ISCTE Business School
Instituto Universitário de Lisboa

HEDGING OF BARRIER OPTIONS

Diogo Monteiro da Costa Soares Justino

Master in Finance

Thesis Supervisor:

Prof. João Pedro Nunes, Associate Professor, ISCTE Business School, Department of Finance

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Resumo

As opções com barreira são opções exóticas cuja gestão do risco pode ser difícil de executar devido à possibilidade de o delta e gamma serem muito elevados nas imediações da barreira. Esta tese procura investigar alternativas para a gestão do risco deste tipo de opções que minimizem os ganhos e perdas da estratégia dinâmica conhecida por *delta hedging*.

Investigamos duas metodologias propostas por Carr (1994) e Derman (1994) que constroiem portfolios estáticos com várias opções *vanilla.* A primeira envolve uma relação designada por simetria *put*-*call* e usa opções com a mesma maturidade e diferentes preços de exercício. A segunda divide o tempo em vários intervalos e procura replicar o valor da opção na barreira através de outras opções com diferentes maturidades.

As estratégias são avaliadas num ambiente de movimento Browniano geométrico e em dados reais do índice S&P 500.

Palavras-chave: Black-Scholes-Merton, Opções com Barreira, Delta Hedging, Static Hedging

Classificação JEL: G12, G13

Abstract

Barrier options are exotic options that can be difficult to hedge due to the possibility of the delta and gamma values being very large close to the barrier. This thesis investigates methods to hedge barrier options that can minimize the profits and losses that arise when using a dynamic technique known as delta hedging.

We present two methods by Carr (1994) and Derman (1994) that create a static portfolio of vanilla options. The first method uses the put-call symmetry relationship and builds a portfolio of different options with the same maturity and different strikes. The second method divides time into intervals and tries to match the barrier option payoff at each of theses intervals with other vanilla options.

The techniques are evaluated in the geometric Brownian motion environment where they were deduced and in real market conditions with time series of the S&P 500 index.

Keywords: Black-Scholes-Merton, Barrier Options, Delta Hedging, Static Hedging

JEL Classification: G12, G13

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Notation

S – asset price

K – exercise price

H – barrier price

R – rebate

r – risk-free interest rate

q – asset yield

t – time instant

T – ending time

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1 Introduction

In recent years, the use and importance of derivatives, financial instruments whose value depends on the value of other underlying variables, has grown enormously as mathematical models to value them appear and more research is devoted to their study. Derivative contracts are widely traded on many exchanges throughout the world and in the over-the-counter market where financial institutions trade between them and some of their biggest clients.

Many of this institutions now employ mathematicians, statisticians, physics and engineers to handle the complex mathematics behind many derivatives. Their job is to trade, value and analyse the risk to the financial institution of trading derivatives.

Options are one type of derivative contract that provide their buyer a right and their seller an obligation. They can be categorized as non-linear derivatives, as their value is a non-linear function of the underlying asset price, and further divided between standard, also termed vanilla, and exotic options. Exotic options appear as the development of mathematical finance provided financial institutions with the tools to meet different market views of clients apart from the typical upside and downside exposition to a financial asset provided by standard options. Barrier options are one type of exotic options where the payoff depends on whether or not the barrier was breached during the life of the option.

Financial institutions that trade options are often only interested in providing a service to their clients and are not expressing a particular view on some underlying asset. They are left with the task of hedging the risk of the positions assumed which can be a complex procedure because of the non-linear characteristics of options.

One of this procedures follows directly from the derivation of the Black-Scholes-Merton model in Black and Scholes (1973) and Merton (1973) and builds a riskless portfolio of the underlying asset and risk-free securities that is continuously rebalanced. This dynamic hedging technique can be applied to barrier options but is subject to substantial error when the portfolio cannot be continuously rebalanced as is often the case.

To overcome the difficulties of dynamic hedging, static hedging techniques were proposed. In Carr (1994), Carr (1997) and Carr (1998) the authors use a relationship between options of different strikes to create a static portfolio that can be used to hedge barrier options. Derman

(1994) took a slightly different approach by constructing a portfolio with many different maturities that replicates a target option boundary payoff.

An evaluation of this techniques is carried out in this paper. The rest of the thesis is organized as follows:

Chapter 2 provides the theoretical background of options and their mathematics. The Black-Scholes-Merton model is introduced and the eight barrier options are valued under the assumptions of this model. We also introduce the risk measures of barrier options, or greeks, and compare them to the greeks of vanilla options.

Chapter 3 introduces methods to hedge barrier options. The process of dynamic hedging is described and two techniques for static hedging, the put-call symmetry and the option boundary replication, are explained.

Chapter 4 tests the hedging models under computer generated data and real market data.

Chapter 5 concludes.

2 Theoretical Background

2.1 Options

Options are contracts where the holder has a right and the seller a liability. A call option gives its holder the right to buy an asset by a certain date at a certain price. A put option gives its holder the right to sell an asset by a certain date at a certain price. The price at which the holder of the option can buy (sell) the asset and at which the seller has to sell (buy) is called the exercise price or strike. The date at which the option rights expire is called the expiration or maturity date. If the holder can only exercise his right at the maturity of the option contract then the option is of the European type. If, on the contrary, the holder can exercise his right at any time before or at maturity, the option it of the American type.
The call option payoff is defined mathematically as

$$c = max(S - K, 0) \tag{1}$$

where S is the asset price at expiration and K is the exercise price. Similarly, the payoff of the put option is defined as

$$p = max(K - S, 0) \tag{2}$$

2.2 Black-Scholes-Merton Model

In 1973 a major breakthrough in the pricing of options was achieved. Fischer Black and Myron Scholes in Black and Scholes (1973) and Robert Merton in Merton (1973) were able to model and develop a closed pricing formula for European options. Their work was later recognized with the Nobel prize for economics award.
The Black-Scholes-Merton model develops on the assumption that an asset price follows a geometric Brownian motion with constant drift and volatility, a continuous-time, continuous-variable stochastic process also called a generalized Wiener process that satisfies the equation

$$\frac{dS}{S}=(r-q)\,dt+\sigma\,dz \tag{3}$$

where S is the asset price, r is the continuously compounded risk-free interest rate, q is the continuously compounded asset yield, σ is the standard deviation of the return of the asset or annualized volatility and dz is a Wiener process. By definition, dz follows a normal distribution with mean zero and variance rate equal to the time instant dt

With this model for the asset price and using a well known result from stochastic calculus called Itô's lemma, we arrive at the stochastic process followed by a function of the asset price from the stochastic process followed by the asset price itself. From Itô's lemma, the process followed by a function G of S and t when S follows the process defined by (3) is

$$dG=\left(\frac{\partial G}{\partial S}(r-q)\,S+\frac{\partial G}{\partial t}+\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt+\frac{\partial G}{\partial S}\sigma\,S\,dz \tag{4}$$

Taking into account expression (4) it follows that the process followed by *ln S* is

$$d\ln S=\left(r-q-\frac{\sigma^2}{2}\right)dt+\sigma\,dz \tag{5}$$

Therefore, the change in *ln S* between time zero and some future time T is normally distributed

$$\ln\frac{S_T}{S_0}\sim\phi\left[\left(r-q-\frac{\sigma^2}{2}\right)T\,,\sigma^2 T\right] \tag{6}$$

where $\phi(m,v)$ is a normal distribution with mean m and variance v.

Integrating both sides of equation (5) between t and T we have that

$$S_T=S_t\exp\left[\left(r-q-\frac{\sigma^2}{2}\right)(T-t)+\sigma\int_t^T dz\right] \tag{7}$$

Knowing that *ln S* follows a normal distribution, the price of an asset at future time T given its price today follows a log-normal distribution. The only source of uncertainty is the Wiener process which is the same for both *S* and *ln S*.

We now construct a portfolio of the asset and a derivative *f* and try to eliminate this source of uncertainty keeping in mind that the derivative *f* must satisfy equation (4). Defining Π as the portfolio value

$$\Pi = -f + \frac{\partial f}{\partial S} S \tag{8}$$

And taking the discrete versions of equations (3) and (4) results in

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t \tag{9}$$

which does not involve the source of uncertainty *dz*. Therefore, the portfolio must be riskless for the period of time Δt and, according to risk neutral valuation, must earn the same as other short-term risk-free securities or an arbitrage opportunity would arise

$$\Delta \Pi = (r - q) \Pi \Delta t \tag{10}$$

When we substitute equations (8) and (9) into equation (10) we obtain the Black-Scholes-Merton partial differential equation

$$\frac{\partial f}{\partial t} + (r - q) S \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{11}$$

This equation has many solutions corresponding to the different derivatives that can be defined on *S*. If we define the following boundary conditions

$$\begin{cases} f(S,T) = max(S - K, 0) & t = T \\ f(S,t) \rightarrow S & t \rightarrow \infty \\ f(0,t) = 0 & \forall t \end{cases} \tag{12}$$

and solve the Black-Scholes-Merton partial differential equation we arrive at the following formula for the time zero price of an European option

$$c = Se^{-qT} N(d_1) - Ke^{-rT} N(d_2) \tag{13}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{14}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{15}$$

and $N(x)$ is the cumulative probability distribution function of the standard normal distribution.

From the put-call parity given by

$$c + K\,e^{-rT} = p + S\,e^{-qT} \tag{16}$$

we obtain the pricing formula for an European put option

$$p = Ke^{-rT} N(-d_2) - Se^{-qT} N(-d_1) \tag{17}$$

The following assumptions apart from the geometric Brownian motion were made while deriving the pricing formulas for European call and put options:

- It is possible to short-sell the asset with full use of the proceeds;
- There are no transaction costs or taxes;
- Securities are perfectly divisible and trading is continuous;
- There are no riskless arbitrage opportunities;
- The risk-free interest rate is constant for all maturities and one can borrow and lend at this rate.

2.3 Barrier Options

While the payoff of standard call and put options only depends on the price of the underlying at maturity, barrier options are path-dependent exotic derivatives whose value depends on the underlying having breached a given level, the barrier, during a certain period of time. The market for barrier options has grown strongly because they are cheaper then corresponding standard options and provide a tool for risk managers to better express their market views without paying for outcomes that they may find unlikely.

We can divide barrier options into knock-in and knock-out options. An European knock-in option is an option whose holder is entitled to receive a standard European option if a given level is breached before expiration date or a rebate otherwise. An European knock-out option is a standard European option that ceases to exist if the barrier is touched, giving its holder the right to receive a rebate. In both cases the rebate can be zero.

The way in which the barrier is breached is important in the pricing of barrier options and, therefore, we can define down-and-in, up-and-in, down-and-out and up-and-out options for both calls and puts, giving us a total of eight different barrier options. There are more complex types of barrier options like double barrier options but we will not cover them in this text.

2.4 Relationships

Suppose that we have a portfolio composed of a down-and-in call and a down-and-out call with identical characteristics and no rebate. If the barrier is never hit the down-and-out call provides us a standard call. If the barrier is hit then the down-and-out call expires worthless but the down-and-in call emerges as a standard call. Either way we end up with a vanilla call so the following relationship between barrier options and vanilla options must hold when the rebate is zero

$$c = c_{di} + c_{do} \tag{18}$$

With a similar reasoning we can reach the same relationships for the other barrier options

$$c = c_{ui} + c_{uo} \tag{19}$$

$$p = p_{di} + p_{do} \tag{20}$$

$$p = p_{ui} + p_{uo} \tag{21}$$

2.5 Restricted Distribution

To price standard options in the Black-Scholes-Merton model we used the density function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi T}}\exp\left[-\frac{(x-\mu T)^2}{2\sigma^2 T}\right] \tag{22}$$

where

$$\mu=r-q-\frac{\sigma^2}{2} \tag{23}$$

which results directly from expression (7) and the probability density function of the standard normal distribution. The distribution (22) is the unrestricted distribution of the underlying asset return because no condition on the path of *S* is imposed.

On the other hand, barrier options are claims conditional on the path of *S* and, therefore, the distribution of the underlying asset return is restricted. This restriction raises the need for a new density function conditioned on the barrier being breached and leads us to the study of absorbing and reflecting barriers.

An absorbing barrier is a barrier which upon touching all particles vanish thus resembling knock-out barrier options. A reflecting barrier is one in which, as stated by the reflection principle, for every sample path that hits level *y* before time *t* but finishes below level *x* at time *t*, there is another equally probable path that hits *y* before time *t* and then travels upwards at least (*y*-*x*) units to finish above level (2*y*-*x*).

