ISCTE Business School
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Portfolio Optimization Methods, Their Application and Evaluation

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Abstract

The submitted master's thesis focuses on practical application of quantitative portfolio optimization in various forms. The thesis is organized in two main parts, theoretical and practical.

The theoretical part introduces the underpinnings of portfolio theory. It describes the optimization process, introduces a number of selected optimization methods, and provides an overview of portfolio management. As a whole, it serves as an underlying for the practical part.

The practical part of the thesis is based on an experiment that put multiple quantitative portfolio optimization methods into a contest. Different optimizers were applied to portfolios composed of identical assets, which were subsequently held under different portfolio management styles over a pre-specified period of time. The performance of each portfolio was measured ex-post, adequately evaluated in accord with the criteria of the experiment, and confronted with the others.

The questions that this master's thesis tried to find answers to were (1) which portfolio optimizer, out of the selected ones, performs the best, and (2) whether it is beneficial to conduct rather an active, or a passive portfolio management.

Keywords: Quantitative Portfolio Management, Optimization, Asset Allocation, Diversification

JEL Classification: C610, G110

Resumo

Esta dissertação de mestrado apresenta uma aplicação prática da otimização quantitativa de um portfólio realizada de diversas formas. A tese está organizada em duas partes principais, uma teórica e uma prática.

A parte teórica introduz os fundamentos da teoria de portfólio. Descreve o processo de otimização, apresenta vários métodos de otimização selecionadas e fornece uma visão geral da gestão de portfólios. Como um todo, serve como base para a parte prática.

A parte prática da tese coloca vários métodos de otimização de portfólio quantitativos em competição. Diferentes optimizadores foram aplicados a carteiras compostas por ativos idênticos que foram subsequentemente mantidos sob diferentes estilos de gestão ao longo de um período de tempo pré-especificado. O desempenho de cada carteira foi medido ex-post, adequadamente avaliado de acordo com os critérios de otimização e comparado com as demais carteiras.

As perguntas para as quais esta tese de mestrado tentou encontrar respostas foram (1) qual é o optimizador de portfólio, dentre os selecionados, tem o melhor desempenho e (2) se é benéfico conduzir uma gestão de portfólio muito ativa ou passiva.

Palavras-chave: Gestão Quantitativa de Portfólios, Otimização, Alocação de Ativos, Diversificação

JEL Classificação: C610, G110

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1. Introduction

"An investment in knowledge pays the best interest". A famous quote pronounced by Benjamin Franklin, an American polymath and one of The Founding Fathers of the United States, is pointing clearly to where every person should invest in the first place. Learning is a never-ending process and shall we look at the profiles of the greatest from the greatest, regardless the field of interest, they have never considered their education as complete. The more we invest into our mind, the higher the return we may expect to come back to us in the future in various forms. Our knowing can be seen as a portfolio of knowledge. Portfolio, which we build in more or less constructive matter.

Our knowing deeply influences the way we approach, understand, handle, and reflect all of the challenges we encounter. Based on our obtained experience and our knowing, we derive theories and philosophies. The investment philosophy has alike origin and, as well as other philosophies, is a subject of evolution. Swensen (2000) describes the investment philosophy as a coherent approach being applied consistently to all aspects of portfolio management process. In his eyes, the philosophical principals represent time-tested insights into investment matters that eventually evolve into lasting professional convictions. The investor's effort to find the most effective way to generate investment returns emanates from those convictions and fundamental beliefs. The investment returns are seen as a product of three tools of portfolio management: (1) asset allocation, (2) security allocation, and (3) market timing. Sophisticated investor then considers contribution of each of the portfolio management tools to costruct portfolios in a conscious manner.

The core focus of this thesis is on the problematics of the first tool of portfolio management, the asset allocation and the various approaches to it. Asset allocation is often understood as the second step of the investment process with the first one being the determination of investor's investment objectives, time preferences, and risk profile. Various empirical studies over the time have shown that asset allocation has the biggest influence on an overall portfolio performance. Asset allocation represents spreading the investor's investment capital across various asset classes such as stocks, bonds, derivatives, properties, commodities, funds, cash etc. in order to achieve a diversified portfolio.

The thesis is primarily structured in two main parts, theoretical and practical. The theoretical part introduces the underpinnings of portfolio theory. It describes the optimization process, introduces a number of selected optimization methods, and provides an overview of portfolio management. As a whole, it serves as an underlying for the practical part.

The practical part of the thesis is based on an experiment that puts multiple quantitative portfolio optimization methods into a contest. Different optimizers are applied to portfolios composed of identical assets, which are subsequently held under different portfolio management styles over a pre-specified period of time. The performance of each portfolio is measured ex-post, adequately evaluated in accord with the criteria of the experiment, and confronted with the others.

The questions that this master's thesis tries to find answers to are (1) which portfolio optimizer, out of the selected ones, performs the best, and (2) whether it is beneficial to conduct rather an active, or a passive portfolio management.

2. Portfolio Theory

This chapter introduces the fundamentals of portfolio theory, and describes some of the essentials regarding the portfolio optimization frameworks.

In a wide interpretation, any portfolio can be viewed as a set of various items. Those items are acquired by the portfolio's owner with accordance to his needs, wants, preferences, predispositions, possibilities, and/or expectations. From the financial perspective, such portfolio is understood as a set of financial assets[1]. The finance universe has two main underlying dimensions: the time dimension and the risk dimension.

2.1. Time Dimension

The importance of time in finance is absolutely crucial and is best explained by the concept of the time value of money (TVM). TVM concerns equivalence between cash flows occurring at different times. One dollar today has a higher value than one dollar one year from now. However, the same dollar invested today grows in value over time. Formally, the value of one dollar invested today grows accordingly to its interest rate (i). An interest rate is a rate of return that reflects the relationship between differently dated cash flows. When considering the time dimension, we distinguish between the discrete and the continuous time. Consequently, when considering the interest rates under the time dimension framework, we distinguish between the discrete and the continuous interest rates. Therefore, the TVM allows us to either obtain the future value (compounding) or the present value (discounting).

2.1.1. Discrete Time

The discrete time considers a variable to occur at distinct, separate points in time. For instance, monthly, quarterly, yearly. The discrete interest is then being accrued to the principal[2] at those separate points in time. The discrete interest has characteristics of a discrete random variable.

- **Discrete compounding**
  - the future value (FV) of $1 invested for n years at the interest rate I compounded once per annum

[1] Financial asset – a tangible liquid asset which gets its value from a contractual claim
[2] Principal - the amount of money originally invested

$$FV = (1 + i)^n \tag{2.1}$$

- FV of $1 compounded m-times per annum for n years

$$FV = (1 + \frac{i}{m})^{mn} \tag{2.2}$$

- **Discrete discounting**
  - the present value (PV) of a future $1 discounted for n years

$$PV = \frac{1}{(1 + i)^n} \tag{2.3}$$

  - PV of a future $1 discounted m-times a year

$$PV = \frac{1}{(1 + \frac{i}{m})^{mn}} \tag{2.4}$$

2.1.2. Continuous Time

The continuous time also considers the variable to occur at specific points in time. However, and in contrast to the discrete time, between two points in time there is an infinite number of other points of time and the distance between them is converging to zero. Thus, the continuous time sees the variable to occur continuously. The continuous interest does have characteristics of a continuous random variable.

The effect of continuous time, when m → ∞, on compounding can be expressed formally using limits as following:

$$\lim_{m \to \infty} (1 + \frac{i}{m})^{mn} = e^{In} \tag{2.5}$$

- **Continuous compounding**
  - FV of $1 continuously compounded for n years

$$FV = e^{in} \tag{2.6}$$

- **Continuous discounting**
  - PV of $1 discounted with continuously compounded i

$$PV = e^{-in} \tag{2.7}$$

2.2. Risk Dimension

The risk dimension relates closely with the time dimension. The time dimension distinguishes, in its fundamental property, between two points in time. For instance, between the presence and the future. The presence is well known. However, the future is, from its very nature, uncertain and thus risky. The riskiness, in such case, is the potential deviation from our expectation about the future. Taking the perspective of an investor, the realized return from his investment at the end of the period may be different from the expected one. Those deviations can be both positive or negative. Logically, only the negative deviations are considered undesirable. The most widespread used measure of riskiness in finance is the square root of variance, the standard deviation (SD). The use of SD is convenient under a number of simplifying assumptions, e.g. normal distribution of returns. Also, SD is preferred over variance due to its equality of units with returns as they both are expressed in percentages. Another risk measures are, for example, beta, semi-variance, value-at-risk (VaR), or expected shortfall (ES)

Having mentioned the term expectation, it is convenient to briefly introduce the concept of utility functions, expected utility criterion, and risk aversion. These play a crucial role in the investment decision making process.

2.2.1. Utility Functions

Utility functions are mathematical representations of investor's attitudes toward risk and return. It is a function that assigns a utility value to all possible investment outcomes. The utility value measures the degree of individual satisfaction the investor receives from any specific level of wealth (W). Such satisfaction may come from several sources, typically from the additional consumption of goods and/or services that the investor can enjoy after selling the investment portfolio. Some investors, on the other hand, may enjoy the satisfaction derived from excitement of the investing game itself (Esch et al., 2005).

The utility functions in general have four important properties:

- Utility is increasing and is always positive
- Marginal utility is decreasing

$$u'(W_1) < u'(W_0) \quad when \quad W_1 > W_0 \tag{2.8}$$

- Utility functions are strictly concave

$$U'' < 0 \tag{2.9}$$

- Different investors have different utility functions
- A utility function, for a given investor and for a given time, is not unique

Examples of the most common utility functions

- **Log utility**

$$u(W) = \ln(W) \tag{2.10}$$

- **Exponential utility**

$$u(W) = 1 - e^{-vW} \tag{2.11}$$

- **Power utility**

$$u(W) = \frac{W^{1-\gamma}}{1-\gamma} \tag{2.12}$$

- **Quadratic utility**

$$u\,(W) = W - \frac{b}{2}W^2 \quad b > 0 \tag{2.13}$$

Where W represents the investor's wealth.

Special cases of utility functions incorporating the risk aversion coefficients are introduced in the following Section 2.2.2.

2.2.1. Expected Utility Criterion

The expected utility (EU) criterion provides a framework for when an individual must make a decision under uncertainty. That being not knowing which future outcome, out of a set of possible outcomes, is going to result from a decision made today. In such situation, one will make decision that offers the highest expected utility. Each outcome is assigned a probability of occurrence. Thus, the EU is a probability-weighted average of utilities over all possible outcomes and over a specific period of time. The final decision also depends on one's risk aversion. To demonstrate a general case, let's consider a lottery L(x,y,π), where outcome x has a probability of occurrence π and y with (1-π). The expected utility is following:

$$E[U(L)] = \pi u(x) + (1 - \pi)u(y) \tag{2.14}$$

The above equation is referred to as the **von Neumann-Morgernstern** (VNM) utility function and represents the **expected utility criterion**. Let's adjust the VNM with respect to asset returns. Let's consider an asset A at time t_0 with possible future returns $r_1, r_2, \dots, r_n$ and with probabilities of occurrence $\pi_{r_1}, \pi_{r_2}, \dots, \pi_{r_n}$ at time t_1, respectively. The expected utility can then be expressed as following:

$$E[U(r_A)] = \pi_{r_1}u(r_1) + \pi_{r_2}u(r_2) + \cdots + \pi_{x_n}u(r_n) = \sum_{i=1}^{n} \pi_{x_i}u(r_i) \tag{2.15}$$

The expected utility criterion confirms that the individual is concerned only with the final payoffs and the cumulative probability associated with achieving them (Levy and Post, 2005).

2.2.2. Risk Aversion

An important fundamental property of the risk dimension under the portfolio theory framework is the investor's relation towards risks. His risk aversion. Every investor is different and has a different perception of risks, or in other words, has a different degree of risk aversion. Naturally, the desire of every investor is the same - to avoid risks, i.e. to smooth their consumption across all states of nature and to avoid variations in the value of their portfolio holdings. The investor is considered to be weakly risk averse when

$$u(W) \geq E[u(W)] \tag{2.16}$$

the utility of current wealth is higher or equal than the expected utility of potential wealth from a gamble. Strict risk aversion is represented by strict inequality (Beck, 2017).

The risk aversion is measured by the risk aversion coefficients:

- **Absolute risk aversion** coefficient

$$\alpha\,(W) = -\frac{u''(W)}{u'(W)} \tag{2.17}$$

- **Relative risk aversion** coefficient

$$\rho\,(W) = W\alpha(W) = -\frac{Wu''(W)}{u''(W)} \tag{2.18}$$

- **Risk tolerance** coefficient

$$\tau\,(W) = \frac{1}{\alpha\,(W)} = -\frac{u'(W)}{u''(W)} \tag{2.19}$$

Utility functions incorporating the risk aversion coefficients:

- **Constant absolute risk aversion** (CARA)
  - Absolute risk aversion is constant at every wealth level
  - CARA is a function of exponential utility function

$$u\,(W) = -e^{-\alpha W} \tag{2.20}$$

- **Constant relative risk aversion** (CRRA)
  - Relative risk aversion is constant at every wealth level
  - CRRA is a function of power utility function

$$u\,(W) = \frac{W^{1-\rho}}{1-\rho} \tag{2.21}$$

2.3. Modern Portfolio Theory

Modern Portfolio Theory (MPT) is a financial portfolio construction theory developed by professor Harry Max Markowitz (1952), which first introduced in his paper *"Portfolio Selection"* published by the Journal of Science in 1952, and for which he was awarded the Nobel Memorial Prize in Economic Sciences in 1990.

In Markowitz's paper, the portfolio selection process is divided in two stages. The first stage begins with observations and ends with some expectations regarding the future performance of observed securities. The second stage begins with those expectations and ends with the choice of final optimal portfolio. In this thesis, only the second stage is presented.

MPT, as well as any other theoretical concept, stands upon a number of underlying assumptions. As such, it provides the groundwork for portfolio composition under the mean-variance framework and for an arbitrary number of risky assets, with or without a risk-free asset.

MPT assumptions:

- Markets are perfectly efficient
- No transaction costs, no taxes
- Assets are perfectly divisible
- Risk-free asset is available to all investors
- Unlimited long and short positions are allowed
- Investors are rational and risk averse
- Utility function is quadratic
- Investors possess homogeneous investment motivation, horizon, and expectation
- Distribution of returns is Gaussian
- Investors are concerned only with asset's mean and variance
- Investors desire to maximize their expected utility
- Investors always seek the maximum portfolio return for varying levels of risk

2.3.1. Canonical Portfolio Problem

When an investor faces a portfolio creation decision he needs to make a choice regarding budgeting the asset allocation. Specifically, how much of his capital is going to be allocated to, and spread across, risky assets and how much to risk-free asset as an alternative to risky assets. The future payoff from risky assets is uncertain. However, the payoff from risk-free asset is always certain. The portfolio problem thus becomes the maximization of the expected payoff. That being done in accord with investor's utility function.

Canonical portfolio problem is well described by Danthine and Donaldson (2015). Let's consider a risk-free asset r_f and a number of risky assets with returns $r_1, r_2, \dots, r_i$. Also, a portion of capital w_f being invested into the risk-free asset and $\left(1 - w_f\right) = \sum_{i=1}^{n} w_i$ among n risky assets with w_i being the individual weight of each risky asset.

$w_f + \sum_{i=1}^{n} w_i = 1.$

The maximization problem becomes following:

$$\max_{(w_1, w_2, \dots, w_n)} \quad E\left\{u\left[w_f(1 + r_f) + \sum_{i=1}^{n} w_i(r_i - r_f)\right]\right\} \tag{2.22}$$

Where the term $\left(r_i - r_f\right)$ represents the risk premium of each risky asset. It is intuitive to assume that the portion of capital invested in risky assets increases with increasing risk premium and decreases when opposite. This intuition can be formally described via the first-order condition (FOC). Under the risk aversion $U''() < 0$, the FOC of the maximization problem becomes

$$E\left\{u'\left[w_f\left(1 + r_f\right) + \sum_{i=1}^{n} w_i(r_i - r_f)\right]\sum_{i=1}^{n}(r_i - r_f)\right\} = 0 \tag{2.23}$$

The FOC allows to describe the relationship between the investor's risk aversion and his portfolio's consumption via the following theorem, Equations 2.24-26, regarding the problem.

For simplification, let's substitute $\sum_{i=1}^{n} w_i = w_r$ where w_r represents the capital allocated to risky assets, and $E(r_R) = \sum_{i=1}^{n} E(r_i) w_i$ where $E(r_R)$ is the expected payoff of the risky assets combined. Let's assume $U''() < 0$ and $U'() > 0$.

$$w_r > 0 \leftrightarrow E(r_R) > r_f \tag{2.24}$$

$$w_r = 0 \leftrightarrow E(r_R) = r_f \tag{2.25}$$

$$w_r < 0 \leftrightarrow E(r_R) < r_f \tag{2.26}$$

The theorem states that a risk averse investor is willing to make a risky investment if, and only if, the expected payoff from the risky investment exceeds the risk-free rate. Or in other words, if the odds are favorable and there is a positive remuneration from the additional risk accepted.

2.3.2. Mean-Variance Criterion

MPT considers mean and variance as sufficient information when making a choice regarding the investment assets. It rests on the presumption that rational investors like the asset's returns and dislikes the return variance. The mean-variance criterion allows the investor to compare various assets between each other while accounting the mean-variance preferences. Let's assume assets A and B, with μ_A, σ_A and μ_B, σ_B, respectively.

Portfolio A dominates portfolio B if, and only if, the following conditions are satisfied:

$$\mu_A \geq \mu_B \quad \text{and} \quad \sigma_A < \sigma_B$$

Or equivalently,

$$\mu_A > \mu_B \quad \text{and} \quad \sigma_A \leq \sigma_B$$

The mean-variance utility function describes the risk-return trade-off when reflecting the investor's degree of risk aversion. The function is based on two pivotal assumptions of MPT, the quadratic utility function and the Gaussian distribution of returns. The quadratic utility is important because it implies mean-variance preferences. The Gaussian distribution is attractive due to its simplicity and properties. When considering only mean and variance, the expected utility has following form:

$$E[U(r)] = u[E(r)] + \frac{1}{2}u''[E(r)]Var(r) \quad (2.27)$$

To assure consistency with the risk-aversion assumption, the u''must be < 0 and hence the positive sign changes to negative. Considering u to be quadratic in form described by Equation 2.13, setting b = 1, and including the investor's risk aversion coefficient to reflect his perception of risks, we arrive to the standard mean-variance utility function that can be found across the literature and has following form:

$$U = E(r) - \frac{1}{2}A\sigma^2 \quad (2.28)$$

Where E(r) is the expected return, σ^2 is the variance, and A is investor's risk aversion coefficient.

