a continuous time approach to Livschitz Theorem oriented to the study of cohomology in dynamical systems. From what is known, it no reference to this theorem pragmatically oriented to the cohomological context exists in the literature, and the only published proof of Livschitz Theorem for flows is thanks to Livschitz himself in References [1,2].

We begin by introducing fundamental notions for the study of cohomology in dynamical systems (Section 2.1). In particular, we introduce the concepts of cocycle, coboundary and cohomology between cocycles. We present cohomological equations in the case of continuous time and discuss the fundamental relationship between the existence of solutions of these equations and the behavior of the cocycles along periodic orbits (Section 2.2). We will go on by presenting a detailed proof of the Livschitz Theorem in a version for hyperbolic flows (Section 3), and then discuss the generalization of this Theorem to suspension flows (Section 4). This generalization allows an alternative proof of the Livschitz Theorem for hyperbolic flows based on Bowen and Ratner's construction of Markov systems for (hyperbolic) flows [3,4]. As far as is known, these last two approaches are new (Section 5).

In the dynamical systems theory several problems of considerable importance can be reduced to solving an equation of the form

$$\varphi = \Phi \circ f - \Phi, \tag{1}$$

where $f : X \to X$ is a dynamical system and $\varphi : X \to \mathbb{R}$ is a function, both known, and $\Phi : X \to \mathbb{R}$ is unknown. The Equation (1) is called a cohomological equation. The study of cohomological equations is related in particular to the study of conjugations to an irrational rotation of circle, the existence of absolutely continuous measures for expanding transformations of circle and the topological stability of hyperbolic automorphisms of torus. Such equations also arise naturally in celestial mechanics and statistical mechanics. Some results established by Livschitz in the 1970s ([1,2]) address precisely the possibility of obtaining solutions of cohomological equations in the context of hyperbolic dynamics.

Given a hyperbolic dynamical system, the Livschitz Theorem provides a necessary and sufficient condition, based only on the information given by periodic orbits, for the existence of Hölder solutions. It is one of the main tools for obtaining global cohomological information from periodic information.

2. From Cohomological to Periodic Information

2.1. Cocycles and Cohomology Defined on a General Group

Let G be a group with identity e. Let $T : G \times X \to X$ be a dynamical system with phase space X and time in G. Given $g \in G$ we define the transformation $T(g) : X \to X$ by $T(g)x = T(g, x)$. We designate by cocycle over T each function $\alpha : G \times X \to \mathbb{R}$ such that

$$\alpha(g_2 g_1, x) = \alpha(g_2, T(g_1)x) + \alpha(g_1, x), \tag{2}$$

whenever $x \in X$ and $g_1, g_2 \in G$. The cocycles over T constitute a linear space. Defining for each $g \in G$ the transformation $\widetilde{T}(g) : X \times \mathbb{R} \to X \times \mathbb{R}$ by

$$\widetilde{T}(g)(x, y) = (T(g)x, y + \alpha(g, x)),$$

the property (2) is equivalent to $\widetilde{T}(g_2 g_1) = \widetilde{T}(g_2) \circ \widetilde{T}(g_1)$.

Each function $\Phi : X \to \mathbb{R}$ induces a cocycle through the expression

$$\alpha(g, x) = \Phi(T(g)x) - \Phi(x). \tag{3}$$

In fact, the function α defined this way satisfies (2) since

$$\begin{aligned}\Phi(T(g_2 g_1)x) - \Phi(x) = \Phi(T(g_2)(T(g_1)x)) - \Phi(x) = [&\Phi(T(g_2)T(g_1)x) - \Phi(T(g_1)x)] \\ &+[\Phi(T(g_1)x) - \Phi(x)],\end{aligned}$$

The cocycles defined by (3) are designated by coboundary.

A natural equivalence relationship between cocycles is the cohomology. Two cocycles α and β over T are cohomologous if they differ by a coboundary, that is, if there is a function $\Phi : X \to \mathbb{R}$ such that

$$\alpha(g, x) - \beta(g, x) = \Phi(T(g)x) - \Phi(x).$$

We note that a cocycle α is a coboundary if and only if α is cohomologous to the trivial cocycle $\beta(g, x) = 0$; in this case it is said that α is cohomologically trivial and that any function Φ satisfying (3) is a trivialization of α. Also, for a cocycle α to be a coboundary it is necessary that $\alpha(g, x) = 0$ for all $g \in G$ e $x \in X$ such that $T(g)x = x$. Equation (3) is said to be a cohomological equation.

2.2. Cocycles in Continuous Time and Relation to Periodic Orbits

Suppose now that $G = \mathbb{R}$. Let us see that in this case the cocycles arise naturally from temporal changes of flows.

Let $\Phi = \{\varphi^t\}_{t \in \mathbb{R}}$ be a flow in X. It is possible to obtain new flows $\Psi = \{\psi^t\}_{t \in \mathbb{R}}$ keeping the orbits of Φ and its orientations but crossing them with a different 'speed'. In a strict way, we say that a flow Ψ is a time change from the flow Φ if

$$\psi^t x = \varphi^{\alpha(t,x)} x \tag{4}$$

for each $t \in \mathbb{R}$ and $x \in X$, for some function $\alpha : \mathbb{R} \times X \to \mathbb{R}$ with $\alpha\,(0, x) = 0$ and $\alpha\,(t, x) \geq 0$ when $t \geq 0$. The group structure to the flow Ψ shows that

$$\psi^{s+t} x = (\psi^s \circ \psi^t)x = \varphi^{\alpha(s,\psi^t x)}(\varphi^{\alpha(t,x)} x) = \varphi^{\alpha(s,\psi^t x) + \alpha(t,x)} x.$$

We then conclude that

$$\alpha(s+t,x) = \alpha(s,\psi^t x) + \alpha(t,x)$$

meaning α is a cocycle over the flow Ψ. So, any time change Ψ of a Φ flow Φ is the same way as in (4) where α is a cocycle over flow Ψ.