The variables *y* and *Y* will be defined as the, respectively, minimum and maximum rate of return of the underlying asset from time *t* to time *T*

$$y_\tau=min\left(\ln\left(\frac{S_\tau}{S_t}\right)\middle|\tau\in[t,T]\right) \tag{24}$$

$$Y_\tau=max\left(\ln\left(\frac{S_\tau}{S_t}\right)\middle|\tau\in[t,T]\right) \tag{25}$$

In Harrison (1985) is shown that the density function of the return of the asset conditional on an upper barrier *U* never being touched is given by

$$\phi\left(x|Y_\tau<\ln\left(\frac{U}{S}\right)\right)=\begin{cases} f(x)-\left(\frac{U}{S}\right)^{\frac{2\mu}{\sigma^2}} f\left(x-2\ln\left(\frac{U}{S}\right)\right) & x<\ln\left(\frac{U}{S}\right) \\ 0 & x\geq\ln\left(\frac{U}{S}\right) \end{cases} \tag{26}$$

which is the same as the solution to a Brownian motion with an absorbing barrier. Making use of the identity that expresses the probability that the barrier is never touched and the probability that the barrier is indeed touched

$$\phi\left(x|Y_\tau<\ln\left(\frac{U}{S}\right)\right)+\phi\left(x|Y_\tau\geq\ln\left(\frac{U}{S}\right)\right)=f(x) \tag{27}$$

we can easily find that the density function of the return of the asset conditional on a upper barrier *U* being touched is given by

$$\phi\left(x|Y_\tau\geq\ln\left(\frac{U}{S}\right)\right)=\begin{cases} \left(\frac{U}{S}\right)^{\frac{2\mu}{\sigma^2}} f\left(x-2\ln\left(\frac{U}{S}\right)\right) & x<\ln\left(\frac{U}{S}\right) \\ f(x) & x\geq\ln\left(\frac{U}{S}\right) \end{cases} \tag{28}$$

Also in Harrison (1985) the density function of the return of an asset conditional on a lower barrier *L* never being touched is shown to be given by

$$\phi\left(x|y_\tau>\ln\left(\frac{L}{S}\right)\right)=\begin{cases} f(x)-\left(\frac{L}{S}\right)^{\frac{2\mu}{\sigma^2}} f\left(x-2\ln\left(\frac{L}{S}\right)\right) & x>\ln\left(\frac{L}{S}\right) \\ 0 & x\leq\ln\left(\frac{L}{S}\right) \end{cases} \tag{29}$$

and using an expression similar to (27)

$$\phi\left(x|y_\tau>\ln\left(\frac{L}{S}\right)\right)+\phi\left(x|y_\tau\leq\ln\left(\frac{L}{S}\right)\right)=f(x) \tag{30}$$

we find that the density function of the return of the asset conditional on a lower barrier L being touched is given by

$$\phi\left(x|y_T\le\ln\left(\frac{L}{S}\right)\right)=\begin{cases}\left(\frac{L}{S}\right)^{\frac{2\mu}{\sigma^2}}f\left(x-2\ln\left(\frac{L}{S}\right)\right) & x>\ln\left(\frac{L}{S}\right)\\ f(x) & x\le\ln\left(\frac{L}{S}\right)\end{cases} \tag{31}$$

Looking closer at the density functions we see that we can replace both U and L for a general barrier H without changing the meaning of the formulas.

When pricing rebates for knock-in barrier options we are interested in the density function of the return of an asset conditional on the barrier never being hit. We can only know if this condition holds at the expiration of the option. Thus we can use the previously developed density functions to price the rebate.

In the presence of a knock-out barrier option the rebate is priced differently because the barrier can be breached at any time until maturity. We need a new density function that takes into account the distribution of the first passage time at a particular point which can be shown to be given by

$$\phi\left(T|\ln\left(\frac{U}{S}\right)>0\right)=\frac{\ln(U/S)}{\sigma\sqrt{2\pi T^3}}\exp\left[\frac{-(\ln(U/S)-\mu T)^2}{2\sigma^2 T}\right] \tag{32}$$

This density function works for both up-and-out and down-and-out barrier options as long as we change the sign of the expression.

2.6 Pricing formulas

Merton (1973) and Rubinstein (1991a) derived the following closed formulas for the pricing of European barrier options. They make use of the restricted, unrestricted and first passage time density functions developed in the previous section and the remaining Black-Scholes-Merton model assumptions.

$$A=\phi Se^{-qT}N(\phi x_1)-\phi Ke^{-rT}N(\phi x_1-\phi\sigma\sqrt{T}) \tag{33}$$

$$B = \phi S e^{-qT} N(\phi x_2) - \phi K e^{-rT} N(\phi x_2 - \phi \sigma \sqrt{T}) \tag{34}$$

$$C = \phi S e^{-qT} \left(\frac{H}{S}\right)^{\frac{2\mu+2}{\sigma^2}} N(\eta y_1) - \phi K e^{-rT} \left(\frac{H}{S}\right)^{\frac{2\mu}{\sigma^2}} N(\eta y_1 - \eta \sigma \sqrt{T}) \tag{35}$$

$$D = \phi S e^{-qT} \left(\frac{H}{S}\right)^{\frac{2\mu+2}{\sigma^2}} N(\eta y_2) - \phi K e^{-rT} \left(\frac{H}{S}\right)^{\frac{2\mu}{\sigma^2}} N(\eta y_2 - \eta \sigma \sqrt{T}) \tag{36}$$

$$E = R e^{-rT} \left[N(\eta x_2 - \eta \sigma \sqrt{T}) - \left(\frac{H}{S}\right)^{\frac{2\mu}{\sigma^2}} N(\eta y_2 - \eta \sigma \sqrt{T}) \right] \tag{37}$$

$$F = R \left[\left(\frac{H}{S}\right)^{\frac{\mu}{\sigma^2}+\lambda} N(\eta z) - \left(\frac{H}{S}\right)^{\frac{\mu}{\sigma^2}-\lambda} N(\eta z - 2\eta \lambda \sigma \sqrt{T}) \right] \tag{38}$$

$$x_1 = \frac{\ln(S/K)}{\sigma \sqrt{T}} + \left(1 + \frac{\mu}{\sigma^2}\right) \sigma \sqrt{T} \tag{39}$$

$$x_2 = \frac{\ln(S/H)}{\sigma \sqrt{T}} + \left(1 + \frac{\mu}{\sigma^2}\right) \sigma \sqrt{T} \tag{40}$$

$$y_1 = \frac{\ln(H^2/SK)}{\sigma \sqrt{T}} + \left(1 + \frac{\mu}{\sigma^2}\right) \sigma \sqrt{T} \tag{41}$$

$$y_2 = \frac{\ln(H/S)}{\sigma \sqrt{T}} + \left(1 + \frac{\mu}{\sigma^2}\right) \sigma \sqrt{T} \tag{42}$$

$$z = \frac{\ln(H/S)}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T} \tag{43}$$

$$\lambda = \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{44}$$

where ϕ and η are variables that can be set to 1 or -1.

Note that we can merge expressions (13) and (17) for the Black-Scholes-Merton price of a call and put option respectively to obtain a general vanilla option pricing formula. If ϕ is set to

1 then expression (33) becomes the call option pricing formula from (13). If ϕ is set to -1 the same expression is equal to the put option pricing formula (17).

2.6.1 Down-and-in Call

The down-and-in call option gives its holder the right to receive a vanilla call option if the barrier *H* is hit or a rebate *R* otherwise. At inception the underlying price *S* is higher than the barrier *H* and has to move down before the regular option becomes active. Expressed mathematically, the payoff function is

$$c_{di} = \begin{cases} max(S-K;0) & \exists t \leq T : S_t \leq H \\ R & \forall t \leq T : S_t > H \end{cases} \tag{45}$$

When $K \geq H$ we value the option using the restricted density function in the interval [*K*; +∞[. If $K \leq H$ we need to split the integration region and use the unrestricted density function in the interval [*K*; *H*] and the restricted density function in the interval [H; +∞[. The price of a down-and-in call is then given by

$$c_{di} = \begin{cases} C+E & K \geq H; \eta=1; \phi=1 \\ A-B+D+E & K \leq H; \eta=1; \phi=1 \end{cases} \tag{46}$$

2.6.2 Up-and-in Call

Like the down-and-in, the up-and-in call option gives its holder the right to receive a regular call option if the barrier *H* is breached or a rebate *R* otherwise, but the underlying price *S* starts below the barrier *H* and has to move up for the regular option to be activated. The payoff function is defined as

$$c_{ui} = \begin{cases} max(S-K;0) & \exists t \leq T : S_t \geq H \\ R & \forall t \leq T : S_t < H \end{cases} \tag{47}$$

If $K \geq H$ the value of the option is simply the value of a standard call option. If, instead, $K \leq H$ we use the restricted density function in the interval [*K*; *H*] and the unrestricted density function in the interval [*H*; +∞[. The price of a up-and-in call is then given by

$$c_{ui}=\begin{cases} A+E & K\geq H\,;\eta=-1\,;\phi=1 \\ B-C+D+E & K\leq H\,;\eta=-1\,;\phi=1 \end{cases} \tag{48}$$

2.6.3 Down-and-in Put

A down-and-in put option gives its holder the right to receive a standard put option if the barrier *H* is hit or a rebate *R* otherwise. The underlying price *S* starts above the barrier *H* and has to move down before the vanilla put is born. The payoff function is expressed mathematically by

$$p_{di}=\begin{cases} max(K-S\,;0) & \exists t\leq T: S_t\leq H \\ R & \forall\, t\leq T: S_t>H \end{cases} \tag{49}$$

If $K\geq H$ we value the option with restricted density function in the interval [*H*; *K*] and the unrestricted density function in the region]-∞; *H*]. When $K\leq H$ the value of the down-and-in put is the value of a vanilla put option. The price of a down-and-in put is given by

$$p_{di}=\begin{cases} B-C+D+E & K\geq H\,;\eta=1\,;\phi=-1 \\ A+E & K\leq H\,;\eta=1\,;\phi=-1 \end{cases} \tag{50}$$

2.6.4 Up-and-in Put

The up-and-in put option gives its holder the right to receive a regular put option if the barrier *H* is hit or a rebate *R* otherwise. At inception the underlying price *S* is lower than the barrier *H* and has to move up for the regular put option to become activated. Expressed mathematically, the payoff function is

$$p_{ui}=\begin{cases} max(K-S\,;0) & \exists t\leq T: S_t\geq H \\ R & \forall\, t\leq T: S_t<H \end{cases} \tag{51}$$

When $K\geq H$ the standard option is born in the money and valued as the sum of the unrestricted density function in the region [*H*; *K*] and the restricted density function in the region]-∞;*H*]. If $K\leq H$ we simply use the restricted density function in the interval]-∞;*K*]. The price of a up-and-in put is then given by

$$p_{ui} = \begin{cases} A - B + D + E & K \geq H\,;\eta = -1\,;\phi = -1 \\ C + E & K \leq H\,;\eta = -1\,;\phi = -1 \end{cases} \tag{52}$$

2.6.5 Down-and-out Call

A down-and-out call option is a regular call option that expires worthless or paying rebate R as soon as the barrier H is hit. At inception the underlying price S is higher than the barrier H. The payoff function is expressed as

$$c_{do} = \begin{cases} max(S - K\,;0) & \forall\, t \leq T : S_t > H \\ R & \exists\, t \leq T : S_t \leq H \end{cases} \tag{53}$$

It is easy to price the down-and-out call by making use of the relationship between barrier and standard options. Substituting expression (46) into expression (18) gives the following result:

$$c_{do} = \begin{cases} A - C + F & K \geq H\,;\eta = 1\,;\phi = 1 \\ B - D + F & K \leq H\,;\eta = 1\,;\phi = 1 \end{cases} \tag{54}$$

2.6.6 Up-and-out Call

The up-and-out call option is a regular call option that expires and pays rebate *R* if the barrier *H*, above the underlying price *S* at inception, is breached. Expressed mathematically, the payoff function is

$$c_{uo} = \begin{cases} max(S - K\,;0) & \forall\, t \leq T : S_t < H \\ R & \exists\, t \leq T : S_t \geq H \end{cases} \tag{55}$$

When $K \geq H$ the value of the option is the rebate because the option is knocked out before getting in the money. Making use of relationship (19) and expression (48) the price of the up-and-out call is given by

$$c_{uo} = \begin{cases} F & K \geq H\,;\eta = -1\,;\phi = 1 \\ A - B + C - D + F & K \leq H\,;\eta = -1\,;\phi = 1 \end{cases} \tag{56}$$

2.6.7 Down-and-out Put

A down-and-out put option is a regular put option while the barrier *H* is not hit. If the barrier is breached then a rebate *R* is payed. In the beginning the underlying price *S* is higher than the barrier. The payoff function is given by

$$p_{do}=\begin{cases} max(K-S;0) & \forall t\leq T: S_t>H \\ R & \exists t\leq T: S_t\leq H \end{cases} \tag{57}$$

When $K\leq H$ the value of the option is the rebate because the option is knocked out before getting in the money. Substituting expression (50) into expression (20) gives the following result: for the value of the down-and-out put

$$p_{do}=\begin{cases} A-B+C-D+F & K\geq H;\eta=1;\phi=-1 \\ F & K\leq H;\eta=1;\phi=-1 \end{cases} \tag{58}$$

2.6.8 Up-and-out Put

The up-and-out put option is a vanilla put option that expires paying a rebate *R* as soon as the barrier *H*, above the underlying price *S* at inception, is hit. Expressed mathematically, the payoff function is

$$p_{uo}=\begin{cases} max(K-S;0) & \forall t\leq T: S_t<H \\ R & \exists t\leq T: S_t\geq H \end{cases} \tag{59}$$

Again, we make use of the relationship between vanilla and barrier options. Substituting expression (52) into (21) gives the value of the up-and-out put as

$$p_{uo}=\begin{cases} B-D+F & K\geq H;\eta=-1;\phi=-1 \\ A-C+F & K\leq H;\eta=-1;\phi=-1 \end{cases} \tag{60}$$

2.7 Greeks

Options are non-linear derivatives whose value depends on the price of an underlying asset. The greeks measure the sensitivities of the option value to changes in the parameters that are used to price the option. They provide crucial information to the risk manager who wishes to protect his portfolio from adverse market movements.

Closed formulas for the greeks of European options are widely known. While not providing closed formulas for the greeks of European barrier options, we can use numeric procedures to study the behaviour of delta, gamma, theta, vega and rho and point out the differences relative to standard options that make their hedging riskier.