2.4. Efficient Frontier

Prior to the introduction of the minimum-variance frontier (MVF) and the efficient frontier (EF), it is convenient to define the portfolio expected return and variance/SD in a matrix form, as they are subsequently used throughout the thesis. Let's consider a portfolio P of n risky assets, with expected returns $r_1, r_2, \dots, r_n$ forming N x 1 vector of returns and portfolio's assets weights $w_1, w_2, \dots, w_n$ forming N x 1 vector of weights. The set of covariances between the assets form the variance-covariance matrix denoted Σ.

- Vector of expected returns

$$E(r) = \mu = \begin{bmatrix} E(r)_1 \\ E(r)_2 \\ \vdots \\ E(r)_n \end{bmatrix}$$

- Vector of portfolio weights

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

- Variance-covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{11}^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn}^2 \end{bmatrix}$$

- Portfolio expected return

$$E(r_p) = \mu_p = w^T \mu \tag{2.29}$$

- Portfolio variance and SD

$$Var(P) = \sigma_P^2 = w^T \Sigma w \tag{2.30}$$

$$SD = \sigma_P = \sqrt{w^T \Sigma w} \tag{2.31}$$

Where the subscript T stands for transposed.

The essential notations being introduced, it is now possible to procced to the introduction of both frontiers. Let's consider a set of $n \geq 2$ assets. An infinite number of portfolios can be formed from such set of assets. This creates a set of feasible portfolios, the feasible set. To evaluate efficiency and compare the portfolios, an investor considers their expected returns and variances with accordance to the mean-variance criterion. The investor will choose his or her optimal portfolio from the feasible set, that

1. Offers highest expected return for varying levels of risk, and
2. Offers lowest level of risk for varying levels of expected returns

The above-mentioned conditions are known as the efficient set theorem (Sharpe, 1995).

The set of portfolios meeting the efficient set theorem are called the efficient portfolios and graphically they plot the EF, which is part of the MVF. EF is the set of frontier portfolios where each portfolio represents the portfolio with the highest expected return for varying levels of risk. Both frontiers are conventionally plotted in an expected return-risk (μ-σ) space. The shape of EF depends whether a risk-free asset is present or not.

2.4.1. Efficient Frontier for Risky Assets

Considering only the risky assets, the graphical representation of MVF and EF is nearly identical yet it is important to distinguish between them. MVF represents the entire curve, EF represents only the non-dominated part of MVF, originating in the global minimum-variance portfolio (GMV). GMV is frontier portfolio with the smallest variance. The shape of MVF is strongly influenced by the correlation (ρ) between the assets as it directly affects the portfolio standard deviation (SD) as descripted by Equation 2.32 for two risky assets A and B, with μ_A, σ_A and μ_B, σ_B, respectively.

$$\sigma_P = \sqrt{w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_A \sigma_B \rho_{A,B}} < w_A\sigma_A + w_B\sigma_B \tag{2.32}$$

The above-mentioned inequality descripts the gain from diversification[3] coming from assets with imperfect correlation. Simply put, the lower the correlation between assets, the better the diversification effect, the more parabolic the shape of MVF. The MVF and EF are depicted on the following Figure 1.

[3] Diversification is further introduced in Section 3.2.1

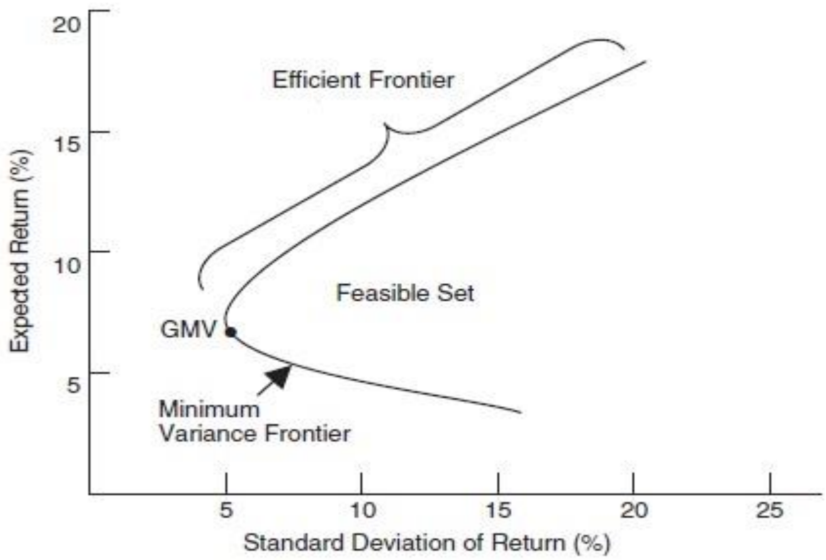

Figure 1: Efficient frontier for risky assets

2.4.2. Efficient Frontier for Risk-Free and Risky Assets

The portfolio of assets where one is risk-free affects the shape of EF. Let's consider two assets, one risky asset A and one risk-free asset, forming a portfolio P. Since the risk-free asset carries no risk, its SD = 0. The SD of such complete portfolio is simply a linear weighted average $SD_P = w_A \sigma_A + w_{rf} \sigma_{rf} = w_A \sigma_A + 0$ where $w_A + w_{rf} = 1$. If short positions on risk-free asset are allowed, then $w_A > 1$ and $w_{rf} < 0$. The efficient frontier of such combined portfolios is a straight line originating in the risk-free rate on axis y.

The EF for a combination of n risky assets and a risk-free asset follows the above-described scenario. The risky assets themselves form a parabolic-shaped MVF. The combination of risky portfolio, originally depicted on MVF, with a risk-free asset forms a straight line, referred to as the capital allocation line (CAL) The only CAL that dominates the parabolic curve in all of its length, as well as the other CALs, is the tangent to the MVF. This tangent CAL represents the efficient frontier as depicted in Figure 2. The tangent point represents the only portfolio which can be made solely of risky assets, usually called the tangency portfolio (T). The tangency portfolio is important and plays a crucial role in the investor's optimal portfolio choice decision process, described in Section 4.1.3.

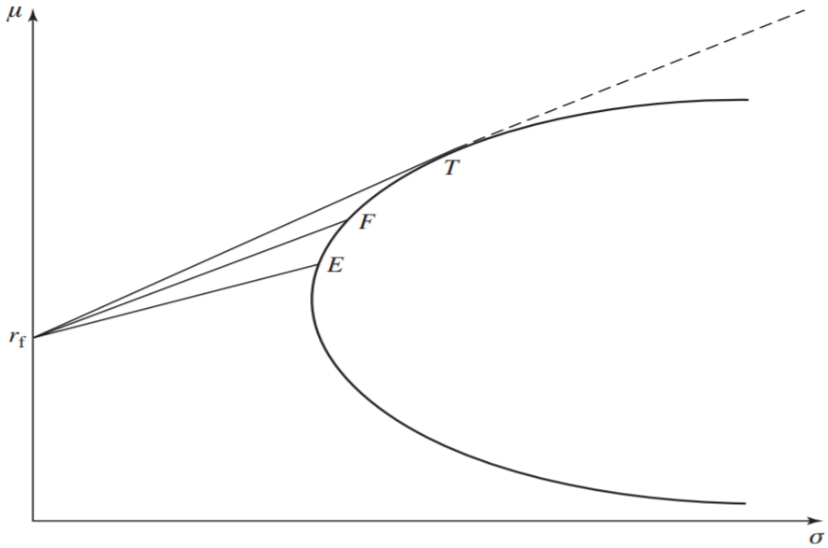

Figure 2: Different CALs for portfolios composed of risk-free and risky assets

3. Return Generating Models

Although asset pricing is not the main subject of interest of this thesis, there are two pricing models that can be considered as essential to the portfolio theory and thus it is convenient for these to be briefly introduced. These models are the capital asset pricing model and the market model. These models represent an alternative to the historical approach when estimating asset's expected returns, variances, and covariances.

3.1. Capital Asset Pricing Model

Capital asset pricing model (CAPM) is an equilibrium pricing model and it has played a pivotal role in the development of quantitative investment management since its introduction. CAPM is derived from MPT and as such shares most of the MPT's assumptions introduced in Section 2.3 with few specifics. As the concept of CAPM is based on the principle of equilibrium, few of the assumptions are worthwhile to review.

CAPM assumptions[4]

- All investors have homogeneous expectations regarding returns, variances, and covariances for all assets
- Risk-free asset is available for an infinite lending or borrowing
- Markets are perfectly efficient
- All assets are tradable and infinitely divisible
- Unlimited short sales are allowed
- Perfect competition, i.e. an individual alone cannot affect the price

Few implications coming from the CAPM assumptions

- MVF is identical for all investors
- EF is identical for all investors as well
- As all investors share homogeneous expectations, they all demand the same assets
- Hence, the tangency portfolio is identical for everyone
- Tangency portfolio becomes the market portfolio

According to Litterman (2003), CAPM describes the market equilibrium in a sense that, if the model is correct and any asset's expected return differs from its equilibrium return, the market forces come into play and restore the relationship suggested by the model. However, CAPM theory goes bit further. As known, risk of a stock can be split between systematic and non-systematic, or specific, risk. If portfolio is large enough, the non-systematic risk can be diversified away[5]. Since every investor holds a combination of market portfolio and risk-free asset, which both theoretically carry zero of specific risk, the specific risk no longer matters. Therefore, CAPM fundamentally describes a relationship of any asset's equilibrium return as a linear function of its systematic risk, measured by β, market risk premium and a risk-free rate. The β of market portfolio is always equal to 1. When an asset carries higher systematic risk than the market, i.e. $\beta > 1$, it should be remunerated by higher return. If the asset carries no systematic risk, thus no specific risk as well, then the equilibrium return should be equal to the risk-free rate. Let's consider an asset A with

[4] Complete list of assumptions is in Section 2.3
[5] See Section 3.2.1.

return r_A, a risk-free asset with return r_f, and a market portfolio M with return r_M. CAPM equation is following:

$$r_A = r_f + \beta_A(r_M - r_f) \tag{3.1}$$

Equivalently, let's substitute the single asset A with a complete portfolio P, then the equation becomes

$$r_P = r_f + \beta_P(r_M - r_f) \tag{3.2}$$

This is the standard CAPM, where β is the systematic risk measure of an asset/portfolio, and the term $(r_M - r_f)$ is the market risk premium.

The graphical representation of CAPM is the security market line (SML). In CAPM world, all portfolios should lie on SML. SML is plotted in the μ-σ space, originating at the risk-free rate on axis Y and going through the market portfolio M. SML is a useful tool for determining whether an asset is overvalued, undervalued, or correctly valued on the market. This can be done mathematically by comparing the equilibrium return suggested by CAPM and the actual return observed on the market, or graphically plotting the asset's return together with the SML in one graph.

3.2. Market Model

Market model belongs to the group of so-called factor models. Factor models represent the building stone of the Arbitrage Pricing Theory (APT), introduced by Ross (1976). The theory is based on an assumption that all asset returns can be determined by a set of factors. It believes that the asset returns are related to each other through their correlations with a limited set of factors. The simplest factor model is the market model. More advanced are, for example, the 3-factor and 5-factor Fama-French models.

The market model (MM) is a one factor model. The factor is the return on market portfolio. MM describes a relationship between the returns on asset and the returns on market portfolio through a classical regression. Let's assume an asset A with return r_A, and the market portfolio M with return r_M. MM regression equation is following (DeFusco et al., 2007):

$$r_A = \alpha_A + \beta_A r_M + \varepsilon_A \tag{3.3}$$

Where α_A is the intercept representing an average return on asset A independent of the market, and ε_A is the error term representing the residual risk. Alternatively, the MM equation can be expressed in terms of excess returns as following:

$$r_A - r_f = \alpha_A + \beta_A(r_M - r_f) + \varepsilon_A \tag{3.4}$$

MM stands upon following assumptions

- $E(\varepsilon_A) = 0$
- $Cov(r_M, \varepsilon_A) = 0$
- $Cov(\varepsilon_A, \varepsilon_B) = 0 \quad A \neq B$

These assumptions partially correspond to the OLS regression model. However, MM does not assume the error term to be normally distributed, as well as the variance of error term being identical across assets. Given these assumptions, three postulates can be made regarding the expected returns, variances, and covariances.

- Expected return of asset A depends on the expected return of market M, A's β towards M, and the independent part of A's return

$$E(r_A) = \alpha_A + \beta_A E(r_M) \tag{3.5}$$

- Variance of asset A depends on the variance of market M, the residual variance of A, and A's β towards M

$$Var(r_A) = \beta_A^2 \sigma_M^2 + \sigma_{\varepsilon_A}^2 \tag{3.6}$$

- Covariance between the returns of asset A and asset B depends on the variance of returns of market M, and A's and B's sensitivities β_A, β_B

$$Cov(r_A, r_B) = \beta_A \beta_B \sigma_M^2 \tag{3.7}$$

3.2.1. Diversification

The market model is helpful in explanation of one of the core features of large financial portfolios and that being the diversification effect. The positive effect of correlation on diversification is introduced already in Section 2.4.1. In this section, the concept of diversification is extended with regard to the number of assets within the portfolio.

Let's assume an asset, e.g. a stock. Each stock's total risk is primarily composed of two main types of risk. The systematic risk and the specific risk.

- **Systematic risk**, or market risk, refers to the risks associated with the macroeconomic events or developments impacting the entire market. Market risk impacts all market participants equally and from its nature cannot be eliminated via diversification. However, its impacts can be eased using an appropriate hedging or asset allocation strategy. Systematic risk is measured by β. Beta of an asset can be interpreted as its sensitivity towards the market. Beta of the market is always equal to 1. Let's assume an asset A and a market portfolio M, with returns r_A and r_M, respectively. β calculation is following:

$$\beta_A = \frac{cov(r_A, r_M)}{var(r_M)} \tag{3.8}$$

Beta of a portfolio is calculated as a weighted average of individual asset betas. Let's assume a portfolio P with n assets. β calculation is following:

$$\beta_P = \sum_{i=1}^{n} w_i \beta_i \tag{3.9}$$

β < 0	Asset returns move the opposite direction compared to the market. If the market return is positive, the asset return is negative and vice versa.
β = 1	Asset returns move identically with the market.
β > 1	Asset returns move the same direction as the market but quicker, both up and down. The asset is riskier than the market.
0 < β < 1	Asset returns move the same direction as the market but slower, both up and down. The asset is less risky than the market.

Table 1: Values of β

- **Specific risk,** or idiosyncratic risk or residual risk, refers to the risks associated with an individual industry, firm, or product. Specific risk can be eliminated via diversification.

The elimination of specific risk from a portfolio is a sought-after benefit. Mathematically, the elimination can be explained by using the properties of the market model. Let's consider the MM's second postulate for an entire portfolio P consisting of n assets as following (Elton et al., 2011):

$$Var(r_p) = \beta_P^2 \sigma_M^2 + \sigma_{\varepsilon_P}^2 \tag{3.10}$$

Where the term $Var(r_p)$ represents the **total risk**, $\beta_P^2 \sigma_M^2$ the **systematic risk**, and $\sigma_{\varepsilon_P}^2$ the **specific risk**. In terms of n individual assets, the equation can be re-written as following:

$$Var(r_p) = \beta_P^2 \sigma_M^2 + \sigma_{\varepsilon_P}^2 = \sum_{i=1}^{n} w_i^2 \beta_i^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{\varepsilon_i}^2 \tag{3.11}$$

For evidential purposes, it is convenient to ignore the systematic risk part and focus solely on the specific one. Moreover, let's assume an equally weighted portfolio where $w_i = \frac{1}{n}$, the diversification effect on the specific part is following:

$$\sum_{i=1}^{n} w_i^2 \sigma_{\varepsilon_i}^2 = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma_{\varepsilon_i}^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{\varepsilon_i}^2 = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \sigma_{\varepsilon_i}^2}{n^2} = 0 \tag{3.12}$$

It is easy to see that with a number of assets increasing to infinity, the specific risk converges towards zero.

4. Portfolio Optimization

Since the introduction of MPT in 1952, a number of portfolio optimization techniques have been developed. All of them, however, more or less build upon the mean-variance optimization (MVO) model with the motivation to overcome some of its main drawbacks. The time has proven that the MVO developed by professor Harry Markowitz has truly become the cornerstone of the portfolio theory. This chapter regarding portfolio optimization introduces only the methods used in the practical part of this thesis.

4.1. Mean-Variance Optimization

This section dedicated to the MVO is a direct continuation of Section 2.4 regarding the efficient frontier. As introduced there, the EF is influenced by the parameters of individual assets. Their means, variances, covariances, and the presence of risk-free asset.

4.1.1. MVO for Risky Assets

To compute portfolios making up the EF considering only the risky assets, the following constrained problem must be satisfied

$$min \quad w^T \Sigma w$$

$$s.t. \;\; w^T \mu = r_R$$

$$w^T I = 1$$

Where I is the N x 1 column vector of ones, and r_R is the required portfolio return[6] demanded by the investor. No non-negativity constrains are present. The problem can be solved by minimizing the Lagrangian

$$\min \quad \mathcal{L} = \frac{1}{2} w^T \Sigma w + \lambda (r_R - w^T \mu) + \gamma (1 - w^T I) \qquad (4.1)$$

[6] The use of required return is convenient as the investor may desire a return different from the expected return. Nonetheless, the expected return may be used in the computations as well. Required return is often referred to as target return.