We see that if Φ and Ψ are C^r flows for some $r \geq 1$ (in variables t and x) and x is not a fixed point, then the Implicit Function Theorem [5] assures that α is a C^r function in both variables in a neighborhood of $(0,x)$.

Considering now Φ and Ψ flows of class C^r for some $r \geq 1$. In this case, an alternative description of time change can be done through vector fields associated with the flows, defined by

$$G(x) = \frac{d(\varphi^t x)}{dt}|_{t=0} \quad \text{e} \quad H(x) = \frac{d(\psi^t x)}{dt}|_{t=0}.$$

The flows Φ and Ψ result, respectively, from the solution of differential equations $x' = G(x)$ and $x' = H(x)$. Consider now Ψ as a time change from Φ. Given that the orbits $\{\varphi^t x\}_{t\in\mathbb{R}}$ e $\{\psi^t x\}_{t\in\mathbb{R}}$ are coincident for each fixed $x \in X$, we have $G(x) = 0$ if and only if $H(x) = 0$ (recalling that the zeros from a vector field are the fixed points from the corresponding flow). We remark that in general $G(x)$ and $H(x)$ can be distinct. However if x is not a fixed point, non-null vectors tangent to curves $\{\varphi^t x\}_{t\in\mathbb{R}}$ e $\{\psi^t x\}_{t\in\mathbb{R}}$ in x are collinear and have the same sense, that is,

$$H(x) = a(x)G(x)$$

for some constant $a(x) > 0$. Following (4) we see that the cocycle α determines univoquely the function $a : X \to \mathbb{R}_0^+$ defined by

$$a(x) = \frac{\partial\alpha}{\partial t}(t,x)\mid_{t=0}. \tag{5}$$

We realize that $a(x) = 0$ whenever x is a fixed point. If Φ does not have fixed points then a is a C^{r-1} function.

Let's consider then a continuous non-negative function $a : X \to \mathbb{R}_0^+$. The function a defines a cocycle (over Ψ) with values in $\mathbb{R}$ by

$$\alpha(t,x) = \int_0^t a(\psi^u x)\, du. \tag{6}$$

In fact, given that

$$\int_0^s a(\psi^u(\psi^t x))\,du + \int_0^t a(\psi^u x)\,du = \int_t^{s+t} a(\psi^u x)\,du + \int_0^t a(\psi^u x)\,du = \int_0^{s+t} a(\psi^u x)\,du.$$

we have $\alpha(s,\psi^t x) + \alpha(t,x) = \alpha(s+t,x)$. Hence we identify a biunivocal correspondence between continuous functions $a : X \to \mathbb{R}_0^+$ and time changes Ψ from Φ defined by a function α as in (6).

We have a particularly simple case of time change when it produces a flow differentially equivalent to the original flow through an equivalency that preserves orbits. We say that two flows $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ and $\Psi = \{\psi^t\}_{t\in\mathbb{R}}$ from class C^r in X are C^m-differentially equivalents if there is a diffeomorphism $h : X \to X$ of class C^m such that

$$\psi^t = h \circ \varphi^t \circ h^{-1} \tag{7}$$

for all $t \in \mathbb{R}$. The orbit preservation requires that

$$hx = \varphi^{\beta(x)} x$$

for all $x \in X$, where $\beta : X \to \mathbb{R}$ is a differentiable function with derivative directed to the flow Φ,

$$\frac{d\beta\left(\varphi^t x\right)}{dt}\,|_{t=0} = \lim_{t\to 0}\frac{\beta(\varphi^t x) - \beta\left(x\right)}{t}.$$

This derivative is positive when x is not a fixed point of the flow Φ, that is $G(x) \neq 0$. Using (7) we obtain

$$(\psi^t \circ h)(x) = (h \circ \varphi^t)(x) = \varphi^{\beta(\varphi^t x)+t} x. \tag{8}$$

On the other hand, following (4), we have

$$(\psi^t \circ h)(x) = \varphi^{\alpha(t,hx)}(hx) = \varphi^{\alpha(t,hx)+\beta(x)} x. \tag{9}$$

From (8) and (9) we can conclude that

$$\alpha\left(t, hx\right) = \beta(\varphi^t x) + t - \beta\left(x\right). \tag{10}$$

In particular when the orbit of x is periodic with period ρ from (8) and (9) it follows that

$$\alpha\left(t, hx\right) = \beta(\varphi^t x) + t - \beta\left(x\right) + k_x \rho$$

for some $k_x \in \mathbb{Z}$, but taking $t = 0$ we obtain $k_x = 0$. We can then state that if a time change arises from a differential equivalency that preserves orbits, in which case it is referred as a trivial time change, there is a differentiable function $\beta : X \to \mathbb{R}$ that induces a cocycle α through identity (10).