As an example in the study of the greeks, consider an up-and-out call and a down-and-in put option with the following characteristics:

Exercise price – 100

Interest rate – 5%

Asset yield – 3%

Volatility – 20%

Up-and-out call barrier – 110

Down-and-in put barrier – 90

Rebate – 0

2.7.1 Delta

The delta of an option measures the rate of change of the option value relative to changes in the underlying asset price. Defined mathematically, the delta is given by

$$\Delta = \frac{\partial f}{\partial S} \tag{61}$$

where f is the price of the option and S is the asset price. This means that when the asset price changes one unit, the price of the option on the asset will change by Δ units. When deducing the Black-Scholes-Merton model, delta was used to find the quantity of the underlying that the portfolio should have to be riskless. It is important to keep in mind that delta changes and the portfolio will be riskless only for an infinitesimal period of time.

Vanilla call options have deltas that range from 0 to 1 and standard put options have deltas that range from -1 to 0 but the delta of a barrier option does not behave in such a convenient way.

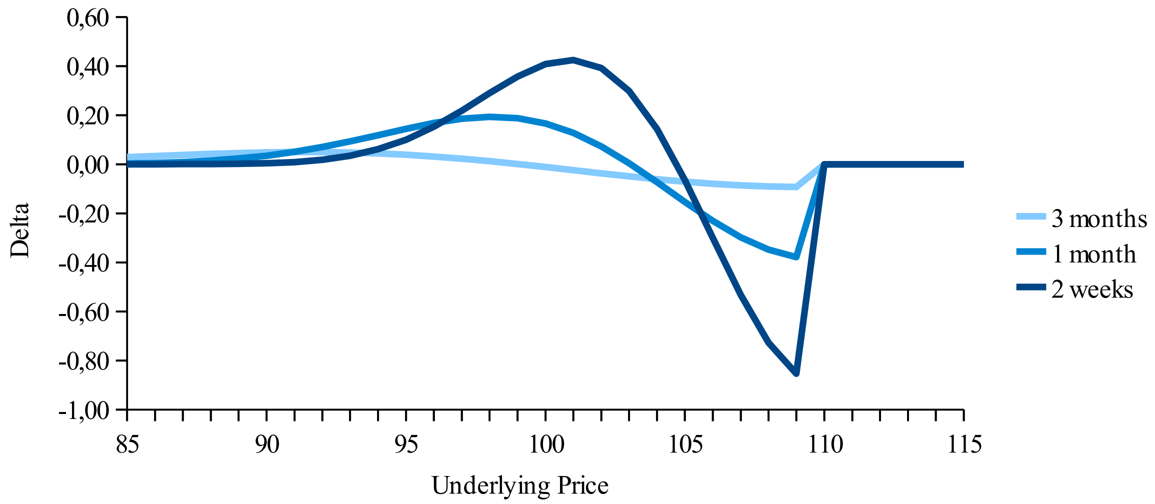

Figure 1: Delta of the up-and-out call option with different times to maturity.

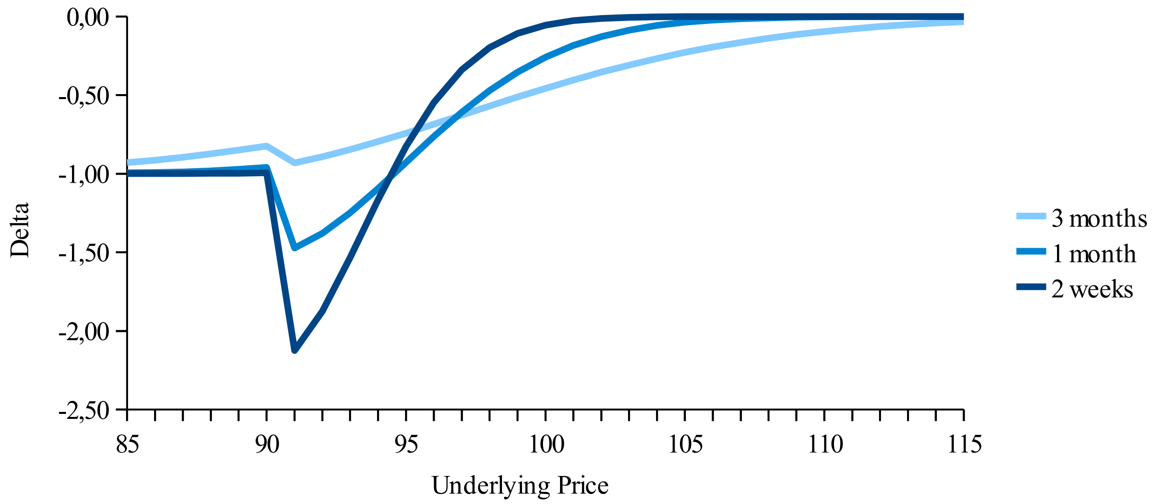

Figure 2: Delta of the down-and-in put option with different times to maturity.

Figure 1 shows the evolution of the delta of the up-and-out call option as the expiration date approaches. It is interesting to note that the delta of the barrier option can be positive or negative depending on the underlying price and that its absolute value grows when we are near the barrier and maturity gets closer. An important characteristic of knock-out barrier options is that if the barrier is touched the delta goes immediately to zero forcing the risk manager to buyback or sell all the underlying asset in his portfolio.

In Figure 2 we see another characteristic of barrier options that is very important to the risk manager which is the absolute value of delta possibly being much larger than one. This can have a enormous impact in the trading of the underlying asset as trading costs rise and the market may not offer appropriate liquidity for the risk manager to correctly rebalance his portfolio.

2.7.2 Gamma

The gamma of an option measures the rate of change of the delta as the underlying asset price changes. It is the second partial derivative of the price of the option relative to the asset price and is defined by

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 f}{\partial S^2} \tag{62}$$

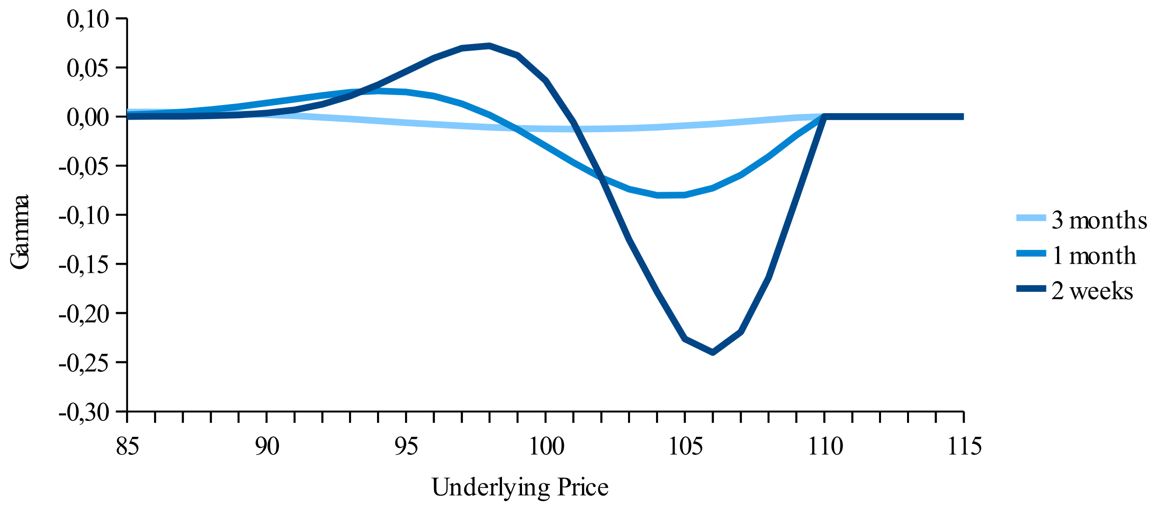

Figure 3: Gamma of the up-and-out call option with different times to maturity.

Gamma is an important measure of risk because if gamma is small then delta changes slowly and keeping a delta neutral portfolio only requires small and infrequent adjustments. If, on the contrary, the absolute value of gamma is large then delta changes quite rapidly as well as the frequency and size of adjustments and, consequently, the profit or loss arising will be much larger. It is also important to note that when gamma is positive the adjustment needed to keep a delta neutral portfolio results in a profit and when gamma is negative the adjustment results in a loss.

Looking at Figure 3 there are two key aspects of the gamma of a barrier option to keep in mind. First, contrary to vanilla options, the gamma can change from positive to negative and vice-versa without one changing from being long or short the option. Second, the absolute value of gamma is usually larger than the gamma of a standard option and can be extremely large near the barrier. The consequence of this behaviour is the enhanced difficulty the risk manager faces when rebalancing his portfolio to keep it delta neutral.

2.7.3 Theta

The theta of an option measures how much the price of an option will change as time passes and is sometimes referred to as the time decay.

$$\Theta=\frac{\partial f}{\partial t} \tag{63}$$

While the theta of a standard option is negative except in a few special cases, the theta of a barrier option can be positive or negative. This means that the holder of a knock-out barrier option can gain value as time passes as can be seen in Figure 4. The rationale is simple: if the option is in-the-money, as time passes, the probability of being knocked-out diminishes and, consequently, its value increases.

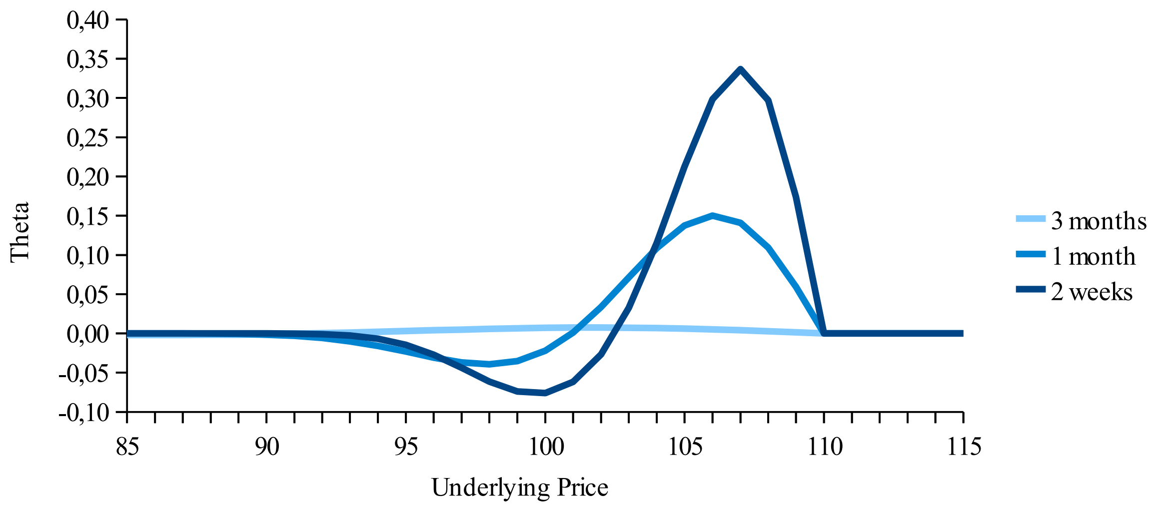

Figure 4: Theta of the up-and-out call option with different times to maturity.

2.7.4 Vega

The Black-Scholes-Merton assumes that the underlying asset volatility is constant. In practice this is seldom true because the implied volatility of an option will change as traders adjust their expectations for future volatility. Vega measures the change in the option price with changes in the implied volatility.

$$VEGA = \frac{\partial f}{\partial \sigma} \tag{64}$$

Volatility is the single most important factor that influences the price of an option. Both vanilla calls and puts rise in value as volatility rises. In barrier options this behaviour is different. Knock-in options will rise in value as volatility rises because the probability of the vanilla option emerging rises. Knock-out barriers will decrease in value as volatility rises because the probability of expiring worthless will rise.

2.7.5 Rho

The rho of an option is the change in the option value when interest rates change

$$P = \frac{\partial f}{\partial r} \tag{65}$$

It is usually not looked into because the value of an option is not very sensitive to interest rate changes except in extreme circumstances.

3 Hedging

A financial institution acts as a market maker to its clients by being present in the market and providing liquidity by showing willingness to buy and sell at specific bid and ask prices. The market maker is trying to profit from the difference between the price he is willing to buy and sell. When an options market maker enters into a transaction he is left with the task of hedging his market risk.

On the other hand, the arbitrageur is trying to profit from the difference between theoretical and market prices of options. Volatility is one of the parameters of an option pricing model but we can only know the realized volatility of an asset at the expiration of the option. When pricing an option we need to estimate future volatility. The arbitrageur can try to profit from the difference between implied volatility in option market prices and the volatility that he believes will be realized by the underlying asset until the maturity date of the contract. He will then take a position in the option that he believes to be miss priced and in the underlying asset to hedge his exposition to market moves.

Both market makers and arbitrageurs need a framework to hedge their risk. This section presents a method of dynamic hedging known as delta hedging that requires the continuously rebalancing of a portfolio composed by the option and the underlying asset. It is followed by two techniques of static hedging: the put-call symmetry and the boundary replication. Static hedging is an attempt to construct a fixed portfolio of vanilla options to replicate a target exotic option. The portfolio is built at the inception of the barrier option and is unwound when the barrier is breached or the option reaches maturity. The technique can be specially useful when hedging exotic options with high gamma, like barrier options, that make dynamic hedging both difficult and inaccurate.

3.1 Delta Hedging

Recall that in the Black-Scholes-Merton model deduction we set up a portfolio of a derivative, an option for example, and a quantity Δ of the underlying asset. It was shown that for an infinitesimal period of time the portfolio is riskless and must earn the risk-free interest rate, such that the gain or loss from the asset position always offsets the gain or loss from the derivative. As the asset price changes and time passes the delta of the option changes. This requires the quantity of the underlying asset in the portfolio to be adjusted so that the portfolio remains riskless. This form of dynamic hedging is called delta hedging.