Where λ and γ are the Lagrange multipliers. The first-order conditions to solve the Lagrangian are following:

$$\frac{\partial \mathcal{L}}{\partial w} = \Sigma w - \lambda\mu - \gamma I = 0 \tag{4.2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = r_R - w^T \mu = 0 \tag{4.3}$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = 1 - w^T I = 0 \tag{4.4}$$

The FOCs with applied constrains can be re-written in terms of portfolio weights as following:

$$w = w_P = \lambda\Sigma^{-1}\mu + \gamma\Sigma^{-1}I \tag{4.5}$$

$$r_R = w^T\mu = \mu^T w = \lambda(\mu^T\Sigma^{-1}\mu) + \gamma(\mu^T\Sigma^{-1}I) \tag{4.6}$$

$$1 = I^T w_P = w_P^T I = \lambda(I^T\Sigma^{-1}\mu) + \gamma(I^T\Sigma^{-1}I) \tag{4.7}$$

For simplification purposes, Danthine and Donaldson (2015) use the following constants

$$A = I^T\Sigma^{-1}\mu = \mu^T\Sigma^{-1}I \tag{4.8}$$

$$B = \mu^T\Sigma^{-1}\mu > 0 \tag{4.9}$$

$$C = I^T\Sigma^{-1}I > 0 \tag{4.10}$$

$$D = BC - A^2 > 0 \tag{4.11}$$

Solving the set of FOCs, with applied substitution, for the Lagrange multipliers, we obtain:

$$\lambda = \frac{Cr_R - A}{D} \quad and \quad \gamma = \frac{B - Ar_R}{D}$$

Finally, substituting for the Lagrange multipliers into the Equation 4.5, the solution for portfolio weights is following:

$$w_P = \frac{Cr_R - A}{D}\Sigma^{-1}\mu + \frac{B - Ar_R}{D}\Sigma^{-1}I \tag{4.12}$$

Re-arranging the terms, the solution can be written in an alternative form as following:

$$w_P = \frac{1}{D}[B(\Sigma^{-1}I) - A(\Sigma^{-1}\mu)] + \frac{1}{D}[C(\Sigma^{-1}\mu) - A(\Sigma^{-1}I)]r_R \tag{4.13}$$

Or,

$$w_P = g + hr_R \tag{4.14}$$

Where g represents the weight vector for portfolio with $r_R = 0$, and g + h represents the weight vector for portfolio with $r_R = 1$

The FOCs are essential in defining the portfolio weights representing any frontier portfolio for a given level of required return. The solution for portfolio weights is highly practical as it delivers the weights of corresponding frontier portfolio for a chosen level of desired return.

The computation of parameters of any frontier portfolio is quite a straightforward matter.

- Expected return

$$E(r_P) = \mu_P = w_P^T\mu \tag{4.15}$$

Where w_P is the portfolio weights, and μ is vector of expected returns of assets.

- Variance

$$Var(P) = \sigma_P^2 = w_P^T\Sigma w_P = \frac{C}{D}\left(\mu_P - \frac{A}{C}\right)^2 + \frac{A}{C} \tag{4.16}$$

The global minimum-variance portfolio (GMV), is the frontier portfolio with the smallest variance. It represents a pivotal point on the MVF, as it splits the MVF between the efficient and non-efficient frontier. The portfolio parameters calculated by Equations 2.29-31 apply for GMV as well. However, it can be calculated in a simpler way:

- Expected return

$$E(r_{gmv}) = \mu_{gmv} = \frac{A}{C} \tag{4.17}$$

- Variance

$$Var(gmv) = \sigma_{gmv}^2 = \frac{1}{C} \tag{4.18}$$

4.1.2. MVO for Risk-free and Risky Assets

The inclusion of a risk-free asset within a portfolio of otherwise risky assets improves the efficiency of the complete portfolio. Let's assume a fraction of capital denoted w invested in a vector of risky assets and $(1 - I^T w)$ in the risk-free asset denoted w_f. The optimization problem is following:

$$min \quad w^T \Sigma w$$

$$s.t. \;\; r_f + \left(\mu - r_f I\right)^T w = r_R$$

Where μ represents the vector of expected returns on risky assets, r_f the return on risk-free asset, and I the vector of ones.

It is worthwhile to mention that the constrain $w^T I = 1$ is no longer present. Thus, $w_f + \sum_{i=1}^{n} w_i \neq 1$. No non-negativity constrains are present. The problem can be solved by minimizing the Lagrangian:

$$\min \quad \mathcal{L} = \frac{1}{2} w^T \Sigma w + \lambda(r_R - r_f - (\mu - r_f I)^T w \tag{4.19}$$

Where λ is the Lagrange multiplier. The FOCs to solve the Lagrangian are following:

$$\frac{\partial \mathcal{L}}{\partial w} = \Sigma w - \lambda(\mu - r_f I) = 0 \tag{4.20}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = r_R - r_f - \left(\mu - r_f I\right)^T w = 0 \tag{4.21}$$

The FOCs can be re-written in terms of portfolio weights as following:

$$w = \lambda \Sigma^{-1}(\mu - r_f I) \tag{4.22}$$

$$w = \frac{r_R - r_f}{\left(\mu - r_f I\right)^T} \tag{4.23}$$

Applying the constrain and solving for the Lagrange multiplier, we obtain

$$\lambda = \frac{r_R - r_f}{\left(\mu - r_f I\right)^T \Sigma^{-1}(\mu - r_f I)} \tag{4.24}$$

For simplification purposes, a new constant H (Danthine and Donaldson, 2015) for replacing the denominator may be used

$$H = \left(\mu - r_f I\right)^T \Sigma^{-1}(\mu - r_f I) \tag{4.25}$$

$$H = B - 2Ar_f + Cr_f^2 > 0 \tag{4.26}$$

Where A, B, C represent the constants introduced in Section 4.1.1.

The solution for the optimal portfolio weights by replacing λ is following:

$$w = \frac{r_R - r_f}{\left(\mu - r_f I\right)^T \Sigma^{-1}\left(\mu - r_f I\right)} \Sigma^{-1}\left(\mu - r_f I\right) = \frac{r_R - r_f}{H} \Sigma^{-1}(\mu - r_f I) \tag{4.27}$$

Where w represents the vector of portfolio weights on risky assets.

This is the formula that delivers the optimal portfolio weights when considering risky assets in combination with a risk-free asset for any level of desired return. Since short selling is allowed, the sum of weights of risky assets may go above 1, implying a short-position on risk-free asset, in order to achieve the desired return. The sum of weights of risky assets below 1 implies a partial long position on risk-free asset. Formally, it can be expressed as following: $\sum_{i=1}^{n} w_i \neq 1$, and $w_p = w_f + \sum_{i=1}^{n} w_i = 1$ where w_p represents the weights of complete portfolio. $w_P \neq w$. If, and only if, $\sum_{i=1}^{n} w_i = 1$ and thus $w_P = w$ with no holdings of risk-free asset, we identify such portfolio as the tangency portfolio. All portfolios lie on the efficient frontier. Tangency portfolio lies on both frontiers.

The computation of parameters of any frontier portfolio combining risky and riskless assets is, again, a straightforward matter.

- Expected return

$$E(r_P) = \mu_P = r_f + \left(\mu - r_f I\right)^T w = r_f + \sigma_P \sqrt{H} \tag{4.28}$$

Where μ is the vector of expected returns, I is the vector of ones, w is the vector of portfolio risky holdings, σ_P is the portfolio's SD, and H is a constant.

- Variance

$$Var(P) = \sigma_P^2 = w^T \Sigma w = \frac{\left(\mu_P - r_f\right)^2}{H} \tag{4.29}$$

The tangency portfolio (T) is a special case of frontier portfolio. It is the only portfolio lying on both MVF and EF and is composed entirely of risky assets. As such, it must solve for both of the optimization problems introduced in Sections 4.1.1 and 4.1.2. It plays an important role in the complete portfolio construction process as the investor first determines the tangency portfolio and then adjusts it accordingly to his individual preferences. The tangency portfolio is determined as following:

- Tangency portfolio weights

$$w_T = \frac{1}{A - Cr_f} \Sigma^{-1}(\mu - r_f I) \tag{4.30}$$

Where $\sum_{i=1}^{n} w_i = 1$ and $w_T = w_P = w$ implying no holdings of risk-free asset.

- Expected return

$$E(r_T) = \mu_T = w_T^T \mu = r_f + \left(\mu - r_f I\right)^T w_T = r_f + \frac{H}{A - Cr_f} \tag{4.31}$$

- Variance

$$Var(T) = \sigma_T^2 = w_T^T \Sigma w_T \tag{4.32}$$

4.1.3. Portfolio Choice

The choice of a complete portfolio within the MPT framework is a subject matter under the mean-variance utility hypothesis. MPT considers all investors to be rational and naturally risk averse. The investor's level of risk aversion is primarily derived from his utility function. When constructing a portfolio, the investor faces the canonical portfolio problem. This is a two-step process. The first step is an identification of optimal risky portfolio regardless the investor's preferences. The second step is allocation of capital between the optimal risky portfolio and the risk-free asset to form the most desired portfolio. This two-step process is formally called the Separation theorem, or Two-fund theorem, (Sharpe, 1995).

The second step is fully done with accordance to investor's utility function and his risk aversion. To make this simpler, MPT assumes all investors to have a quadratic utility function. The investor's objective is therefore same for all and that being the maximization of his mean-variance utility. Let's assume a complete portfolio P with expected return μ_P and variance σ_P^2. The maximization problem then becomes following:

$$\max \quad U = \mu_P - \frac{1}{2} A\sigma_P^2 \tag{4.33}$$

Where A is the risk aversion coefficient representing the degree of investor's risk aversion. It is defined as the additional marginal return the investor demands for accepting more risk. It is easy to see that the value of utility function rewards higher expected return and penalizes portfolio risk.

Potential values of A

Risk aversion	A > 0
Risk neutrality	A = 0
Risk seeking	A < 0

Table 2: Values of A

Alternatively, let's consider a capital allocated to portfolio of risky assets denoted as w_r and $(1 - w_r) = w_f$ allocated to risk-free asset, then the mean-variance utility equation can be re-written as following:

$$\max \quad U = \mu_P - \frac{1}{2}A\sigma_P^2 = w_r^T\mu + (1 - w_r^T I)r_f - \frac{1}{2}Aw_r^T \Sigma w_r \tag{4.34}$$

Solving the maximization problem by setting the first derivative with respect to w_r equal to zero, we obtain

$$\frac{\partial U}{\partial w_r} = \mu - r_f I - A\Sigma w_r = 0 \tag{4.35}$$

$$\vdots$$

$$w_r = \frac{\mu - r_f I}{A\Sigma} = \frac{1}{A}\Sigma^{-1}(\mu - r_f I) \quad or \quad w_r = \frac{\mu_P - r_f}{A\sigma_P^2} \tag{4.36}$$

Where w_r represents the capital allocation to risky assets, μ is the vector of expected returns on risky assets, I is the vector of ones, rf is the risk-free rate, A is the investor's risk aversion coefficient, and Σ is the covariance matrix.

Another method of selecting the most desirable portfolio involves the use of indifference curves.

The indifference curves are graphical representation of investor's preferences for risk and return, and are conventionally plotted in two dimensional, risk and return space. Each investor possesses an infinite set of unique indifference curves creating so-called map of indifference curves. Each indifference curve represents all combinations of portfolios that provide the investor the desired level of satisfaction equally. All that being done with respect to investor's utility function. However, and with reference to the MPT assumptions presented in Section 2.3, the MPT assumes all investors to have a quadratic utility function. Indifference curves under the quadratic utility assumption thus too have a quadratic form of convex shape in the relevant area of the μ-σ space. The steepness of the curve is influenced by the investor's risk aversion coefficient. The higher the coefficient, the more risk averse the investor, the steeper the curve. With accordance to the separation theorem, the investor first finds the tangency portfolio and then adjusts the portfolio with risk-free asset to meet the desired characteristics, i.e. to reach the point where the investor's indifference curve meets the efficient frontier.

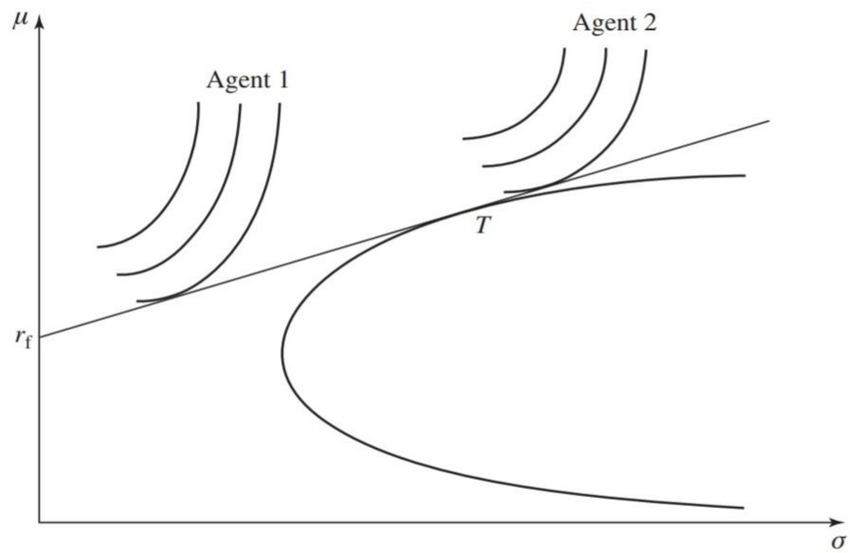

Figure 3: Optimal portfolios for agents with different risk aversion

4.1.4. MPT Limitations

Although the MPT has become the cornerstone of portfolio theory and as such has its sovereign position within quantitative finance, it possesses a number of shortcomings which make the model being criticized from today's perspective. In a theoretical world, the MPT is correct and performs well. However, the assumptions under which the MPT operates usually do not hold in reality. These matters of fact have been empirically proven by a number of studies conducted over the time in various fields of study, e.g. behavioral economics or applied econometrics. Alongside the research, some assumptions are simply not true from its very nature, e.g. no transaction costs or taxes. This section provides a non-exhaustive list of the most significant limitations of the mean-variance optimization (Michaud and Michaud, 2008).

- MVO overuses statistically estimated information resulting in a high input sensitivity. Even a small change of inputs delivers a major impact on the optimal portfolio holdings. Consequently, it tends to maximize the estimation error[7].
- MVO tends to deliver unintuitive, highly concentrated portfolios
- Return distributions in real world are rarely normal. In fact, distributions are usually leptokurtic (excess kurtosis) and skewed.
- Under non-normality, symmetric risk measures perform poorly and asymmetric risk measures, such as semi-deviation or value-at-risk, are more adequate
- MVO assumes a single-period framework only, while investors usually have long term, multi-period investment horizons
- Quadratic utility function exhibits increasing absolute risk aversion (IARA) which is unrealistic
- Investors' expectations are not homogenous as every investor is somehow biased
- Investors being able to buy or sell any quantity of assets doesn't hold as investors often have a credit limit. Moreover, some assets have the minimum order size and can't be traded in fractions
- Transaction costs, fees, and taxes exist in real world
- Correlations across assets are never stable and fixed

[7] Is the difference between the true values of parameters (mean, var, cov) and their estimated values.

4.2. Treynor-Black

The Treynor-Black model (TB) is an optimization method developed by Jack Treynor and Fischer Black (1973), and was originally published in Journal of Business in 1973. The model is based on a presumption that only securities showcasing abnormal returns are worth adding to an otherwise most efficient portfolio, the market portfolio. If such securities occur and are not yet included in the market portfolio, the market portfolio is no longer efficient. The optimal portfolio suggested by TB is thus a combination of the market portfolio and the active portfolio composed of selected securities with positive abnormal returns. Since the number of securities within the active portfolio is usually limited, the incorporation of the market portfolio also significantly improves the overall diversification. The ability to predict abnormal returns is critical within the TB framework, so to avoid any possible inconsistencies coming from using a variety of different security analyses, the TB assumes the use of the market model characterized by the Equation 3.4. In MM, the abnormal return is represented by non-zero alpha, i.e. $\alpha \neq 0$. Any rational investor desires and seeks $\alpha > 0$, which delivers superior return to the portfolio. This inequality is important in order to maintain the positive risk-return trade-off as the security always increases the portfolio risk through its own residual variance. The ultimate goal of TB optimization is the maximization of the optimal portfolio's Sharpe ratio[8]. The majority of MVO assumptions apply for the TB model as well (Kane et al., 2003).

Let's assume n+1 assets, where n is the number of securities with abnormal returns forming an active portfolio A and +1 represents the market index as a passive portfolio M, both together forming an optimal portfolio P. The estimates of alpha, beta, and residual variance coefficients on portfolio level are following (Bodie et al., 2018):

$$\alpha_P = \sum_{i=1}^{n+1} w_i \alpha_i \quad ; \quad \alpha_M = 0 \tag{4.37}$$

$$\beta_P = \sum_{i=1}^{n+1} w_i \beta_i \quad ; \quad \beta_M = 1 \tag{4.38}$$

[8] $SR = \frac{E(r_P) - r_f}{\sigma_P}$

$$\sigma_{\varepsilon_P}^2 = \sum_{i=1}^{n+1} w_i^2 \sigma_{\varepsilon_i}^2 \quad ; \quad \sigma_{\varepsilon_{n+1}}^2 = \sigma_{\varepsilon_M}^2 = 0 \tag{4.39}$$

The formula for optimal weight allocated to the active portfolio A is following:

$$w_A = \frac{E(\overline{r_A})\sigma_M^2 - E(\overline{r_M})\sigma_{AM}}{E(\overline{r_A})\sigma_M^2 + E(\overline{r_M})\sigma_A^2 - [E(\overline{r_A}) + E(\overline{r_M})]\sigma_{AM}} \tag{4.40}$$

Where $\bar{r}$ stands for risk premium. Therefore,

$E(\overline{r_A}) = E(r_A) - r_f = \alpha_A + \beta_A[E(r_M) - r_f]$

$E(\overline{r_M}) = E(r_M) - r_f$

$\sigma_{AM} = \beta_A \sigma_M^2$

$\sigma_A^2 = \beta_A^2 \sigma_M^2 + \sigma_{\varepsilon_A}^2$

After plugging all together and proceeding algebraic simplifying manipulations, the allocation to portfolio A gets following:

$$w_A = \frac{w_0}{1 + (1 - \beta_A)w_0} \quad ; \quad w_M = 1 - w_A \tag{4.41}$$

Where

$$w_0 = \frac{\alpha_A / \sigma_{\varepsilon_A}^2}{E(\overline{r_M}) / \sigma_M^2} \tag{4.42}$$

is the initial allocation to A if $\beta_A = 1$.