From (10) it follows that

$$\lim_{t\to 0}\frac{\alpha\left(t, hx\right) - t}{t} = \lim_{t\to 0}\frac{\beta\left(\varphi^t x\right) - \beta\left(x\right)}{t}.$$

So

$$\lim_{t\to 0}\frac{\alpha\left(t, hx\right) - \alpha(0, hx)}{t} - 1 = \frac{d\beta\left(\varphi^t x\right)}{dt}\,|_{t=0}\,,$$

which allows us to conclude, using (5), that it is achieved the cohomological equation

$$a(hx) - 1 = \frac{d\beta\left(\varphi^t x\right)}{dt}\,|_{t=0}\,. \tag{11}$$

To show that cohomological Equation (11) has a solution is equivalent to show that the cocycle induced by the function $a \circ h - 1$ is a coboundary related to flow Φ. In fact, if the Equation (11) is satisfied by β then

$$\begin{aligned} \int_0^t \left[a\left(h(\varphi^u x)\right) - 1\right] du &= \int_0^t \left(\frac{d\beta\left(\varphi^s(\varphi^u x)\right)}{ds}\,|_{s=0}\right) du = \int_0^t \left(\frac{d\beta\left(\varphi^s(\varphi^u x)\right)}{du}\right) du\,|_{s=0} \\ &= \beta(\varphi^t x) - \beta(x) \end{aligned} \tag{12}$$

and $a \circ h - 1$ is a coboundary. On the other hand, if α is a cocycle that satisfies (10) (that is equivalent to (12)) then

$$a(hx) - 1 = \frac{\partial}{\partial t}\alpha\left(t, hx\right)|_{t=0} - 1 = \frac{d\beta\left(\varphi^t x\right)}{dt}\,|_{t=0}$$

and the Equation (11) is satisfied.

Let us presume now that the cohomological Equation (11) has a solution and let's consider the cocycle α defined by (6). If x belongs to a periodic orbit of flow Φ with period ρ, from (10) we get

$$\alpha(\rho, hx) = \rho. \tag{13}$$

Using (6) and (7) this identity is equivalent to

$$\int_0^\rho [a(h(\varphi^u x)) - 1]\, du = 0. \tag{14}$$

So, there is a solution of cohomological Equation (11), if it is satisfied the identity (13) (equivalent to (14)) for all point x in a periodic orbit of Φ with period ρ. This necessary condition is also sufficient, since when these identities are satisfied we can choose a point x in each orbit of Φ, and arbitrarily a $\beta(x) \in \mathbb{R}$, and define then $\beta : X \to \mathbb{R}$ by

$$\beta(\varphi^t x) = \beta(x) + \int_0^t [a(h(\varphi^u x)) - 1]\, du.$$

The function β satisfies the cohomological Equation (11).

3. Formulating and Demonstrating Livschitz Theorem to Hyperbolic Flows

Let $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ be a C^1 flow in a Riemannian manifold M and $\Lambda \subset M$ is Φ-invariant set (i.e., $\varphi^t\Lambda = \Lambda$ for all $t \in \mathbb{R}$). A Φ-invariant compact set $\Lambda \subset M$ is hyperbolic to Φ if for each $x \in \Lambda$ there is a continuous decomposition of tangent space

$$T_x M = E^0(x) \oplus E^s(x) \oplus E^u(x)$$

and constants $C > 0$ and $\tau \in (0, 1)$ such that for each $x \in \Lambda$ the following properties are valid:

Property 1. $\frac{d}{dt}\left(\varphi^t x\right)|_{t=0}$ *gererates* $E^0(x)$.

Property 2. $d_x\varphi^t E^s(x) = E^s(\varphi^t x)$ *and* $d_x\varphi^t E^u(x) = E^u(\varphi^t x)$ *for each* $t \in \mathbb{R}$.

Property 3. $\left\|d_x\varphi^t v\right\| \leq C\tau^t \left\|v\right\|$ *for each* $v \in E^s(x)$ *and* $t > 0$.

Property 4. $\left\|d_x\varphi^{-t} v\right\| \leq C\tau^t \left\|v\right\|$ *for each* $v \in E^u(x)$ *and* $t > 0$.

The Property 1 implies that the flow does not have fixed points in Λ. If there is an open neighborhood V of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \varphi^t V$$

then we say Λ is locally maximal to Φ.

We describe a version from Anosov closing lemma to hyperbolic flows that ensure that there are always periodic orbits in the neighborhood of orbits that turn close enough of themselves. This result gives also an estimate to the distance between the corresponding points in the initial orbit and the periodic orbit (regarding this, see for instance References [6,7]).

Lemma 1 (Anosov Closing Lemma)**.** *Let M be a Riemannian manifold, $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ a C^1 flow and $\Lambda \subset M$ a compact hyperbolic set locally maximal to Φ. Then for all large enough $\alpha \in (0, 1)$ there is an open neighborhood V of Λ and constants $C, \delta > 0$ such that if $x \in \Lambda$ verifies $d(\varphi^s x, x) < \delta$ then there is a periodic orbit $\{\varphi^t y : 0 \leq t \leq T\}$ with $y \in \Lambda$ and $|T - s| \leq C\delta$ such that*

$$d(\varphi^t x, \varphi^t y) \leq C\alpha^{\min\{t,s-t\}} d\left(\varphi^s x, x\right)$$

for $0 \leq t \leq s$.