Imagine that a call option is sold with the following characteristics:

Asset price – 100

Exercise price – 100

Time to maturity – 15 days

Risk-free interest rate – 5%

Asset yield – 3%

Volatility – 20%

Rebate – 0

Each call option contract is on 100 units of the underlying asset. Disregard weekends and assume that the portfolio rebalancing is done once a day at the close of the market. Table 1 illustrates the delta hedging procedure for a possible sample path of the asset price.

Days to maturity	**Asset price**	**Option delta**	**Quantity of asset purchased**	**Cost of asset purchased**	**Interest and yield**	**Total cost**
15	100,00	0,52	52	5200,00	0,26	5034,00
14	100,83	0,60	8	806,64	0,30	5840,90
13	100,10	0,53	-7	-700,70	0,27	5140,50
12	101,07	0,63	10	1010,70	0,32	6151,47
11	99,64	0,47	-16	-1594,24	0,24	4557,55
10	99,01	0,39	-8	-792,08	0,20	3765,71
9	98,60	0,34	-5	-493,00	0,17	3272,91
8	100,59	0,59	25	2514,75	0,31	5787,83
7	101,80	0,75	16	1628,80	0,39	7416,94
6	101,54	0,73	-2	-203,08	0,38	7214,25
5	102,68	0,88	15	1540,20	0,46	8754,83
4	103,12	0,93	5	515,60	0,48	9270,89
3	102,82	0,94	1	102,82	0,49	9374,19
2	102,41	0,95	1	102,41	0,50	9477,09
1	103,20	1,00	5	516,00	0,52	9993,59
0	104,35	1,00	0	0,00		9994,11

Table 1: Simulation of the delta hedging procedure.

The initial Black-Scholes-Merton call option price is 1,66 and the risk manager receives 1,66 x 100 = 166 upfront. The delta of the call is initially 0,52 so the hedger buys 52 units of the underlying asset for 100 x 52 = 5200. Total cost for first day of hedging is 5200 minus the premium received from the option buyer for a total of 5034 borrowed at the risk-free interest rate to finance the asset purchase.

On the second day the asset price has moved to 100,83 and the new option delta is 0,60. The risk manager finds himself short of 8 underlying assets and buys them in the market for 100,83 x 8 = 806,64 that he borrows. At the same time, the risk manager has incurred in interest costs from the loan he took to buy the underlying asset in day 1 and received the corresponding yield from the assets in his portfolio. The result is a total cost at the end of day 2 of 5034 + 806,64 + 0,26 = 5840,90.

Next day the asset price falls to 100,10. The risk manager has 60 underlying assets, more than the new call option delta of 0,53, and sells 7 in the market realizing 100,10 x 7 = 700,70 that he uses to payback the money he borrowed after paying interest and receiving the asset yield. His total cost for the hedge is now 5840,90 – 700,70 + 0,30 = 5140,50.

The risk manager follows this procedure every day to keep a riskless portfolio. On the expiration day, the asset price is 104,35. As the option is in-the-money, the risk manager sells all the underlying asset in his portfolio realizing 104,35 x 100 = 10435 from which he uses (104,35 – 100) x 100 = 435 to fulfil his obligation to the buyer of the option and the remainder to payback the borrowed 9994,11. His net result is a profit of 5,89.

If the risk manager followed the Black-Scholes-Merton model and rebalanced his portfolio continuously as the underlying asset evolved to the expiration date, realizing the same volatility used as input to the model, he would be guaranteed by the model that the hedging cost incurred in the portfolio rebalancing would be exactly the initial value of the option, no matter the path the asset price took, resulting in profit and loss (P&L) of zero.

In practice it is impossible to continuously rebalance the portfolio because markets close, trading has costs, liquidity is finite and buying and selling is not always done at the desired price. The risk manager will have to choose discrete times to adjust the underlying asset position of his portfolio. As the previous example of delta hedging shows, the risk manager will end up with some P&L. In Kamal (1999) is shown that the standard deviation of the P&L, or hedging error, is directly proportional to the frequency of rebalancing but the average result will be zero as if the hedging was carried out continuously. Figure 5 shows the P&L distribution for 1000 simulations of the delta hedging procedure for the previous call option which resulted in an average P&L around zero.

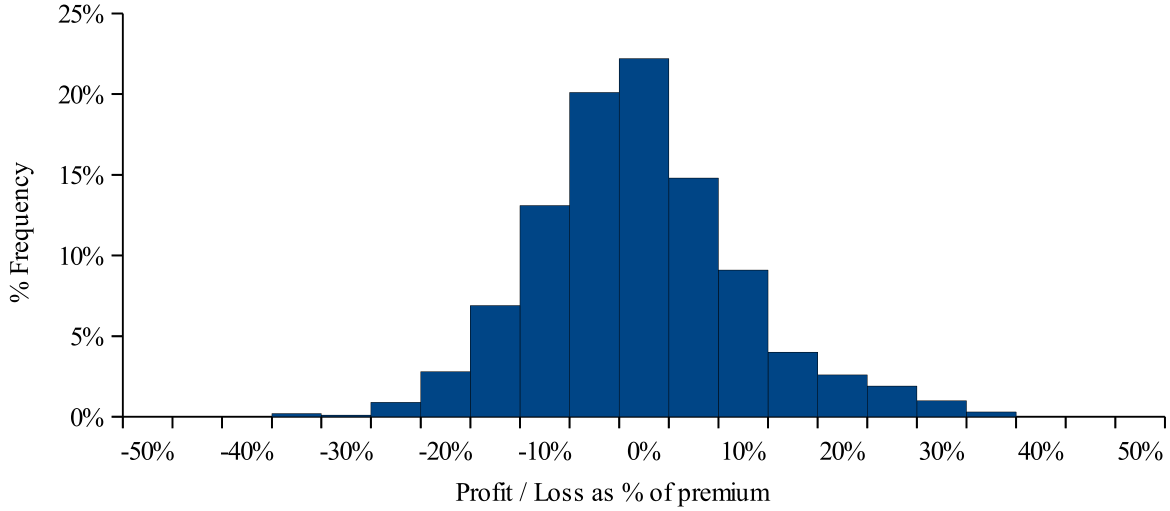

Figure 5: Profit and loss distribution of 1000 simulations of the delta hedging procedure.

As mentioned before, the gamma of an option measures the change in delta as the underlying asset moves. As hedging cannot be carried out continuously, the gamma of the option will play a crucial role in the final P&L of the delta hedging strategy. When the absolute value of gamma is high, the risk manager will make a higher profit if long gamma and a higher loss if short gamma when the portfolio is rebalanced because the delta that needs to be adjusted is higher.

One of the inputs of the Black-Scholes-Merton model is the volatility and the model assumes that volatility is constant. In practice, volatility is seldom constant. Unlike the remaining parameters, with the exception of the asset yield that might be uncertain, volatility cannot be directly observed in the market. We can only calculate the asset return volatility realized over a period of time at the end of that time, therefore, volatility has to be estimated. The risk manager should note that the delta hedging procedure assumes that one can correctly estimate the underlying asset future realized volatility over the life of the option.

In standard call and put options, if one sells an option with implied volatility lower than the future realized volatility, a loss will be incurred as selling the option translates into a short volatility position. If an option is sold with implied volatility greater than the future realized volatility, then a gain will be made.

The explanation is that the premium of an option is the amount the buyer of the option pays the seller to cover the expected loss made in the delta hedging procedure. If an option is sold with greater implied volatility than the future asset return volatility, the seller of the option will not lose as much money as the premium received because the expected adjustments to the replication portfolio are smaller, resulting in an expected positive difference.

When we are dealing with barrier options this behaviour is not so simple because, as we have seen before, a barrier option can have positive and negative vega over its life. If we have sold the barrier option and at a specific time the option vega is positive, money will be made if the underlying asset volatility rises. On the contrary, if vega becomes negative, the rise in volatility will lead to a loss.

3.2 Put-Call Symmetry

The put-call symmetry is a relationship between calls and puts with different strikes first studied by Bates (1988). It can be viewed as both an extension and a restriction on the put-call parity relationship between calls and puts of the same strike. Under the assumptions of the Black-Scholes-Merton model and further enforcing the drift rate of the underlying asset to be zero, the following relationship holds

$$\frac{C(K_1)}{\sqrt{K_1}} = \frac{P(K_2)}{\sqrt{K_2}} \tag{66}$$

where the geometric mean of the call strike K_1 and the put strike K_2 is the forward price F

$$\sqrt{K_1 K_2} = F \tag{67}$$

The zero drift assumption implies that the forward price must be equal to the spot price thus requiring zero carrying cost for options written on the spot price of the underlying asset.

This relationship can be shown to hold with restrictions even if we relax the Black-Scholes-Merton model assumption that the volatility of the asset is a known constant. Carr (1998) shows that if the volatility of the forward price F is a known function of time $\sigma(F_t, t)$ of the forward asset price F_t and time t, the put-call symmetry still holds when

$$\sigma(F_t, t) = \sigma(F^2/F_t, t) \qquad \forall F_t \geq 0 \wedge t \in [0, T] \tag{68}$$

This means that the volatility of the forward price F at any future date is the same for any two levels with the same geometric mean thus providing support for a symmetric volatility smile. As an example consider an asset whose current forward price is 100 and the volatility is 20%. The value of a call option on the asset with strike price 125 and volatility 25% is the same as the value of 1,25 put options with strike 80 as long as the volatility at this strike is also 25%.

In Carr (1994) the authors are able to use this symmetry to create a static replication portfolio for barrier options. Given the previous assumptions, the sale of a down-and-in call option when the strike K and the barrier H are equal is hedged by buying a standard put option with strike K. If the barrier is touched, put-call parity implies that, when the asset drift rate is zero, the value of the put option is equal to the value of one call option with the same strike and maturity. One can then sell the put option bought and use the proceeds to buy the call option that knocks-in without any out of pocket expenses. If the barrier is never breached the put options expire worthless as does the down-and-in call.

When the barrier H is below the strike K and the barrier is breached we need to buy an out-of-the-money call option. In this case the put in the replication portfolio is at-the-money which translates into very different values between this put option and the emerging call option. Nevertheless the same hedging strategy can be used provided that we replace the put-call parity with the put-call symmetry. The relationship ensures that by going long K/H puts with strike H^2/K we will have the funds needed to buy the call option with strike K if the barrier is touched. On the other hand, the puts, as the down-and-in call, will expire worthless if the barrier is not touched.

If the barrier H is above the strike K the hedging strategy has to be different because the emerging call is in-the-money and thus has intrinsic value. Define a down-and-in bond as a security that pays one monetary unit if at any time until maturity the asset price is equal to or below the strike K. We can construct a replication portfolio for the down-and-in call by buying $(H - K)$ down-and-in bonds with strike H that provide the intrinsic value of the emerging call at the barrier and a standard put option with strike K that provides the time value.

The down-and-in bond can be valued using European binary and vanilla put options. Binary options are valued in Rubinstein (1991b). An European binary cash-or-nothing put option pays one unit of cash if the spot is below the strike at maturity. Its present value is given by

$$BP = \mathrm{e}^{-rT} N(-x_1 + \sigma\sqrt{T}) \tag{69}$$

It can also be valued and statically hedged as a portfolio constructed by a large number of long standard put options struck just above the strike and short the same number of standard puts below the strike as follows

$$BP(K) = \lim_{n\to+\infty} \frac{n}{2}\left[P\left(K+\frac{1}{n}\right) - P\left(K-\frac{1}{n}\right)\right] \tag{70}$$

where P(*K*) is a standard put with strike *K* and *BP*(*K*) is an European binary cash-or-nothing put option with strike *K*.

Given that when the forward is at the barrier each binary put has approximately 50% probability to finish in-the-money when discounting the positive skew of the price distribution, it can be shown that the value of a down-and-in bond with strike *K* is given by

$$B_{di}(K) = 2\mathrm{BP}(K) - \frac{1}{K}P(K) \tag{71}$$

The down-and-out call is constructed by buying a standard call option with strike *K* and selling the portfolio of the down-and-in call as stated by relationship (18).

We have seen that, when the barrier is below the strike, the value of an up-and-in call is equal to the value of a standard call option and, consequently, a short position in the up-and-in call is hedged by going long a standard call option. When the barrier is above the strike, the emerging call is in-the-money and the static hedging strategy has to be split, as in the down-an-in call, into a replication of the intrinsic value and time value.

Since we are in the presence of an up barrier, we need to introduce the up-and-in bond with strike *K* as a security that pays one monetary unit if at any time until maturity the asset price is equal to or above the strike *K*. The up-and-in bond value is given by

$$B_{ui}(K) = 2\mathrm{BC}(K) + \frac{1}{K}C(K) \tag{72}$$

where $C(K)$ is a standard call with strike K and $BC(K)$ is an European binary cash-or-nothing call with present value given by

$$BC = e^{-rT} N(x_1 - \sigma\sqrt{T}) \tag{73}$$

As with the European binary cash-or-nothing put, the European binary cash-or-nothing call can be statically hedged with the following portfolio of standard calls

$$BC(K) = \lim_{n\to+\infty} \frac{n}{2}\left[C\left(K-\frac{1}{n}\right)-C\left(K+\frac{1}{n}\right)\right] \tag{74}$$

The up-and-in call can be hedged with $(H - K)$ up-and-in bonds that provide the intrinsic value of the emerging call at the barrier. One could think that a long position in put options would provide the time value as in the replication of the down-and-in call but this could lead to problems at expiration if the put finishes in-the-money instead of out-of-the-money. The elegant solution is to go long an up-and-in put as this security can only have value at expiration if the barrier is touched and, at that moment, it will be instantly sold. To hedge the up-and-in put we can follow the same reasoning used for the down-and-in call to see that, when the barrier H is higher then the strike K, the put-call symmetry guarantees that the value of the emerging put option is the same as the value of K/H calls struck at H^2/K.

