The Early Exercise Boundary under the Jump to Default Extended CEV Model*

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Abstract

This paper proves the existence, uniqueness, monotonicity and continuity of the early exercise boundary attached to American-style standard options under the *jump to default extended constant elasticity of variance* model of Carr and Linetsky (2006).

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1. Introduction

American-style standard call and put options on (defaultable) stocks were first listed in the United States (hereafter, U.S.) by the Chicago Board Options Exchange (hereafter, CBOE) in 1973 and 1977, respectively. Such contracts are nowadays actively traded throughout the world on several options exchanges. For instance, the market statistics report of the CBOE for the year 2015 documents that about 370 million equity contracts were traded on the CBOE during that year, representing options on about 37 billion shares of underlying stock. At year-end 2015, the open interest in equity options on the CBOE was about 192 million contracts—107.5 million calls and 84.5 million puts.

Given that American-style options on equity are frequently traded on exchanges, the valuation of such contingent claims has become prominent in the theory of modern finance and has received much attention in the literature.[1] Several alternative valuation methodologies have been developed, ranging from numerical solution methods to analytical approximations.[2]

Until relatively recently, the literature on stock options and the literature on corporate bonds and credit risk developed almost independently of each other. The vast majority of the proposed equity options pricing models has been generally concerned with modeling implied volatility smiles and typically ignored the possibility of default of the underlying stock. In contrast, the credit risk literature has been essentially devoted to modeling bankruptcy and credit spreads, ignoring the information available in the equity options market.

It has been known for a long time, however, that the possibility of default has relevance for the pricing of equity options. Merton (1976) is the first to recognize the impact of corporate default on the stock price process by assuming a model where the stock price of a firm follows

[1] The valuation of American-style contingent claims has a long history and a complete literature review on the topic is outside the scope of the present paper. A general overview of this literature may be found, for example, in the survey papers of Myneni (1992), Broadie and Detemple (2004) and Barone-Adesi (2005), as well as in the monographs of Shreve (2004, Chapter 8) and Detemple (2006, Chapters 3, 4 and 8).

[2] A comparison of the different methods is available, for instance, in Broadie and Detemple (1996), Huang, Subrahmanyam, and Yu (1996), Ju (1998), Nunes (2009) and Ruas, Dias, and Nunes (2013).

a geometric Brownian motion (hereafter, GBM) punctuated with a single jump that takes the stock price from a positive value to zero (i.e. to the default or bankruptcy state). Such jump to default event evolves according to a Poisson process with constant default intensity (or arrival rate), which is independent of the firm's stock price. However, the economic rationale and the accumulated empirical evidence suggest that the probability of a jump to default increases at lower stock prices and decreases at higher stock prices. Hence, the modeling of the default intensity as a decreasing function of the stock price should clearly be much more realistic.

This is in line with the relatively recent developments in both the credit and the equity derivatives markets—in particular with the observed close linkages between credit default swaps (hereafter, CDS) and stock options on the same reference company. For instance, market participants start observing repeatedly that sharp stock price decreases coupled with increases in implied volatilities of stock options tend to occur simultaneously with sharp increases in market credit spreads on corporate debt and CDS spreads. The past decade revealed also that every time the credit markets become seriously concerned about the possibility of default of a firm, the open interest, the daily volume of trading, and the implied volatility of deep-out-of-the-money puts on the firm's stock explode many times over their historical average.

Perhaps the most prominent story supporting this riddle is the bankruptcy event of Lehman Brothers, that still remains as the largest bankruptcy filing in U.S. history: Lehman filed for Chapter 11 bankruptcy protection on September 14, 2008. In September 9, 2008, as the beleaguered investment bank's stock plummeted for a second session, the implied volatility of Lehman's put options attained stratospheric levels: September puts had implied volatilities of about 500%. Such stock declines with accompanying increases in implied volatilities of stock options were probably a reaction to news from Seoul that South Korea's government-owned Korea Development Bank had withdrawn its investment interest in the U.S. investment bank. Trading volumes were extraordinary heavy in puts with strike prices of \$7.50, \$5.00, and even \$2.50. For example, trades reported through Bloomberg showed that a total of 45,668 contracts for the deep-out-of-the-money put option with a strike price of \$2.50 and expiring on the 19th of September were traded during the session of September 9 (even though the underlying stock was still trading at a closing price of \$8.98). In light of this, traders were essentially buying "catastrophe" puts whose value is mostly derived from the probability of bankruptcy that will render equity worthless. This type of trading clearly indicates that investors believed profoundly that bad news could be in store for Lehman.

To accommodate the aforementioned stylized facts, a new generation of hybrid credit-equity models has emerged in the literature to value and hedge all securities related with a given firm, including equity and credit derivatives, in a unified modeling framework. Linetsky (2006) proposes an extension of the Black and Scholes (1973) and Merton (1973) model with bankruptcy, where the hazard rate of default is a negative power of the stock price, and obtains closed-form solutions for both corporate bonds and European-style stock options. This model establishes a link between the implied volatility of stock options and the probability of default, and avoids the unrealistic constant default intensity assumption of Merton (1976). However, since the local diffusion volatility of the stock price process remains constant, the probability of default in Linetsky (2006) model is assumed to explain all of the volatility skew.

Carr and Linetsky (2006) relaxed the latter assumption by modeling the stock price dynamics through the *jump to default extended constant elasticity of variance* (hereafter, JDCEV) process, where prior to default the stock price follows a diffusion process with a constant elasticity of variance (hereafter, CEV).[3] The default event is formally defined as the equity becoming worthless, i.e. the stock price dropping to zero. This can happen in one of two ways. Either the stock price process hits zero via diffusion, or a jump to default occurs that takes the stock price from a positive value to zero, whichever comes first. The default intensity (or hazard rate) of the jump to default event is modeled as an affine function of the local variance. This allows the linkage between the default intensity, the stock volatility and

[3]See, for instance, Cox (1975), Emanuel and MacBeth (1982), Schroder (1989), Davydov and Linetsky (2001, 2003), Linetsky (2004) and Larguinho, Dias, and Braumann (2013) for background on the CEV process.

the stock price, since the local volatility in the CEV model is a (negative) power function of the stock price.

We recall that the CEV volatility specification exhibits the so-called *leverage effect*—i.e. the noticed tendency of a negative relationship between stock returns and equity volatility—and leads to the implied volatility skew across different strike prices frequently revealed in the prices of individual stock options. However, the event of default under the CEV model can only happen via continuous diffusion of the stock price toward zero. Therefore, there is no element of surprise, i.e. there is no possibility of a jump to the bankruptcy state from a positive stock value. The appealing feature of the JDCEV framework is its ability to link equity and credit markets. In summary, the JDCEV model is able to accommodate not only the *leverage effect*—documented, for instance, in Black (1976), Christie (1982) and Bekaert and Wu (2000)—and the stock option implied volatility skew—highlighted, for example, in Dennis and Mayhew (2002) and Bakshi, Kapadia, and Madan (2003)—, but also the positive correlation between default probabilities or CDS spreads and equity volatilities observed in the credit markets, as empirically shown in many relatively recent works, e.g. Campbell and Taksler (2003), Bakshi, Madan, and Zhang (2006), Cremers, Driessen, Maenhout, and Weinbaum (2008), Zhang, Zhou, and Zhu (2009) and Carr and Wu (2010).

Carr and Linetsky (2006) obtain closed-form solutions for European-style plain-vanilla options, survival probabilities, CDS spreads, and corporate bonds in the JDCEV model by exploring the powerful link between CEV and Bessel processes.[4] Several other recent papers consider also the hybrid credit-equity JDCEV architecture modeling framework. For instance, Nunes (2009) and Ruas, Dias, and Nunes (2013) value standard option contracts possessing early exercise features through the *optimal stopping* and *static hedging portfolio* approaches, respectively. Mendoza-Arriaga and Linetsky (2011) price equity default swaps under a time-homogeneous version of the JDCEV model, and obtain an analytical solution to the first passage time of the JDCEV process with killing. More recently, Dias, Nunes, and Ruas (2015) show that the stopping time and static hedging portfolio approaches can

[4]See, for example, Borodin and Salminen (2002), Göing-Jaeschke and Yor (2003), Jeanblanc, Yor, and Chesney (2009, Chapter 6) and Katori (2016) for background on Bessel processes.

be easily extended to efficiently price and hedge European-style (single and double) barrier option contracts under the JDCEV model, whereas Nunes, Ruas, and Dias (2015) generalize both approaches for the valuation and hedging of American-style (single and double) knock-in options under the same JDCEV setup and highlight that American-style down-and-in puts with a sufficiently low knock-in barrier level may be viewed as a credit protection contract.

Notwithstanding the valuation of standard American-style options under the JDCEV model has been already treated in Nunes (2009) and Ruas, Dias, and Nunes (2013), the literature still lacks a rigorous analytical characterization of the corresponding optimal stopping boundary separating the so-called *continuation* and *stopping* regions of such free boundary problem. Moreover, while Ruas, Dias, and Nunes (2013) have been able to characterize the asymptotic behavior of the early exercise boundary near the option's expiry date, the existence of the early exercise boundary under the JDCEV model was never formally proved before in the literature. This is precisely the main aim of this paper.

For this purpose, we recall that the pricing of American-style contingent claims boils down to a boundary value problem in a domain whose boundary is not fully known and, hence, must be also determined. The solution to such mathematical problem has been initially provided by Kolodner (1956) in the context of free boundary problems appearing in mathematical physics. Inspired by the (discounted) warrant pricing problem of Samuelson (1965), McKean (1965) provides the earliest rigorous mathematical analysis on the pricing of American-style options by transforming the option pricing problem into a free boundary problem for the heat equation. By solving the latter, McKean (1965) is able to write the American option price V explicitly up to knowing a certain function E (the optimal stopping boundary). Van Moerbeke (1976) further extended this early work by studying several properties of the optimal stopping boundary, while Bensoussan (1984) and Karatzas (1988) provide an economic motivation for the optimal stopping problem attached to American-style contingent claims using hedging arguments and no-arbitrage conditions. A clear economic insight to the American-style option pricing problem has appeared in the beginning of the 1990s when Kim (1990), Jacka (1991), Carr, Jarrow, and Myneni (1992) and Jamshidian (1993) independently arrived at a nonlinear integral equation for the time-dependent boundary E that is closely linked to the early exercise premium representation of V. This implies that the price of an American-style option can be written as the sum of the corresponding European-style counterpart and a nonlinear integral term involving its early exercise boundary. The decomposition offered by such *integral representation method* has become then standard in the option pricing literature.

The main contribution of this paper is to prove the existence, uniqueness, monotonicity and continuity of the early exercise boundary attached to American-style standard put options under the JDCEV setup. Even though our main focus is on the American-style put, we start by proving—in Proposition 4.1 and through Detemple and Tian (2002, Proposition 1)—the existence and uniqueness of the early exercise boundary for call options. This is accomplished by: (i) Replacing the state dependent interest rate process in Detemple and Tian (2002, Equation 1) with an adjusted interest rate process composed by a short-term risk-free (deterministic) interest rate coupled with a state dependent default intensity possessing a negative relationship with the stock price; and (ii) Replacing the state dependent dividend yield process in Detemple and Tian (2002, Equation 1) with a state independent (but possibly time dependent) dividend yield. The proof of the monotonicity and right-continuity of the early exercise boundary for call options is then straightforward as shown in Remark 4.1. We stress that, mathematically, the optimal stopping problem for put options under the JDCEV model is significantly more difficult than the corresponding one for calls due to the killing and recovery features associated to put option contracts. To prove the existence and uniqueness—in Proposition 5.5—and the monotonicity and continuity—in Propositions 5.6, 5.7 and 5.8—of the early exercise boundary for put options, we follow Jacka (1991, Proposition 2.1), Lamberton and Mikou (2008, Theorem 4.2), and Monoyios and Ng (2011, Theorem 3.3), while using some well known properties of Bessel processes.