This result contains crucial information to the demonstration of Livschitz Theorem for flows that we will now describe.

Theorem 1 (Livschitz Theorem for flows)**.** *Let M be a Riemannian manifold and $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ a C^1 flow in M. Supposing that:*

Hypothesis 1. *$\Lambda \subset M$ is a locally maximal compact hyperbolic set such that $\Phi|_\Lambda$ is topologically transitive.*

Hypothesis 2. *$g : \Lambda \to \mathbb{R}$ is a Hölder function such that for each point $x = \varphi^T x$ we have $\int_0^T g(\varphi^t x)\, dt = 0$.*

Then there is a Hölder function $G : \Lambda \to \mathbb{R}$, with at least the same Hölder exponent that g and unique in less than an additive constant, such that

$$g(x) = \lim_{t\to 0} \frac{G(\varphi^t x) - G(x)}{t}. \tag{15}$$

Proof. Given that, for each $t \in \mathbb{R}$, the function $\varphi^t|_\Lambda$ is topologically transitive, there is a point $x_0 \in \Lambda$ whose orbit is dense in Λ. Considering then a real function G defined in the dense orbit of x_0 by

$$G(\varphi^t x_0) = \int_0^t g(\varphi^u x_0)du + G(x_0),$$

in which $G(x_0) \in \mathbb{R}$ is an arbitrary fixed value. Let's see that the function G here defined is Hölder with the same exponent as in g in the orbit of x_0. Given $\delta > 0$, being $t_1 < t_2$ such that

$$d\left(\varphi^{t_1}(x_0), \varphi^{t_2}(x_0)\right) = \delta.$$

By Lemma 1 there is a point $y \in \Lambda$ with T-periodic orbit such that $|T - (t_2 - t_1)| < C\delta$ and

$$d\left(\varphi^{t_1+t}x_0, \varphi^t y\right) \leq C\alpha^{\min\{t, t_2-t_1-t\}} d(\varphi^{t_2}x_0, \varphi^{t_1}x_0) \tag{16}$$

for $0 \leq t \leq t_2 - t_1$. Naming the difference $G(\varphi^{t_2}x_0) - G(\varphi^{t_1}x_0)$ by A, we then have

$$|A| = \left|\int_0^{t_2} g(\varphi^t x_0)dt - \int_0^{t_1} g(\varphi^t x_0)dt\right| = \left|\int_0^{t_2-t_1} g(\varphi^{t+t_1}x_0)dt\right|.$$

So

$$|A| \leq \left|\int_0^{t_2-t_1} g(\varphi^{t+t_1}x_0)dt - \int_0^{t_2-t_1} g(\varphi^t y)dt\right| + \left|\int_0^{t_2-t_1} g(\varphi^t y)dt\right|.$$

With Hypothesis 2 on periodic points we have

$$\begin{aligned} |A| &\leq \left|\int_0^{t_2-t_1} [g(\varphi^{t+t_1}x_0) - g(\varphi^t y)]dt\right| + |T - (t_2 - t_1)| \max|g| \\ &\leq \int_0^{t_2-t_1} \left|g(\varphi^{t+t_1}x_0) - g(\varphi^t y)\right| dt + C\delta \max|g|. \end{aligned}$$

Since g is continuous Hölder with exponent $\theta \in (0,1]$ there is $K > 0$ such that

$$|g(x_1) - g(x_2)| \leq Kd(x_1, x_2)^\theta.$$

We then have

$$|A| \leq \int_0^{t_2-t_1} Kd(\varphi^{t+t_1}x_0, \varphi^t y)^\theta dt + C\delta \max|g|.$$

For the inequality (16) we get

$$
\begin{aligned}
|A| &\leq \int_0^{t_2-t_1} K(C\alpha^{\min\{t,t_2-t_1-t\}} d(\varphi^{t_2}x_0, \varphi^{t_1}x_0))^\theta dt \;+\; C\delta \max|g| \\
&\leq 2KC^\theta d(\varphi^{t_2}x_0, \varphi^{t_1}x_0)^\theta \int_0^{t_2-t_1} \alpha^{\theta t} dt \;+\; C\delta \max|g| \\
&= 2KC^\theta \frac{1}{\theta \ln \alpha}(\alpha^{\theta(t_2-t_1)} - 1) d(\varphi^{t_2}x_0, \varphi^{t_1}x_0)^\theta \;+\; C\delta \max|g| \\
&< 2KC^\theta \frac{\alpha^{\theta(t_2-t_1)} - 1}{\theta \ln \alpha} d(\varphi^{t_2}x_0, \varphi^{t_1}x_0)^\theta \;+\; C\delta^\theta \max|g|\,.
\end{aligned}
$$

So we have the inequality

$$
|A| < \left(2KC^\theta \frac{\alpha^{\theta(t_2-t_1)} - 1}{\theta \ln \alpha} + C\max|g|\right) d(\varphi^{t_2}x_0, \varphi^{t_1}x_0)^\theta.
$$

Since G is Hölder in the orbit of x_0 and this orbit is dense in Λ, the function G can be uniquely extended to a Hölder function in Λ (with exponent θ) which we denote by G.