The allocation to n individual securities within the portfolio A is following:

$$w_i = w_A * \frac{\frac{\alpha_i}{\sigma_{\varepsilon_i}^2}}{\sum_{i=1}^{n} \frac{\alpha_i}{\sigma_{\varepsilon_i}^2}} \tag{4.43}$$

As mentioned, the end goal of TB optimization is maximization of the optimal portfolio’s Sharpe ratio (SR). Therefore, as optimal portfolio is a combination of market portfolio and a portfolio of securities with superior expected returns, the overall SR must exceed the one of the market. The exact relationship is following:

$$SR_P = \sqrt{SR_M^2 + \left[\frac{\alpha_A}{\sigma_{\varepsilon_A}}\right]^2} = \sqrt{SR_M^2 + \sum_{i=1}^{n}\left[\frac{\alpha_i}{\sigma_{\varepsilon_i}}\right]^2} \tag{4.44}$$

Where the ratio of alpha to its residual SD is called the information ratio.

Parameters of optimal portfolio

- Risk premium

$$E(\overline{r_P}) = (w_M + w_A\beta_A)E(\overline{r_M}) + w_A\alpha_A \tag{4.45}$$

- Variance

$$\sigma_P^2 = (w_M + w_A\beta_A)^2\sigma_M^2 + (w_A\sigma_{\varepsilon_A})^2 \tag{4.46}$$

4.3. Black-Litterman

The Black-Litterman model (BL) is an optimization method developed by Fischer Black and Robert Litterman (1992), and was originally published in Financial Analysts Journal in 1992. Over the time and due to the popularity of BL approach, a number of extensions to BL have been developed. In this thesis, only the original BL model is introduced and used. BL is based on a combination of inverse optimization and Bayesian statistics. It assumes that the optimal portfolio asset weights are known, represented by their weighting in the market index, and then these weights are subjects of adjustments in accord to the investor's unique views about the future performance of these assets. This is in contrast with MVO, in which the estimates of expected returns are used as a starting point in derivation of optimal weights. Such approach overcomes the major shortcomings of MVO – input sensitivity, high concentration, and estimation error maximization. This brief introduction of BL is based upon the works of Idzorek (2002) and Walters (2014).

The starting point of inverse optimization under BL framework is the derivation of implied equilibrium excess returns, denoted Π, which is a N x 1 column vector resulting from following expression:

$$\Pi = \lambda\Sigma w_{mkt} \tag{4.47}$$

Where $\lambda = \frac{E(r_M) - r_f}{\sigma^2_{of\ M\ excess\ returns}}$ is the risk-aversion coefficient of market portfolio, Σ is the covariance matrix of excess returns, and w_{mkt} is the market capitalization weight N x 1 column vector of the assets.

As in MVO, the optimization goal of BL is the maximization of investor's mean-variance utility

$$\max \quad U = w^T\mu - \frac{1}{2}\lambda w^T \Sigma w \tag{4.48}$$

$$\vdots$$

$$w = (\lambda\Sigma)^{-1}\mu \tag{4.49}$$

Where μ is any vector of excess returns. If $\mu = \Pi$, then $w = w_{mkt}$

If an investor possesses no specific views about the future performance of the assets, he should then hold the portfolio with weights derived from the vector of implied equilibrium returns, i.e. w_{mkt} , which is the view-neutral starting point of the BL model.

The original BL formula is:

$$E(\bar{\mu}) = [(\tau\Sigma)^{-1} + P^T\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q] \tag{4.50}$$

Where

$E(\bar{\mu})$ is the posterior combined return vector (N x 1)

τ is scalar

Σ is covariance matrix of excess returns

P is a (K x N) matrix identifying the assets involved in the views, where K is the number of views and N the number of assets

Ω is a diagonal (K x K) matrix representing the residual variance associated with the expressed views

Q is a (K x 1) column vector of views

Idzorek (2002: 13) describes the BL model as a "*complex weighted average of the implied equilibrium return vector Π and the view vector Q, in which the relative weightings are a function of the scalar τ and the uncertainty of the views Ω.*" Although the BL model doesn't require one to specify any views, the possible incorporation of investor's views within the model is perhaps the most attractive feature of the BL model. The views can be expressed either in an absolute or relative form. The absolute view expresses an idea about an absolute return on an asset, e.g. 5%. The relative view expresses an idea about an asset under- or outperforming relatively to some other asset, e.g. asset A outperforms asset B by 25 b.p. The views form Q (K x 1) matrix. The uncertainty about the views is expressed in the error term vector denoted ε, where each error term $\varepsilon \sim N(0, \sigma^2)$.

$$Q + \varepsilon = \begin{bmatrix} Q_1 \\ \vdots \\ Q_k \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_k \end{bmatrix} \tag{4.51}$$

The expressed views are linked to the assets in question via the matrix P (K x N)

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,n} \end{bmatrix} \tag{4.52}$$

Where each row is associated with one specific view. If the view is positive, the associated weight has a positive sign, e.g. +1, if negative then -1. The sum of weights in each row must be equal to 0 in case of relative views, and equal to 1 in case of absolute views. The actual weighting used in practice is where multiple versions of the BL model differ. Some weighting schemes use equal weighting, market capitalization weighting, or confidence level based weighting expressed as percentage on an intuitive scale 0-1.

The error terms enter the BL formula in form of its variance, denoted ω, and expressed in the Ω matrix

$$\Omega = \begin{bmatrix} \tau\omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \tau\omega_k \end{bmatrix} \tag{4.53}$$

Where

$$\omega_k = P_k \Sigma P_k^T \tag{4.54}$$

The scalar τ should be more or less inversely proportional to the relative weight given to Π. However, its recommended value differs across literature and its variation is one of the ways how to calibrate the model for specific needs. Black and Litterman recommend to use values close to zero, such as often recommended τ=0.0025.

The last step of BL optimization is to obtain the combined return vector $E(\bar{\mu})$, Equation 4.50, and plug it into the Equation 4.49, which returns the optimal portfolio weights.

4.4. Naïve Optimization

Naïve portfolio optimization methods are methods that may be used when a little, or none, statistical information about the assets, e.g. their means, variances, and/or correlations, is available to the investor. When no information is available and, at the same time, the investor possesses no knowledge of the capital market theory, the intuitive way to create a diversified optimal portfolio is to spread his available wealth across chosen assets equally. With increasing awareness and available information, more naïve methods come into play. This thesis introduces three naïve optimizers, the equal weighting, the Sharpe ratio based method, and the most diversified portfolio method.

4.4.1. Equal Weighting

Equal weighting is a logical starting point for any investor, who desires to allocate his available capital across multiple assets which he has no information about, and/or is unaware of the capital market theory. Without any information, all the assets should look exactly the same to the investor and thus he should allocate his capital across them equally. Assuming n assets, the individual asset weighting is following:

$$w_i = \frac{1}{n} \tag{4.55}$$

The equal weighting approach appears to be quite popular among the investors. Potentially due to its simplicity, or since it requires no estimations of parameters, it does not suffer the estimation error. Another strong argument in favor of equal weighting is its empirical evidence, for instance, DeMiguel et al. (2009) and Playkha et al. (2012), showcasing that such portfolios usually strongly outperform portfolios build under the mean-variance framework. Thus, it raises a question whether practitioners using 1/N approach are unaware of the information regarding the assets, or are aware of the outcomes of such empirical studies and have decided to exploit them (Kinlaw et al., 2017).

4.4.2. Sharpe Ratio Model

Sharpe ratio based asset allocation model (SRM) is slightly more sophisticated than the equal weighting as it requires the parameter estimates, yet still quite naïve in its fundamental nature. Sharpe ratio (SR) is a simple measure providing an information about asset's risk premium per one unit of total risk measured by standard deviation. Simply put, the higher the SR the better the investment. It is fair to note that high SR doesn't necessarily mean the highest return or the lowest risk. SR and SR based allocation are calculated as following (Amenc and Le Sourd, 2003):

$$SR = \frac{\mu - r_f}{\sigma} \tag{4.56}$$

Where the numerator represents the asset's risk premium and the denominator its standard deviation.

Assuming n assets within a portfolio, the SR based allocation is following:

$$w_i = \frac{SR_i}{\sum_{i=1}^{n} SR_i} \tag{4.57}$$

This naïve optimization approach based on Sharpe ratios of individual assets as a starting point for asset allocation is slightly different than the Sharpe ratio maximization problem for complete portfolios and defined as

$$\max \quad \frac{w^T \mu - r_f}{\sqrt{w^T \Sigma w}}$$

$$w^T I = 1$$

Where r_f is the mean risk-free rate and I is the column vector of ones.

This maximization problem seeks the highest SR of a complete portfolio. In theory, the highest SR is guaranteed for portfolios lying on the CML, i.e. the market portfolio (M) with possible long/short position in risk-free asset. Should one consider rather a subset of n risky assets instead of M, the highest SR is then guaranteed for portfolios lying on the CAL, i.e. the tangency portfolio (T) with possible long/short position in risk-free asset. Eventually, the CAL can be steeper, and therefore have a higher SR, than CML[9]. The optimal weights of T can be easily obtained via MVO as described in Section 4.1.2. However, such MVO weights may suffer the shortcomings of MVO, such as extreme long/short positions or allocation only to few assets. These shortcomings do not apply within the SRM. On the other hand, it may suffer from the cumulative estimation error caused, for instance, by insufficient data sets.

4.4.3. Most Diversified Portfolio

Most diversified portfolio (MDP) is a naïve optimization method focused on maximization of the diversification ratio (DR). The maximization problem for a portfolio of n assets is following (Scherer, 2015):

$$\max DR = \max \frac{\sum_{i=1}^{n} w_i \sigma_i}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{i,j}}} = \max \frac{w^T \sigma}{\sqrt{w^T \Sigma w}} \quad (4.58)$$

Where σ is a N x 1 column vector of asset volatilities. DR itself can be interpreted in a following way: the lower the correlation between the assets, the higher the ratio. If all assets were perfectly correlated, the ratio equals 1.

The solution to the maximization problem in order to obtain the optimal portfolio weights is following:

$$w_{mdp} = \frac{\Sigma^{-1} \sigma}{I^T \Sigma^{-1} \sigma} \quad (4.59)$$

Where I is the N x 1 column vector of ones, and Σ^{-1} is the N x N inverse covariance matrix.

The MDP represents a mean-variance portfolio, where it is assumed that all asset returns are proportional to their standard deviations. The level of proportionality is usually defined by a constant SR.

$$\mu_i = SR \sigma_i \quad (4.60)$$

[9] Beating the market in terms of performance is the ultimate goal of active portfolio management

5. Portfolio Performance

Performance measurement of investments is an essential part of any investment process. Performance analysis can be done both ex-ante and ex-post. Ex-ante analysis may be of a help to an investor before making an investment. Whether conducting a scenario analysis or analysis of historical data, such obtained values should only be taken with reserve as they do not possess a real predicting value but rather only orientational. Or put differently, the historical performance never guarantees the future performance due to the risks associated with the investment. During the holding period, the value to the portfolio is being added through a variety of sources such as superior asset allocation, security selection, market timing, transaction executions etc. Logically, any investor is curious and wants to know how well his investments have been doing. Performing a periodical performance analysis may serve as an underlying evidence for potential changes in his investment strategy. After the holding period, therefore ex-post, it is possible to conduct an overall, exact, risk-adjusted performance assessment to see how the portfolio performed. Then, the answers to questions such as What is the total return? or Why the portfolio performed that way? can be answered. Portfolio's risk-adjusted performance measures can also be used for comparing mutually exclusive portfolios between themselves. Although there are dozens of portfolio performance measures available, this chapter introduces only the ones that are used within its practical part. All presented measures can be found in Bacon (2008).

5.1.1. Time-Weighted Return

Time-weighted rate of return (TWR) is a measure of a per-period compounded return on an investment where each time period is given an equal weight regardless the money invested[10]. It is calculated as a geometric mean of a series of n realized returns. TWR is useful as an average return in a long-term perspective.

$$TWR = \sqrt[n]{(1 + r_1)(1 + r_2) \dots (1 + r_n)} - 1 = \sqrt[n]{\prod_{i=1}^{n}(1 + r_i)} - 1 \qquad (5.1)$$

[10] Measure where each time period is weighted by the money invested is called the money-weighted return.

5.1.2. Effective Annual Return

Effective annual return (EAR) is a measure of total return over a multi-period time frame that reflects compounding[11]. Alongside the daily frequency, returns are often expressed on a weekly, monthly, or yearly basis. Therefore, the average per-period return requires adjustment to meet the criteria. EAR is calculated as following:

$$EAR = (1 + r)^n - 1 \quad (5.2)$$

Where r is the average per-period return, and n is the number of periods

5.1.3. Standard Deviation

Variance (σ^2) is a statistical measure of total dispersion of a random variable, such as asset returns, around its mean. It can be calculated both ex-ante, as expected variance, and ex-post, as historical variance. Its square root, the standard deviation (σ or SD) or volatility, is the most common measure of asset's risk. Historical sample SD is calculated in following way:

$$SD = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2} \quad (5.3)$$

Alternatively, it is possible to use the properties of the market model and obtain SD as

$$SD = \sqrt{\beta^2 \sigma_M^2 + \sigma_\varepsilon^2} \quad (5.4)$$

Where σ_M^2 is the market variance and σ_ε^2 is the variance of residuals.

The residual Var/SD [Var(e)/SD(e)] represents the asset's specific risk. It is introduced in a more detailed way in Section 3.2 regarding the market model and diversification.

SD can be adjusted to longer periods as $SD_T = SD\sqrt{T}$ where T represents the number of periods, e.g. from daily to monthly or yearly. This adjustment is, however, only an approximation.

[11] The measure of total, multi-period return that does not reflect compounding is called Annualized Percentage Rate (APR) and is calculated $APR = r * n$

5.1.4. Beta

Beta as a measure of asset's systematic risk is closely introduced in Chapter 3 and its Section 3.2.1 about diversification.

5.1.5. Sharpe Ratio

Sharpe ratio (SR), or reward-to-volatility ratio, is a risk adjusted measure of portfolio performace. It measures the amount of portfolio's excess return per one unit of total risk measured by SD. SR is the slope of the capital allocation line. The larger the SR, the better.

$$SR = \frac{r_P - r_f}{\sigma_P} \tag{5.5}$$

Where r_p is the return on portfolio P, r_f is the risk-free rate, and σ_P is the SD of P. In ex-ante calculations, mean values are used. In ex-post, the actual realized values are used.

5.1.6. Treynor Ratio

Treynor ratio (TR) is a risk adjusted measure of portfolio performance. It is similar to SR, but with the difference that it considers systematic risk only. It assumes that all portfolios are well diversified, and thus the idiosyncratic risk no longer matters. It measures the amount of portfolio's excess return per one unit of systematic risk measured by beta. TR is the slope of the security market line. The larger the TR, the better.

$$TR = \frac{r_P - r_f}{\beta_P} \tag{5.6}$$

Where β_P is the systematic risk of P. In ex-ante calculations, mean values are used. In ex-post, the actual realized values are used.

Few comments regarding the TR need to be mentioned. TR is a theoretical concept assuming that all portfolios are well diversified, thus with no idiosyncratic risk. However, real portfolios usually carry some idiosyncratic risk. Therefore, portfolios that have the same amount of systematic risk but differ in total risk, will have the same TR.

5.1.7. Jensen's Alpha

Jensen's alpha (α) is a risk-adjusted measure of superior performace. α is the intercept of the standard CAPM regression equation. It compares the portfolio's realized excess return with the risk-adjusted excess return. The measure adjusts for systematic risk. The regression equation is following:

$$r_{P,t} - r_{f,t} = \alpha_{P,t} + \beta_P(r_{M,t} - r_{f,t}) + \varepsilon_{P,t} \tag{5.7}$$

Where t represents a time point within the time series.

In ex-post analysis, the error term can be ignored and alpha is then calculated by using the actual realized returns

$$\alpha_P = r_P - r_f - \beta_P(r_M - r_f) \tag{5.8}$$

This is called the Jensen's alpha, or Jensen's measure, or Jensen's differential return, or ex-post alpha.

Positive α indicates a superior risk-adjusted return, i.e. the portfolio's return is higher than what it should be in accord to the level of undertaken risk. In CAPM universe, $\alpha = 0$. Positive α therefore lies above the SML and is desired. Positive α can be, for instance, due to the portfolio manager's superior security selection or timing skills. Jensen's α is a measure often used to evaluate portfolio managers. However, it does not evaluate the manager's ability to diversify as it accounts the systematic risk only.

5.1.8. Information Ratio

Information ratio (IR) is a measure of portfolio's excess return (er) relative to market portfolio per one unit of tracking error, which is the standard deviation of those excess returns. Positive IR represents a superior performance. IR is a key statistic and is extensively used as it evaluates the portfolio manager's abilities to generate excess returns. The higher the IR, the better.

$$IR = \frac{r_P - r_M}{\sigma_{er}} \tag{5.9}$$

Where r_P is the return on P, r_M is the return on market index used as a benchmark, and σ_{er} is the tracking error calculated as following:

$$\sigma_{er} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(er_t - \overline{er})^2} \tag{5.10}$$

Where $er_t = r_{P,t} - r_{M,t}$, and $\overline{er}$ is the mean excess return.

The tracking error can also be seen as a function of portfolio's SD and the correlation between the portfolio and the market, calculated as following:

$$\sigma_{er} = \sigma_P \sqrt{(1 - \rho_{P,M}^2)} \tag{5.11}$$

IR can be obtained in an alternative way by regressing the standard CAPM, Equation 3.4.

$$IR_P = \frac{\alpha_P}{\sigma_{\varepsilon_P}} \tag{5.12}$$

Where α_P is the Jensen's alpha, and σ_{ε_P} is the standard error of regression.

In ex-ante IR calculations, mean values are used. In ex-post, the realized values are used.

The IR can also be used in calculation of portfolio's SR by using Equation 4.44.