The remainder of this paper is organized as follows. For the sake of completeness, Section 2 summarizes the modeling assumptions of the JDCEV model. Section 3 presents the Snell envelop for an American-style option pricing problem with killing and shows that the *exercise region* is non-empty. Sections 4 and 5 prove the existence, uniqueness, monotonicity and continuity of the early exercise boundary attached to American-style standard call and put options, respectively, under the JDCEV model. Section 6 numerically illustrates the behavior of the early exercise boundary for put options on defaultable stocks, and shows that traders may incorrectly follow a premature exercise strategy when ignoring the possibility of default as a surprise event. Finally, Section 7 contains some concluding remarks.

2. The JDCEV model

For the analysis to remain self-contained, this section summarizes the main features of the hybrid credit-equity pricing model proposed by Carr and Linetsky (2006). From now on, and during the trading interval $[t_0, T]$, for some fixed time T $(> t_0)$, uncertainty is generated by a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, where the martingale measure $\mathbb{Q}$ (associated to the numéraire money-market account) is taken as given.

2.1. Predefault stock price

Before the random time of default ζ, Carr and Linetsky (2006) assume that the time-t price S_t of the underlying stock is described, under the martingale measure $\mathbb{Q}$, through the following stochastic differential equation:

$$\frac{dS_t}{S_t} = [r(t) - q(t) + \lambda(t, S_t)]\, dt + \sigma(t, S_t)\, dW_t, \tag{2.1}$$

where $r(t)$ (≥ 0) denotes the time-t risk-free and short-term (deterministic) interest rate, $q(t)$ (≥ 0) represents the time-t (deterministic) dividend yield, $\lambda(t, S_t) \in \mathbb{R}_+$ is a default intensity, $\sigma(t, S_t) \in \mathbb{R}_+$ corresponds to the time-t instantaneous volatility of asset returns, and $\{W_t, t \geq t_0\}$ is a standard Brownian motion defined under measure $\mathbb{Q}$ and generating the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq t_0\}$.

Note that the inclusion of the hazard rate $\lambda(t, S_t)$ in the drift of equation (2.1) compensates the stockholders for default (with zero recovery, due to the assumed *absolute priority rule* in the event of default) and insures, under the risk-neutral measure $\mathbb{Q}$, an expected rate of return equal to the risk-free interest rate.

2.2. Default time

The underlying stock price can diffuse to zero at the first passage time

$$\tau_0 := \inf\{t > t_0 : S_t = 0\}. \tag{2.2}$$

Alternatively, the stock price can also jump to zero at the first jump time $\tilde{\zeta}$ of a doubly-stochastic Poisson process with intensity $\lambda(t, S_t)$. Therefore, the random time of default is simply given by[5]

$$\zeta = \tau_0 \wedge \tilde{\zeta}. \tag{2.3}$$

2.3. Defaultable stock price

At time ζ, the stock price process is killed and sent to a *coffin* (i.e. bankruptcy) state Δ, where it remains forever. Hence, and following, for instance, Karlin and Taylor (1981, Equation 12.30) or Borodin and Salminen (2002, Page 28), the defaultable stock price process $\{S_t^\Delta, t \geq t_0\}$ can be summarized as

$$S_t^\Delta = \begin{cases} S_t \Longleftarrow t < \zeta \\ 0 \Longleftarrow t \geq \zeta \end{cases}. \tag{2.4}$$

Alternatively, and following Linetsky and Mendoza-Arriaga (2011, Page 558), the defaultable stock price process can be also represented as

$$\frac{dS_t^\Delta}{S_{t^-}^\Delta} = [r(t) - q(t)]\,dt + \sigma\left(t, S_{t^-}^\Delta\right) dW_t - dM_t, \tag{2.5}$$

where

$$M_t = \mathcal{D}_t - \int_0^{t\wedge\zeta} \lambda\left(u, S_u^\Delta\right) du, \tag{2.6}$$

$\{\mathcal{D}_t, t \geq t_0\}$ is a default indicator process, with $\mathcal{D}_t = \mathbb{1}_{\{t>\zeta\}}$, and $t^- := \lim_{\varepsilon\downarrow 0}(t-\varepsilon)$. Clearly, the defaultable stock price process $\{S_t^\Delta, t \geq t_0\}$ is adapted not to the filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq$

[5] For any two real numbers x and y, we denote by $x \vee y$ and $x \wedge y$, respectively, their maximum and minimum.

$t_0\}$ generated by the predefault process $\{S_t, t \geq t_0\}$, but rather to the enlarged filtration $\mathbb{G} = \{\mathcal{G}_t : t \geq t_0\}$, obtained as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.

In summary, the defaultable stock price process $\{S_t^{\Delta}, t \geq t_0\}$ is a time-inhomogeneous and Markov diffusion process with killing at rate $\lambda(t, S_t)$, and with the same infinitesimal mean and variance as the predefault process $\{S_t, t \geq t_0\}$.

2.4. JDCEV assumptions

To accommodate the leverage effect and the implied volatility skew stylized features, Carr and Linetsky (2006, Equation 4.1) adopt an extended CEV-type specification for the instantaneous stock volatility:

$$\sigma(t, S_t) = a(t)\, S_t^{\bar{\beta}}, \tag{2.7}$$

where $\bar{\beta} < 0$ is the volatility elasticity parameter and $a(t) > 0$, $\forall t$, is a deterministic volatility scale function. Additionally, and to be consistent with the empirical evidence of a positive correlation between default probabilities and equity volatility, Carr and Linetsky (2006, Equation 4.2) also assume that the default intensity is an increasing affine function of the instantaneous stock variance (implying, therefore, a negative relation between default intensity and stock prices):[6]

$$\lambda(t, S_t) = b(t) + c\,\sigma^2(t, S_t), \tag{2.8}$$

where $c \geq 0$, and $b(t) \geq 0$, $\forall t$, is a deterministic function of time.

Since $\bar{\beta} < 0$ and both c and $a(t)$ are nonnegative, equations (2.7) and (2.8) imply that $\lambda(t, S_t) \to \infty$ as $S \to 0$. Therefore, and as argued by Carr and Linetsky (2006, Page 311), zero is an unattainable boundary for S, since the defaultable stock price process would be killed from a positive value before it could ever reach zero via diffusion. Consequently,

$$\zeta = \tilde{\zeta} < \tau_0 \tag{2.9}$$

[6] The default intensity specification as the negative power of the stock price has become also popular for pricing convertible bonds and other hybrid securities. See, for example, Das and Sundaram (2007).

a.s., and following, for instance, Andersen and Buffum (2003, Equation 1), the stochastic differential equation (2.5) can be restated, for $t \leq \tilde{\zeta}$, as

$$\frac{dS_t^{\Delta}}{S_{t^-}^{\Delta}} = \left[r\left(t\right) - q\left(t\right) + \lambda\left(t, S_{t^-}^{\Delta}\right)\right] dt + \sigma\left(t, S_{t^-}^{\Delta}\right) dW_t - d\mathcal{D}_t, \tag{2.10}$$

meaning that $\{\mathcal{D}_t, t \geq t_0\}$ can be taken as a Cox process and $\tilde{\zeta} := \inf\{t > t_0 : \mathcal{D}_t = 1\}$.[7]

3. Snell envelopes

Our goal is to prove the existence, uniqueness, monotonicity and continuity of the early exercise boundary associated to an American-style option on the stock price S^{Δ}, with strike price K, and with maturity date T. The time-t ($\leq T$) value of the American-style option will be denoted by $V_t\left(S^{\Delta}, K, T; \phi\right)$, where $\phi = -1$ for a call option or $\phi = 1$ for a put option.

Assuming that $\zeta > t_0$, since the defaultable stock price process $\{S_t^{\Delta}, t \geq t_0\}$ is a Markov process (killed at the zero boundary), and because the American-style option can be exercised at any time during its lifetime, it is well known—see, for example, Zhang (1994, Equation 1.2) or Pham (1997, Page 148)—that its time-t_0 ($\leq T$) price can be represented by the following *Snell envelope*:

$$V_{t_0}\left(S^{\Delta}, K, T; \phi\right) = \operatorname*{ess\,sup}_{\theta \in \mathbb{G}[t_0, T]} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\theta} r(l)dl} \left(\phi K - \phi S_{\theta}^{\Delta}\right)^{+} \middle| \mathcal{G}_{t_0}\right], \tag{3.1}$$

where $\mathbb{G}A$ denotes the set of all $\mathbb{G}$-stopping times taking values in $A \subseteq \mathbb{R}$, and $\alpha^{+} \equiv \alpha \vee 0$ is the positive part of $\alpha \in \mathbb{R}$.

For American-style puts, it is easy to show that the option (if still alive) shall be exercised upon default of the underlying stock. For this purpose, and following Detemple and Kitapbayev (2017), we first note that even though the (discounted) payoff function

$$\Psi\left(t_0, u, S^{\Delta}; \phi\right) := e^{-\int_{t_0}^{u} r(l)dl} \left(\phi K - \phi S^{\Delta}\right)^{+}, \tag{3.2}$$

for $u \geq t_0$, is not C^2 with respect to the third argument, it is still a convex function in S^{Δ}. Therefore, the Meyer-Itô formula—see, for instance, Protter (2005, Theorem 70 and

[7]Intuitively, at time $\tilde{\zeta}$, $\mathcal{D}$ jumps from 0 to 1, $dS_t^{\Delta} = -S_{t^-}^{\Delta}$, and the stock price falls to 0 where it remains forever.

its Corollary 1, Chapter IV)—can be applied to the semimartingale S^{Δ} defined through the stochastic differential equation (2.10), yielding, for any $\mathbb{G}$-stopping time θ, and with $\phi = 1$,

$$\begin{aligned}
&\Psi\left(t_0, \theta, S_\theta^{\Delta}; 1\right) - \Psi\left(t_0, t_0, S_{t_0}^{\Delta}; 1\right) \\
= & -\int_{t_0}^{\theta} e^{-\int_{t_0}^{u} r(l)dl} \left\{ r(u)\left(K - S_u^{\Delta}\right) + S_u^{\Delta}\left[r(u) - q(u) + \lambda\left(u, S_u^{\Delta}\right)\right]\right\} \mathbb{1}_{\{S_u^{\Delta} < K\}} du \\
& -\int_{t_0}^{\theta} e^{-\int_{t_0}^{u} r(l)dl} \mathbb{1}_{\{S_u^{\Delta} < K\}} S_u^{\Delta} \sigma\left(u, S_u^{\Delta}\right) dW_u \\
& +\int_{t_0}^{\theta} \left[\Psi\left(t_0, u, 0; 1\right) - \Psi\left(t_0, u, S_u^{\Delta}; 1\right)\right] d\mathcal{D}_u \\
& +\frac{1}{2}\int_{t_0}^{\theta} e^{-\int_{t_0}^{u} r(l)dl} dL_u^K\left(S^{\Delta}\right),
\end{aligned} \tag{3.3}$$

where, following Peskir (2005, Equation 1.6),

$$L_u^K\left(S^{\Delta}\right) := \mathbb{Q} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{t_0}^{u} \mathbb{1}_{\left\{K-\varepsilon < S_l^{\Delta} < K+\varepsilon\right\}} \left(S_l^{\Delta}\right)^2 \sigma^2\left(l, S_l^{\Delta}\right) dl \tag{3.4}$$

represents the *local time* of S^{Δ} at level K, and $dL_u^K\left(S^{\Delta}\right)$ refers to the integration with respect to the continuous and increasing function $u \to L_u^K\left(S^{\Delta}\right)$. Applying conditional expectations to both sides of equation (3.3), and since both the Itô's integral—second term on the right-hand side of equation (3.3)—and the compensator process (2.6) are $\mathbb{Q}$-martingales, then