The uniqueness is a consequence of the fact that choosing $G(x_0)$ determines G in an unique way. The identity (15) follows from the exposed on Subsection 2.2. □

4. Livschitz Theorem for Suspension Flows

We now consider suspension flows and obtain a version of Livschitz Theorem for these flows. As shown at the end of Section 5, this result also allows recovering Theorem 1.

Let $f : X \to X$ be a bi-Lipschitz homeomorphism from the compact metric space (X, d_X) and $\tau : X \to (0, \infty)$ a Lipschitz function. Consider the space

$$
Y = \{(x,s) \in X \times \mathbb{R} : 0 \leq s \leq \tau(x)\} \tag{17}
$$

with the points $(x, \tau(x))$ and $(f(x), 0)$ identified for each $x \in X$. The suspension flow over f with height function τ is the flow $\Psi = \{\psi^t\}_{t\in\mathbb{R}}$ in Y com $\psi^t : Y \to Y$ defined by

$$
\psi^t(x,s) = (x, s+t)\,. \tag{18}
$$

We can insert in a natural way a topology in Y that turns it into a compact topological space. This topology is induced by the Bowen–Walters distance defined in Reference [8]. In what follows we describe this distance that is necessary to be able to consider Hölder functions. Without generality loss we assume that the diameter of X, diam X, is at most equal to 1. If this is not the case we can divide by diam X because X is compact.

First of all, we assume that $\tau(x) = 1$ for all $x \in X$ and we insert the Bowen-Walters distance d_1 in the corresponding space Y. For that it is firstly considered the horizontal and vertical segments and then their length is defined. Given $x, y \in X$ and $t \in [0,1]$ the length of the horizontal segment $[(x,t), (y,t)]$ is given by

$$
\rho_h((x,t),(y,t)) = (1-t)d_X(x,y) + t d_X(f(x), f(y)). \tag{19}
$$

On the other hand, given $(x,t), (y,s) \in Y$ in the same orbit, we define the length of the vertical segment $[(x,t), (y,s)]$ by

$$
\rho_v((x,t),(y,s)) = \inf\{|r| : \psi^r(x,t) = (y,s) \text{ and } r \in \mathbb{R}\}\,. \tag{20}
$$

Finally, given two points $(x,t), (y,s) \in Y$, the distance $d_1((x,t),(y,s))$ is given by the infimum of the lengths of paths between (x,t) and (y,s) constituted by a finite number of horizontal and vertical segments. In a stricter way, for each $n \in \mathbb{N}$ we consider all the finite chains $z_0 = (x,t), z_1, \ldots, z_{n-1}, z_n =$

(y,s) of points in Y such that, for each i, or z_i and z_{i+1} are in the same segment $X \times \{t\}$ for some $t \in [0,1]$ (case in which $[z_i, z_{i+1}]$ is a horizontal segment), or z_i and z_{i+1} are in the same orbit flow (case in which $[z_i, z_{i+1}]$ is a vertical segment). If $[z_i, z_{i+1}]$ is simultaneously a horizontal and vertical segment its length is calculated considering it as a horizontal segment. The length of the chain from z_0 to z_n is finally defined as the sum of the segments' length $[z_i, z_{i+1}]$ for $i = 0, 1, \ldots, n-1$ as defined in (19) and (20).

Assuming now the case of an arbitrary function $\tau : X \to (0, \infty)$ we will introduce the Bowen–Walters distance d_Y in the space Y. Given the points $(x,t), (y,s) \in Y$ we consider

$$d_Y((x,t),(y,s)) = d_1\left(\left(x, \frac{t}{\tau(x)}\right), \left(y, \frac{s}{\tau(s)}\right)\right)$$

where d_1 is the Bowen–Walters distance defined above. Given $(x,t), (y,s) \in Y$ we also define

$$d_\pi((x,t),(y,s)) = \min\left\{\begin{array}{l} d_X(x,y) + |t-s|, \\ d_X(f(x),y) + \tau(x) - t + s, \\ d_X(x, f(y)) + \tau(y) - s + t \end{array}\right\}, \tag{21}$$

which is not necessarily a distance in Y. As exposed in Reference [9], there is a constant $C > 1$ such that for each $x, y \in Y$ the following relationship between d_π and d_Y

$$C^{-1} d_\pi(x,y) \leq d_Y(x,y) \leq C d_\pi(x,y). \tag{22}$$

is valid.

Let us now consider the extension of τ to a function $\tau : Y \to \mathbb{R}$ by the expression

$$\tau(y) = \min\{t > 0 : \psi^t y \in X \times \{0\}\},$$

and the extension of f to a function $f : Y \to X \times \{0\}$ given by

$$f(y) = \psi^{\tau(y)} y.$$

Since there is no danger of a misunderstanding we continue using symbols τ and f for the extensions. In order to apply the following result of Barreira and Saussol [9], given a continuous function $g : Y \to \mathbb{R}$ we define a new function $I_g : Y \to \mathbb{R}$ by

$$I_g(y) = \int_0^{\tau(y)} g(\psi^s y)\, ds.$$

Theorem 2. *If $\Psi = \{\psi^t\}_{t \in \mathbb{R}}$ is a suspension flow in Y over the homeomorphism $f : X \to X$, and $g : Y \to \mathbb{R}$, $h : Y \to \mathbb{R}$ and $q : Y \to \mathbb{R}$ are continuous functions, then the following properties are equivalent:*