With similar reasoning we can find the replication portfolio of the remaining barrier options. Table 2 is a summary of the hedging strategy provided by the put-call symmetry when the rebate is zero, where $C(X)$ is a call with strike X, $P(X)$ is a put with strike X, $B_{ui}(X)$ is an up-and-in bond with strike X and $B_{di}(X)$ is a down-and-in bond with strike X. When the rebate is not zero one can use the in bonds developed to statically hedge its risk.

As mentioned before, the put-call symmetry implies that the drift rate of the asset is zero. If this is not the case than the value of the replication portfolio is no longer ensured to be zero along the barrier, in case of knock-out options, or equal to the funds required to buy the emerging option, in case of knock-in options. The reason is that the forward value at the barrier is different from the spot which implies that call and put values with the same geometric mean are no longer equal. Even so, Carr (1994) is able to deduce tight bounds for barrier option values with static hedges when the zero drift assumption is relaxed.

Option	Replication	
Down-and-in call	$c_{di}=\begin{cases}\frac{K}{H}P\left(\frac{H^2}{K}\right) & H\le K\\ (H-K)B_{di}(H)+P(K) & H\ge K\end{cases}$	(75)
Down-and-out call	$c_{do}=\begin{cases}C(K)-\frac{K}{H}P\left(\frac{H^2}{K}\right) & H\le K\\ C(K)-(H-K)B_{di}(H)-P(K) & H\ge K\end{cases}$	(76)
Up-and-in call	$c_{ui}=\begin{cases}C(K) & H\le K\\ \frac{K}{H}C\left(\frac{H^2}{K}\right)+(H-K)B_{ui}(H) & H\ge K\end{cases}$	(77)
Up-and-out call	$c_{uo}=\begin{cases}0 & H\le K\\ C(K)-\frac{K}{H}C\left(\frac{H^2}{K}\right)-(H-K)B_{ui}(H) & H\ge K\end{cases}$	(78)
Down-and-in put	$p_{di}=\begin{cases}\frac{K}{H}P\left(\frac{H^2}{K}\right)+(K-H)B_{di}(H) & H\le K\\ P(K) & H\ge K\end{cases}$	(79)
Down-and-out put	$p_{do}=\begin{cases}P(K)-\frac{K}{H}P\left(\frac{H^2}{K}\right)-(K-H)B_{di}(H) & H\le K\\ 0 & H\ge K\end{cases}$	(80)
Up-and-in put	$p_{ui}=\begin{cases}(K-H)B_{ui}(H)+C(K) & H\le K\\ \frac{K}{H}C\left(\frac{H^2}{K}\right) & H\ge K\end{cases}$	(81)
Up-and-out put	$p_{uo}=\begin{cases}P(K)-(K-H)B_{ui}(H)-C(K) & H\le K\\ P(K)-\frac{K}{H}C\left(\frac{H^2}{K}\right) & H\ge K\end{cases}$	(82)

Table 2: Put-call symmetry hedging strategy for barrier options with zero rebate.

As this technique uses options to build a replication portfolio instead of dealing directly with the underlying asset, the portfolio is also somewhat protected against moves in the implied volatility of the options used in the portfolio. As long as the volatility at two levels with the same geometric mean is the same, the portfolio will always match the barrier option. This means that positive and negative shifts in the skew and in the term structure of volatility do not impact the replication portfolio value at the barrier.

3.3 Boundary Replication

In Derman (1994) a technique of static hedging using standard options as building blocks is explained. Given a particular target option, the authors are able to construct a portfolio of vanilla options with fixed weights that will replicate the target option value for a range of future times and market levels without further adjustments and within the Black-Scholes-Merton framework.

In the model, an option can be hedged with a position in the underlying asset and its theoretical value is the discounted expected payoff at the option maturity. This payoff is subject to the predefined boundary conditions of the option. If we are able to construct a portfolio of standard options that has the same value everywhere in the boundary and the same cash flows within the boundary, the model guarantees that the replication portfolio and the target option will have the same value everywhere inside the boundary.

The principle of static replication states that it is possible to replicate a target option for all future underlying asset prices and times within some boundary by constructing a portfolio of standard options with the same net cash flows within this boundary and the same values on the boundary.

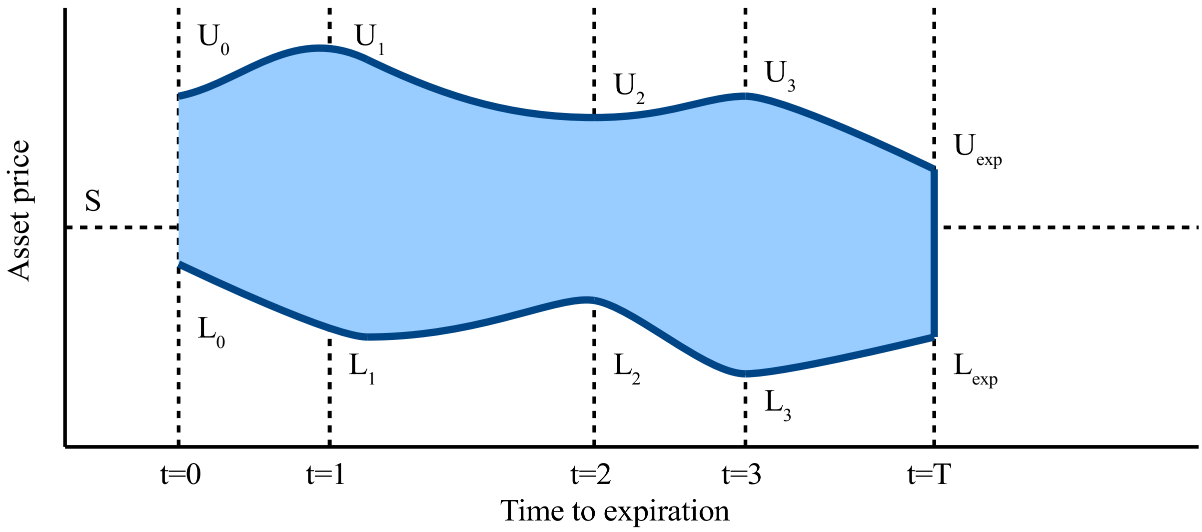

Figure 6: Boundary conditions of a general option.

To illustrate the procedure, Figure 6 shows an example of a target option with an upper boundary, a lower boundary and an expiration boundary for which we try to construct a replication portfolio with standard options. The first step is to replicate the expiration boundary *t*=*T* by matching it with a combination of options with different strikes that expire at this maturity. Moving back one time step to *t*=3, we compute the theoretical value at this time of the options used to match the expiration boundary condition in the upper and lower boundary, which will probably be different than the theoretical value of the target option. Then we choose a new combination of standard options that, when added to the existing replication portfolio, will match the theoretical value of the target option. For the upper boundary, we enter into a position on call options with expiration date on *t*=*T* and exercise price U_{exp} or more. The new calls will expire out-of-the-money below asset price U_{exp} and not alter the payoff already achieved for that time step. The same reasoning leads to a position on put options with maturity *t*=*T* and exercise price L_{exp} or lower.

Having achieved the desired payoff for time steps *t*=*T* and *t*=3, we move back to *t*=2 and add more call options with expiration *t*=3 and strikes above U_3 and put options with strike below L_3. And so on until we reach time step *t*=0.

The more points in time that we match the target option, the better replication we can achieve. If an infinite number of matching point were used then the replication would be perfect.

It is important to note that, if the asset price hits the boundary, the replication portfolio needs to be unwound and replaced with the security that produces the target option value on expiration.

We can use the method to replicate barrier options. Consider the following up-and-out call option:

Asset price – 100

Exercise price – 100

Barrier – 120

Time to expiration – 1 year

Risk-free interest rate – 5%

Asset yield – 3%

Volatility – 20%

Rebate – 0

Theoretical value – 1,11

The up-and-out call is a regular call option if the barrier is never breached until maturity. This is the expiration boundary condition and can be replicated by buying a standard call option with exercise price 100 and time to expiration 1 year.

Moving back 6 months the value of this option when the asset price is at the barrier is 21,29, much higher than the theoretical value of the up-and-out call at this boundary which is zero. The solution is to eliminate this value by selling 2,94235 vanilla call options with strike 120 and one year to maturity. The quantity 2,94235 is the number of call options with strike 120 that, when the asset price is 120 and the time remaining to maturity is 6 months are valued at 7,24 each, are needed to eliminate the portfolio value of 21,29 created by the call option with strike 100. The call options sold will not alter the portfolio value already achieved for the expiration boundary because they will expire out-of-the-money inside the boundary. The sum of the values of the call options sold and the call already in the replication portfolio is now zero when we are 6 months away from maturity.

At the inception of the up-and-out call, the previous replication portfolio has a negative value of -7,51 which is lower than the up-and-out call value of zero at the barrier. To achieve the theoretical value of zero we buy 1,03796 call options struck at 120 and time to maturity 6 months so that we do not alter the replication already achieved for the final 6 months of the option life. The value of such an option is 7,24 and thus 1,03796 options are needed to void the 7,51 portfolio value.

The complete replication portfolio is given in Table 3.

Quantity	**Option**	**Strike**	**Expiration**	**Value S=100**
1,00000	Call	100	1 year	8,65253
-2,94235	Call	120	1 year	-7,27248
1,03796	Call	120	6 months	0,84980
			Total	2,23

Table 3: Replication portfolio of the up-and-out call with matching every 6 months.

Notice that at inception the replication portfolio value is 2,23, much higher than the theoretical value of the up-and-out call option which is 1,11. Also, the portfolio suffers from substantial replication error as can be seen in Figure 7 that charts the difference between the replication portfolio value and the theoretical up-and-out call value along the barrier.

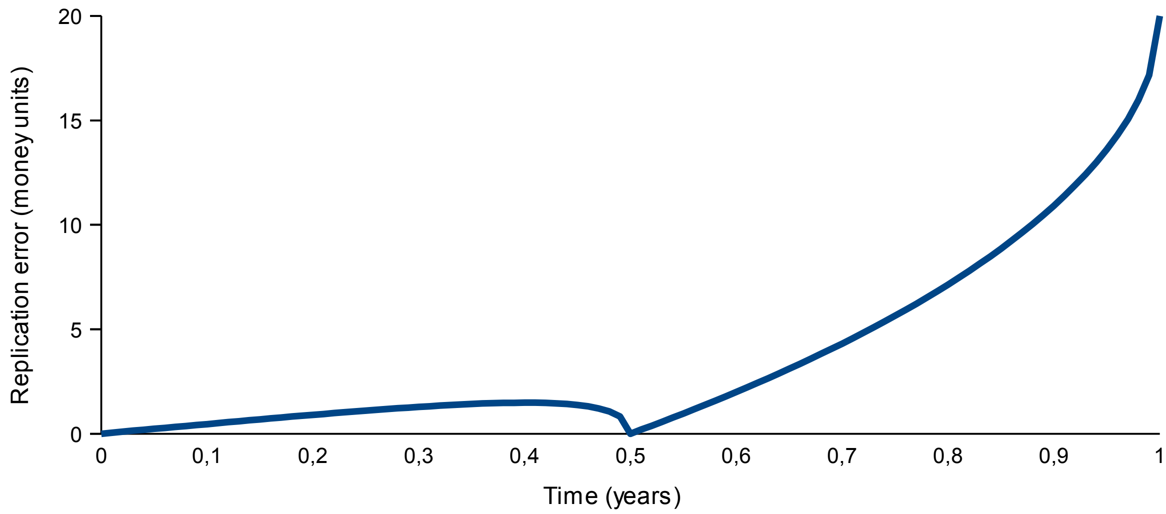

Figure 7: Up-and-out call replication error along the barrier, in money units, with matching every 6 months.

We can minimize the replication error by modelling the barrier boundary at more discrete times. Table 4 shows the replication portfolio when the boundary condition is modelled every two months. The replication portfolio value at inception is now 1,49, much closer to the 1,11 theoretical value, and the replication error is also minimized as can be seen in Figure 8.

Quantity	Option	Strike	Expiration	Value S=100
1,00000	Call	100	1 year	8,65253
-4,96280	Call	120	1 year	-12,26631
2,04107	Call	120	10 months	3,90234
0,65318	Call	120	8 months	0,88475
0,30681	Call	120	6 months	0,25119
0,17626	Call	120	4 months	0,06135
0,11444	Call	120	2 months	0,00508
			Total	1,49

Table 4: Replication portfolio of the up-and-out call with matching every 2 months.

If we model the barrier boundary every month, the difference between the replication portfolio value at inception and the theoretical up-and-out call value is even further minimized as shown in Table 5. At inception the value of the replication portfolio is 1,30, higher than the 1,11 theoretical value of the up-and-out call.