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\theta} r(l)dl} \left(K - S_\theta^{\Delta}\right)^+ \middle| \mathcal{G}_{t_0}\right] \\
= & \left(K - S_{t_0}^{\Delta}\right)^+ + \mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{\theta} e^{-\int_{t_0}^{u} r(l)dl} H\left(u, S_u^{\Delta}\right) du \middle| \mathcal{G}_{t_0}\right] \\
& +\frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\int_{t_0}^{\theta} e^{-\int_{t_0}^{u} r(l)dl} dL_u^K\left(S^{\Delta}\right) \middle| \mathcal{G}_{t_0}\right],
\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
&H\left(u, S_u^{\Delta}\right) \\
= & \left[-r(u)K + q(u)S_u^{\Delta} - \lambda\left(u, S_u^{\Delta}\right)S_u^{\Delta}\right]\mathbb{1}_{\{S_u^{\Delta} < K\}} + \left[K - \left(K - S_u^{\Delta}\right)^+\right]\lambda\left(u, S_u^{\Delta}\right) \\
= & \left[q(u)S_u^{\Delta} - r(u)K\right]\mathbb{1}_{\{S_u^{\Delta} < K\}} + K\lambda\left(u, S_u^{\Delta}\right)\mathbb{1}_{\{S_u^{\Delta} \geq K\}}
\end{aligned} \tag{3.6}$$

measures the instantaneous benefit of postponing the exercise of the put.

Upon default, $S_u^{\Delta} = 0$ for all $u \geq \zeta$ (since 0 is an absorbing state), and, thus, $H\left(u, S_u^{\Delta}\right) = -r(u)K \leq 0$ (assuming nonnegative interest rates) while the last (local time) term on the right-hand side of equation (3.5) is equal to zero (since $K > 0$). Hence, and as expected, it is optimal to stop (i.e. to exercise the American-style put) at the default date. Intuitively, after the default date, the exercise payoff generated by the American-style put is always the highest one and the same (i.e. the strike price)—since the stock price process will remain forever at the zero (bankruptcy) level. Consequently, postponing the exercise decision beyond time ζ would be equivalent to losing the interest on the strike price of the put option (if interest rates are nonzero). Note that even in the extreme case of $r(u) = 0$ for all $u \in [t_0, T]$, ζ can still be taken as an optimal stopping time because it would be indifferent to exercise at the default time or later: the payoff is always the same (i.e. the strike price), and no interest gain or loss will occur since interest rates are equal to zero.

Given that it is optimal to exercise the American-style put at the default time ζ, and since both calls and puts can only be exercised until the expiry date T, the optimal stopping problem (3.1) can be rewritten as

$$\begin{aligned} V_{t_0}\left(S^{\Delta}, K, T; \phi\right) &= \operatorname*{ess\,sup}_{\tau \in \mathbb{G}[t_0,\infty]} \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau\wedge\zeta} r(l)dl} \left(\phi K - \phi S^{\Delta}_{T\wedge\tau\wedge\zeta}\right)^{+} \middle| \mathcal{G}_{t_0}\right] \\ &= \operatorname*{ess\,sup}_{\tau \in \mathbb{G}[t_0,\infty]} \left\{ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau} r(l)dl} \left(\phi K - \phi S_{T\wedge\tau}\right)^{+} \mathbb{1}_{\{\zeta > T\wedge\tau\}} \middle| \mathcal{G}_{t_0}\right] \right. \\ &\quad \left. + \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{\zeta} r(l)dl} \left(\phi K\right)^{+} \mathbb{1}_{\{\zeta \leq T\wedge\tau\}} \middle| \mathcal{G}_{t_0}\right] \right\}, \end{aligned} \tag{3.7}$$

where the second equality follows from identity (2.4). Equation (3.7) corresponds exactly to Nunes (2009, Equation 53). Note also that the second term on the right-hand side of equation (3.7) is equal to zero for $\phi = -1$, meaning that the American-style call becomes worthless upon the default event.

Even though both Snell envelopes (3.1) and (3.7) are equivalent for

$$\theta = T \wedge \tau \wedge \zeta, \tag{3.8}$$

the latter representation has the advantage of being easily rewritten under the restricted filtration $\mathbb{F}$, with respect to which the predefault stock price S behaves as a pure diffusion process with continuous sample paths. Using, for instance, Øksendal (1995, Equation 8.17) and Carr and Linetsky (2006, Equation 3.4), and since $\tilde{\zeta}$ is the first jump time of a Cox process with intensity $\lambda\left(t,S_t\right)$, then equation (3.7) yields

$$V_{t_0}\left(S^{\Delta},K,T;\phi\right)=\operatorname*{ess\,sup}_{\tau\in\mathbb{F}[t_0,\infty]}\mathbb{E}_{\mathbb{Q}}\left[\left.Y\left(t_0,\tau\right)\right|\mathcal{F}_{t_0}\right],\tag{3.9}$$

where

$$\begin{aligned}Y\left(t_0,\tau\right):=\quad & e^{-\int_{t_0}^{T\wedge\tau}(r(l)+\lambda(l,S_l))dl}\left(\phi K-\phi S_{T\wedge\tau}\right)^{+}\mathbb{1}_{\{\tau_0>T\wedge\tau\}}\\ & +\int_{t_0}^{T\wedge\tau}e^{-\int_{t_0}^{u}(r(l)+\lambda(l,S_l))dl}\lambda\left(u,S_u\right)\mathbb{1}_{\{\tau_0>u\}}du.\end{aligned}\tag{3.10}$$

Since the predefault stock price process $\{S_t,t\geq t_0\}$ is nonnegative and possesses right-continuous sample paths, then Karatzas and Shreve (1998, Theorem D.9) implies that the optimal stopping time τ^* for the problem (3.9) is

$$\tau^*:=\inf\left\{t\in[t_0,T\wedge\zeta[\,:V_t\left(S^{\Delta},K,T;\phi\right)=Y\left(t,t\right)=\left(\phi K-\phi S_t\right)^{+}\right\},\tag{3.11}$$

with $\inf\emptyset=\infty$, and

$$\begin{aligned}V_{t_0}\left(S^{\Delta},K,T;\phi\right)&=\mathbb{E}_{\mathbb{Q}}\left[\left.Y\left(t_0,\tau^*\right)\right|\mathcal{F}_{t_0}\right]\\ &=\mathbb{E}_{\mathbb{Q}}\left[\left.\Psi\left(t_0,T\wedge\tau^*\wedge\zeta,S^{\Delta}_{T\wedge\tau^*\wedge\zeta};\phi\right)\right|\mathcal{G}_{t_0}\right],\end{aligned}\tag{3.12}$$

where the second equality follows again from Øksendal (1995, Equation 8.17) and Carr and Linetsky (2006, Equation 3.4).

Using equality (3.8), the optimal stopping time for the problem (3.1) can be stated as

$$\begin{aligned}\theta^*&=T\wedge\tau^*\wedge\zeta\\ &=\inf\left\{t\in[t_0,T\wedge\zeta]:V_t\left(S^{\Delta},K,T;\phi\right)=\Psi\left(t,t,S^{\Delta}_t;\phi\right)=\left(\phi K-\phi S^{\Delta}_t\right)^{+}\right\},\end{aligned}\tag{3.13}$$

where again the convention $\inf\emptyset=\infty$ is adopted. While τ^* represents the optimal time for early exercise strictly before the default event, the hitting time θ^* corresponds to the optimal exercise date (through default or not). In both cases, and as usual—see, for instance, Jacka (1991, Theorem 2.1) or Pham (1997, Equation 2.5)—the optimal stopping time is the first time that the option price is equal to its intrinsic value.

Given equation (3.13), and since standard no-arbitrage restrictions imply that

$$V_t\left(S^{\Delta},K,T;\phi\right)\geq\left(\phi K-\phi S^{\Delta}_t\right)^{+},\tag{3.14}$$

for $t \in [t_0, T]$, we can divide the set $\left\{\left(S^{\Delta}, t\right) \in [0, \infty[\times [t_0, T]\right\}$ into the *exercise* (or *stopping*) region

$$\mathcal{E} := \left\{\left(S^{\Delta}, t\right) \in [0, \infty[\times [t_0, T] : V_t\left(S^{\Delta}, K, T; \phi\right) = \left(\phi K - \phi S_t^{\Delta}\right)^+\right\}, \tag{3.15}$$

and the *continuation* (or *holding*) region

$$\mathcal{C} := \left\{\left(S^{\Delta}, t\right) \in [0, \infty[\times [t_0, T] : V_t\left(S^{\Delta}, K, T; \phi\right) > \left(\phi K - \phi S_t^{\Delta}\right)^+\right\}. \tag{3.16}$$

Note that the *exercise* region is non-empty because we have already shown that $\left(S^{\Delta}, t\right) = (0, t) \in \mathcal{E}$, for all $t \in [t_0, T]$.

Our main goal now is to prove that there exists—at each time $t \in [t_0, T]$—a (unique) critical asset price

$$E(t) := \inf\left\{S^{\Delta} \geq 0 : V_t\left(S^{\Delta}, K, T; 1\right) > \left(K - S^{\Delta}\right)^+\right\} \tag{3.17}$$

for $\phi = 1$, or $E(t) := \sup\left\{S^{\Delta} \geq 0 : V_t\left(S^{\Delta}, K, T; -1\right) > \left(S^{\Delta} - K\right)^+\right\}$ for $\phi = -1$, below (resp., above) which the American-style put (resp., call) price equals its intrinsic value and, therefore, early exercise should occur. If this is the case, then the optimal policy should be to exercise the American-style option when the underlying asset price first enters the exercise region and, hence, the stopping region (3.15) can be rewritten as

$$\mathcal{E} = \left\{\left(S^{\Delta}, t\right) \in [0, \infty[\times [t_0, T] : \phi S_t^{\Delta} \leq \phi E(t)\right\}, \tag{3.18}$$

whereas the corresponding continuation region (3.16) becomes equal to

$$\mathcal{C} := \left\{\left(S^{\Delta}, t\right) \in [0, \infty[\times [t_0, T] : \phi S_t^{\Delta} > \phi E(t)\right\}. \tag{3.19}$$

4. American-style calls

Since there is no payoff upon default attached to the American-style call, and because S behaves as a pure diffusion process with respect to the filtration $\mathbb{F}$, the uniqueness and existence of the early exercise boundary $t \to E(t)$ will arise easily from Detemple and Tian (2002, Proposition 1).