Property 5. *g is Ψ-cohomological to h in Y with*

$$g(y) - h(y) = \lim_{t \to 0} \frac{q(\psi^t y) - q(y)}{t} \;\; \text{for each} \;\; y \in Y.$$

Property 6. *I_g is f-cohomological to I_h in Y with*

$$I_g(y) - I_h(y) = q(f(y)) - q(y) \;\; \text{for each} \;\; y \in Y.$$

Property 7. *$I_g \big|_{X \times \{0\}}$ is f-cohomological to $I_h \big|_{X \times \{0\}}$ in $X \times \{0\}$ with*

$$I_g(y) - I_h(y) = q(f(y)) - q(y) \;\; \text{for each} \;\; y \in X \times \{0\}.$$

This result shows that each cohomological class in the basic space X induces a cohomological class in all Y, and that all classes of cohomology in Y are obtained this way. It allows us to establish a version from Livschitz Theorem for suspension flows over diffeomorphisms with a locally maximal compact hyperbolic set.

Theorem 3 (Livschitz Theorem for suspension flows)**.** *Let $f : M \to M$ be a diffeomorphism with a locally maximal compact hyperbolic set $\Lambda_f \subset M$ such that $f\Big|_{\Lambda_f}$ is topologically transitive and $\tau : M \to (0, \infty)$ is a Lipschitz function. Let $\Psi = \{\psi^t\}_{t\in\mathbb{R}}$ be a suspension flow in Y over f with length function τ and for the set*

$$\Lambda = \Big\{(x, s) \in \Lambda_f \times \mathbb{R} : 0 \le s \le \tau(x)\Big\}$$

assuming that $g : \Lambda \to \mathbb{R}$ is a Hölder function such that for each point $y = \psi^T y$ we have $\int_0^T g(\psi^t y)dt = 0$. Then there is a Hölder function $G : \Lambda \to \mathbb{R}$, with at least the same Hölder exponent as g, and unique up to an additive constant, such that

$$g(y) = \lim_{t\to 0} \frac{G(\psi^t y) - G(y)}{t}. \tag{23}$$

Proof. Using Theorem 2, to establish (23) for each $y \in \Lambda$, it is sufficient to note that the function $I_g : \Lambda \to \mathbb{R}$ verifies

$$I_g(y) = G(f(y)) - G(y)$$

for each $y \in \Lambda$, for some function $G : \Lambda \to \mathbb{R}$.

Since by hypothesis $f\Big|_{\Lambda_f}$ is topologically transitive, we conclude that $\Psi|_\Lambda$ is topologically transitive.

Bearing in mind to use the Livschitz Theorem in discreet time we will see that being g a Hölder function in Λ with exponent $\theta \in (0, 1]$ the same is verified with I_g. Given $x, y \in \Lambda$ with $\tau(x) \ge \tau(y)$, designating the difference $I_g(x) - I_g(y)$ by I, we have

$$\begin{aligned} |I| &= \left|\int_0^{\tau(x)} g(\psi^s x)ds - \int_0^{\tau(y)} g(\psi^s y)ds\right| = \left|\int_0^{\tau(y)} g(\psi^s x)ds + \int_{\tau(y)}^{\tau(x)} g(\psi^s x)ds - \int_0^{\tau(y)} g(\psi^s y)ds\right| \\ &= \left|\int_{\tau(y)}^{\tau(x)} g(\psi^s x)ds + \int_0^{\tau(y)} (g(\psi^s x) - g(\psi^s y))ds\right|. \end{aligned}$$

So

$$\begin{aligned} |I| &\le \int_{\tau(y)}^{\tau(x)} |g(\psi^s x)|\, ds + \int_0^{\tau(y)} |g(\psi^s x) - g(\psi^s y)|\, ds \\ &\le \int_{\tau(y)}^{\tau(x)} \sup |g|\, ds + \int_0^{\sup\tau} \sup_{s\in(0,\tau(y))} |g(\psi^s x) - g(\psi^s y)|\, ds \\ &= \sup |g|\, |\tau(x) - \tau(y)| + \sup\tau \sup_{s\in(0,\tau(y))} |g(\psi^s x) - g(\psi^s y)| \\ &\le \sup |g|\, L d_Y(x, y) + K \sup\tau \sup_{s\in(0,\tau(y))} d_Y((x, s), (y, s))^\theta, \end{aligned}$$

for some constants $K, L > 0$. It follows then from (21) and from (22) that

$$|I| \le \sup |g|\, L d_Y(x, y) + K \sup\tau C^\theta \sup_{s\in(0,\tau(y))} d_\pi((x, s), (y, s))^\theta \le \Big[\sup |g|\, L + K C^\theta \sup\tau\Big] d_Y(x, y)^\theta.$$

This shows Hölder continuity of function I_g with the same Hölder exponent as in g.