Since IR is extensively used for comparing different portfolios, it is essential to maintain consistency of how the individual elements are calculated, e.g. the frequency of data, overall time period, arithmetic or geometric means and excess returns, T or T-1, ex-post or ex-ante. The IR is often expressed on an approximate annualized basis as $IR = \sqrt{T} IR_P$.

5.1.9. M^2

M^2 is a measure of risk-adjusted return relative to the market. It compares the hypothetical return on portfolio with adjusted SD to match the SD of market with the return on market. The adjusted risk of portfolio is achieved by long/short positions in the risk-free asset. M^2 is a highly useful measure for comparing portfolios with different levels of risk. The higher the M^2, the better.

$$M^2 = r_P + SR_P(\sigma_M - \sigma_P) \tag{5.13}$$

Or, alternatively

$$M^2 = r_f + \frac{\sigma_M}{\sigma_P}(r_P - r_f) \tag{5.14}$$

In ex-ante M^2 calculations, mean values are used. In ex-post, the actual realized values are used. M^2 of market is always equal to its return.

6. Portfolio Management

Portfolio management is a sophisticated discipline that can be viewed as a process undertaken in a consistent manner to create and maintain investment portfolios that meet the investor's objectives. The objectives are defined beforehand and provide an underlying framework for the portfolio management. In a professional world, the investor's objectives are stated in a document so-called the Investment Policy Statement (IPS). IPS contains information regarding the investor's return expectations, risk profile, time horizon, along with possible constrains such as liquidity needs or tax concerns, among others.

Portfolio management consists of three main steps conducted along the way: (1) planning step, (2) execution step, and (3) feedback step. In the planning step, the IPC is created, market expectations are formed, and investment strategy is established. In the execution step, the investment portfolio is constructed and managed accordingly to the investment strategy. In the feedback step, the portfolio performance is monitored on a constant basis and compared with the IPC. Each of the steps deserves a closer look. This thesis, however, provides only a brief description. For more complex description, see Maginn et al. (2007).

6.1.1. Planning Step

As stated, the planning step consists of the creation of IPC, formation of market expectations, and establishing an investment strategy. The investment strategy involves the strategic asset allocation (SAA) reflecting both IPC and market expectations. Investment strategies are passive, active, or semi-active

1) Passive Strategy

Passive, or not reacting, strategy represents a portfolio management that doesn't react anyhow to market fluctuations. Two most used forms of passive strategy management are indexing and buy-and-hold.

- Indexing

Portfolio is designed to replicate the performance of a specific market index as accurately as possible. The replication can be either physical or synthetic. In physical replication, the exposure to assets is direct and is either full, i.e. holding all constituents with identical weighting as does the market index, or sample, i.e. holding only some of the index constituents. In synthetic replication, the exposure to the index constituents is indirect through derivatives.

- Buy-and-hold

A strategy under which the selected assets are bought and held long term in order to profit from the capital gains and/or additional sources of income such as dividends.

2) Active Strategy

In contract to passive strategy, the active strategy does acknowledge the market fluctuations and tries to construct such optimal portfolio that exploits them as much as possible in order to achieve superior risk-adjusted portfolio performance relative to the market, i.e. to achieve positive alpha.

3) Semi-Active Strategy

Semi-active, or risk-controlled active, or enhanced index approach, is a combination of active and passive approach. The strategy seeks positive alpha as well, and at the same keeps tight control over portfolio's risk relative to the benchmark.

6.1.2. Execution Step

The execution step turns plans into reality. It consists the construction of actual portfolio in accord to what is established in the planning step. The portfolio composition must reflect the investor's objectives and risk profile. Once the optimal portfolio is established, it must be managed accordingly to the SAA. However, if the investment horizon is long-term, some deviations from SAA can be made, usually done on purpose and for limited time. These purposefully made deviations are called the tactical asset allocation (TAA). The prime difference between SAA and TAA is in the time length. While SAA involves the long-term objectives, the TAA involves short-term adjustments to SAA in order to exploit the expected market fluctuations, and, consequently, realize additional superior returns. TAA is thus based on a constant monitoring of both the market and the portfolio. Monitoring and rebalancing belong to the third step of the portfolio management process, the feedback step. Therefore, the execution step and the feedback step are closely related.

6.1.3. Feedback Step

The fluctuations in market values of individual assets create deviations of the current asset allocation from the SAA. Although these deviations may not matter much in short term, their cumulative effect may significantly impact the overall portfolio performance, and may cause significant deviations from the investor's long term objectives. The way how portfolio manager approaches these deviations is the core difference between passive and active portfolio management. Whilst the passive management fundamentally ignores the deviations in allocation, the active management reacts to them by rebalancing the current asset allocation to make the allocation consistent with the SAA. In order to properly rebalance, the portfolio manager must monitor both market and portfolio on continuous basis. Monitoring, rebalancing, and performance evaluation are core elements of the feedback step.

1) Monitoring

Monitoring is an essential part of the feedback step. Being aware, or not, about all possible influences that may have an impact on the portfolio is the difference whether the investor's objectives are going to be reached or not. Therefore, the portfolio manager should keep a constant eye over the investor's circumstances, market and economic changes, and the portfolio itself. Any changes must be dealt with in an appropriate manner.

2) Rebalancing

Portfolio rebalancing represents adjustments in current asset allocation as a reaction to (1) fluctuations in market values of assets in order to be consistent with SAA, (2) changes in investor's objectives, constrains, or market expectations, and (3) tactical asset allocation. In this thesis, only the scenario 1 is considered and introduced bit further. In scenario 1, the portfolio manager sells appreciated assets and buys depreciated assets in case of long positions, or buys appreciated assets and sells depreciated assets in case of short positions, to make the actual composition consistent with SAA. In practice, two most common rebalancing practices are calendar rebalancing and percentage-of-portfolio rebalancing.

- Calendar rebalancing
  - Rebalancing happens on a periodic basis, e.g. monthly, quarterly, semi-annually, or annually.
- Percentage-of-portfolio rebalancing (or percent range or interval rebalancing)
  - Rebalancing is triggered when an asset's weight crosses a pre-specified corridor or tolerance band. Let's assume three assets A, B, and C with SAA, in percentages, 40/40/20, respectively. Assets A and B have corridor $\pm$ 5%, and asset C $\pm$ 1,5%. If the weighting of any asset exceeds the tolerance corridor, the rebalancing is triggered and the initial weighting 40/40/20 is re-established.

Although rebalancing does have its undoubtful benefits, e.g. it reduces the present value of expected utility loss coming from not tracking the optimum, it does have its shortcomings as well, e.g. transaction costs and/or tax costs[12] applied on sale of the appreciated assets. These shortcomings, however, may be reduced by imposing constrains on them in the portfolio optimization process.

3) Performance evaluation

Portfolio performance evaluation is described in Chapter 5.

[12] Tax liability depends on a particular jurisdiction and on whether the investor is a subject to taxation or not.

7. Practical Experiment

The practical part of this master's thesis is based on an experiment that put multiple quantitative portfolio optimization methods into a contest. Different optimizers were applied to portfolios composed of identical assets, which were subsequently held under different portfolio management styles over a pre-specified period of time. The performance of each portfolio was measured ex-post, adequately evaluated in accord with the criteria of the experiment, and confronted with the others.

The assets included in the experiment's portfolio P (P) are 30 US blue-chip stocks, US 4-week Treasury bill (T-bill), and S&P 500 market index. The stocks were selected intuitively without any equity analysis done beforehand. Nonetheless, they were picked with a sense for diversification and represent all leading industries. The complete list of all portfolio components with a brief description is presented in annex at the end of the thesis.

The optimization models selected for the experiment represent 3 sophisticated and 3 naïve models. The first group includes the Mean-variance optimization model (MVO), the Treynor-Black model (TB), and the Black-Litterman model (BL). The latter includes the Equal-weighting model (1/N), the Sharpe ratio based model (SRM), and the Most diversified portfolio model (MDP). The models are presented and descripted in the theoretical part. The models were exercised on the portfolio P which resulted in 6 different suggested optimal asset allocations, thus 6 different portfolios.

Two different portfolio management styles were used in the experiment, active and passive. To assure fair starting point between them, each portfolio was created twice, resulting in two equal sets of 6 portfolios. The market portfolio included in the experiment served as a benchmark to these portfolios. Therefore, the total number of portfolios was 13, 6 active, 6 passive and 1 market portfolio. The portfolios under the active management were rebalanced on daily basis in accord to the asset's daily adjusted closing prices to maintain the initial optimal allocation suggested by the models. The portfolios under the passive management were untouched throughout the experiment in accord to the passive portfolio management strategy buy-and-hold. Both management styles are descripted in Chapter 6.

The total time period of the experiment was from 2/1/2013 to 29/6/2018, and can be split into two smaller periods with 31/12/2017 being the breaking point. The period from 2/1/2013 to 29/12/2017, ex-ante, served as an estimation period to obtain in-sample estimates of inputs for the models. The period from 2/1/2018 to 29/6/2018, the holding period (HLDP), is the period over which the portfolios were held and managed. The appropriate time series for both periods were collected retrospectively.

After the holding period, ex-post, the performance of each portfolio was measured by the appropriate portfolio measures presented in Chapter 5. Each measure was treated equally, and therefore each measure does have an equal weight within the grading system used as a final evaluation tool. The grading system is a simple methodology that gives points to each portfolio accordingly to its ranking within the chart of each measure. The higher the ranking, the more points the portfolio receives. The maximum number of points for each measure was 12. As 9 measures (EAR, SD, Residual SD, α, β, SR, TR, IR, M^2) were used, the theoretical maximum number of points a portfolio could have collected was 108. The portfolio that collected the most points won.

The stock's and market's daily data, i.e. the adjusted closing prices, were obtained from http://finance.yahoo.com. The daily rates on US 4 Week Treasury bill were obtained from http://www.quandl.com.

To assure feasibility of the experiment, the following assumptions were assumed:

- Markets are perfectly efficient
- Returns are Gaussian
- No transaction costs, no taxes
- All assets are perfectly divisible and can be bought/sold in fractions
- Unlimited long and short positions are allowed
- Agent is risk tolerant, therefore his risk aversion played no role in the experiment
- Although risk tolerant, agent does have mean-variance preferences
- Agent accepts the optimal portfolio suggested by each model
- Agent has no specific views about the future
- Agent's initial capital for each portfolio was $ 1 000 000
- S&P 500 index monetization followed 1 index point = $ 1 parity.

7.1. Optimal Portfolios

This section presents the optimal portfolios suggested by each optimization model. The suggested relative and absolute allocation to the assets in the portfolios is presented in tables at the end of this section.

1) MVO

The optimal portfolio suggested by MVO is the tangency portfolio, as it is the only efficient portfolio composed only of risky assets. Market portfolio and risk-free asset are not included.

2) TB

The optimal portfolio suggested by TB is a combination of market portfolio and portfolio P. The risk-free asset is not included.

3) BL

In accordance with the experiment's assumptions, the agent has no views about the future[13] and thus the matrices Q, P, and Ω are omitted. Consequently, the optimal portfolio suggested by BL is the portfolio derived from the vector of implied equilibrium excess returns. The suggested weight vector is identical to the weight vector based on market capitalizations. The market cap of each firm considered corresponds to its value on 29/12/2017. The market caps are presented in the annex within the description of each firm. Market portfolio and risk-free asset are not included.

4) 1/N

The optimal portfolio suggested by 1/N model is the portfolio P with equal allocation across all stocks. Market portfolio and risk-free asset are not included.

5) SRM

The optimal portfolio suggested by SRM is the portfolio P with allocation proportional to SR of each stock. Market portfolio and risk-free asset are not included.

[13] This assumption was made to assure consistency with other models

6) MDP

The optimal portfolio suggested by MDP is the portfolio P with allocation proportional to the inverse of individual volatility of the stocks. Market portfolio and risk-free asset are not included.

Optimal relative asset allocation suggested by the models

	MVO	TB	BL	1/N	SRM	MDP
AAPL	0.06355	0.19750	0.15801	0.03333	0.04063	0.08491
AMT	0.07397	0.12338	0.01107	0.03333	0.03270	0.00292
AMZN	0.08082	0.22512	0.10155	0.03333	0.04755	0.04234
APD	0.10798	0.21117	0.00646	0.03333	0.04133	-0.03210
BA	0.40654	0.54200	0.03151	0.03333	0.06414	0.03191
C	-0.34041	0.19323	0.03665	0.03333	0.02326	-0.03834
CAT	0.02037	0.00740	0.01683	0.03333	0.02797	0.03651
DWDP	0.06091	0.10465	0.01575	0.03333	0.03603	-0.00708
EA	0.16266	0.29388	0.00586	0.03333	0.05749	0.07185
EQR	-0.06446	-0.01698	0.00423	0.03333	0.01928	0.08322
FSLR	-0.05178	-0.01903	0.00127	0.03333	0.01319	0.04638
GS	-0.02297	-0.06653	0.01782	0.03333	0.03007	-0.03868
HD	0.42196	0.63630	0.04039	0.03333	0.06276	-0.02105
INTC	0.09178	0.17357	0.03921	0.03333	0.03915	0.01536
JNJ	0.43291	0.59027	0.06779	0.03333	0.05354	-0.05837
K	-0.13020	-0.04688	0.00424	0.03333	0.01719	0.08613
KIM	-0.23431	-0.17522	0.00140	0.03333	0.00771	0.03471
KO	-0.22498	-0.01007	0.03537	0.03333	0.02269	0.04720
MAR	0.21148	0.44607	0.00914	0.03333	0.05697	-0.03784
MCD	0.25269	0.48702	0.02520	0.03333	0.05008	0.07741
NKE	0.03792	0.22499	0.01845	0.03333	0.04151	0.06404
NUE	-0.09132	0.11537	0.00367	0.03333	0.01766	0.05203
PFE	-0.06458	0.01210	0.03894	0.03333	0.02747	0.09399
PG	0.06074	0.07348	0.04235	0.03333	0.02799	0.04512
REGI	0.02498	-0.00736	0.08253	0.03333	0.01190	0.07860
T	-0.01267	-0.02802	0.04315	0.03333	0.02092	0.10972
TSLA	0.09298	0.12748	0.00939	0.03333	0.04222	0.05388
WBA	-0.05692	0.06584	0.01325	0.03333	0.02815	0.08190
WFC	0.10855	0.01789	0.05444	0.03333	0.03357	0.00694
XOM	-0.41820	-0.48442	0.06406	0.03333	0.00483	-0.01362
SP500	0	-2.39700	0	0	0	0
SUM	1	1	1	1	1	1

Table 3: Optimal relative asset allocation suggested by the models

Optimal absolute asset allocation suggested by the models

	MVO	TB	BL	1/N	SRM	MDP
AAPL	$ 63,552.22	$ 197,500.88	$ 158,009.76	$ 33,333.33	$ 40,633.89	$ 84,906.96
AMT	$ 73,974.06	$ 123,375.87	$ 11,068.33	$ 33,333.33	$ 32,703.14	$ 2,916.14
AMZN	$ 80,821.10	$ 225,116.20	$ 101,552.78	$ 33,333.33	$ 47,550.59	$ 42,341.61
APD	$ 107,975.82	$ 211,171.52	$ 6,464.21	$ 33,333.33	$ 41,331.87	$ (32,098.17)
BA	$ 406,543.44	$ 542,003.26	$ 31,511.21	$ 33,333.33	$ 64,142.60	$ 31,914.03
C	$ (340,408.94)	$ (193,230.76)	$ 36,646.78	$ 33,333.33	$ 23,263.73	$ (38,337.44)
CAT	$ 20,373.73	$ 7,395.06	$ 16,834.78	$ 33,333.33	$ 27,968.58	$ 36,511.34
DWDP	$ 60,912.73	$ 104,648.80	$ 15,750.18	$ 33,333.33	$ 36,033.43	$ (7,077.69)
EA	$ 162,661.01	$ 293,879.63	$ 5,862.26	$ 33,333.33	$ 57,486.70	$ 71,846.00
EQR	$ (64,462.90)	$ (16,982.91)	$ 4,233.55	$ 33,333.33	$ 19,283.46	$ 83,222.26
FSLR	$ (51,776.13)	$ (19,028.88)	$ 1,274.40	$ 33,333.33	$ 13,193.82	$ 46,375.75
GS	$ (22,970.83)	$ (66,534.09)	$ 17,816.34	$ 33,333.33	$ 30,073.55	$ (38,682.82)
HD	$ 421,960.39	$ 636,300.19	$ 40,386.84	$ 33,333.33	$ 62,764.67	$ (21,050.12)
INTC	$ 91,775.43	$ 173,565.81	$ 39,210.05	$ 33,333.33	$ 39,151.34	$ 15,363.25
JNJ	$ 432,912.35	$ 590,266.44	$ 67,789.23	$ 33,333.33	$ 53,541.59	$ (58,365.94)
K	$ (130,202.70)	$ (46,882.69)	$ 4,240.78	$ 33,333.33	$ 17,192.95	$ 86,129.74
KIM	$ (234,309.23)	$ (175,216.01)	$ 1,397.32	$ 33,333.33	$ 7,710.40	$ 34,713.33
KO	$ (224,980.81)	$ (10,071.02)	$ 35,374.19	$ 33,333.33	$ 22,693.62	$ 47,199.86
MAR	$ 211,483.07	$ 446,070.05	$ 9,137.74	$ 33,333.33	$ 56,970.62	$ (37,844.13)
MCD	$ 252,686.17	$ 487,024.65	$ 25,202.46	$ 33,333.33	$ 50,084.13	$ 77,413.71
NKE	$ 37,916.29	$ 224,993.35	$ 18,447.22	$ 33,333.33	$ 41,511.93	$ 64,043.36
NUE	$ (91,320.34)	$ (115,369.39)	$ 3,671.37	$ 33,333.33	$ 17,664.81	$ 52,025.72
PFE	$ (64,576.11)	$ 12,102.58	$ 38,938.90	$ 33,333.33	$ 27,467.74	$ 93,990.56
PG	$ 60,741.35	$ 73,478.71	$ 42,353.58	$ 33,333.33	$ 27,989.78	$ 45,117.94
REGI	$ 24,984.71	$ (7,355.47)	$ 82,527.11	$ 33,333.33	$ 11,896.86	$ 78,601.01
T	$ (12,666.17)	$ (28,022.63)	$ 43,152.57	$ 33,333.33	$ 20,923.34	$ 109,717.79
TSLA	$ 92,977.53	$ 127,482.25	$ 9,392.62	$ 33,333.33	$ 42,223.63	$ 53,877.12
WBA	$ (56,920.79)	$ 65,835.33	$ 13,251.99	$ 33,333.33	$ 28,149.87	$ 81,903.29
WFC	$ 108,546.43	$ 17,894.67	$ 54,439.62	$ 33,333.33	$ 33,565.07	$ 6,942.76
XOM	$ (418,202.86)	$ (484,416.07)	$ 64,061.82	$ 33,333.33	$ 4,832.30	$ (13,617.21)
SP500	$ -	$ (2,396,995.32)	$ -	$ -	$ -	$ -
SUM	$ 1,000,000.00	$ 1,000,000.00	$ 1,000,000.00	$ 1,000,000.00	$ 1,000,000.00	$ 1,000,000.00