Taking $\phi = -1$ and using equation (2.9), equation (3.7) yields

$$\begin{aligned} V_{t_0}\left(S^{\Delta}, K, T; -1\right) &= \operatorname*{ess\,sup}_{\tau \in \mathbb{G}[t_0,\infty]} \left\{ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau} r(l)dl} \left(S_{T\wedge\tau} - K\right)^{+} \mathbb{1}_{\left\{\tilde{\zeta} > T\wedge\tau\right\}} \middle| \mathcal{G}_{t_0} \right] \right\} \\ &= \operatorname*{ess\,sup}_{\tau \in \mathbb{F}[t_0,\infty]} \left\{ \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau} (r(l)+\lambda(l,S_l))dl} \left(S_{T\wedge\tau} - K\right)^{+} \middle| \mathcal{F}_{t_0} \right] \right\}, \end{aligned} \tag{4.1}$$

where the last line follows, for instance, from Øksendal (1995, Equation 8.17) or Bielecki and Rutkowski (2002, Corollary 5.1.1). Equation (4.1) is exactly equivalent to Detemple and Tian (2002, Equation 2) as long as we take $r(t) + \lambda(t, S_t)$ as the state dependent interest rate process in Detemple and Tian (2002, Equation 1).

Using Detemple and Tian (2002) notation, the stochastic differential equation (2.1) can be cast into Detemple and Tian (2002, Equation 1) by taking

$$r(S_t, t) = r(t) + \lambda(t, S_t) \tag{4.2}$$

as a state dependent interest rate process, and

$$\delta(S_t, t) = q(t) \tag{4.3}$$

as a state independent (but possibly time dependent) dividend yield. Therefore, the existence of a unique early exercise boundary attached to the American-style contract (4.1) follows from Detemple and Tian (2002, Proposition 1) as long as we can show that both functions (4.2) and (4.3) satisfy the requirements enunciated by Detemple and Tian (2002, Page 920).

Proposition 4.1 *Under the JDCEV model, there exists a unique function $t \to E(t)$ such that the exercise region of the American-style call is given by equation (3.18) for $\phi = -1$.*

Proof. Proposition 4.1 arises after applying Detemple and Tian (2002, Proposition 1) to the value function (4.1), which can be done because the following two conditions are met by the JDCEV model under analysis:

1. The state dependent interest rate process (4.2) is a nonincreasing function of S. Combining equations (2.7), (2.8) and (4.2), then

$$r(S_t, t) = r(t) + b(t) + c\, a^2(t)\, S_t^{2\bar{\beta}}. \tag{4.4}$$

Since $\bar{\beta} < 0$ and $c \geq 0$, equation (4.4) yields an inverse relationship between $r(S_t, t)$ and S_t, as desired.

2. The process $\delta(S, t) S$ is a nondecreasing function of S. This follows immediately from equation (4.3) because $q(t)$ is nonnegative.

Given that the two previous conditions are met by the JDCEV model, Detemple and Tian (2002, Lemma 1) is satisfied by the value function (4.1), and, therefore, it is easy to show that the exercise region (3.15) is up-connected. ■

Remark 4.1 *Following Detemple and Tian (2002, Proposition 1), it is also possible to show that the early exercise boundary $t \to E(t)$ is a nonincreasing and right-continuous function, as long as the deterministic functions of time $r(t)$, $q(t)$, $a(t)$, and $b(t)$ are specified in such a way that the predefault price process (2.1) satisfies the following time monotonicity condition: For $v \in [t_0, T]$ and $h \geq 0$, $S_v^0 \geq S_v^h$, where S_v^h is the solution of the stochastic differential equation (2.1) with initial condition $S_{t_0} = S$, at time t_0, and time-translated parameters $r(t+h)$, $q(t+h)$, $\lambda(t+h, S)$, and $\sigma(t+h, S)$. Note that this monotonicity condition is clearly satisfied by the time-homogeneous version of the JDCEV model.*

5. American-style puts

For American-style puts, the proof of the existence, monotonicity, and continuity of the early exercise boundary $t \to E(t)$ will be based on Jacka (1991, Proposition 2.1), Lamberton and Mikou (2008, Theorem 4.2), and Monoyios and Ng (2011, Theorem 3.3). For this purpose, some preliminary results are required and stated in the next four propositions.

5.1. Preliminary results

The first two results concern the monotonicity of the default time ζ and of the predefault stock price S with respect to the initial value of the latter.

Proposition 5.1 *Let $\tau_0(x)$ represent the stopping time (2.2) and $S_t(x)$ denote the time-t realization of the predefault stock price process $\{S_t, t \geq t_0\}$ when such process is initialized at $S_{t_0} = x$. Under the JDCEV model and if $y > x > 0$, then $S_t(y) > S_t(x)$ for all $t \in [t_0, \tau_0(x) \wedge \tau_0(y)[$.*

Proof. Carr and Linetsky (2006, Proposition 5.1) show that the predefault stock price process defined by equations (2.1), (2.7), and (2.8) can be stated as

$$S_t(x) = e^{\int_{t_0}^t \alpha(l)dl} \left(\left|\bar{\beta}\right| R^{(v)}_{\gamma(t_0,t)} \left(\frac{1}{\left|\bar{\beta}\right|} x^{\left|\bar{\beta}\right|} \right) \right)^{\frac{1}{\left|\bar{\beta}\right|}}, \tag{5.1}$$

where $\{R_t^{(\delta)}(a), t \geq 0\}$ represents a Bessel process of index δ and started at a,

$$\gamma(t_0, t) := \int_{t_0}^t a^2(s)\, e^{-2\left|\bar{\beta}\right| \int_{t_0}^s \alpha(l)dl} ds \tag{5.2}$$

is a deterministic time change,

$$\alpha(l) := r(l) - q(l) + b(l), \tag{5.3}$$

and $v = \frac{c - \frac{1}{2}}{\left|\bar{\beta}\right|}$, for all $t \geq t_0$ if $c \geq \frac{1}{2}$, or only for $t \in [t_0, \tau_0(x)[$ if $c \in \left]0, \frac{1}{2}\right[$.

Given that $\frac{1}{\left|\bar{\beta}\right|} y^{\left|\bar{\beta}\right|} > \frac{1}{\left|\bar{\beta}\right|} x^{\left|\bar{\beta}\right|}$ whenever $y > x$, and because a Bessel process is an increasing function of its starting value (until the first hitting time of zero),[8] then

$$R^{(v)}_{\gamma(t_0,t)} \left(\frac{1}{\left|\bar{\beta}\right|} y^{\left|\bar{\beta}\right|} \right) > R^{(v)}_{\gamma(t_0,t)} \left(\frac{1}{\left|\bar{\beta}\right|} x^{\left|\bar{\beta}\right|} \right) \tag{5.4}$$

for all $\gamma(t_0, t) \in \left[0, \tau_0^R \left(\frac{1}{\left|\bar{\beta}\right|} x^{\left|\bar{\beta}\right|} \right) \wedge \tau_0^R \left(\frac{1}{\left|\bar{\beta}\right|} y^{\left|\bar{\beta}\right|} \right) \right[$, i.e., and since $\frac{d\gamma(t_0,t)}{dt} = a^2(t)\, e^{-2\left|\bar{\beta}\right| \int_{t_0}^t \alpha(l)dl} > 0$, also for all $t \in [t_0, \tau_0(x) \wedge \tau_0(y)[$, where

$$\tau_0^R \left(\frac{1}{\left|\bar{\beta}\right|} x^{\left|\bar{\beta}\right|} \right) := \inf \left\{ \gamma(t_0, t) > 0 : R^{(v)}_{\gamma(t_0,t)} \left(\frac{1}{\left|\bar{\beta}\right|} x^{\left|\bar{\beta}\right|} \right) = 0 \right\}. \tag{5.5}$$

Since equation (5.1) expresses the predefault stock price as an increasing function of a time-changed Bessel process, inequality (5.4) implies that $S_t(y) > S_t(x)$ for all $t < \tau_0(x) \wedge \tau_0(y)$. ■

[8] See, for instance, Katori (2016, Page 28).

Proposition 5.2 *Let $\zeta(x)$ represent the default time (2.3) when the predefault stock price process $\{S_t, t \geq t_0\}$ is initialized at $S_{t_0} = x$. Under the JDCEV model, $\zeta(x) \leq \zeta(y)$ for all $y > x > 0$.*

Proof. Based on definition (2.3), and for all $y > x > 0$, we will show that

$$\tau_0(x) \wedge \tilde{\zeta}(x) \leq \tau_0(y) \wedge \tilde{\zeta}(y), \tag{5.6}$$

where the first jump time of the Cox process is defined as

$$\tilde{\zeta}(x) := \inf\left\{t > t_0 : \frac{1}{\mathbb{1}_{\{\tau_0(x)>t\}}} \int_{t_0}^{t} \lambda(l, S_l(x))\, dl \geq \Theta\right\}, \tag{5.7}$$

with Θ representing a random variable (independent of $\{W_t, t \geq t_0\}$) following a (unit mean) exponential distribution.

Starting with the first hitting time of zero through diffusion, and following, for instance, Katori (2016, Theorem 1.2), it is well known that, for a Bessel process, such stopping time is a nondecreasing function of its initial state, i.e.

$$\tau_0^R\left(\frac{1}{|\bar{\beta}|} x^{|\bar{\beta}|}\right) \leq \tau_0^R\left(\frac{1}{|\bar{\beta}|} y^{|\bar{\beta}|}\right), \tag{5.8}$$

for all $y > x > 0$. Using definition (5.5) and since the new "clock" (5.2) is an increasing function of calendar time (i.e. $\frac{d\gamma(t_0,t)}{dt} > 0$), equation (5.8) implies that

$$\tau_0(x) = \gamma^{-1}\left(t_0, \tau_0^R\left(\frac{1}{|\bar{\beta}|} x^{|\bar{\beta}|}\right)\right) \leq \tau_0(y) = \gamma^{-1}\left(t_0, \tau_0^R\left(\frac{1}{|\bar{\beta}|} y^{|\bar{\beta}|}\right)\right), \tag{5.9}$$

where $\gamma^{-1}(\cdot)$ is the inverse function of the deterministic time change (5.2).

Concerning the intensity of the Cox process, the inverse relation between the default intensity and stock prices—defined by equations (2.7) and (2.8)—together with Proposition 5.1 imply that

$$\lambda(l, S_l(y)) < \lambda(l, S_l(x)), \tag{5.10}$$

for all $l \geq t_0$. Hence, inequalities (5.9) and (5.10) yield

$$\frac{1}{\mathbb{1}_{\{\tau_0(x)>t\}}} \int_{t_0}^{t} \lambda(l, S_l(x))\, dl \geq \frac{1}{\mathbb{1}_{\{\tau_0(y)>t\}}} \int_{t_0}^{t} \lambda(l, S_l(y))\, dl$$

a.s., and, therefore, definition (5.7) implies that

$$\tilde{\zeta}(x) \leq \tilde{\zeta}(y), \tag{5.11}$$

for all $y > x > 0$.

Combining inequalities (5.9) and (5.11), inequality (5.6) follows immediately.■

Remark 5.1 *Combining Propositions 5.1 and 5.2, we immediately conclude that the defaultable stock price is also an increasing function of its initial level:*

$$S_t^{\Delta}(y) := S_t(y)\,\mathbb{1}_{\{\zeta(y)>t\}} > S_t^{\Delta}(x) := S_t(x)\,\mathbb{1}_{\{\zeta(x)>t\}}, \tag{5.12}$$

for all $y > x > 0$, and where $S_t^{\Delta}(x)$ denotes the time-t realization of the defaultable stock price process $\{S_t^{\Delta}, t \geq t_0\}$ when such process is initialized at $S_{t_0}^{\Delta} = x$.