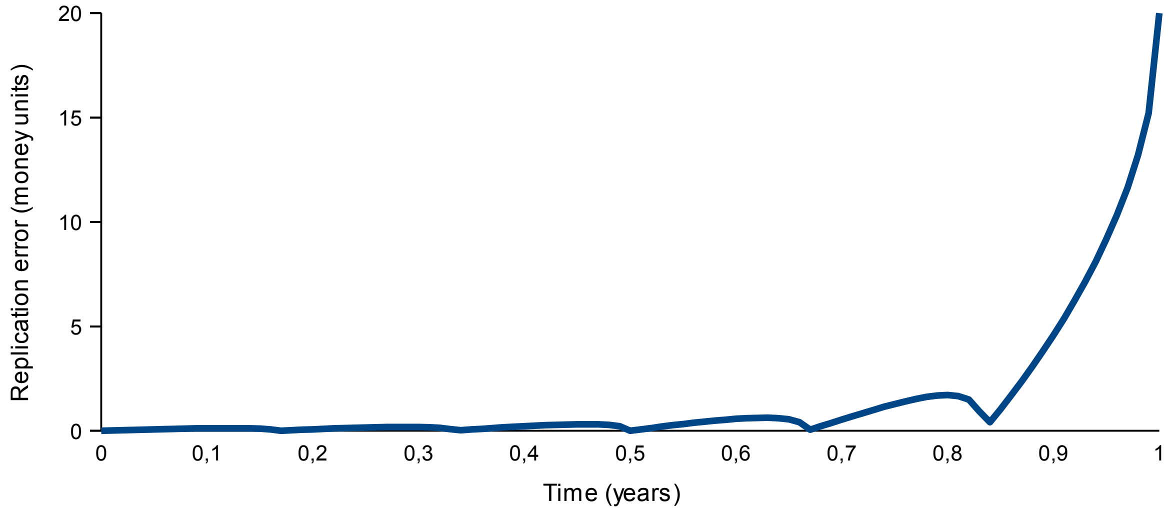

Figure 8: Up-and-out call replication error along the barrier, in money units, with matching every 2 months.

Quantity	Option	Strike	Expiration	Value S=100
1,00000	Call	100	1 year	8,65253
-7,04591	Call	120	1 year	-17,41505
2,97941	Call	120	11 months	6,53146
1,00695	Call	120	10 months	1,92520
0,49120	Call	120	9 months	0,80168
0,28843	Call	120	8 months	0,39069
0,18953	Call	120	7 months	0,20508
0,13425	Call	120	6 months	0,10991
0,10030	Call	120	5 months	0,05725
0,07797	Call	120	4 months	0,02714
0,06249	Call	120	3 months	0,01034
0,05131	Call	120	2 months	0,00228
0,04296	Call	120	1 month	0,00007
			Total	1,30

Table 5: Replication portfolio of the up-and-out call with matching every month.

Matching	**2 months**	**1 month**	**2 weeks**	**1 week**	**Theoretical**
Value	1,49	1,30	1,20	1,15	1,11

Table 6: Comparison of the replication portfolio value with different matching intervals with the theoretical up-and-out call value.

Table 6 shows how quickly the replication portfolio value approaches the theoretical option value as the boundary matching interval is minimized. Using 26 options to match the boundary the present value of the portfolio is 1,20. With 52 different options the value is 1,15. Figure 9 shows the replication portfolio error along the barrier boundary when the up-and-out call value is matched every month and every two weeks. It can be seen that the biggest problem is modelling the target barrier option close to expiration.

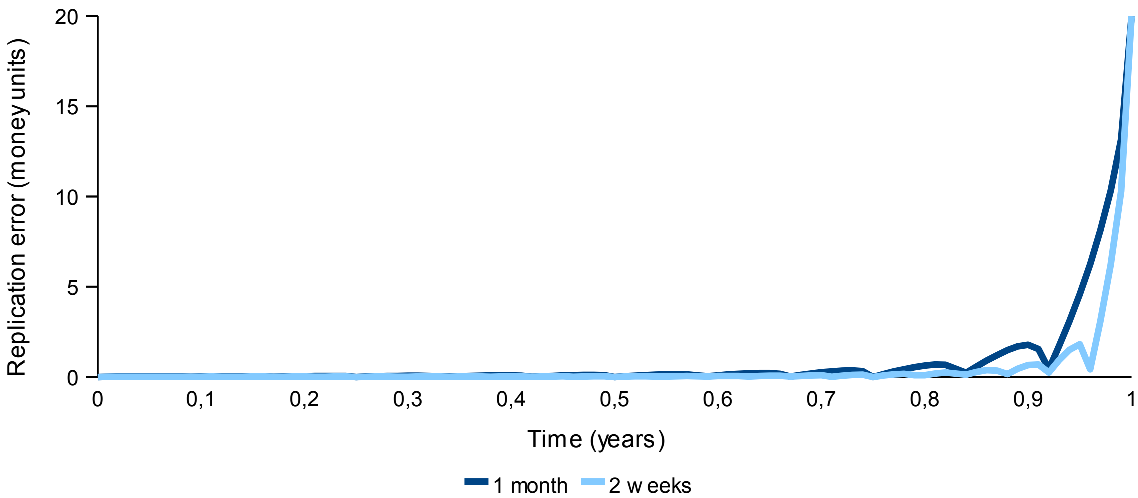

Figure 9: Up-and-out call replication error along the barrier, in money units, with matching every month and every two weeks.

If at any time the barrier is hit, the portfolio is unwound and the theoretical value of zero should be realized. In practice, the value of the replication error along the barrier, which in this case is always positive, will be realized resulting in a profit to the risk manager.

Now consider an up-and-in call option with the same characteristics as the up-and-out. The up-and-in option is worth nothing at the expiration if the barrier is never breached so the replication of the expiration boundary is any empty portfolio. Working back one month we have a theoretical value of zero for the replication portfolio because there are no options in it while we should have the theoretical value of 20,12 that corresponds to the value of a call option with strike 100 and maturity in 1 month. To match the boundary value we add call options with time to expiration 1 year and strike 120 until their value at that time is 20,12. We go back another month and repeat the process until we reach the initial time and have a complete replication portfolio.

We can also recall equation (19) which states that an up-and-in call is the same as buying a standard call option and selling an up-and-out call to see that the replication portfolio shown in Table 7 is just the inverse position in the options of Table 5 without buying the call with strike 100 used to match the expiration boundary .

Quantity	Option	Strike	Expiration	Value S=100
7,04591	Call	120	1 year	17,41505
-2,97941	Call	120	11 months	-6,53146
-1,00695	Call	120	10 months	-1,92520
-0,49120	Call	120	9 months	-0,80168
-0,28843	Call	120	8 months	-0,39069
-0,18953	Call	120	7 months	-0,20508
-0,13425	Call	120	6 months	-0,10991
-0,10030	Call	120	5 months	-0,05725
-0,07797	Call	120	4 months	-0,02714
-0,06249	Call	120	3 months	-0,01034
-0,05131	Call	120	2 months	0,00228
-0,04296	Call	120	1 month	-0,00007
			Total	7,35

Table 7: Replication portfolio of the up-and-in call with matching every month.

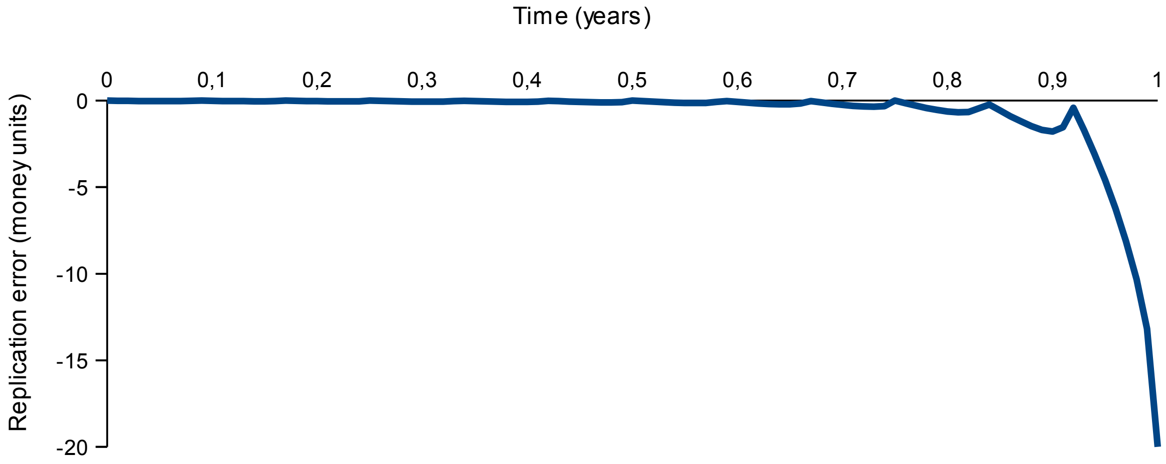

Figure 10: Up-and-in call replication error along the barrier, in money units, with modelling every month.

If the barrier is hit at any time until expiration the portfolio is unwound and the theoretical value needed to buy the call option with strike 100 is realized. In practice, the value realized is subject to the replication error which, in this case, is negative as shown in Figure 10. This means that the replication portfolio may not provide sufficient funds to buy the emerging option in case the barrier is breached.

Matching	2 months	1 month	2 weeks	1 week	Theoretical
Value	7,16	7,35	7,46	7,5	7,55

Table 8: Comparison of the replication portfolio value with different matching intervals with the theoretical up-and-in call value.

Table 8 show the convergence of the replication portfolio to the theoretical option price as the number of matching times increases.

Now consider the following down-and-out call option:

Asset price – 100

Exercise price – 100

Barrier – 90

Time to expiration – 1 year

Risk-free interest rate – 5%

Asset yield – 3%

Volatility – 20%

Rebate – 0

Theoretical value – 7,08

The down-and-out call option is a regular option while the barrier is not breached so the expiration boundary is a vanilla call option with exercise price 100 and expiration in one year. The difference in the replication portfolio relative to the up-and-out call is the presence of a lower boundary which will force us to use put options with exercise price 90 or lower to match the down-and-out on the barrier. An example of a replication portfolio is shown on Table 9. The theoretical value of the portfolio at inception is 7,07, only 0,01 away from the theoretical value of the theoretical value of the down-and-out call. If the barrier is hit the portfolio is unwound for a theoretical value of zero.

Using equation (18) we can quickly find the replication portfolio for the down-and-in call by reversing the positions in the put options of Table 9 and forgetting about the call option because the expiration boundary is zero if the barrier is never touched. If, at any time until expiration, the barrier is indeed breached then the hedger ought to unwind the replication portfolio and use the realized value to buy a call option with strike 100 and expiration on the maturity of the down-and-in call.

Quantity	Option	Strike	Expiration	Value S=100
1,00000	Call	100	1 year	8,65253
-0,03950	Put	90	1 year	-0,11594
-0,14064	Put	90	11 months	-0,38427
-0,12851	Put	90	10 months	-0,32366
-0,10195	Put	90	9 months	-0,23380
-0,08191	Put	90	8 months	-0,16832
-0,06775	Put	90	7 months	-0,12214
-0,05758	Put	90	6 months	-0,08840
-0,05005	Put	90	5 months	-0,06264
-0,04430	Put	90	4 months	-0,04217
-0,03979	Put	90	3 months	-0,02544
-0,03618	Put	90	2 months	-0,01185
-0,03323	Put	90	1 month	-0,00226
			Total	7,07

Table 9: Replication portfolio for the down-and-out call option with matching every month.

For barrier puts the reasoning is similar. The down-and-out put replication portfolio is constructed by a put option to model the expiration boundary and put options with exercise prices equal or lower than the barrier and different maturities to match the payoff at the barrier. The down-and-in is, according to equation (20), just the opposite of the down-and-out portfolio without the put option corresponding to the expiration boundary. The up-and-out put barrier option uses a standard put to capture the expiration boundary payoff and, as it has an upper barrier, positions in call options with exercise prices equal to or higher then the barrier. Finally, according to equation (21), the replication portfolio for the up-and-in put option is constructed with the reverse positions of the up-and-out portfolio without the put option corresponding to the expiration boundary.

As in all barrier options, if the barrier is hit the replication portfolio is unwound with a theoretical realization of zero for the knock-out barrier options and the value needed to buy the put option in case of a knock-in barrier option. In practice the risk manager will have a profit when replicating a knock-out and a loss when replicating a knock-in option.

In the example given the options had zero rebate. When the barrier option pays a rebate the hedging strategy does not change and one just needs to match the value of the rebate at the boundary.

Recall from Table 6 that the replication portfolio value for the up-and-out call option when the barrier boundary is matched every month is 1,30. If the barrier is not breached then the call option that models the expiration boundary condition can be sold at maturity to pay the buyer of the option and the risk manager will have a final P&L of zero. On the other hand, if the barrier is breached, the final P&L is given by the replication error at the barrier boundary as seen in Figure 9. The expected value of the replication portfolio P&L when running multiple asset price paths simulated by geometric Brownian motion is equal to 0,19, exactly the difference between the replication portfolio value and the up-and-out call theoretical value at inception. The conclusion is that if the risk manager had sold the up-and-out call option for its theoretical value of 1,11 and bought the replication portfolio of Table 5 that costs 1,30 to set up by borrowing the remaining funds, he should still expect to average a P&L of zero. This is valid for every knock-out option.

The same simulation can be made with the up-and-in replication portfolio of Table 7 that costs 7,35 to set up while receiving 7,55 from the theoretical option value. The 0,20 difference is exactly the expected loss from running multiple geometric Brownian motion paths that simulate the asset price evolution through time. The conclusion is similar to the previous: the risk manager will have an average P&L of zero despite not spending all the funds received from the buyer of the knock-in barrier option.

4 Experimental Results

In this chapter the frameworks developed to hedge barrier options are tested. In the first section they are subject to the geometric Brownian motion assumptions where the models were deduced, followed by results experienced in real market conditions with time series data of the S&P 500 index daily closing values and the corresponding exchange traded options settlement prices.