Table 4: Optimal absolute asset allocation suggested by the models

The graphical standings of each portfolio's expected per-period return and volatility before the experiment in comparison with the Markowitz's efficient frontier:

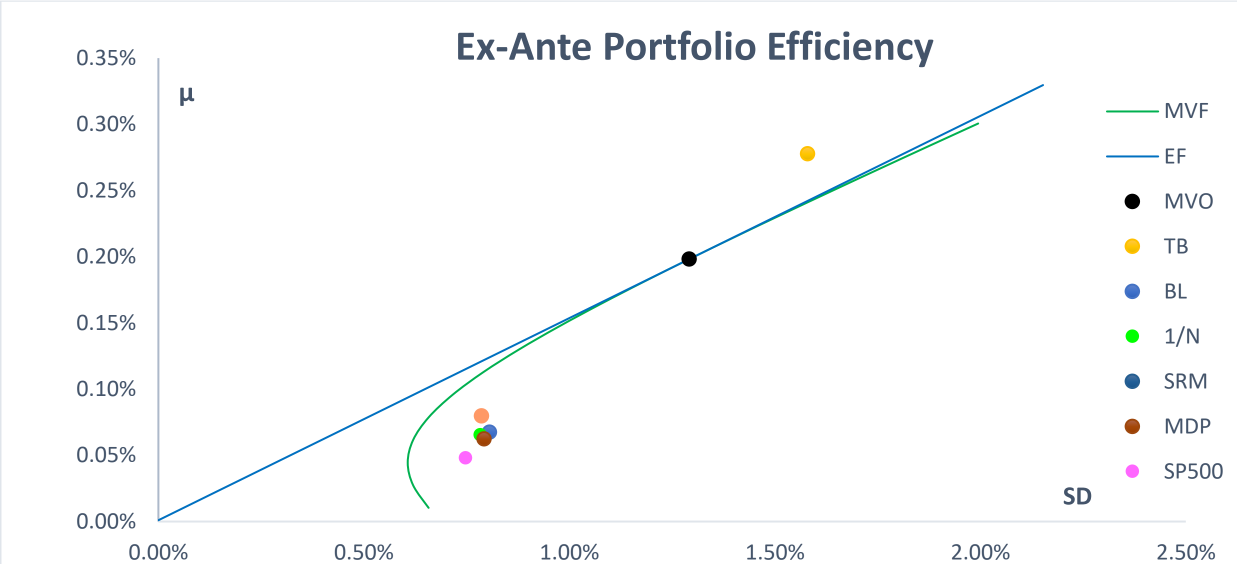

Figure 4: Ex-ante portfolio efficiency

As this was the starting point of the experiment, there is no distinguishing between actively and passively managed portfolios. However, from now onwards such distinguishing is necessary. Therefore, the actively managed portfolios have an attribute (A) to their name, e.g. MVO (A). Likewise, the passively managed portfolios have an attribute (P), e.g. MVO (P).

7.2. Results

This chapter presents the results the portfolios achieved at the end of the holding period (HLDP). During the HLDP, active portfolios were daily rebalanced to maintain the initial optimal allocation suggested by the models, and passive portfolios were left untouched on their own according to the buy-and-hold strategy. After the HLDP, performance measures used within the evaluation system, introduced in Chapter 7, were calculated. Each measure was calculated by using the 'annualized' values. The 'annualization' was made to make the values consistent with the holding period, which was 124 trading days. The holding period return on US 4 Week Treasury bill, used as the risk-free rate, was 0.4611%. The holding period return on S&P 500, used as the market return, was 0.8368%. This chapter is organized as a presentation of the individual results with follow up commentary and is closed with general discussion regarding the final rankings.

7.2.1. Presentation

In this section, each measure with individual results and corresponding portfolio rankings is presented one by one. As risk is, generally, undesired, the charts of risk measures are sorted from min to max to award the less risky portfolios on account of the riskier ones. As the other measures are desired rather higher than lower, the corresponding charts are sorted from max to min. Follow up commentary regarding the individual results and the chart with final rankings are presented at the end.

1) EAR

Portfolios sorted by realized EAR from max to min. Max being the best.

EAR		
1th	TB (P)	24.472%
2nd	TB (A)	19.388%
3rd	MVO (P)	16.744%
4th	MVO (A)	14.026%
5th	BL (P)	7.596%
6th	MDP (P)	7.155%
7th	BL (A)	5.749%
8th	MDP (A)	4.995%
9th	SRM (P)	3.099%
10th	1/N (P)	1.277%
11th	SRM (A)	1.209%
12th	S&P 500	0.837%
13th	1/N (A)	-0.564%

Table 5: EAR

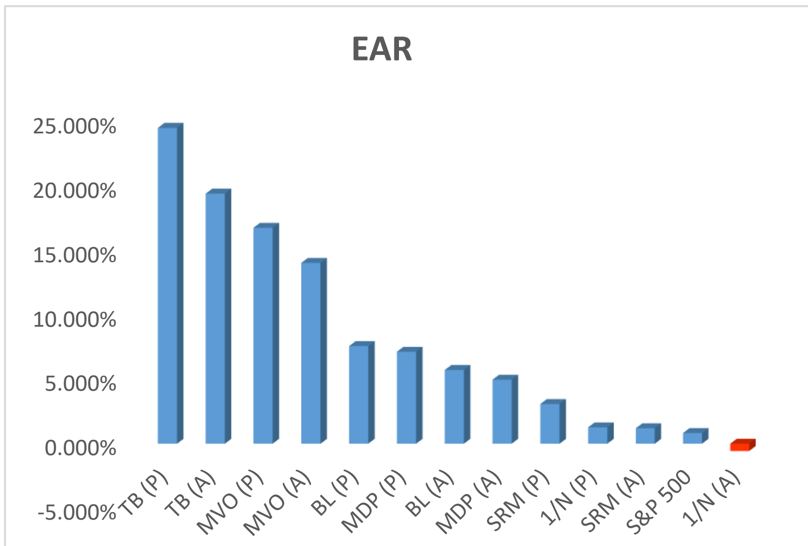

Figure 5: EAR

2) Standard Deviation

Portfolios sorted by their SD from min to max. Min being the best.

Standard Deviation		
1th	MDP (P)	10.709%
2nd	MDP (A)	10.728%
3rd	1/N (P)	11.331%
4th	1/N (A)	11.381%
5th	BL (P)	11.415%
6th	BL (A)	11.462%
7th	S&P 500	11.557%
8th	SRM (P)	12.135%
9th	SRM (A)	12.188%
10th	MVO (P)	20.448%
11th	MVO (A)	20.554%
12th	TB (P)	22.685%
13th	TB (A)	22.863%

Table 6: Standard deviation

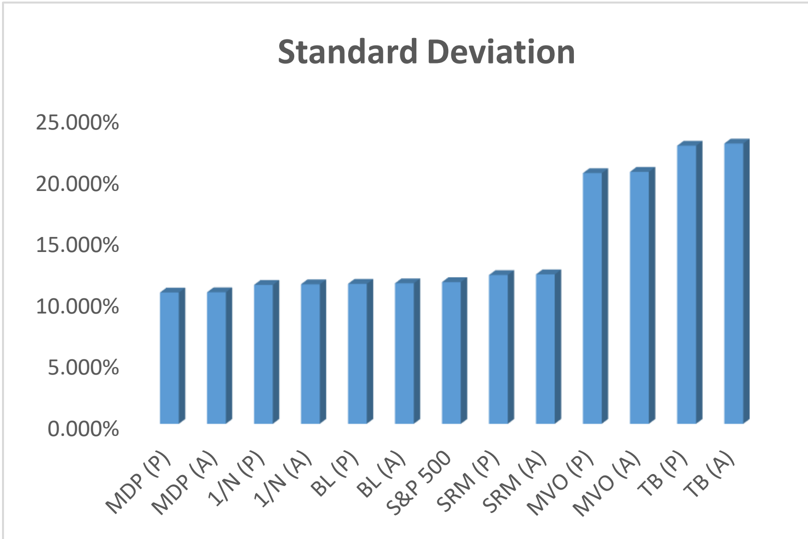

Figure 6: Standard deviation

3) Residual Standard Deviation

Portfolios sorted by their residual SD from min to max. Min being the best.

Residual SD		
1th	S&P 500	0.000%
2nd	1/N (A)	2.652%
3rd	1/N (P)	2.662%
4th	SRM (A)	2.746%
5th	SRM (P)	2.759%
6th	BL (A)	2.944%
7th	BL (P)	2.971%
8th	MDP (A)	5.180%
9th	MDP (P)	5.199%
10th	MVO (P)	12.378%
11th	MVO (A)	12.426%
12th	TB (P)	16.808%
13th	TB (A)	16.975%

Table 7: Residual SD

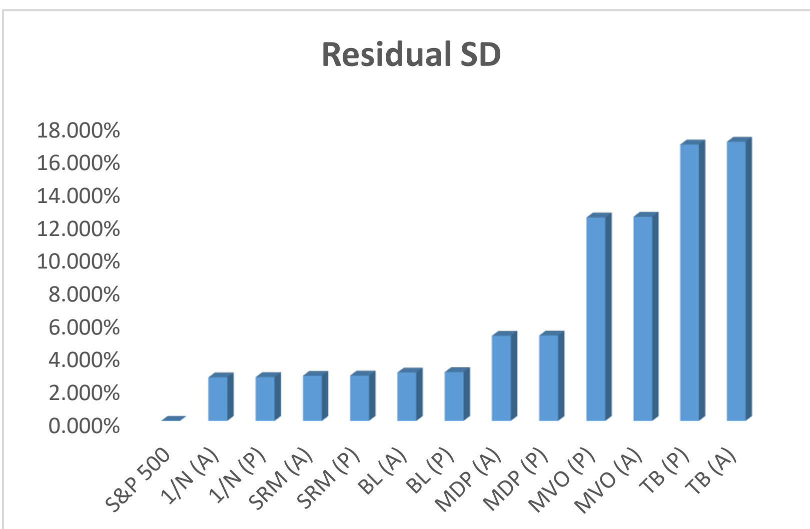

Figure 7: Residual SD

4) Jensen's Alpha

Portfolios sorted by their alpha from max to min. Max being the best.

Jensen's α		
1th	TB (P)	23.516%
2nd	TB (A)	18.429%
3rd	MVO (P)	15.754%
4th	MVO (A)	13.033%
5th	BL (P)	6.777%
6th	MDP (P)	6.390%
7th	BL (A)	4.928%
8th	MDP (A)	4.228%
9th	SRM (P)	2.253%
10th	1/N (P)	0.457%
11th	SRM (A)	0.362%
12th	S&P 500	0.000%
13th	1/N (A)	-1.385%

Table 8: Jensen's alpha

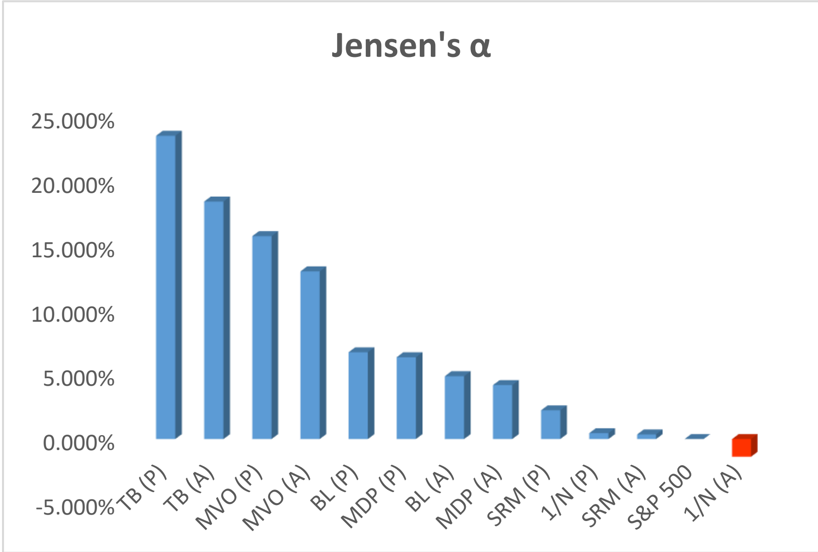

Figure 8: Jensen's alpha

5) Beta

Portfolios sorted by their beta from min to max. Max being the best.

β		
1th	MDP (P)	0.810
2nd	MDP (A)	0.813
3rd	1/N (P)	0.953
4th	BL (P)	0.954
5th	1/N (A)	0.958
6th	BL (A)	0.958
7th	S&P 500	1.000
8th	SRM (P)	1.022
9th	SRM (A)	1.027
10th	TB (P)	1.318
11th	TB (A)	1.325
12th	MVO (P)	1.408
13th	MVO (A)	1.417

Table 9: Beta

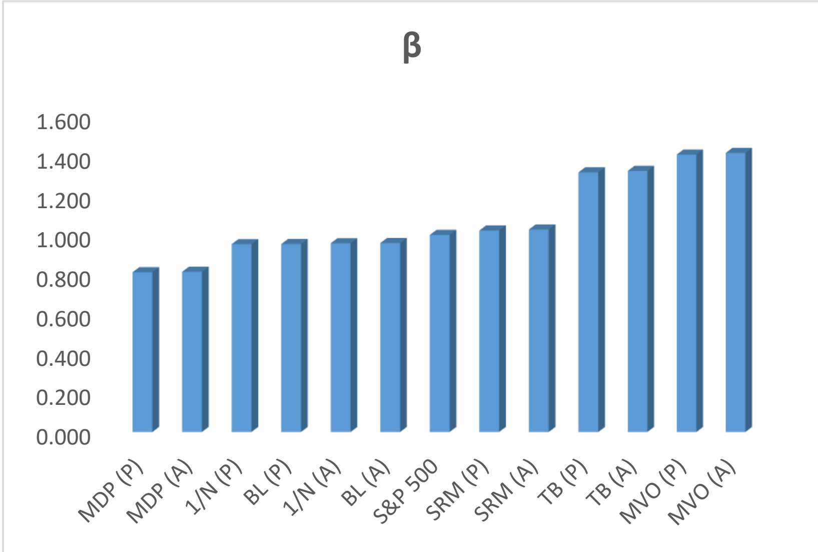

Figure 9: Beta

6) Sharpe Ratio

Portfolios sorted by their Sharpe ratio from max to min. Max being the best.

Sharpe Ratio		
1th	TB (P)	1.058
2nd	TB (A)	0.828
3rd	MVO (P)	0.796
4th	MVO (A)	0.660
5th	MDP (P)	0.625
6th	BL (P)	0.625
7th	BL (A)	0.461
8th	MDP (A)	0.423
9th	SRM (P)	0.217
10th	1/N (P)	0.072
11th	SRM (A)	0.061
12th	S&P 500	0.033
13th	1/N (A)	-0.090

Table 10: Sharpe ratio

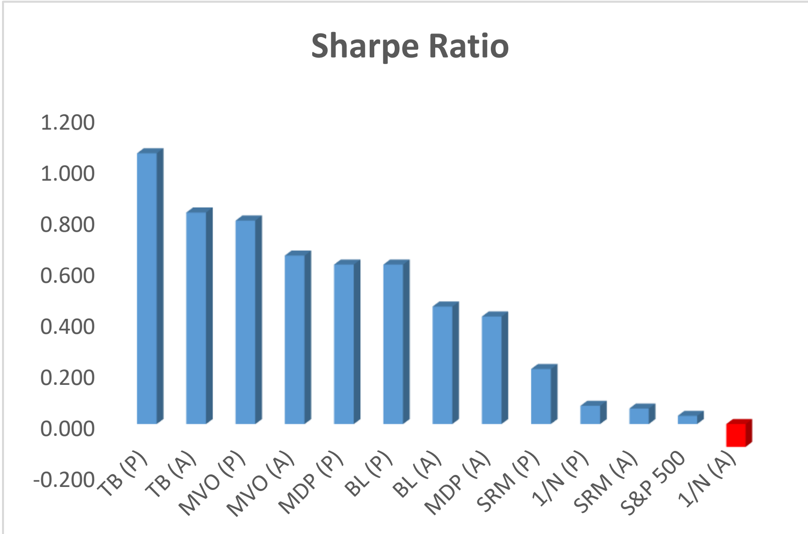

Figure 10: Sharpe ratio

7) Treynor Ratio

Portfolios sorted by their Treynor ratio from max to min. Max being the best.