Next proposition simply shows that, as expected, the discounted cum-dividend defaultable stock price process is a $\mathbb{Q}$-martingale.

Proposition 5.3 *Under the JDCEV model, and for any stopping time $\tau \in \mathbb{G}\,[t_0, \infty[$, the stopped process*

$$z_{t\wedge\tau}^{\Delta}(x) := e^{-\int_{t_0}^{t\wedge\tau} r(l)dl} S_{t\wedge\tau}^{\Delta}(x) + \int_{t_0}^{t\wedge\tau} e^{-\int_{t_0}^{v} r(l)dl} q(v)\, S_v^{\Delta}(x)\, dv \tag{5.13}$$

is a $\mathbb{Q}$-martingale for all $t \geq t_0$.

Proof. Using the stochastic differential equation (2.10), and applying Itô's formula to the process (5.13), it follows that, for any time $t \geq t_0$,

$$dz_t^{\Delta}(x) = e^{-\int_{t_0}^{t} r(l)dl} \left[S_{t^-}^{\Delta} \lambda\left(t, S_{t^-}^{\Delta}\right) dt + S_{t^-}^{\Delta} \sigma\left(t, S_{t^-}^{\Delta}\right) dW_t - S_{t^-}^{\Delta} d\mathcal{D}_t\right]. \tag{5.14}$$

Since, besides the Brownian motion, the compensator $M_t = \mathcal{D}_t - \int_0^t \lambda\left(u, S_u^{\Delta}\right) du$ is also a $\mathbb{Q}$-martingale, equation (5.14) can be finally written with no drift,

$$dz_t^{\Delta}(x) = e^{-\int_{t_0}^{t} r(l)dl} \left[S_{t^-}^{\Delta} \sigma\left(t, S_{t^-}^{\Delta}\right) dW_t - S_{t^-}^{\Delta} dM_t\right],$$

and the stopped process (5.13) inherits the same martingale property from Doob's optional sampling theorem.■

Our last preliminary result concerns the positiveness of the American-style put price.

Proposition 5.4 *Under the JDCEV model, the American-style put price process is strictly positive, i.e.*

$$V_t\left(S^{\Delta}, K, T; 1\right) > 0, \tag{5.15}$$

for every $\left(S^{\Delta}, t\right) \in [0, \infty[\times [t_0, T]$.

Proof. A lower bound for the optimal stopping problem (3.7) with $\phi = 1$ is given by

$$\begin{aligned} v_{t_0}\left(S^{\Delta}, K, T; 1, \zeta\right) &:= \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\zeta} r(l)dl}\left(K - S^{\Delta}_{T\wedge\zeta}\right)^{+}\Big|\,\mathcal{G}_{t_0}\right] \\ &= v^0_{t_0}\left(S^{\Delta}, K, T; 1\right) + v^D_{t_0}\left(S^{\Delta}, K, T; 1, \zeta\right), \end{aligned} \tag{5.16}$$

where

$$v^0_t\left(S^{\Delta}, K, T; 1\right) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(l)dl}\left(K - S_T\right)^{+}\mathbb{1}_{\{\zeta > T\}}\Big|\,\mathcal{G}_t\right] \tag{5.17}$$

is the time-t value of a European-style put contract on the stock price S^{Δ}, with strike price K, maturity date T, and whose payoff is conditional on the survival of the underlying stock until the maturity date T, while

$$v^D_t\left(S^{\Delta}, K, T; 1, \zeta\right) := K\mathbb{E}_{\mathbb{Q}}\left(e^{-\int_t^{\zeta} r(l)dl}\mathbb{1}_{\{\zeta \leq T\}}\Big|\,\mathcal{G}_t\right) \tag{5.18}$$

is the time-t value of the "recovery" payment (equal to the strike price) that occurs at the default time ζ.

We note that for traded European-style puts, the "recovery" payoff K is paid not at the default time ζ but rather at the expiry date of the put contract (time T) because the put can only be exercised at that time. Therefore, the lower bound (5.16) is not exactly a plain-vanilla European-style put. Nevertheless, equations (3.7) and (5.16) imply that[9]

$$V_t\left(S^{\Delta}, K, T; 1\right) \geq v^0_t\left(S^{\Delta}, K, T; 1\right) + v^D_t\left(S^{\Delta}, K, T; 1, \zeta\right). \tag{5.19}$$

[9] A strict inequality is not obtained because it is possible that $\theta^* = T \wedge \zeta$.

We further note that

$$v_t^D\left(S^\Delta, K, T; 1, \zeta\right) > 0, \tag{5.20}$$

since it is easy to show that all the terms inside the integrand function of Carr and Linetsky (2006, Equation 5.15)—including all the arguments of the standard gamma and Kummer confluent hypergeometric functions (of the first kind) involved—are strictly positive unless $b(u) = c = 0$ for all u.

Using inequality (5.20), and since $v_t^0\left(S^\Delta, K, T; 1\right) \geq 0$, then inequality (5.19) yields

$$\begin{aligned} V_t\left(S^\Delta, K, T; 1\right) &\geq v_t^D\left(S^\Delta, K, T; 1, \zeta\right) \\ &> 0, \end{aligned}$$

and inequality (5.15) arises. ■

Remark 5.2 *In Section 3, and using equations (3.5) and (3.6), we have shown that the exercise region is non-empty because* $\left(S^\Delta, t\right) = (0, t) \in \mathcal{E}$*, for all* $t \in [t_0, T]$*. Similarly, Proposition 5.4 and definition (3.16) imply that* $\left\{\left(S^\Delta, t\right) \in [K, \infty[\times [t_0, T]\right\} \subset \mathcal{C}$*, i.e. all stock price levels above the strike price belong to the continuation set. Therefore, it follows that the continuation region* $\mathcal{C}$ *is also non-empty.*

5.2. Existence and uniqueness

Using Propositions 5.1 to 5.4, and following Jacka (1991, Proposition 2.1), we can now prove our main result: the existence of a unique early exercise boundary for the American-style put under the most general time-inhomogeneous version of the JDCEV model.

Proposition 5.5 *Under the JDCEV model, there exists a unique function* $t \to E(t)$ *such that the continuation region of the American-style put is given by equation (3.19) for* $\phi = 1$*.*

Proof. Following Jacka (1991, Equation 2.2), for each $t \in [t_0, T]$, the t section of $\mathcal{C}$ is given by $\mathcal{C}_t := \left\{S^\Delta : \left(S^\Delta, t\right) \in \mathcal{C}\right\}$. Hence, we just need to prove that the continuation region is up-connected, i.e. that $(x \in \mathcal{C}_t) \Longrightarrow (y \in \mathcal{C}_t)$ for any $y > x > 0$, and for each $t \in [t_0, T]$.

Let

$$\theta^*(x) = T \wedge \tau^*(x) \wedge \zeta(x), \tag{5.21}$$

where

$$\tau^*(x) := \inf\left\{u \in [t_0, T \wedge \zeta[\,:\, \left(S_u^\Delta(x), u\right) \notin \mathcal{C}\right\}, \tag{5.22}$$

be the optimal stopping time for the optimal stopping problem (3.1), when the process $\{S_t^\Delta, t \geq t_0\}$ is initialized at $S_{t_0}^\Delta = x$. Since $\theta^*(x)$ is only a feasible (but not necessarily optimal) stopping time when the process $\{S_t^\Delta, t \geq t_0\}$ is initialized at $S_{t_0}^\Delta = y$, equation (3.12) yields

$$\begin{aligned}
& V_{t_0}(y, K, T; 1) - V_{t_0}(x, K, T; 1) \\
= \; & V_{t_0}(y, K, T; 1) - \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} \left(K - S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(x)\right)^+ \middle| \mathcal{G}_{t_0}\right] \\
\geq \; & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} \left(K - S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(y)\right)^+ \middle| \mathcal{G}_{t_0}\right] \\
& -\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} \left(K - S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(x)\right)^+ \middle| \mathcal{G}_{t_0}\right] \\
= \; & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(x) \middle| \mathcal{G}_{t_0}\right] \\
& -\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(y) \middle| \mathcal{G}_{t_0}\right] \\
& +\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} \left(K \vee S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(y)\right) \middle| \mathcal{G}_{t_0}\right] \\
& -\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} \left(K \vee S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(x)\right) \middle| \mathcal{G}_{t_0}\right],
\end{aligned} \tag{5.23}$$

where the last equality arises because $(a-b)^+ = a \vee b - b$, for any $a, b \in \mathbb{R}$.

Equation (5.12) implies that the sum of the last two terms on the right-hand side of inequality (5.23) is nonnegative for any $y > x$, and, therefore, we are left with

$$\begin{aligned}
& V_{t_0}(y, K, T; 1) - V_{t_0}(x, K, T; 1) \\
\geq \; & \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(x) \middle| \mathcal{G}_{t_0}\right] \\
& -\mathbb{E}_{\mathbb{Q}}\left[e^{-\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} r(l)dl} S_{T\wedge\tau^*(x)\wedge\zeta(x)}^\Delta(y) \middle| \mathcal{G}_{t_0}\right].
\end{aligned} \tag{5.24}$$

Furthermore, Proposition 5.3 allows us to rewrite each term on the right-hand side of inequality (5.24) in terms of the initial state of the stock price,

$$\begin{aligned} & V_{t_0}(y,K,T;1) - V_{t_0}(x,K,T;1) \\ \geq\ & x - \mathbb{E}_{\mathbb{Q}}\left[\left.\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} e^{-\int_{t_0}^{v} r(l)dl} q(v)\, S_v^{\Delta}(x)\, dv\right| \mathcal{G}_{t_0}\right] \\ & -y + \mathbb{E}_{\mathbb{Q}}\left[\left.\int_{t_0}^{T\wedge\tau^*(x)\wedge\zeta(x)} e^{-\int_{t_0}^{v} r(l)dl} q(v)\, S_v^{\Delta}(y)\, dv\right| \mathcal{G}_{t_0}\right] \\ \geq\ & x - y, \end{aligned} \tag{5.25}$$

and the last inequality follows again from equation (5.12) and from the assumed nonnegativeness of the dividend yield.

Given that inequality (5.25) is similar to Jacka (1991, Equation 2.4), the rest of the proof follows immediately from Jacka (1991). More specifically, and assuming that $(x \in \mathcal{C}_{t_0})$, definition (3.16) implies that

$$V_{t_0}(x,K,T;1) > (K-x)^+ . \tag{5.26}$$

Hence, inequalities (5.25) and (5.26) can be combined into

$$\begin{aligned} V_{t_0}(y,K,T;1) &> (K-x)^+ + x - y \\ &\geq K - y. \end{aligned} \tag{5.27}$$

Consequently, and since $V_{t_0}(y,K,T;1) > 0$ from Proposition 5.4, then $V_{t_0}(y,K,T;1) > (K-y)^+$, and $(y \in \mathcal{C}_{t_0})$ as well. ■

Proposition 5.5 proves the up-connectedness of the *continuation* region $\mathcal{C}$: $(x \in \mathcal{C}_t) \Longrightarrow (y \in \mathcal{C}_t)$ for any $y > x > 0$, and for all $t \in [t_0, T]$. Hence, $(y \notin \mathcal{C}_t) \Longrightarrow (x \notin \mathcal{C}_t)$ as well, and since $\mathcal{E}$ is the complement of $\mathcal{C}$, then we can also conclude that the *stopping* region $\mathcal{E}$ is down-connected (and, thus, closed by the orthogonal lines $S^{\Delta} = 0$ and $t = T$). Furthermore, note that the existence and uniqueness of the early exercise boundary was proved in Proposition 5.5 under the most general time-inhomogeneous formulation of the JDCEV model, i.e. without the need of imposing any parameter restrictions.