4.1 Geometric Brownian Motion

The geometric Brownian motion procedure simulates the price path of an asset by sampling a normal distribution. It is assumed that each sample is the logarithmic change of the asset price in a certain day. Applying the many price changes to an initial asset price results in a time series.

Call	**Down-and-in call**	**Down-and-out call**	**Up-and-in call**	**Up-and-out call**
4,20	0,06	4,14	3,56	0,64
Put	**Down-and-in put**	**Down-and-out put**	**Up-and-in put**	**Up-and-out put**
3,71	2,89	0,82	0,10	3,60

Table 10: Vanilla and barrier option theoretical prices.

Consider both standard and barrier options with the following characteristics whose theoretical values are shown in Table 10:

Asset price – 100

Exercise price – 100

Time to maturity – 90 days

Risk-free interest rate – 5%

Asset yield – 3%

Volatility – 20%

Up barrier – 110

Down barrier – 90

Rebate – 0

4.1.1 Delta Hedging

Working on the Black-Scholes-Merton model assumptions, we run 1000 simulations of the delta hedging procedure for each of the ten options with portfolio rebalancing performed once a day. Table 11 summarises the statistical results expressed as a percentage of the option premium and Table 12 as a percentage of the asset price.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Call	-0,48%	10,42%	0,43	0,71	-36,91%	36,93%
Put	-0,21%	4,75%	0,11	3,2	-24,93%	22,80%
Up-and-in call	0,55%	22,38%	-0,13	43,57	-286,47%	199,88%
Up-and-out call	-2,37%	94,80%	0,07	22,36	-836,28%	935,75%
Down-and-in call	9,36%	140,31%	14,47	361,04	-960,17%	18,09%
Down-and-out call	-0,04%	4,49%	0,42	2,71	-24,24%	20,83%
Up-and-in put	-1,35%	76,81%	-2,72	58,42	-965,03%	724,55%
Up-and-out put	-0,50%	4,71%	-0,31	4,24	-31,90%	18,81%
Down-and-in put	0,70%	24,41%	5,68	65,9	-135,77%	345,22%
Down-and-out put	-1,00%	67,15%	-1,75	20,57	-566,07%	431,61%

Table 11: Delta hedging P&L distribution as a percentage of the option premium for the geometric Brownian motion.

As expected the average P&L of both standard and barrier options is around zero. The interesting point is the standard deviation and the kurtosis of the distribution of hedging errors which are much higher for barrier options than for standard options. This is explained by the higher values of gamma for barrier options and because hedging is not carried out continuously, resulting in higher profits and losses that are evenly distributed around zero.

In this particular example, the large hedging errors of Table 11 as measured by the standard deviation for both the down-and-in call and the up-and-in put are explained by the fact that the premium of each of this options is very low and a small hedging error represents a large percentage of this small premium. Taking a look at Table 12 where the hedging error is expressed as a percentage of the asset price we see that this two options do not show a great hedging risk because the standard deviation of the error is low.

	Average	**Standard Deviation**	**Skew**	**Kurtosis**	**Min**	**Max**
Call	0,00%	0,17%	0,68	1,52	-0,59%	0,74%
Put	-0,01%	0,18%	0,11	3,35	-1,00%	0,87%
Up-and-in call	0,02%	0,80%	-0,13	43,57	-10,19%	7,11%
Up-and-out call	-0,02%	0,61%	0,07	22,36	-5,38%	6,01%
Down-and-in call	0,00%	0,06%	-1,51	38,5	-0,60%	0,64%
Down-and-out call	0,00%	0,19%	0,4	2,81	-0,98%	0,92%
Up-and-in put	0,00%	0,07%	-1,7	26,29	-0,67%	0,51%
Up-and-out put	-0,02%	0,17%	-0,4	4,62	-1,19%	0,70%
Down-and-in put	0,02%	0,69%	5,45	64,51	-4,12%	9,65%
Down-and-out put	-0,01%	0,54%	-1,25	19,83	-4,08%	3,76%

Table 12: Delta hedging P&L distribution as a percentage of the asset price for the geometric Brownian motion.

The down call options and the up put options can be considered to be the least riskier to hedge because when the barrier is hit the emerging or vanishing option is out-of-the money and only has residual delta that changes very slowly.

The riskier barrier options to hedge are the up call options and the down put options as can be observed in both tables. This options are in-the-money when the barrier is touched and, consequently, can have very high absolute values of gamma. The result is a P&L distribution that has fat tails as explained by the high value of kurtosis and the minimum and maximum error. In this cases there are a lot of occurrences where a large profit or loss is experienced from the hedging procedure. They can be especially risky when the asset price is close to the barrier and maturity approaches.

The risk manager should keep in mind that while delta hedging the down calls and up puts does not seem to represent a greater problem than delta hedging vanilla options, one should be very careful when trying to use the procedure with up calls and down puts.

4.1.2 Put-Call Symmetry

Under the assumptions of geometric Brownian motion and enforcing the drift rate of the asset to be zero, the put-call symmetry method guarantees that the replication is perfect. In this example, the drift rate of the underlying asset is positive because the risk-free rate is higher than the asset yield. The result of 5000 simulations are hedging errors as shown in Table 13 expressed in amount, in Table 14 expressed as a percentage of the option premium and in Table 15 expressed as a percentage of the asset price.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Up-and-in call	0,02	0,03	0,84	-1,23	0	0,08
Up-and-out call	-0,02	0,03	-0,92	-1,08	-0,08	0
Down-and-in call	-0,01	0,02	-2,59	5,83	-0,11	0
Down-and-out call	0,01	0,02	2,41	4,95	0	0,11
Up-and-in put	0,02	0,03	2,09	3,31	0	0,15
Up-and-out put	-0,02	0,03	-2,1	3,29	-0,15	0
Down-and-in put	-0,02	0,03	-1,19	-0,52	-0,09	0
Down-and-out put	0,02	0,03	1,25	-0,4	0	0,09

Table 13: Put-call symmetry P&L distribution in amount for the geometric Brownian motion.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Up-and-in call	0,59%	0,87%	0,84	-1,23	0,00%	2,16%
Up-and-out call	-3,06%	4,69%	-0,92	-1,08	-11,98%	0,00%
Down-and-in call	-13,39%	32,50%	-2,59	5,83	-162,23%	0,00%
Down-and-out call	0,24%	0,53%	2,41	4,95	0,00%	2,67%
Up-and-in put	15,38%	31,44%	2,09	3,31	0,00%	148,17%
Up-and-out put	-0,44%	0,93%	-2,1	3,29	-4,27%	0,00%
Down-and-in put	-0,66%	1,14%	-1,19	-0,52	-2,94%	0,00%
Down-and-out put	2,27%	4,05%	1,25	-0,4	0,00%	10,43%

Table 14: Put-call symmetry P&L distribution as a percentage of the option premium for the geometric Brownian motion.

It is interesting to note that hedging errors only arise in case the barrier is touched. On the other cases the hedging portfolio exactly matches the final payoff of the target barrier option. This hedging errors result from the in bond whose value at the barrier is only one at maturity. Other interesting observation is that the average hedging error of the up-and-in call expressed in money units, shown in Table 13, is exactly the opposite of the average hedging error of the up-and-out call. This happens for all the other option pairs as expected by the relationship between knock-in, knock-out and vanilla options. If the drift rate were negative, the average hedging error would be the opposite sign for each of the barrier options.

The risk manager could adjust the price at which he sells or buys the barrier option by discounting the expected hedging error of the put-call symmetry strategy of Table 13, borrowing or lending the remaining funds necessary to build the replication portfolio at inception because, on average, he would receive or pay those funds at the end of the hedging strategy thus eliminating the hedging error.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Up-and-in call	0,02%	0,03%	0,84	-1,23	0,00%	0,08%
Up-and-out call	-0,02%	0,03%	-0,92	-1,08	-0,08%	0,00%
Down-and-in call	-0,01%	0,02%	-2,59	5,83	-0,11%	0,00%
Down-and-out call	0,01%	0,02%	2,41	4,95	0,00%	0,11%
Up-and-in put	0,02%	0,03%	2,09	3,31	0,00%	0,15%
Up-and-out put	-0,02%	0,03%	-2,1	3,29	-0,15%	0,00%
Down-and-in put	-0,02%	0,03%	-1,19	-0,52	-0,09%	0,00%
Down-and-out put	0,02%	0,03%	1,25	-0,4	0,00%	0,09%

Table 15: Put-call symmetry P&L distribution as a percentage of the asset price in the geometric Brownian motion environment.

4.1.3 Boundary Replication

To test the boundary replication method in the geometric Brownian motion environment we build a replication portfolio with boundary matching interval of one month and simulate 5000 paths for the underlying asset price. The theoretical value of the replication portfolio at inception and the difference to the theoretical barrier option value is shown in Table 16.

	Theoretical value	Portfolio value	Difference
Up-and-in call	3,17	3,56	-0,39
Up-and-out call	1,03	0,64	0,39
Down-and-in call	0,08	0,06	0,02
Down-and-out call	4,12	4,14	-0,02
Up-and-in put	0,12	0,1	0,02
Up-and-out put	3,58	3,6	-0,02
Down-and-in put	2,52	2,89	-0,37
Down-and-out put	1,18	0,82	0,36

Table 16: Barrier option replication portfolio value at inception.

Table 17 shows the results in money units of the geometric Brownian motion simulation with the replication portfolios created. The first important observation is that the average P&L of the hedging strategy matches the difference between the replication portfolio and the theoretical option value at inception when properly discounted to the barrier hitting time or the option expiration.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Up-and-in call	-0,38	1,16	-4,68	25,24	-10	0
Up-and-out call	0,40	1,23	4,55	22,92	0	10
Down-and-in call	0,01	0,03	1,63	1,12	0	0,09
Down-and-out call	-0,01	0,03	-1,68	1,35	-0,09	0
Up-and-in put	0,02	0,03	1,28	0,05	0	0,09
Up-and-out put	-0,02	0,03	-1,28	0,05	-0,09	0
Down-and-in put	-0,38	1,24	-4,79	25,49	-10	0
Down-and-out put	0,36	1,15	4,74	25,21	0	10

Table 17: Boundary matching P&L distribution in amount in the geometric Brownian motion environment.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Up-and-in call	-10,70%	32,54%	-4,68	25,24	-281,08%	0,00%
Up-and-out call	62,75%	191,86%	4,55	22,92	0,00%	1555,70%
Down-and-in call	22,06%	41,22%	1,63	1,12	0,00%	138,51%
Down-and-out call	-0,33%	0,63%	-1,68	1,35	-2,17%	0,00%
Up-and-in put	15,66%	25,96%	1,28	0,05	0,00%	83,00%
Up-and-out put	-0,45%	0,75%	-1,28	0,05	-2,39%	0,00%
Down-and-in put	-13,10%	43,07%	-4,79	25,49	-346,07%	0,00%
Down-and-out put	44,30%	141,31%	4,74	25,21	0,00%	1225,63%

Table 18: Boundary matching P&L distribution as a percentage of the option premium in the geometric Brownian motion environment.

The results suggest that one can set up the replication portfolio using the theoretical option premium received and borrow or lend at the risk-free rate the remaining funds of the portfolio construction. The funds that are in excess will cover the expected loss of the replication strategy and the funds that are borrowed will be returned with the expected gain from the strategy. Taking into account this effect the hedging errors of Table 18 and Table 19 would be around zero.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Up-and-in call	-0,38%	1,16%	-4,68	25,24	-10,00%	0,00%
Up-and-out call	0,40%	1,23%	4,55	22,92	0,00%	10,00%
Down-and-in call	0,01%	0,03%	1,63	1,12	0,00%	0,09%
Down-and-out call	-0,01%	0,03%	-1,68	1,35	-0,09%	0,00%
Up-and-in put	0,02%	0,03%	1,28	0,05	0,00%	0,09%
Up-and-out put	-0,02%	0,03%	-1,28	0,05	-0,09%	0,00%
Down-and-in put	-0,38%	1,24%	-4,79	25,49	-10,00%	0,00%
Down-and-out put	0,36%	1,15%	4,74	25,21	0,00%	10,00%

Table 19: Boundary matching P&L distribution as a percentage of the asset price in the geometric Brownian motion environment.

4.2 Market Data

The models are evaluated with real market data using time series of the daily returns of the S&P 500 index from 1990 to 2009. The S&P 500 is a market free float weighted stock index composed by 500 large caps trading on the United States of America. It is one of the most followed stock market indexes.

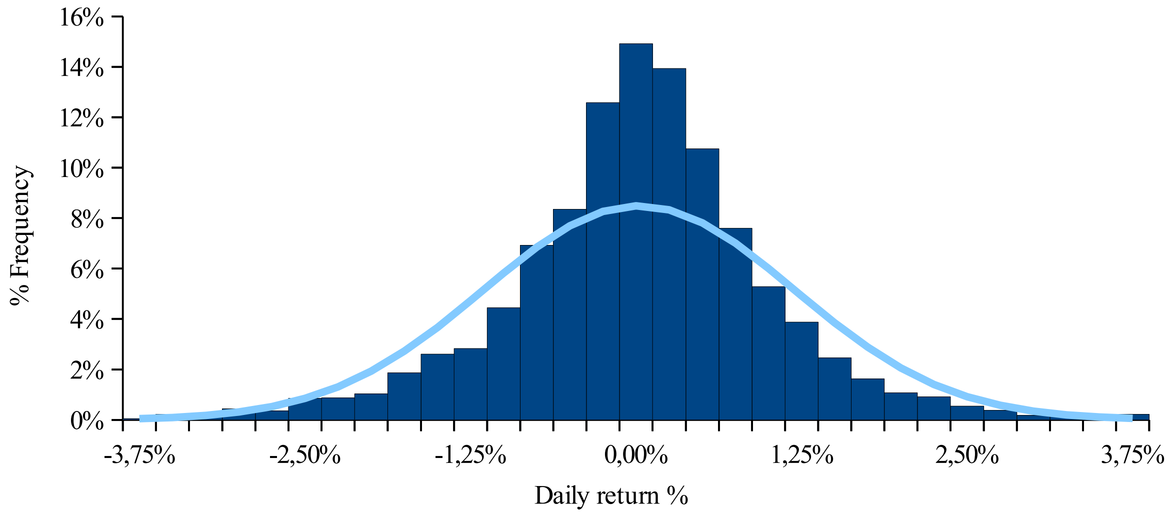

Figure 11: Distribution of the S&P 500 daily returns in the range of -3.75% to +3.75%.