Treynor Ratio		
1th	TB (P)	0.182
2nd	TB (A)	0.143
3rd	MVO (P)	0.116
4th	MVO (A)	0.096
5th	MDP (P)	0.083
6th	BL (P)	0.075
7th	MDP (A)	0.056
8th	BL (A)	0.055
9th	SRM (P)	0.026
10th	1/N (P)	0.009
11th	SRM (A)	0.007
12th	S&P 500	0.004
13th	1/N (A)	-0.011

Table 11: Treynor ratio

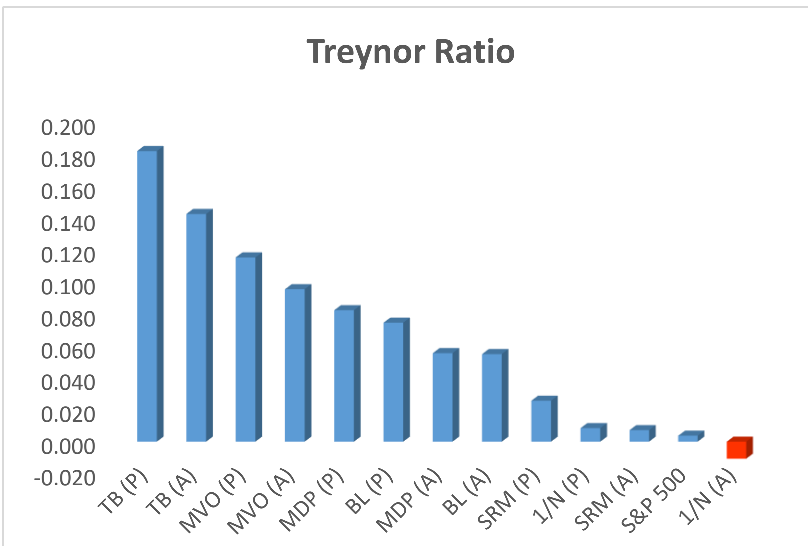

Figure 11: Treynor ratio

8) Information Ratio

Portfolios sorted by their information ratio from max to min. Max being the best.

Information Ratio		
1th	BL (P)	2.281
2nd	BL (A)	1.674
3rd	TB (P)	1.399
4th	MVO (P)	1.273
5th	MDP (P)	1.229
6th	TB (A)	1.086
7th	MVO (A)	1.049
8th	SRM (P)	0.817
9th	MDP (A)	0.816
10th	1/N (P)	0.172
11th	SRM (A)	0.132
12th	S&P 500	0.000
13th	1/N (A)	-0.522

Table 12: Information ratio

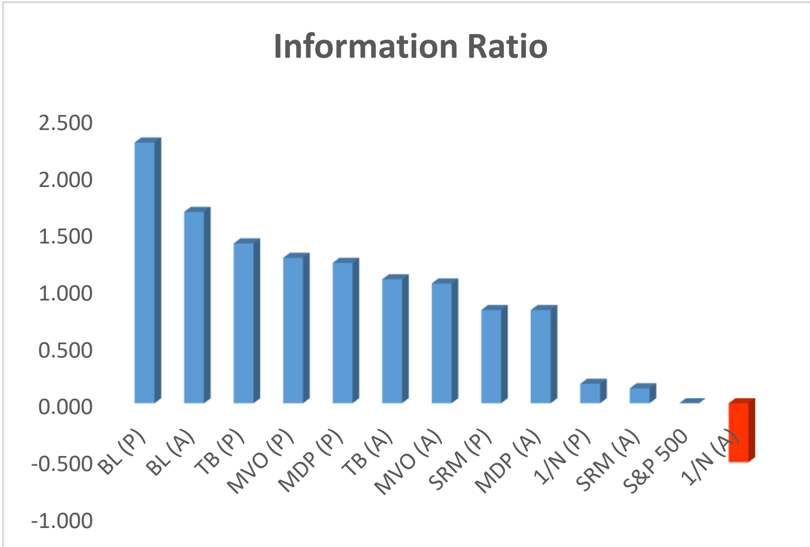

Figure 12: Information ratio

9) M^2

Portfolios sorted by their M^2 measure from max to min. Max being the best.

M^2		
1th	TB (P)	12.694%
2nd	TB (A)	10.029%
3rd	MVO (P)	9.665%
4th	MVO (A)	8.089%
5th	MDP (P)	7.685%
6th	BL (P)	7.685%
7th	BL (A)	5.793%
8th	MDP (A)	5.345%
9th	SRM (P)	2.973%
10th	1/N (P)	1.293%
11th	SRM (A)	1.171%
12th	S&P 500	0.837%
13th	1/N (A)	-0.580%

Table 13: M^2

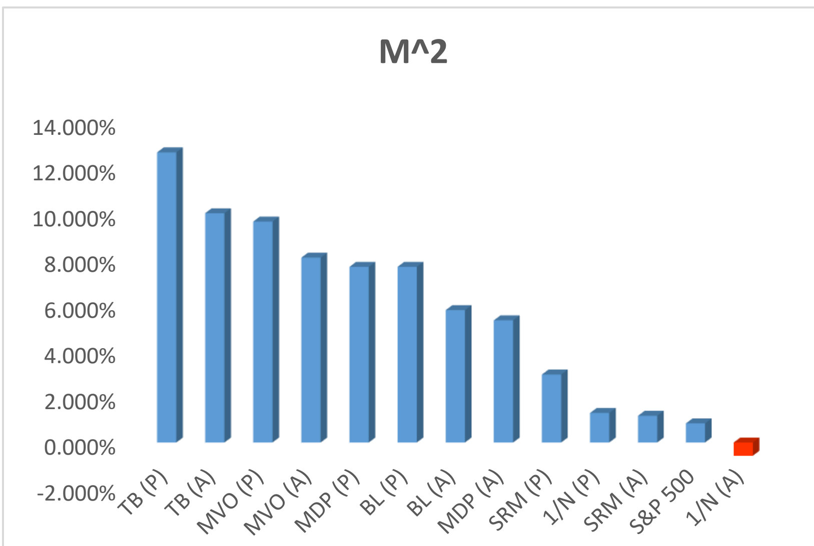

Figure 13: M^2

7.2.2. Commentary regarding the individual results

In this section, a brief commentary regarding the results across the performance measures is provided. Assessment of overall rankings with discussion is presented in the next section.

It is convenient to begin with the risk measures as everything else is more or less connected to them. The total risk of each portfolio was measured by its standard deviation. By far the most volatile portfolios were the ones suggested by MVO and TB, regardless the management. Yet, the difference of approximately 2 percentage points (p.p.) in favor of MVO counts for good. It is interesting to see that this difference of ca. 2 p.p. represents an entire spread between the rest of portfolios. Both MDP portfolios scored the lowest volatility and lived up to their name. The portfolios on 3^{rd} – 7^{th} place manifested more or less identical volatility regardless the significant differences in their allocations.

The level of carried systematic risk was measured by each portfolio's beta. The chart of individual betas corresponds, with small differences, to the one of SD. This is not surprising as systematic risk is part of total risk. Interesting observation, S&P 500 with $\beta = 1$ ended up exactly in the middle of beta chart, that is 6 portfolios with higher β and 6 with lower.

Specific risk was measured by each portfolio's residual standard deviation. As any market portfolio is considered to be perfectly diversified and thus to carry zero specific risk, it is logical that S&P 500 won this category. Nonetheless, shall one compare all three risk measures, it is interesting to see the diversification power of 1/N model as it scores high in all of them. As for the residual SD, little bit disappointing are the results of MDP portfolios as they carry nearly double of residual risk than other naïve portfolios plus BL. Unsurprisingly, MVO and TB portfolios ended up dead last with residual volatility multiple times higher than its competitors. Taking a look at all three risk measures, it is interesting to see that different portfolio management played insignificant role as both (A) and (P) portfolios scored similar values per each method.

In the mean-variance framework, there should always be a positive trade-off between risk and return. More risk should provide an adequate extra return. The total return of each portfolio was measured by EAR. By far the highest return was achieved by the second riskiest portfolio TB (P), followed by the riskiest TB (A). The absolute difference of 5.084 p.p. between them is a value that nearly half of all portfolios didn't even reach at all. 1/N (A) is the only portfolio that scored a negative return.

Although measuring total return and risk is interesting, in order to be consistent with the portfolio theory it is necessary to include risk-adjusted measures in the performance analysis as they provide better informative value about how much was achieved with respect to the undertaken risk.

Both Sharpe and Treynor ratios measure amount of excess return per one unit of risk, differing in the risk measure used. SD and β, respectively. The winning portfolios in both measures are TB. Since the main purpose of TB optimization is Sharpe ratio maximization, the model lived up to its purpose. Winning in TR then comes hand in hand. TB portfolios are followed by MVO, MDP and BL portfolios. In both ratios, 1/N (A) scored negative due to its negative excess return.

Jensen's Alpha is a measure of superior performance that compares realized excess returns with risk-adjusted excess returns. Positive α is what everybody seeks and the higher, the better. The highest α was achieved by TB portfolios, followed by MVO. All portfolios achieved positive α but one, 1/N (A). From obvious reasons, S&P 500 has α = 0.

Information ratio measures portfolio's excess return relative to market per one unit of tracking error. Positive IR means the portfolio has beat the market. All portfolios achieved positive IR but one, 1/N (A). By far the winning portfolio was BL (P), followed by BL (A). TB (P) ranked 3rd, followed by MVO portfolios. From obvious reasons, S&P 500 has IR = 0

M^2 is a hypothetical measure of risk-adjusted return relative to market. It says what the return should be if the portfolio's SD equaled the market's SD. Market M^2 = market return. Therefore, portfolio desires M^2 return ≥ market return. The highest M^2 was achieved by TB (P), followed by TB (A), MVO (P), and MVO (A), which are the portfolios that achieved the highest EAR. The M^2 chart matches the Sharpe ratio chart. The only portfolios with M^2 below market return is 1/N (A).

Portfolios sorted by their results in individual performance measures

	max to min	min to max	min to max	max to min	min to max	max to min	max to min	max to min	max to min
	EAR	**SD**	**Res SD**	**α**	**β**	**SR**	**TR**	**IR**	**M^2**
1 th	TB (P)	MDP (P)	S&P 500	TB (P)	MDP (P)	TB (P)	TB (P)	BL (P)	TB (P)
2 nd	TB (A)	MDP (A)	1/N (A)	TB (A)	MDP (A)	TB (A)	TB (A)	BL (A)	TB (A)
3 rd	MVO (P)	1/N (P)	1/N (P)	MVO (P)	1/N (P)	MVO (P)	MVO (P)	TB (P)	MVO (P)
4 th	MVO (A)	1/N (A)	SRM (A)	MVO (A)	BL (P)	MVO (A)	MVO (A)	MVO (P)	MVO (A)
5 th	BL (P)	BL (P)	SRM (P)	BL (P)	1/N (A)	MDP (P)	MDP (P)	MDP (P)	MDP (P)
6 th	MDP (P)	BL (A)	BL (A)	MDP (P)	BL (A)	BL (P)	BL (P)	TB (A)	BL (P)
7 th	BL (A)	S&P 500	BL (P)	BL (A)	S&P 500	BL (A)	MDP (A)	MVO (A)	BL (A)
8 th	MDP (A)	SRM (P)	MDP (A)	MDP (A)	SRM (P)	MDP (A)	BL (A)	SRM (P)	MDP (A)
9 th	SRM (P)	SRM (A)	MDP (P)	SRM (P)	SRM (A)	SRM (P)	SRM (P)	MDP (A)	SRM (P)
10 th	1/N (P)	MVO (P)	MVO (P)	1/N (P)	TB (P)	1/N (P)	1/N (P)	1/N (P)	1/N (P)
11 th	SRM (A)	MVO (A)	MVO (A)	SRM (A)	TB (A)	SRM (A)	SRM (A)	SRM (A)	SRM (A)
12 th	S&P 500	TB (P)	TB (P)	S&P 500	MVO (P)	S&P 500	S&P 500	S&P 500	S&P 500
13 th	1/N (A)	TB (A)	TB (A)	1/N (A)	MVO (A)	1/N (A)	1/N (A)	1/N (A)	1/N (A)

Table 14: Portfolio rankings by categories

Final ranking

	EAR	SD	Res SD	α	β	SR	TR	IR	M^2
TB (P)	12	1	1	12	3	12	12	10	12
MDP (P)	7	12	4	7	12	8	8	8	8
BL (P)	8	8	6	8	9	7	7	12	7
MVO (P)	10	3	3	10	1	10	10	9	10
TB (A)	11	0	0	11	2	11	11	7	11
BL (A)	6	7	7	6	7	6	5	11	6
MDP (A)	5	11	5	5	11	5	6	4	5
MVO (A)	9	2	2	9	0	9	9	6	9
1/N (P)	3	10	10	3	10	3	3	3	3
SRM (P)	4	5	8	4	5	4	4	5	4
S&P 500	1	6	12	1	6	1	1	1	1
SRM (A)	2	4	9	2	4	2	2	2	2
1/N (A)	0	9	11	0	8	0	0	0	0

Table 15: Final point chart

	SUM of Points	Final Rank
TB (P)	75	1st
MDP (P)	74	2nd
BL (P)	72	3rd
MVO (P)	66	4th
TB (A)	64	5th
BL (A)	61	6th
MDP (A)	57	7th
MVO (A)	55	8th
1/N (P)	48	9th
SRM (P)	43	10th
S&P 500	30	11th
SRM (A)	29	12th
1/N (A)	28	13th

Table 16: Final ranking

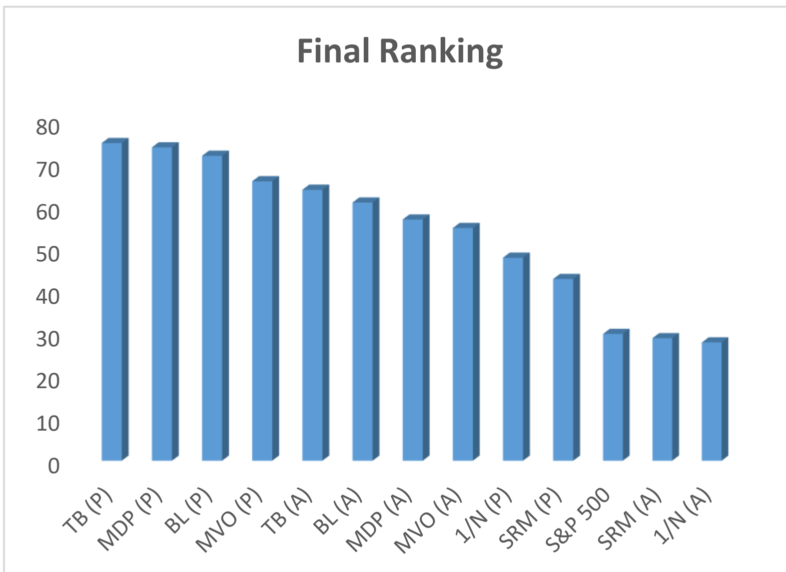

Figure 14: Final ranking

7.3. Discussion

This section provides a general discussion regarding the final results the portfolios achieved in the experiment. The final results are presented in Table 16 on previous page.

The gold medal for winning the experiment goes to TB (P) portfolio, silver medal to MDP (P), and bronze medal to BL (P). The imaginary ‘potato medal’ goes to MVO (P). It is pleasant to see that first five positions are occupied by portfolios suggested by different models. The initial expectation that active management leads to a superior performance has been disproved by the experiment as the passive portfolios occupy the first four spots. This, however, is not a big surprise should one take a closer look. In this experiment, the active management happened on daily basis which turned out to be quite cumbersome as it showcased a number of shortcomings. Firstly, assuming the initial optimal allocation static was wrong. The optimal allocation evolves over the time as much as the values of assets themselves. Therefore, the rebalancing should not be happening to match the initial optimal allocation, but rather the new optimum. Secondly, finding new optimum and rebalancing to it on continuous basis is a challenging process with no guarantee. Thirdly, daily rebalancing is costly. Should this experiment had considered the transaction costs, the actively managed portfolios would have ended up probably dead last with profits completely erased. These factors lead to a conviction that daily rebalancing makes no real sense. Rebalancing happening at longer periods, such as quarterly, semi-annually, or annually, and with respect to the new optimum, sounds more reasonable. Either or, the passive strategy buy-and-hold proved to be more efficient throughout the experiment. These findings correspond to the outcomes of similar studies regarding the issues of active versus passive portfolio management, for example Pace et al. (2016) and Cox (2017).

The selected optimizers provided a balanced mix of sophisticated and naïve models. The sum of points for sophisticated models is 393 versus 279 for naïve models. Nonetheless, this victory does possess no informative value and can be considered as irrelevant. Each model deserves an independent assessment.

All sophisticated models used in this thesis have few things in common, e.g. they require an extensive collection of historical data. Since future is, from its very nature, unknown, any estimation of inputs based solely on historical values is deceptive. Input estimates are therefore almost always a pure noise but the models accept them as true and simply deliver a solution. This is misleading as one may think the delivered solution is the ‘true optimum’. With this in mind, the Black-Litterman model is the only model that somehow deals with it and allows the investor to combine the observed market information with his or hers views about the future, which makes the model somewhat prospective.

On the other hand, the issue of estimation error is well known and there are approaches for its minimization. These models are usually based on Bayesian shrinkage, resampling, or robust optimization. It would had been interesting to include these models in the experiment to see how they would do.

Although the portfolios suggested by TB did well in the experiment, their results need to be taken with a reserve as they are biased with the selected monetization of the index. In reality, index cannot be bought just like that and an index tracking mutual fund, for example, is included instead. Both TB and MVO optimizers suggested portfolios with quite large long/short positions. This may not be an issue for institutional investors, but for individuals most certainly yes.

The most naïve optimizer, 1/N, performed well as a diversification tool. It scored high in all three risk related measures. Nonetheless, both 1/N portfolios performed otherwise poorly which is rather disappointing. Especially should one consider its success in other studies.

Both SRM portfolios showcased poor performance in all categories. Both delivered low returns and carried considerable amount of risk, which, combined, resulted in low scores in all risk-adjusted measures.

The last naïve optimizer, MDP, happened to be a pleasant surprise. Both portfolios achieved solid scores in all risk-adjusted measures. In all but one categories MDP (P) was superior to MDP (A). MDP (P) happened to be the least risky portfolio, but still delivered a reasonable return of 7.155% with alpha 6.39%. Scoring relatively high in all the categories has brought the MDP (P) to the final second place. However, keeping in mind the biasness of TB portfolios, this silver medal has a golden glow.

The total amount of imaginary money invested into all 12 portfolios at the beginning of the experiment was $ 12 000 000, with $ 1 000 000 each. How the portfolio values were developing during the holding period is depicted on the following Figure 15.