5.3. Monotonicity and continuity

Next propositions further characterize the early exercise boundary of the American-style put as a nondecreasing and continuous function of calendar time, as long as some parameter restrictions are satisfied by the JDCEV process.

Proposition 5.6 *Under the JDCEV model, the early exercise boundary $t \to E(t)$ of the American-style put is a nondecreasing function of calendar time if the following four conditions are met:*

$$\frac{dq(t)}{dt} \leq [r(t) - r(t-u)]\, q(t), \tag{5.28}$$

$$\frac{dr(t)}{dt} \geq [r(t) - r(t-u)]\, r(t), \tag{5.29}$$

$$\frac{db(t)}{dt} \leq [r(t) - r(t-u)]\, b(t), \tag{5.30}$$

and

$$\frac{da(t)}{dt} \leq \frac{1}{2}[r(t) - r(t-u)]\, a(t), \tag{5.31}$$

for all $t \in [t_0, T]$ and $u \in]0, t-t_0]$.

Proof. Definition (3.17) implies that it is only necessary to show that the map $t \to V_t\left(S^\Delta, K, T; 1\right)$ is nonincreasing, under conditions (5.28) to (5.31). For this purpose, equation (3.5) can be restated as

$$\mathbb{E}_\mathbb{Q}\left[e^{-\int_{t_0}^{\theta} r(l)dl}\left(K - S_\theta^\Delta\right)^+ \middle| \mathcal{G}_{t_0}\right] = \left(K - S_{t_0}^\Delta\right)^+ + \mathbb{E}_\mathbb{Q}\left[\int_{t_0}^{\theta} e^{-\int_{t_0}^{u} r(l)dl}\bar{H}\left(u, S_u^\Delta\right)du \middle| \mathcal{G}_{t_0}\right], \tag{5.32}$$

with

$$\bar{H}\left(u, S_u^\Delta\right) = H\left(u, S_u^\Delta\right) + \frac{1}{2}\delta\left(S_u^\Delta - K\right)\left(S_u^\Delta\right)^2\sigma^2\left(u, S_u^\Delta\right), \tag{5.33}$$

because the *local time* (3.4) can be rewritten (in the distributional sense) as[10]

$$L_u^K\left(S^\Delta\right) = \int_{t_0}^{u} \delta\left(S_u^\Delta - K\right)\left(S_l^\Delta\right)^2\sigma^2\left(l, S_l^\Delta\right)dl, \tag{5.34}$$

[10]See, for instance, Protter (2005, Page 220).

where $\delta(\cdot)$ is the Dirac's delta (generalized) function.

Following Monoyios and Ng (2011, Theorem 3.3), let $t+v^*(x)$, with $v^*(x) \geq 0$, denote the optimal stopping time (3.13) for the initial state $(x,t) \in \mathcal{C}$, with $t \in]t_0, T]$. Combining equations (3.12), (3.13) and (5.32), then

$$\begin{aligned} 0 < V_t(x,K,T;1) - (K-x)^+ &= \mathbb{E}_{\mathbb{Q}}\left[\left.\int_t^{t+v^*(x)} e^{-\int_t^u r(l)dl} \bar{H}\left(u, S_u^{\Delta}\right) du \right| \mathcal{G}_t\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\left.\int_0^{v^*(x)} e^{-\int_t^{t+w} r(l)dl} \bar{H}\left(t+w, S_{t+w}^{\Delta}\right) dw \right| \mathcal{G}_t\right], \end{aligned} \tag{5.35}$$

where the inequality arises because $x \in \mathcal{C}_t$. Furthermore, and since the hitting time $t_0+v^*(x)$ might be sub-optimal for the starting state (x,t_0), equations (3.12), (3.13) and (5.32) yield

$$\begin{aligned} V_{t_0}(x,K,T;1) &\geq \mathbb{E}_{\mathbb{Q}}\left[\left.\Psi\left(t_0, t_0+v^*(x), S_{t_0+v^*(x)}^{\Delta}; 1\right)\right| \mathcal{G}_{t_0}\right] \\ &\geq (K-x)^+ + \mathbb{E}_{\mathbb{Q}}\left[\left.\int_{t_0}^{t_0+v^*(x)} e^{-\int_{t_0}^u r(l)dl} \bar{H}\left(u, S_u^{\Delta}\right) du \right| \mathcal{G}_{t_0}\right] \\ &\geq (K-x)^+ + \mathbb{E}_{\mathbb{Q}}\left[\left.\int_0^{v^*(x)} e^{-\int_{t_0}^{t_0+w} r(l)dl} \bar{H}\left(t_0+w, S_{t_0+w}^{\Delta}\right) dw \right| \mathcal{G}_{t_0}\right]. \end{aligned} \tag{5.36}$$

Therefore, if we can show that the integrand function

$$M\left(h,u,S^{\Delta}\right) := e^{-\int_h^{h+u} r(l)dl} \bar{H}\left(h+u, S^{\Delta}\right) \tag{5.37}$$

is nonincreasing in h (for all $u \in [0,\infty[$), then the second term on the right-hand side of inequality (5.36) will be not smaller than the right-hand side of equation (5.35), meaning that

$$V_{t_0}(x,K,T;1) - (K-x)^+ \geq V_t(x,K,T;1) - (K-x)^+ \tag{5.38}$$

for $t_0 < t$, and, hence, that the map $t \rightarrow V_t(S,K,T;1)$ is nonincreasing.[11]

From definition (5.37), and to test the monotonicity of function $M\left(h,u,S^{\Delta}\right)$, it follows that

$$\frac{\partial M\left(h,u,S^{\Delta}\right)}{\partial h} = e^{-\int_h^{h+u} r(l)dl}\left\{\left[-r(h+u)+r(h)\right]\bar{H}\left(h+u,S^{\Delta}\right) + \frac{\partial \bar{H}\left(h+u,S^{\Delta}\right)}{\partial h}\right\}. \tag{5.39}$$

[11]Note that, in both cases, the stock price is initialized at the same level—i.e. $S_t^{\Delta} = x$ and $S_{t_0}^{\Delta} = x$—and, therefore, the monotonicity of function (5.37) is a sufficient condition to yield the inequality (5.38).

Furthermore, multiplying both sides of equation (5.39) by $e^{\int_h^{h+u} r(l)dl}$ and using equations (2.7), (2.8), (3.6) and (5.33), we obtain, after some tedious algebra,

$$
\begin{aligned}
(5.40)\quad & e^{\int_h^{h+u} r(l)dl} \frac{\partial M\left(h,u,S^{\Delta}\right)}{\partial h} \\
= \quad & \left\{ S^{\Delta}_{h+u} \left[\left(r\left(h\right) - r\left(h+u\right)\right) q\left(h+u\right) + \frac{dq\left(h+u\right)}{dh} \right] \right. \\
& \left. -K \left[\left(r\left(h\right) - r\left(h+u\right)\right) r\left(h+u\right) + \frac{dr\left(h+u\right)}{dh} \right] \right\} \mathbb{1}_{\left\{S^{\Delta}_{h+u} < K\right\}} \\
& +K \left\{ \left[\left(r\left(h\right) - r\left(h+u\right)\right) b\left(h+u\right) + \frac{db\left(h+u\right)}{dh} \right] \right. \\
& \left. +ca\left(h+u\right) \left(S^{\Delta}_{h+u}\right)^{2\bar{\beta}} \left[\left(r\left(h\right) - r\left(h+u\right)\right) a\left(h+u\right) + 2\frac{da\left(h+u\right)}{dh} \right] \right\} \mathbb{1}_{\left\{S^{\Delta}_{h+u} \geq K\right\}} \\
& +\frac{1}{2}\delta\left(S^{\Delta}_{h+u} - K\right) a\left(h+u\right) \left(S^{\Delta}_{h+u}\right)^{2+2\bar{\beta}} \left[\left(r\left(h\right) - r\left(h+u\right)\right) a\left(h+u\right) + 2\frac{da\left(h+u\right)}{dh} \right].
\end{aligned}
$$

Therefore, using the change of variables $t = h + u$, and since $S^{\Delta}, K, c,$ and $a\left(\cdot\right)$ are all nonnegative, equation (5.40) implies that $\frac{\partial M\left(h,u,S^{\Delta}\right)}{\partial h} \leq 0$ if conditions (5.28) to (5.31) are all met.■

Remark 5.3 *Note that equations (5.28) to (5.31) are only sufficient (but not necessary) conditions for the monotonicity of the early exercise boundary.*

Remark 5.4 *Given the deterministic interest rate setup adopted, the time-t_0 discount factor for maturity at time t ($\geq t_0$) can be stated as $P\left(t_0, t\right) := \exp\left[-\int_{t_0}^{t} r\left(l\right) dl\right]$. Therefore, and since the short-term interest rate is assumed to be nonnegative, then the discount function is surely nonincreasing: $\frac{\partial P(t_0,t)}{\partial t} = -r\left(t\right) P\left(t_0, t\right) \leq 0$. Moreover, condition (5.29) allows the discount function to be either convex or concave as $\frac{\partial^2 P(t_0,t)}{\partial t^2} = \left[r^2\left(t\right) - \frac{dr(t)}{dt}\right] P\left(t_0, t\right)$ can be nonnegative or nonpositive.*

Remark 5.5 *If the convexity of the discount function is further imposed, then the short-term interest rate function must be such that $\frac{dr(t)}{dt} \leq r^2\left(t\right)$, and condition (5.29) implies that $\frac{dr(t)}{dt}$ can be positive, negative or zero. Therefore, and even though the functions $t \rightarrow b\left(t\right)$ and $t \rightarrow a\left(t\right)$ are nonnegative, the right-hand side of both inequalities (5.30) and (5.31) can be positive or negative, and, hence, equations (2.7) and (2.8) imply that the default intensity (2.8) can be both an increasing or a decreasing function of calendar time.*

Remark 5.6 *Conditions (5.28) to (5.31) are trivially satisfied for constant parameters* r*,* q*,* a*, and* b*. Consequently, and as expected, the early exercise boundary* $t \to E(t)$ *of the American-style put is surely nondecreasing under the time-homogeneous JDCEV specification defined in Carr and Linetsky (2006, Remark 5.1).*

As usual, and following, for instance, Detemple and Kitapbayev (2017, Page 12), the nondecreasing nature of the map $t \to E(t)$ and the fact that $\mathcal{E}$ is closed both yield the right-continuity of the early exercise boundary.

Proposition 5.7 *Under the JDCEV model, the early exercise boundary* $t \to E(t)$ *of the American-style put is a right-continuous function of calendar time if conditions (5.28) to (5.31) are met.*

Proof. We need to show that $E(t+) = E(t)$, for any $t \in [t_0, T[$, where $E(t+) := \lim_{u \downarrow t} E(u)$ is the right limit of E at t. For this purpose, let $\{t_n\}_{n \geq 1}$ be a decreasing sequence of dates such that $t_n \downarrow t$ as $n \to \infty$. Since $(E(t_n), t_n) \in \mathcal{E}$, for all $n \geq 1$, and $(E(t_n), t_n) \to (E(t+), t)$ as $n \to \infty$, then the closedness of $\mathcal{E}$ implies that $(E(t+), t) \in \mathcal{E}$. Therefore, $E(t+) \leq E(t)$ by definition (3.18)—with $\phi = 1$.