Given $m \in \mathbb{N}$ we define now the function $\tau_m : \Lambda \to \mathbb{R}$ by

$$\tau_m(y) = \sum_{j=0}^{m-1} \tau(f^j y).$$

Using the group structure from the flow Ψ, we have $f^i y = \psi^{\tau_i(y)} y$. Besides that, for $y \in \Lambda$ and $m \in \mathbb{N}$ we have

$$\begin{aligned} \int_0^{\tau_m(y)} g(\psi^s y) ds &= \sum_{i=0}^{m-1} \int_{\tau_i(y)}^{\tau_{i+1}(y)} g(\psi^s y) ds = \sum_{i=0}^{m-1} \int_0^{\tau(f^i y)} g(\psi^{s+\tau_i(y)} y) \\ &= \sum_{i=0}^{m-1} \int_0^{\tau(f^i y)} g(\psi^s(f^i y)) ds = \sum_{i=0}^{m-1} I_g(f^i y). \end{aligned} \tag{24}$$

Let be $y \in Y$ such that $f^m y = y$. We then have $\psi^{\tau_m(y)} y = y$, that is to say, y belongs to a periodic orbit from the flow Ψ with period $\tau_m(y)$. By the hypothesis on periodic points we have $\int_0^{\tau_m(y)} g(\psi^s y) ds = 0$ equivalent, by (24) to $\sum_{i=0}^{m-1} I_g(f^i y) = 0$.

We can then conclude by the Livschitz Theorem for diffeomorphisms [1,2] that there is a Hölder function $G : \Lambda \to \mathbb{R}$, with at least the same Hölder exponent as g and unique up to an additive constant, such that $I_g(y) = G(f(y)) - G(y)$ for each $y \in \Lambda$. From Theorem 2, the intended result follows. □

5. Markov Systems

We will now briefly show how the Livschitz Theorem for suspension flows applied in the last section can somehow be considered a generalization of the Livschitz Theorem for hyperbolic flows. Reducing one to the other is based on inserting Markov systems that constitute an appropriate version of Markov partitions in the case of flows.

Let $\Phi = \{\varphi^t\}_{t \in \mathbb{R}}$ be a C^1 flow and $\Lambda \subset M$ a compact hyperbolic set locally maximal for Φ such that $\Phi|_\Lambda$ is topologically transitive. Let $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ be the stable and unstable local manifolds of size ε in point $x \in \Lambda$. For each $\varepsilon > 0$ sufficiently small, there is $\delta > 0$ such that if $x, y \in \Lambda$ are at a distance $d(x, y)$ inferior to δ then there is a unique instant of time $t = t(x, y) \in [-\varepsilon, \varepsilon]$ for which the set $[x, y] = W^s_\varepsilon(\varphi^t x) \cap W^u_\varepsilon(y)$ consists of a single point and $[x, y] \in \Lambda$.

Consider now an open disc $D \subset M$ with dimension $\dim M - 1$ transversal to flow Φ. For each $x \in D$ there is a diffeomorphism from $D \times (-\varepsilon, \varepsilon)$ over an open neighbourhood $U(x)$ of x. The A projection function $\pi_D : U(x) \to D$ defined by $\pi_D(\varphi^t y) = y$ is differentiable. A closed set $R \subset \Lambda \cap D$ is said to be a rectangle if $R = \overline{\text{int } R}$ (where the interior is calculated relative to the topology of $\Lambda \cap D$) and $\pi_D[x, y] \in R$ every time that $x, y \in R$.

Let $R_1, \ldots, R_p \subset \Lambda$ be a rectangle collection (each one inside in some open disc D_i transversal to the flow) with $R_i \cap R_j = \partial R_i \cap \partial R_j$ to/for $i \neq j$ (i.e., the rectangles only eventually intersect at the borders) such that there is $\varepsilon > 0$ with:

1. $\Lambda = \bigcup_{t \in [0,\varepsilon]} \varphi^t \left(\bigcup_{i=1}^p R_i \right)$;
2. for each $i \neq j$ we have $\varphi^t R_i \cap R_j = \emptyset$ for all $t \in [0, \varepsilon]$ or $\varphi^t R_j \cap R_i = \emptyset$ for all $t \in [0, \varepsilon]$.

We define a function $\tau : \Lambda \to [0, \infty)$ by the expression

$$\tau(x) = \min \left\{ t > 0 : \varphi^t x \in \bigcup_{i=1}^p R_i \right\}$$

Let $T : \Lambda \to \bigcup_{i=1}^{p} R_i$ be also a transference function given by $Tx = \varphi^{\tau(x)}x$ (i.e., T marks the value of the orbit of x in the first rectangle R_i reached by the orbit). We realize that the restriction of T to the union $\bigcup_{i=1}^{p} R_i$ is invertible. We say that rectangles $R_1, \ldots, R_p$ form a Markov system for Φ in Λ if

$$T(\text{int}\,(W_\varepsilon^s(x) \cap R_i)) \subset \text{int}\,(W_\varepsilon^s(Tx) \cap R_j)$$

and

$$T^{-1}(\text{int}\,(W_\varepsilon^u(Tx) \cap R_j)) \subset \text{int}\,(W_\varepsilon^u(x) \cap R_i),$$

whenever $x \in \text{int}\,TR_i \cap \text{int}\,R_j$. Any compact hyperbolic set locally maximal relating a C^1 flow possesses Markov systems of an arbitrarily small diameter (see References [3,4]). Besides that, the function τ is Hölder in each continuity domain and $0 < \inf_{x\in\Lambda} \tau \leq \sup_{x\in\Lambda} \tau < \infty$.

Given a continuous function $g : \Lambda \to \mathbb{R}$ and a Markov system for the flow $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ we define a new function $I_g : \Lambda \to \mathbb{R}$ by

$$I_g(x) = \int_0^{\tau(x)} g(\varphi^s x)\, ds. \tag{25}$$

$I_g(x)$ in Equation (25) appears in the following version from Theorem 2 in the context of Markov Systems.