We know that one of the assumptions of the Black-Scholes-Merton model is that the asset price follows a geometric Brownian motion which means that the returns are normally distributed. The S&P 500 time series data shows that the daily returns are also normally distributed but slightly skewed and with a leptokurtic profile as can be seen in Figure 11.

Key statistics for the time series are shown in Table 20. The return distribution has a slightly positive average of 0,02%, standard deviation of 1,17% and is negatively skewed which means that there are more days where the index falls than days where the index rises. The biggest difference relative to the geometric Brownian motion assumptions is the presence of fat tails as shown by the very large kurtosis. Most of the returns are concentrated around the average, more than in the standard normal distribution, but there are many days when there is a big drop or a big rise in the index. This can also be observed by noticing that the minimum and maximum values of the distribution are almost ten standard deviations.

Average	Standard Deviation	Skew	Kurtosis	Maximum	Minimum
0,02%	1,17%	-0,2	9,22	-9,47%	10,96%

Table 20: Statistics for the S&P 500 time series.

Option prices of the S&P 500 show that, contrary to the geometric Brownian motion assumption of constant volatility, the volatility surface of the index is not static.

In the following section we investigate the impact of this return distribution and stochastic volatility on the models developed so far.

4.2.1 Delta Hedging

The delta hedging procedure is run on the S&P 500 index closing value time series for both vanilla and barrier options with the following characteristics:

Strike – At-the-money

Time to maturity – 90 days

Risk-free interest rate – continuously compounded overnight risk-free interest rate

Asset yield – continuously compounded daily dividend yield

Volatility – future realized volatility

Up barrier – Strike x (1 + 50% Volatility)

Down barrier – Strike x (1 - 50% Volatility)

Rebate – 0

The delta rebalancing is performed at the end of each day at the closing value of the index. The procedure is run each five days yielding approximately 1000 iterations. The options are born at-the-money and it is assumed that future volatility is correctly estimated. If this were not the case the risk manager would face gains or losses resulting from selling volatility above or below the realized volatility.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Call	0,81%	8,21%	-0,26	3,08	-48,08%	29,05%
Put	0,90%	8,97%	-0,51	4,03	-56,94%	32,44%
Up-and-in call	9,21%	29,78%	2,27	30,64	-284,77%	269,08%
Up-and-out call	-44,30%	161,69%	-2,88	29,85	-1550,06%	1403,53%
Down-and-in call	7,97%	227,53%	7,07	61,93	-362,77%	2483,12%
Down-and-out call	0,81%	8,12%	-0,41	3,56	-49,20%	31,41%
Up-and-in put	13,47%	78,12%	-6,67	95,63	-1177,89%	387,31%
Up-and-out put	0,65%	9,33%	-0,67	4,55	-60,38%	33,79%
Down-and-in put	-9,48%	39,29%	-5,81	49,04	-477,34%	104,08%
Down-and-out put	38,14%	143,13%	5,48	43,39	-274,95%	1683,36%

Table 21: Delta hedging P&L distribution as a percentage of the option premium for the S&P 500.

	Average	Standard Deviation	Skew	Kurtosis	Min	Max
Call	0,07%	0,43%	0,88	7,98	-2,96%	2,43%
Put	0,08%	0,43%	0,89	7,89	-2,95%	2,42%
Up-and-in call	0,36%	1,04%	4,17	29,61	-4,14%	11,32%
Up-and-out call	-0,28%	1,02%	-4,63	35,38	-11,64%	4,36%
Down-and-in call	0,01%	0,18%	7,68	71,98	-0,30%	2,18%
Down-and-out call	0,07%	0,41%	0,62	8,47	-2,99%	2,32%
Up-and-in put	0,02%	0,07%	-1,5	37,27	-0,92%	0,47%
Up-and-out put	0,07%	0,42%	0,52	7,97	-3,03%	2,21%
Down-and-in put	-0,27%	1,54%	-4,45	57,17	-21,32%	10,70%
Down-and-out put	0,36%	1,66%	4,63	54,91	-11,19%	22,84%

Table 22: Delta hedging P&L distribution as a percentage of the index value for the S&P 500.

Table 21 summarises the results of the hedging procedure as a percentage of the option premium and Table 22 as a percentage of the index value at the inception of the option.

The first thing that comes to attention is that the standard deviation and the kurtosis of the hedging error distribution era greater than in the geometric Brownian motion environment. This is explained by the leptokurtic distribution of returns of the index that causes many days of big profits or losses due to the difference in delta and portfolio rebalancing.

The difficulty of hedging the up calls and down puts is especially noticed in the S&P 500 index data. The standard deviation of the hedging error distribution is high and the presence of fat tails is especially noticed but the most important observation is that the average hedging error for this options type of barrier options is not zero because of the different return distribution profile.

4.2.2 Put-Call Symmetry

In real market conditions volatility is not constant as in the geometric Brownian motion environment. Market prices for options will change as traders adjust their expectations for future volatility, asset returns and interest rates. The put-call symmetry method has the benefit that the replication portfolio can be decomposed into a portfolio of options with different strikes. One does not need to trade the underlying asset and thus there is no need to estimate future volatility. Nevertheless, the volatility surface may exhibit a smile or skew that will change over time. While the strategy can accommodate shifts in volatility and in the skew as long as the volatility is the same for two points with the same geometric mean, in practice this is seldom the case and results in substantial hedging errors.

Imagine a possible situation where the S&P 500 is trading at 1000 with a slight negative skew and the risk manager sells a down-and-out put with strike 1000 and barrier 900. The value of the portfolio is shown in Table 23.

The option knocks-out 29 days later and because the market is falling the skew has become more negative. The value of the replication portfolio should be zero at the barrier but due to the different skew has now a negative value and the risk manger realizes a loss. The situation is shown in Table 24.

We have seen before that the zero drift assumption can be relaxed and properly priced. This example show that while the risk manager is somewhat protected from volatility changes by using only one maturity in the replication portfolio he can still run into problems with the put-call strategy if the skew changes in an asymmetric way as is often the case in real markets.

Option	Strike	Maturity	Implied Volatility	Quantity	Total Value
Put	1000	90 days	20,00%	1	5,05
Put	810	90 days	21,00%	1,11	-0,44
Down-and-in Bond	900	90 days	20,50%	10	-4,4
				Total	0,21

Table 23: Replication portfolio at inception.

Option	Strike	Maturity	Implied Volatility	Quantity	Total Value
Put	1000	61 days	25,00%	1	11,21
Put	810	61 days	27,00%	1,11	-2
Down-and-in Bond	900	61 days	24,00%	10	-9,49
				Total	-0,28

Table 24: Replication portfolio at knock-out after skew change.

4.2.3 Boundary Replication

The boundary replication method also has the benefit of using a portfolio of options to replicate the target barrier option thus eliminating the need to estimate volatility. Despite this characteristic, the model cannot adapt itself to the dynamics of real markets because one often takes the opposite position in volatility in the replication portfolio and the volatility surface changes over time.

Consider the following example of real market data where the risk manager sells a down-and-out put option:

Date – 10/12/2008

Closing value of the S&P 500 – 899,24

Exercise price – 900

Barrier – 750

Maturity date – 19/03/2009

The replication portfolio is built with exchange traded options and is shown in Table 25. A quick inspection into the total value of the portfolio shows a negative value. The reason is that we are pricing the option with a negative skew, buying the 900 strike at a lower volatility than we are selling the 750 strike.

Option	Strike	Maturity Date	Value	Implied Volatility	Quantity	Total Value
Put	900	19-03-2009	95	52,30%	1,00000	95,00
Put	750	19-03-2009	45	61,26%	-3,15036	-141,77
Put	750	19-02-2009	32,2	60,99%	1,13250	36,47
Put	750	15-01-2009	15,5	62,70%	0,22789	3,53
Put	750	18-12-2008	0,8	63,36%	0,03036	0,02
					Total	-6,74

Table 25: Replication portfolio at inception.

On 23/02/2009 the barrier is breached and the index closes at 743,33. The portfolio value at this date is shown in Table 26.

Option	Strike	Maturity Date	Value	Implied Volatility	Quantity	Total Value
Put	900	19-03-09	159,2	52,51%	1,00000	159,2
Put	750	19-03-09	38	45,78%	-3,15036	-119,71
Put	750	19-02-09	Expired	-	1,13250	0,00
Put	750	15-01-09	Expired	-	0,22789	0,00
Put	750	18-12-08	Expired	-	0,03036	0,00
					Total	39,49

Table 26: Replication portfolio value when the barrier is breached.

On this date the portfolio has a positive value of 39,49, higher than the expected value of zero for a knock-out option at the barrier. One could think this was due to the date 23/02/2009 not being a matching date and thus subject to hedging error but the date is close to 19/02/2009 where the boundary matching was performed. The explanation is simply that we are at that time short volatility at the 750 strike because we sold 3,15036 put options. As the implied volatility in this option fell from 61,26% to 45,78%, the risk manager realizes a gain because, near the barrier, the down-and-out put option is short vega. If volatility had risen the portfolio at the barrier would be negative instead of zero and the risk manager would realize a loss.

This example shows that the method has limited applicability in real market conditions because of the effects of stochastic volatility.

5 Conclusions

Barrier options can be split in two different categories according to their risk profile. Down calls and up puts are as easy to delta hedge as vanilla options because they knock out-of-the money and with residual delta. The risk manager should expect to have an average P&L of zero when delta hedging many transactions on this options under geometric Brownian motion and in real market conditions despite the slightly different return distribution of the S&P 500 index. The only condition imposed is the ability to correctly estimate future volatility. On the other side we have the risky up calls and the down puts that knock in-the-money. They exhibit high delta and gamma values near the barrier resulting in the highest hedging errors. The risk manager should not attempt to delta hedge this barrier options.

Static hedging techniques are proposed to enable less hedging error for barrier options. While this methods show promising results in the geometric Brownian motion assumptions where they were developed we show that they are prone to hedging errors in real market condition where volatility is stochastic.

The put-call symmetry strategy will degenerate in hedging errors when the drift rate of the asset is not zero but one can still properly price the barrier options with this factor in mind. One improvement that could be arranged is substituting the in bonds with American binary options so that the value at the barrier is guaranteed to be one. The biggest problem comes when the volatility surface changes in manners different than parallel shifts. Nevertheless, and when taken into account all this factors, seems to be a better hedging technique as it provides small hedging errors and some vega protection.

Boundary replication techniques prove their theoretical value but run into problems when the volatility surface changes and leads to situations where the replication portfolio value at the barrier is not what was expected.

Extending this techniques to other types of barrier options should be easy. One possibility it to combine the developed techniques with concepts such as gamma hedging and vega hedging. Further research into the hedging of American binary options should provide interesting results as barrier options can be almost split into plain vanillas and American binaries. These are topics that are left for future work.

6 Bibliography

Bates, D. (1988), *The Crash Premium: Option Pricing under asymmetric processes, with applications to options on the Deutschemark futures*.

Black, F. and M. Scholes (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637–654.

Carr, P, and J. Bowie (1994), Static Simplicity, *Risk* August 1994, 45-49.

Carr, P, and A. Chou (1997), Breaking Barriers, *Risk* September 1997, 139-145.

Carr, P., K. Ellis and V. Gupta (1998), Static Hedging of Exotic Options, *The Journal of Finance* Volume LIII No. 3, 1165-1190.

Derman, E., D. Ergener and I. Kani (1994), Static Options Replication, *Goldman Sachs Quantitative Strategies Research Notes* May 1994.

Derman, E and I. Kani (1996), The Ins and Outs of Barrier Options: Part 1, *Derivatives Quarterly* Winter 1996, 55-67.

Derman, E and I. Kani (1997), The Ins and Outs of Barrier Options: Part 2, *Derivatives Quarterly* Spring 1997, 73-80.

Harrison, J.M. (1985), *Brownian Motion and Stochastic Flow Systems*, Wiley.

Hull, J. C. (2008), *Options, Futures and other Derivatives*, 7th Edition, Prentice Hall.

Kamal, M. (1999), When You Cannot Hedge Continuously: The Corrections of Black-Scholes, *Risk* January 1999, 82-85.

Merton, R. C. (1973), Theory of Rational Option Pricing, *Bell Journal of Economics and Management Science* 4, 141–183.

Nunes, J. P. V. (2003), *Lecture Notes: Complementos de Opções*, ISCTE.

Rubinstein, M. and E. Reiner (1991a), Breaking Down the Barriers, *Risk* September 1991, 28-35.

Rubinstein, M. and E. Reiner (1991b), Unscrambling the binary code, *Risk* October 1991, 75-83.

Taleb, N. (1997), *Dynamic Hedging: Managing Vanilla and Exotic Options*, Wiley.

Zhang, P. G. (1998), *Exotic Options: Guide to Second Generation Options*, 2nd Edition, World Scientific Publishing.