However, by Proposition 5.6, and since conditions (5.28) to (5.31) are assumed to be met, the map $t \to E(t)$ is nondecreasing, and, hence, $E(t+) \geq E(t)$. Consequently, we must have $E(t+) = E(t)$, for all $t \in [t_0, T[$.■

The proof of the left-continuity of the early exercise boundary will be based on Lamberton and Mikou (2008, Theorem 4.2) and, therefore, will have to be restricted to a JDCEV setup with strictly positive interest rates. This result is important because the *optimal stopping approach*—followed, for instance, by Nunes (2009, Proposition 5)—assumes the continuity of the early exercise boundary in order to recover the first passage time density of the underlying stock price through the stopping region.

Proposition 5.8 *Under the JDCEV model, the early exercise boundary* $t \to E(t)$ *of the American-style put is a left-continuous function of calendar time if conditions (5.28) to (5.31) are met and if* $r(u) > 0$ *for all* $u \in [t_0, T[$*.*

Proof. Using the stochastic differential equation (2.5), and applying the Itô's formula to any function $f \in C^{1,2}$, it follows that

$$df\left(t, S_t^{\Delta}\right) = \mathcal{A}f\left(t, S_t^{\Delta}\right) dt + S_t^{\Delta}\sigma\left(t, S_t^{\Delta}\right) dW_t - \left[f\left(t^-, 0\right) - f\left(t^-, S_{t^-}^{\Delta}\right)\right] dM_t, \tag{5.41}$$

where

$$\begin{aligned}\mathcal{A}f(t,x) := \quad & \frac{\partial f(t,x)}{\partial t} + \left[r(t) - q(t) + \lambda(t,x)\right] x \frac{\partial f(t,x)}{\partial x} \\ & + \frac{1}{2} x^2 \sigma^2(t,x) \frac{\partial^2 f(t,x)}{\partial x^2} + \left[f(t,0) - f(t,x)\right] \lambda(t,x)\end{aligned} \tag{5.42}$$

is the infinitesimal generator of S^{Δ}. Taking $f\left(t, S_t^{\Delta}\right) = V_t\left(S^{\Delta}, K, T; 1\right)$ and since the discounted price process of an American-style option must be a $\mathbb{Q}$-martingale in the continuation region $\mathcal{C}$, equations (5.41) and (5.42) yield the following partial integro-differential equation (PIDE, hereafter):

$$\frac{\partial V_t\left(S^{\Delta}, K, T; 1\right)}{\partial t} + \mathcal{L}V_t\left(S^{\Delta}, K, T; 1\right) + K\lambda\left(t, S_t^{\Delta}\right) = 0 \tag{5.43}$$

for $\left(S^{\Delta}, t\right) \in \mathcal{C}$, and where $\mathcal{L}$ is the differential operator

$$\mathcal{L} := \left[r(t) - q(t) + \lambda\left(t, S_t^{\Delta}\right)\right] S_t^{\Delta} \frac{\partial}{\partial S_t^{\Delta}} + \frac{1}{2}\left(S_t^{\Delta}\right)^2 \sigma^2\left(t, S_t^{\Delta}\right) \frac{\partial^2}{\partial S_t^{\Delta 2}} - \left[r(t) + \lambda\left(t, S_t^{\Delta}\right)\right]. \tag{5.44}$$

Of course, it is well known that the American put price is not smooth, and, therefore, the PIDE (5.43) must be understood in a weaker sense—for instance, in the viscosity sense of Pham (1998, Theorem 3.1).

We can now prove the left-continuity of the map $t \to E(t)$ by contradiction. For this purpose, suppose that the early exercise boundary is (left) discontinuous at time $u \in [t_0, T[$, i.e.

$$E(u-) < E(u), \tag{5.45}$$

where $E(u-) := \lim_{l \uparrow u} E(l)$ is the left limit of E at u. Using definition (3.19) and Proposition 5.6, it follows that the set $\mathcal{U} := \left\{\left(S^{\Delta}, t\right) \in]E(u-), E(u)[\times [t_0, u[\right\}$ belongs to the continuation region, and, therefore, equation (5.43) is valid for any $\left(S^{\Delta}, t\right) \in \mathcal{U} \subset \mathcal{C}$. Moreover, the nonincreasing nature of the map $t \to V_t\left(S^{\Delta}, K, T; 1\right)$ implies that

$$\mathcal{L}V_t\left(S^{\Delta}, K, T; 1\right) + K\lambda\left(t, S_t^{\Delta}\right) = -\frac{\partial V_t\left(S^{\Delta}, K, T; 1\right)}{\partial t} \geq 0 \tag{5.46}$$

for any $\left(S^{\Delta}, t\right) \in \mathcal{U}$. Since both

$$V_u\left(x, K, T; 1\right) = K - x, \tag{5.47}$$

and inequality (5.46) hold for $x \in \left]E\left(u-\right), E\left(u\right)\right[$, then definition (5.44) yields

$$\begin{aligned} & \left[r\left(u\right) - q\left(u\right) + \lambda\left(u, x\right)\right] x \times (-1) - \left[r\left(u\right) + \lambda\left(u, x\right)\right]\left(K - x\right) + K\lambda\left(u, x\right) \\ = \; & q\left(u\right) x - r\left(u\right) K \\ \geq \; & 0 \end{aligned} \tag{5.48}$$

for $x \in \left]E\left(u-\right), E\left(u\right)\right[$.

In opposition, and given the nondecreasing nature of the map $t \to E\left(t\right)$, the set $\mathcal{V} := \left\{\left(S^{\Delta}, t\right) \in \left]0, E\left(u\right)\right[\times \left]u, T\right[\right\}$ belongs to the exercise region, and, hence, we must have

$$\begin{aligned} \mathcal{L}V_t\left(S^{\Delta}, K, T; 1\right) + K\lambda\left(t, S_t^{\Delta}\right) & \leq -\frac{\partial V_t\left(S^{\Delta}, K, T; 1\right)}{\partial t} \\ & \leq 0 \end{aligned} \tag{5.49}$$

for any $\left(S^{\Delta}, t\right) \in \mathcal{V} \subset \mathcal{E}$, because, in $\mathcal{E}$, the discounted price process of an American-style option must be a supermartingale under the risk-neutral measure and the *theta* of the American-style put is zero.[12] Since both equations (5.47) and (5.49) hold on the set $\mathcal{V}$, then definition (5.44) yields

$$q\left(u\right) x - r\left(u\right) K \leq 0 \tag{5.50}$$

for $x \in \left]0, E\left(u\right)\right[$.

Combining inequalities (5.48) and (5.50), then

$$q\left(u\right) x = r\left(u\right) K \tag{5.51}$$

for $x \in \left]E\left(u-\right), E\left(u\right)\right[$. Consequently, and if $q\left(u\right) > 0$, then function $x \to q\left(u\right) x$ is strictly increasing in $\left[x, E\left(u\right)\right[$, which contradicts equation (5.51). Otherwise (i.e. if $q\left(u\right) = 0$), and since $r\left(u\right) > 0$, then equation (5.51) can not prevail either.[13]$\blacksquare$

[12] Again, please note that the partial integro-differential inequality (5.49) should be interpreted in a distributional—e.g. Jaillet, Lamberton, and Lapeyre (1990)—or in a viscosity sense—see, for instance, Pham (1998).

[13] Left-continuity has only been proved for strictly positive interest rates because equation (5.51) would be trivially satisfied if $r\left(u\right) = q\left(u\right) = 0$.

In both Propositions 5.7 and 5.8, the analysis was restricted to the time interval $[t_0, T[$ but the early exercise boundary is also continuous at the maturity date because

$$E(T) = \lim_{u \uparrow T} E(u). \tag{5.52}$$

Moreover, next proposition shows that $E(T)$ possesses the usual structure obtained, for instance, in Van Moerbeke (1976).

Proposition 5.9 *Under the JDCEV model, the early exercise boundary at the maturity date of the American-style put is equal to*

$$E(T) = K \wedge \frac{r(T)}{q(T)} K, \tag{5.53}$$

as long as $t \to r(t)$, $t \to q(t)$, $t \to a(t)$ and $t \to b(t)$ are all continuous functions of time and if $r(u) > 0$ for all $u \in [t_0, T]$.

Proof. This proof follows closely the proof of Lamberton and Mikou (2008, Theorem 4.4).

Since it is not rational to exercise an out-of-the-money option (that would yield a zero payoff), then we must have

$$E(T) \leq K. \tag{5.54}$$

Additionally, and following the same steps as in the proof of Proposition 5.8, it is easy to show that inequality (5.50) is valid in the exercise region $\mathcal{E}$, i.e. for $u \in]t_0, T[$ and $x \in]0, E(u)[$. Hence, and using the continuity of the functions $t \to r(t)$ and $t \to q(t)$, it follows that

$$q(T)x - r(T)K \leq 0, \tag{5.55}$$

for all $x \in]0, E(T)[$.

In opposition, and as also shown in the proof of Proposition 5.8, inequality (5.46) is valid in the continuation region $\mathcal{C}$, i.e. for $t \in]t_0, T[$ and $x \in]E(t), \infty[$. Therefore, and since

$$\lim_{t \uparrow T} \left[\mathcal{L}V_t\left(S^\Delta, K, T; 1\right) + K\lambda\left(t, S_t^\Delta\right) \right] = \mathcal{L}\left(K - S_T^\Delta\right)^+ + K\lambda\left(T, S_T^\Delta\right)$$

follows (in the sense of distributions) from the continuity of the functions $t \to a(t)$ and $t \to b(t)$, we must have

$$\mathcal{L}(K-x)^{+} + K\lambda(T,x) \geq 0, \tag{5.56}$$

for $x \in \left]E(T), \infty\right[$. Furthermore, for $x \in [0, K[$, and using definition (5.44), it follows that

$$\begin{aligned} \mathcal{L}(K-x)^{+} &= \mathcal{L}(K-x) \\ &= q(T)\,x - r(T)\,K - K\lambda(T,x), \end{aligned}$$

and, hence, inequality (5.56) yields

$$q(T)\,x - r(T)\,K \geq 0, \tag{5.57}$$

for all $x \in \left]E(T), \infty\right[\cap [0, K[$.