Theorem 4. *Let $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ be a C^1 flow and $\Lambda \subset M$ a compact hyperbolic set locally maximal for Φ such that $\Phi|_\Lambda$ is topologically transitive. Let $g : \Lambda \to \mathbb{R}$, $h : \Lambda \to \mathbb{R}$ and $q : \Lambda \to \mathbb{R}$ be continuous functions and τ the transference function from any Markov system for Φ in Λ. Then the following properties are equivalent:*

Property 8. *g is Φ-cohomological to h in Λ With*

$$g(x) - h(x) = \lim_{t\to 0} \frac{q(\varphi^t x) - q(x)}{t}$$

for each $x \in \Lambda$.

Property 9. *I_g is T-cohomological to I_h in Λ with*

$$I_g(x) - I_h(x) = q(Tx) - q(x)$$

for each $x \in \Lambda$.

This result allows, through an approach analogous to that considered in the proof of Theorem 3 for suspension flows, to obtain an alternative proof of Livschitz Theorem for hyperbolic flows $\Phi = \{\varphi^t\}_{t\in\mathbb{R}}$ (Theorem 1). Being $R_1, \ldots, R_p$ a Markov system for Φ in Λ to prove that

$$g(x) = \lim_{t\to 0} \frac{G(\varphi^t x) - G(x)}{t}$$

it suffices, given Theorem 4, to prove that the function $I_g : \Lambda \to \mathbb{R}$ defined by (25) satisfies $I_g(x) = G(Tx) - G(x)$ for each $x \in \Lambda$.

First, let us observe that being g a Hölder function with exponent $\theta \in (0, 1]$ the same happens with I_g. Indeed,

$$\begin{aligned} |I_g(x) - I_g(y)| &= \left| \int_0^{\tau(x)} g(\varphi^s x) ds - \int_0^{\tau(y)} g(\varphi^s y) ds \right| \\ &= \left| \int_0^{\tau(y)} g(\varphi^s x) ds + \int_{\tau(y)}^{\tau(x)} g(\varphi^s x) ds - \int_0^{\tau(y)} g(\varphi^s y) ds \right| \end{aligned}$$

for $x, y \in \Lambda$ placed in the same continuity domain of τ with $\tau(y) < \tau(x)$. So

$$\begin{aligned} |I_g(x) - I_g(y)| &= \left| \int_{\tau(y)}^{\tau(x)} g(\varphi^s x) ds + \int_0^{\tau(y)} (g(\varphi^s x) - g(\varphi^s y)) ds \right| \\ &\leq \int_{\tau(y)}^{\tau(x)} |g(\varphi^s x)| \, ds + \int_0^{\tau(y)} |g(\varphi^s x) - g(\varphi^s y)| \, ds \\ &\leq C \, |\tau(x) - \tau(y)| + C \sup_{s \in (0, \sup \tau)} d(\varphi^s x, \varphi^s y)^\theta \end{aligned}$$

for a certain constant $C = \max \{\sup |g|, k \sup \tau\} > 0$ where $k > 0$ is such that $|g(x) - g(y)| \leq kd(x, y)^\theta$. We have then, for $K > 0$ such that $|\tau(x) - \tau(y)| \leq Kd(x, y)^\theta$,

$$\begin{aligned} |I_g(x) - I_g(y)| &\leq CKd(x, y)^\theta + C \sup_{s \in (0, \sup \tau), z \in M} \|d_z \varphi^s\|^\theta \, d(x, y)^\theta \\ &= C \left(K + \sup_{s \in (0, \sup \tau), z \in M} \|d_z \varphi^s\|^\theta \right) d(x, y)^\theta \end{aligned}$$

as intended.

On the other hand, defining for $m \in \mathbb{N}$ the function $\tau_m : \Lambda \to [0, \infty)$ by

$$\tau_m(x) = \sum_{j=0}^{m-1} \tau(T^j x),$$

we can easily verify that

$$\int_0^{\tau_m(x)} g(\varphi^s x) ds = \sum_{i=0}^{m-1} I_g(T^i x),$$

an equality analogous to (24). We can then use a similar approach to the one used in demonstrating Theorem 3 to establish a Livschitz Theorem for hyperbolic flows.

6. Conclusions

Although this article does not focus on Anosov Closing lemma, it is worth emphasizing that this result is crucial in the statement of the Livschitz Theorem and consequently in ensuring the existence of sufficiently regular solutions of cohomological equations. For flows with hyperbolic sets, this lemma establishes how the distance between corresponding points of an initial orbit and the constructed periodic orbits is controlled. It formalizes how the combination of local hyperbolicity, coming from the linearized dynamical systems analysis, with nontrivial recurrence tends to produces an abundance of periodic orbits. The important class of hyperbolic dynamical systems contains several examples of invertible smooth dynamical systems with complicated orbit structure, namely hyperbolic toral automorphisms, their C^1-perturbations, as well as expanding maps of the circle. The use of Anosov Closing Lemma in continuous time allowed us to present a proof of Livschitz's theorem for hyperbolic flows and to generalize this theorem to suspension flows, which are of significant importance from an application point of view in dynamic systems. Finally, it also happened to be possible to prove the Livschitz Theorem for hyperbolic flows based on Markov systems.

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