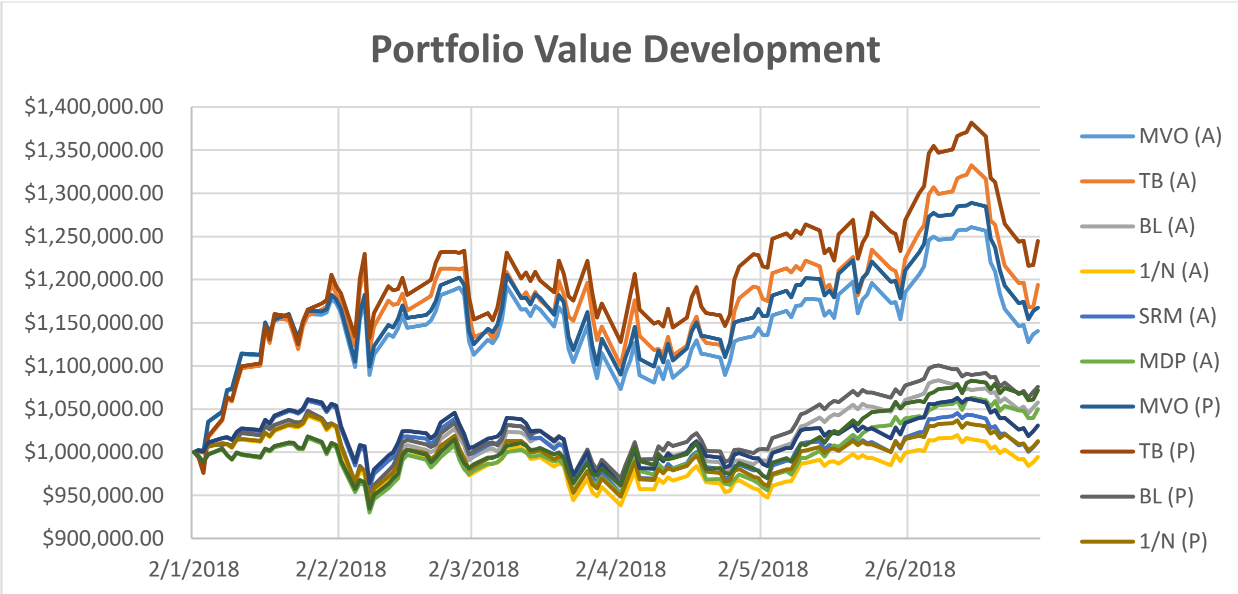

Figure 15: Portfolio value development

As can be deciphered from the graph, all portfolios exhibit strong co-integration. This is no surprise as all portfolios are composed of identical assets. The steep start of TB and MVO portfolios is caused by their relatively higher betas in comparison to the rest of portfolios. The effect of individual betas is also responsible for the spread variations throughout the entire holding period. All portfolios but one, 1/N (A), show increasing trend. The final value of each portfolio with the total return realized from the experiment is summarized in the following table.

	Initial Value	Final Value
MVO (A)	$ 1,000,000.00	$ 1,140,262.18
TB (A)	$ 1,000,000.00	$ 1,193,876.01
BL (A)	$ 1,000,000.00	$ 1,057,493.73
1/N (A)	$ 1,000,000.00	$ 994,355.69
SRM (A)	$ 1,000,000.00	$ 1,012,094.80
MDP (A)	$ 1,000,000.00	$ 1,049,947.67
MVO (P)	$ 1,000,000.00	$ 1,167,443.93
TB (P)	$ 1,000,000.00	$ 1,244,722.63
BL (P)	$ 1,000,000.00	$ 1,075,961.58
1/N (P)	$ 1,000,000.00	$ 1,012,766.18
SRM (P)	$ 1,000,000.00	$ 1,030,987.77
MDP (P)	$ 1,000,000.00	$ 1,071,553.88
SUM	$ 12,000,000.00	$ 13,051,466.05
Profit		$ 1,051,466.05
Return		8.762%

Table 17: Final values

8. Conclusion

The master's thesis with a topic 'Portfolio Optimization Methods, Their Application and Evaluation' focused on practical application of various quantitative portfolio optimization methods, their performance, usefulness, pros and cons. The quantitative portfolio management is a complete, data-driven process in which an investor builds, optimizes, holds, and adjusts portfolios in order to achieve superior risk-adjusted returns with respect to the applied constrains, such as his or her risk aversion, budget limitations, turnover constrains etc. The pivotal point of the thesis was to conduct an experiment in which a number of selected optimizers were put in a contest.

The underlying theory on which the experiment was built is presented in the first part of the thesis. It introduces the underpinnings of portfolio theory, describes the optimization process, introduces the selected optimization methods, and provides an overview of portfolio management. The experiment itself, together with the results, is presented in the second part of the thesis, called 'Practical Experiment'.

The selected optimizers represented both sophisticated and naïve models. The sophisticated included the classical Markowitz's mean-variance model, the Treynor-Black model, and the Black-Litterman model. The naive included the 1/N model, the Sharpe ratio based model, and the Most diversified portoflio model. The optimizers were applied to portfolios composed of identical assets and held under different portfolio management styles over a pre-specified period of time. The performance of each portfolio was measured ex-post, adequately evaluated in accord with the criteria of the experiment, and confronted with the others.

The questions that this master's thesis tried to find answers to were (1) which portfolio optimizer, out of the selected ones, performs the best, and (2) whether it is beneficial to conduct rather an active, or a passive portfolio management.

As presented in Section 7.2 devoted to the results of the experiment, the passively management portfolios demonstrated superiority to the actively managed ones as they occupy the first four spots in the final ranking. The expected superiority of sophisticated models was disproved as both of the portfolios suggested by MDP ranked among them. Moreover, the passively managed MDP portfolio left behind all portfolios but one, the Treynor-Black passive portfolio. The TB (P) won the experiment with a final score of 75 points. However, the overall performance of TB portfolios got biased by the selected monetization of the market index, and thus their results should be taken with reserve. With respect to this bias, the logical move is to look at the second place where shines the MDP (P) portfolio with a final score of 74 points, only 1 point behind the TB (P).

Is therefore the MDP the best optimizer and the buy-and-hold strategy the best management? No and no.

Regarding the optimizers, the selected models created a balanced mix but represent only a tip of an iceberg. Due to the steep development of quantitative finance since approximately the 1990’, the set of available optimizers has enlarged significantly. Models based on, for example, resampled efficiency, robust optimization, Bayesian shrinkage, or on various downside risk measures, have their undoubtful charm. Furthermore, should one consider the number of possible ways for input estimations, the set of optimizers starts to grow exponentially. To make a relevant statement, all the optimizers with all their possible adjustments would had needed to be included in the experiment as well. Nonetheless, in order to conclude the experiment the MDP (P) approach happened to be the most efficient one.

Regarding the management, the passive form has its undoubtful perks such as difficulty and costs. It is easy to do and it has no turnover costs coming from the rebalancing. However, the optimal allocation is dynamic and changes over time. Therefore, buy-and-hold strategy might deviate too much from the optimum in the long run and can get possibly behind its potential. Replicating an index may be a way, however full replication is expensive. Daily rebalancing is unnecessarily complicated and expensive as well, which makes such management style ineffective. The turnover costs can get eased by the additional income from dividends, yet way too frequent rebalancing almost guarantees the costs > dividends inequality. Rebalancing at longer periods, e.g. quarterly, semi-annualy, or annualy, and with respect to the new optimum, seems to be more reasonable. Either or, the used management style should be periodically evaluated and adjusted accordingly.

The world of quantitative portoflio management is evolving at a pace faster than ever. This thesis has only scratched the surface, and even though this final chapter is called Conclusion, it is rather a personal beginning.

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Annex

Apple Inc. (AAPL)

Apple Inc. is an American multinational technology company with its main area of business in designing, developing, and selling of consumer electronics, computer software, and online services. The company was founded in 1976 and has its headquarters in Cupertino, California. Since its foundation, Apple has become one of the largest companies in the world. It is a component of S&P 100, S&P 500, DJIA, and Nasdaq 100 indices. The company is listed on Nasdaq and its market capitalization on December 29, 2017 was $ 874,11bn.

Electronic Arts Inc. (EA)

Electronic Arts Inc. is an American company engaged in the video gaming industry. EA was founded in 1982 and has its headquarters in Redwood city, California. EA is a component of Nasdaq 100 and S&P 500 indices, is listed on Nasdaq, and its market capitalization on December 29, 2017 was $ 32,43bn.

Intel Corporation (INTC)

Intel Corporation is an American multinational technology company specialized in invention, innovation, and production of computer hardware such as processors, motherboards, network interface controllers and so on. Intel was founded in 1968 and has its headquarters in Santa Clara, California. INTC is a component of the S&P 100, S&P 500, DJIA, and Nasdaq 100 indices, is listed on Nasdaq, and its market capitalization on December 29, 2017 was $ 216,91bn.

American Tower Corporation (AMT)

American Tower Corporation is a global provider of wireless and broadcast communications infrastructure. The company was founded in 1995 and has its headquarters in Boston, Massachusetts. AMT is a component of the S&P 500 index, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 61,23bn.

Equity Residential (EQR)

Equity Residential is an American real estate investment company. The company’s investment model is focused on acquisition, development, and management of rental apartment buildings. The company was founded in 1969 and is headquartered in Chicago, Illinois. EQR is a component of the S&P 500 index, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 23,42bn

Kimco Realty Corporation (KIM)

Kimco Realty Corporation is an American real estate investment trust with specialization in the segment of shopping malls. The trust was founded in 1958 and is headquartered in New Hyde Park, New York. KIM is a component of the S&P 500 index and is listed on NYSE. The market capitalization on December 29, 2017 was $ 7,73bn.

Johnson & Johnson (JNJ)

Johnson & Johnson is an American multinational company specialized in production of pharmaceuticals, health care products, and medical devices. The company was founded in 1886 and is headquartered in New Brunswick, New Jersey. JNJ is a component of the S&P 100, S&P 500, and DJIA indices. Its stock is listed on NYSE and its market capitalization on December 29, 2017 was $ 375,01bn.

Pfizer Inc. (PFE)

Pfizer Inc. is an American multinational pharmaceutical company and one of the global leaders within its field. The company was founded in 1849 and has its headquarters in New York City, New York. PFE is a component of the S&P 100, S&P 500, and DJIA indices, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 215,41bn.

Procter & Gamble Co. (PG)

Procter & Gamble Co. is an American multinational company with specialization in production of personal care products, beauty care products, health care products, cleaning agents, and similar. The company was founded in 1837 and has its headquarter in Cincinnati, Ohio. PG is a component of the S&P 100, S&P 500, and DJIA indices, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 234,30bn.

Exxon Mobil Corporation (XOM)

Exxon Mobil Corporation is an American multinational company specialized in oil and natural gas industry. The company was founded in 1999 and is headquartered in Irving, Texas. XOM is a component of the S&P 100, S&P 500, and DJIA indices, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 354,39bn.

Renewable Energy Group Inc. (REGI)

Renewable Energy Group Inc. is an American multinational company focused on renewable energy with specialization in biofuel, biomass-based diesel, renewable chemicals, and carbon lowering solutions. The company was founded in 1996 and is headquartered in Ames, Iowa. REGI is listed on Nasdaq and its market capitalization on December 29, 2017 was $ 456,54bn.

First Solar Inc. (FSLR)

First Solar Inc. is an American multinational company involved in the photovoltaic industry with focus on production of solar panels, rigid thin film modules, and similar. The company was founded in 1999 and has its headquarters in Tempe, Arizona. FSLR is a component of the S&P 400 index, is listed on Nasdaq, and it market capitalization on December 29, 2017 was $ 7,05bn.

Citigroup Inc. (C)

Citigroup Inc. is an American multinational bank providing banking and financial services. It is one of the biggest banks worldwide. It was founded in 1813 and is headquartered in New York City, New York. C is a component of the S&P 100 and S&P 500 indices, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 202,73bn.

Wells Fargo & Company (WFC)

Wells Fargo & Company is an American multinational company providing financial services of all kinds. The company was founded in 1852 and is headquartered in New York City, New York. WFC is a component of the S&P 100 and S&P 500 indices, is listed on NYSE, and its market capitalization on December 29, 2017 was $ 301,16bn.

The Goldman Sachs Group Inc. (GS)

The Goldman Sachs Group Inc. is an American multinational investment bank and financial services provider. The company was founded in 1869 and has its headquarters in New York City, New York. GS is a component of the S&P 100, S&P 500, and DJIA indices. Its stock is listed on NYSE and its market capitalization on December 29, 2017 was $ 98,56bn.

Tesla Inc. (TSLA)

Tesla Inc. is an American multinational company specializing in electromobility, lithium-ion energy battery storage systems, and solar panel production. The company was founded in 2003 and is headquartered in Palo Alto, California. TSLA is a component of Russell 1000 and Nasdaq 100 indices. The company is listed on Nasdaq and its market capitalization on December 29, 2018 was $ 51,96bn.

The Boeing Company (BA)

The Boeing Company is an American multinational company that specializes in manufacturing of airplanes, rotorcraft, rockets, satellites, and missiles. The company was founded in 1916 and is headquartered in Chicago, Illinois. BA is a component of the S&P 100, S&P 500, and DJIA indices. Its stock is listed on NYSE and its market capitalization on December 29, 2017 was $ 174,32bn.

Caterpillar Inc. (CAT)

Caterpillar Inc. is an American multination company specializing mainly in invention and production of heavy machinery and engines. However, it is also a provider of insurance and financial services via its global site of subsidiaries. The company was founded in 1925 and has its headquarter in Deerfield, Illinois. CAT is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 93,13bn.

The Kellogg Company (K)

The Kellogg Company is an American multinational company specializing in the food processing industry. The company was founded in 1906 and has its headquarters in Battle Creek, Michigan. K is a component of the S&P 500 index and is listed on NYSE. The market capitalization on December 29, 2017 was $ 23,46bn.

The Coca Cola Company (KO)

The Coca Cola Company is an American multinational company specializing in production of non-alcoholic beverages and syrups. The company was founded in 1886 and has its headquarters in Atlanta, Georgia. KO is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 195,69b.

McDonald’s (MCD)

McDonald’s is an American multinational company operating in the fast food and real estate industry. The company was founded in 1940 and has its headquarters in Chicago, Illinois.

MCD is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 139,42b.

AT&T Inc. (T)

AT&T Inc. is an American multinational company engaged in the telecommunications and mass media industry. It is the world's largest provider of telecommunications services and world's second largest provider of mobile services. The company was founded in 1983 and has its headquarters in Dallas, Texas. T is a component of the S&P 100 and S&P 500 indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 238,72b.

Amazon.com Inc. (AMZN)

Amazon.com Inc. is an American multinational company operating in the e-commerce and cloud services business. Amazon.com is the biggest internet retailer in the world. The company was founded in 1994 and has its headquarters in Seattle, Washington. AMZN is a component of the S&P 100, S&P 500, and Nasdaq 100 indices. The company is listed on Nasdaq and its market capitalization on December 29, 2017 was $ 561,79b.

Walgreens Boots Alliance Inc. (WBA)

Walgreens Boots Alliance Inc. is an American holding company created after a merge between a US based The Walgreens Company and a Switzerland based Alliance Boots. The company operates in the pharmaceutical industry as one of the largest drug and health care product retail reseller. The Walgreens Company was founded in 1901 and the merge took place in 2014. The company is headquartered in Deerfield, Illinois. WBA is a new ticker reflecting the merge but otherwise it is a continuation of The Walgreens Company's stock. WBA is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on Nasdaq and its market capitalization on December 29, 2017 was $ 73,31b.

Air Products & Chemicals Inc. (APD)

Air Products & Chemicals Inc. is an American multinational company operating within the chemical and gas industry. The company sells its products primarily to other businesses. The business was founded in 1940 and has its headquarters in Allentown, Pennsylvania. APD is a component of the S&P 500 index, is listed on NYSE and its market capitalization on December 29, 2017 was $ 35,76bn.

DowDuPont Inc. (DWDP)

DowDuPont Inc. is an American multinational company operating in the chemical industry. The company is a merge product between The Dow Chemical Co. and E. I. du Pont de Nemours and Company and the merge took place in 2017. The original companies were founded in 1897 and 1802, respectively. The current DowDuPont Inc. has two headquarters in Midland, Michigan and Wilmington, Delaware. DWDP is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 87,13b.

Nucor Corporation (NUE)

Nucor Corporation is an American company operating within the heavy metals industry with specialization in production of steel and steel products. The company was founded in 1940 and has its headquarters in Charlotte, North Carolina. NUE is a component of the S&P 500 index, and is listed NYSE. The market capitalization on December 29, 2017 was $ 20,31bn.

Nike Inc. (NKE)

Nike Inc. is an American multinational corporation operating within the textile industry. Its main area of business is in design, development, manufacture, marketing, and sales of clothes of all kinds including footwear, apparel, accessories etc. The company was founded in 1964 and has its headquarters in Washington County, Oregon. NKE is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 102,05b.

Marriott International (MAR)

Marriott International is an American multinational company operating within the hospitality industry. It is one of the world's biggest operators of hotels and related lodging facilities. The company was founded in 1927 and has its headquarters in Bethesda, Maryland. MAR is a component of the S&P 500 and Nasdaq 100 indices. The company is listed on Nasdaq and its market capitalization on December 29, 2017 was $ 50,55b.

The Home Depot Inc. (HD)

The Home Depot Inc. is an American multinational retail company specialized in home improvement supplies, construction products, tools, and related services. The company was founded in 1978 and has its headquarters in Cobb County, Georgia. HD is a component of the S&P 100, S&P 500, and DJIA indices. The company is listed on NYSE and its market capitalization on December 29, 2017 was $ 223,42b.

Standard & Poor's 500 (S&P500, ^GSPC)

Standard & Poor's 500 is an American stock market index operated by the S&P Dow Jones Indices LLP. The index comprises 500 US large cap public companies across all industries. S&P 500 is a market cap weighted, free float adjusted index. For the purposes of this thesis, the S&P 500 serves as a market indicator.

US 4 Week Treasury Bill (T-bill)

US 4 Week Treasury Bill is an American short-term government bond with maturity of 4 weeks. The bond has a standard denomination of $1000. The bond is rated AAA and thus is considered risk-free. For the purposes of this thesis, the US 4 Week Treasury Bill serves as the risk-free asset.