Hereafter, the left-hand side of inequalities (5.55) and (5.57) will be represented by the function $\eta_T(x) := q(T)\,x - r(T)\,K$. If $q(T) \leq r(T)$ and $q(T) > 0$, then $\eta_T(K) = q(T)\,K - r(T)\,K \leq 0$. Moreover, and since $\frac{\partial \eta_T(x)}{\partial x} = q(T) > 0$, it follows that $\eta_T(x) < 0$ for all $x \in [0, K[$. Therefore, equations (5.54) and (5.57) imply that $E(T) = K$. Likewise, if $q(T) \leq r(T)$ but $q(T) = 0$, then $\eta_T(x) = -r(T)\,K < 0$ (as $r(T) > 0$) for all $x \in [0, K]$, and again equations (5.54) and (5.57) imply that $E(T) = K$. In opposition, if $q(T) > r(T)$, then $\eta_T(K) > 0$ and $\eta_T(0) = -r(T)\,K < 0$. Consequently, and since $\frac{\partial \eta_T(x)}{\partial x} = q(T) > 0$ (as $r(T) > 0$), it follows that equation $\eta_T(x) = 0$ possesses a unique solution in $]0, K[$. Moreover, equations (5.55) and (5.57) imply that such unique solution must be obtained at $x = E(T)$, i.e. must be equal to $E(T) = \frac{r(T)}{q(T)}K$.■

6. Numerical examples

We now give numerical examples of early exercise boundaries under the simpler time-homogeneous version of the JDCEV model that nests, as a special case, the well known CEV specification. Similarly to Carr and Linetsky (2006), Ruas, Dias, and Nunes (2013), Dias, Nunes, and Ruas (2015), and Nunes, Ruas, and Dias (2015), we calibrate the (constant) volatility scale parameter a such that the initial instantaneous volatility is the same across different models. More specifically, we assume an initial stock price reference level $S_{t_0} = 100$ and a volatility (at that reference level) equal to $\sigma(t_0, S_{t_0}) \equiv \sigma_{t_0} = 0.20$. Then, the volatility scale parameter a to be used in our set of applications with different $\bar{\beta}$ values is adjusted to $a = \sigma_{t_0} S_{t_0}^{-\bar{\beta}}$. Moreover, we assume the options contracts expire in one year ($T - t_0 = 1$), the strike price is 100 ($K = 100$), the risk-free rate is 6% ($r = 0.06$), and the dividend yield is 3% ($q = 0.03$).

Our main interest is in the dependence of the early exercise boundary on the parameters $\bar{\beta}$, b and c governing the local volatility function (2.7) and the default intensity (2.8). This will allow us to shed some economic insights on the early exercise behavior of traders. To accomplish this purpose, we deploy three values of $\bar{\beta}$ to show its effect on the early exercise boundary: $\bar{\beta} \in \{-0.5, -1.0, -1.5\}$; then, we obtain $a \in \{2, 20, 200\}$, respectively. Furthermore, and for each $\bar{\beta}$ value, we consider five different combinations of the two parameters b and c. Therefore, a constellation of fifteen option contracts is obtained. The standard CEV model (with $b = c = 0$) is considered for comparative purposes. We further consider the cases with $b = 0$ or $b = 0.02$ (adding, in the latter specification, 2% per annum to the default intensity) and $c = 0.5$ or $c = 1$. For instance, the case with $c = 1$ (coupled with the initial reference levels $S_{t_0} = 100$ and $\sigma_{t_0} = 0.20$) provides a contribution to the default intensity due to the variance term $c\,\sigma^2(t_0, S_{t_0})$ of 0.04. Following this line of reasoning, we easily get the set of five initial default intensity values for each chosen $\bar{\beta}$: $\lambda \in \{0, 0.02, 0.04, 0.04, 0.06\}$. We recall that as the stock price falls (resp., increases), the implied volatility increases (resp., decreases) and the default intensity also increases (resp., decreases). Hence, such variance term is intended to capture the positive correlation between default probabilities (or CDS spreads) and equity volatilities observed in the credit markets.

[Please insert Figure 1 about here.]

Figure 1 plots the early exercise boundary of standard American-style put option contracts as a function of calendar time. We note that each early exercise boundary is obtained through the *static hedge portfolio* procedure offered by Ruas, Dias, and Nunes (2013, Section 3.2) using 256 evenly-spaced time points. At the maturity date $T = 1$ the early exercise boundary does not depend on the parameters $\bar{\beta}$, b and c and is simply equal to the strike price K (given the choice made for r and q)—cf. Proposition 5.9. Moreover, near expiry there are no markedly differences between the different early exercise boundaries. This is consistent with the asymptotic analysis offered by Chung and Shih (2009, Section 4) and Ruas, Dias, and Nunes (2013, Section 4) when deriving the early exercise boundary near expiration under the standard CEV and JDCEV models, respectively. However, as we move farther away from the maturity date T to the inception date of each contract ($t_0 = 0$) we observe that the choice of the parameters $\bar{\beta}$, b and c influences significantly the level of the early exercise boundary, thus providing important economic insights about its behavior.

Let us first analyze the impact of $\bar{\beta}$ on the early exercise boundary (maintaining b and c fixed). As expected, Figure 1 reveals that as the $\bar{\beta}$ value falls the early exercise boundary decreases. The economic rationale for this result is justified by the observation that as the $\bar{\beta}$ parameter departs from the limiting geometric Brownian motion process (i.e. $\bar{\beta} = 0$), the probability of the predefault stock price hitting the zero default boundary—via diffusion only under the standard CEV process or accommodating the possibility of a sudden jump to default under the JDCEV modeling setup—becomes higher. Hence, the early exercise boundary falls as a consequence of the increasing killing probability. We recall also that since all the contracts were calibrated such that they possess the same initial volatility, the differences found throughout the numerical analysis stem purely from the effect of the relationship between volatility and price levels, which is captured by the CEV volatility specification (2.7).

While the standard CEV model is able to address the volatility smile effect commonly found in equity options markets, the default probability is unrealistically small for empirically reasonable values of the parameters $\bar{\beta}$ and a attached to the CEV stock price volatility function. The default extended CEV stock price process provides a much more reasonable modeling framework mainly because it endogenizes the hazard rate (2.8) by assuming that it is affine in a negative power of the defaultable stock price and, therefore, it captures several stylized features such as the negative relation between equity prices and equity volatility, the negative relationship between default intensity and equity prices, and the positive correlation between default probability and equity volatility. Hence, it is also noteworthy to consider the impact of the b and c parameters on the early exercise boundary (while keeping $\bar{\beta}$ fixed).

Figure 1 highlights that the early exercise boundary under the standard CEV model specification (i.e. with $b = c = 0$, and, hence, $\lambda = 0$) is clearly above the remaining early exercise boundaries, i.e. early exercise under the CEV model occurs sooner. For any given $\bar{\beta}$ value, we observe that increasing b and c decreases the early exercise boundary (contracts #3 and #4 for each $\bar{\beta}$ parameter are almost indistinguishable since they possess the same default intensity, i.e. $\lambda = 0.04$). We recall that even though the CEV process (with $\bar{\beta} < 0$) can hit the zero default boundary with positive probability, such killing probability (via diffusion only) is generally quite small. In contrast, this default probability is substantially increased under the JDCEV framework, since default can also arrive as an unexpected event. Therefore, increasing b and c augments the default probability and, as a result, the early exercise boundary falls. This suggests that a trader may incorrectly follow a premature exercise strategy when ignoring the possibility of default as a surprise event. Consequently, the trader may sacrifice much of the value of the option contract by exercising the American-style put too soon.

7. Conclusion

The valuation of American-style standard options under the JDCEV framework is already well established in the literature, but the existence of the associated early exercise boundary has never been proved. This paper fills this gap. For calls, the existence of the early exercise boundary is based on Detemple and Tian (2002, Proposition 1). For put options, the existence, uniqueness, monotonicity, and continuity of the early exercise boundary follows from Jacka (1991, Proposition 2.1), Lamberton and Mikou (2008, Theorem 4.2), Monoyios and Ng (2011, Theorem 3.3), and from well known properties of Bessel processes.

The numerical tests run show that ignoring the possibility of default as a surprise event will lead to suboptimal exercise strategies.

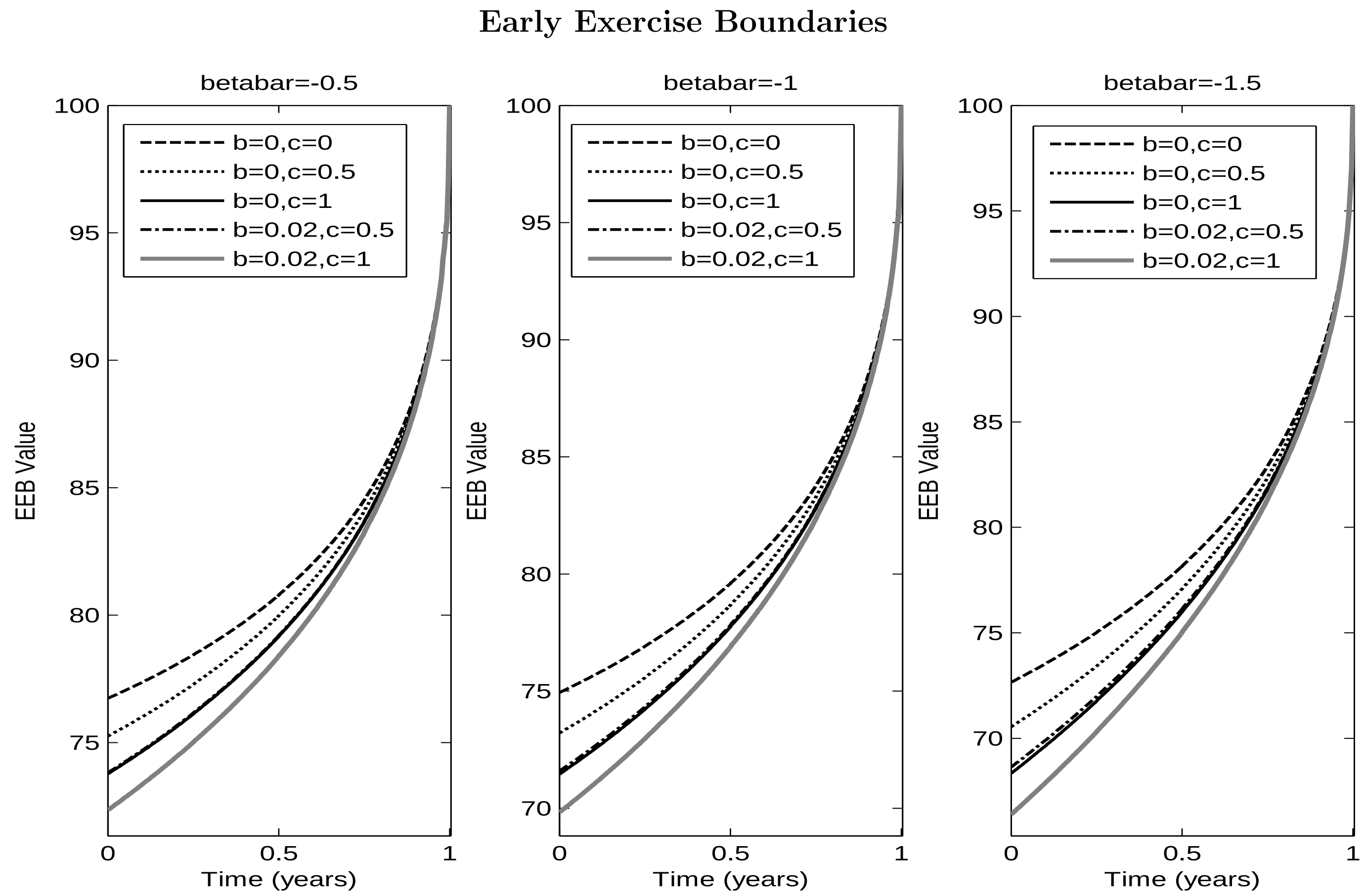

Figure 1: Early exercise boundaries of standard American-style put options under the time-homogeneous JDCEV stock price process. Parameter values: $S_{t_0} = 100$, $\sigma_{t_0} = 0.20$, $K = 100$, $T - t_0 = 1$, $\bar{\beta} \in \{-0.5, -1.0, -1.5\}$, $b \in \{0, 0.02\}$, $c \in \{0, 0.5, 1\}$, $r = 0.06$, and $q = 0.03$.

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