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Bootstrap Tests for Time Varying Cointegration

Luis F. Martins[*]

Department of Quantitative Methods, ISCTE-IUL, Portugal, and

Centre for International Macroeconomic Studies, University of Surrey, UK

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Abstract

This paper proposes wild and the i.i.d. parametric bootstrap implementations of the time-varying cointegration test of Bierens and Martins (2010, Econometric Theory 26, 1453-1490). The bootstrap statistics and the original likelihood ratio test share the same first-order asymptotic null distribution. Monte Carlo results suggest that the bootstrap approximation to the finite-sample distribution is very accurate, in particular for the wild bootstrap case. The tests are applied to study the purchasing power parity hypothesis for twelve OECD countries and we only find evidence of a constant long-term equilibrium for the U.S.-U.K. relationship.

Keywords: Time-Varying Cointegration, Bootstrap, Likelihood Ratio Test, Purchasing Power Parity Hypothesis.

J.E.L. Classification: C12, C32.

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1 Introduction

Structural change is of key importance in economics and econometrics, especially for cointegration analysis, as it normally involves long term historical trends, which, consequently, are likely to display breaks in their equilibrium relationship. In a recent paper, Bierens and Martins (2010) proposed a vector error correction model in which the cointegration vectors change smoothly over time and, from that model, a likelihood ratio test for standard time-invariant cointegration. Despite the simplicity of the test, the asymptotic chi-square distribution appears to be a poor approximation to the relevant finite-sample distribution. In particular, the test falsely indicates the existence of time-varying cointegration too often.

To address this problem, we propose in this paper a bootstrap algorithm for obtaining critical values and show that this alternative approach does lead to test procedures that have, approximately, the correct size. As in many other estimation and inference contexts, in our case the bootstrap distribution is also an accurate approximation to the finite-sample one under the null hypothesis of constant cointegration vectors, as initially presented by Johansen (1988, 1991 and 1995).

It has extensively been shown in the literature that bootstrap methods provide higher order asymptotic refinements and, thus, better results in bias reduction, confidence interval construction and hypothesis testing in finite-samples, even in the cases where analytical results are known. See, for example, Jeong and Maddala (1993), Horowitz (2001), as well as Li and Maddala (1996) and Härdle, Horowitz and Kreiss (2001) surveys, the last two in the time series context. As an example in the structural breaks literature, Diebold and Chen (1996) demonstrate the benefits of bootstraping the 'supremum' tests of Andrews (1993).

The likelihood ratio bootstrap tests we suggest follow very closely those already proposed in the literature for testing and determining the cointegration rank in VAR models. Swensen (2006) consider an i.i.d. bootstrap version of the pseudo LR trace test statistic whereas Cavaliere, Rahbek and Taylor (2010a) advocate the usage of the wild bootstrap instead as it provides better results under conditional heteroskedasticity when compared to the i.i.d. procedure. Moreover, Cavaliere, Rahbek and Taylor (2012) propose a bootstrap scheme that improves Swensen's approach to create bootstraped data once the VECM is

estimated under the null hypothesis and not under both hypothesis as in Swensen (2006).[1]

The contents of the paper are as follows. In Section 2, we review the time-varying VECM and the original LR test for time-invariant cointegration and introduce the wild and i.i.d. parametric bootstrap versions of the pseudo LR test, showing their consistency because the three share the same chi-square asymptotic distribution. Section 3 sets out the designs of the Monte Carlo experiments and suggests that the bootstrap approximation to the finite-sample distribution is very accurate, especially for the wild bootstrap. Finally, Section 4 discusses the purchasing power parity hypothesis in the context of time-varying cointegration and reports results from an application of the methodology to international prices and nominal exchange rates.

2 Bootstrap Time-Varying Cointegration Tests

2.1 The Time-Varying Cointegration Model

As in Bierens and Martins (2010), consider the time-varying VECM(p) with a drift,

$$\triangle Y_t = \mu + \alpha \beta_t' Y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \triangle Y_{t-j} + \varepsilon_t, \ t = 1, ..., T, \tag{1}$$

where $Y_t \in \mathbb{R}^k$, $\varepsilon_t \in \mathbb{R}^k$, μ, α, Ω and the Γ_j's are fixed coefficients, the β_t's are time-varying (TV) $k \times r$ matrices of cointegrating vectors, and T is the number of observations. [2] The initial values $Y_t, t = 0, -1, ..., -p + 1$, are assumed to be fixed. The LR test is defined for the null hypothesis of standard time-invariant (TI) cointegration, $\beta_t = \beta$ for all t,

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[1] In the context of cointegrated VAR models, see also the bootstrap approaches of Trenkler (2009) - i.i.d. bootstrap where in a first stage the deterministic terms are estimated by a feasible GLS procedure - and Cavaliere, Rahbek and Taylor (2010b) - wild bootstrap under non-stationary volatility. van Giersbergen (1996), Harris and Judge (1998) and Mantalos and Shukur (2001) focus on the properties of the bootstrap by means of Monte Carlo simulations. For bootstrapping methods in standard unit root testing, see, for example, Park (2003) and Paparoditis and Politis (2003).

[2] In this Bierens and Martins (2010) baseline model, there is a drift in Y_t which is denoted by μ. This does not cover all the usual leading cases for the deterministic components within standard cointegration test frameworks. In particular, we do not consider the cases of the intercept being absorbed into the cointegration relation and the existence of a linear trend. Accounting for a trend is not straightforward in time-varying cointegration as pointed out by Bierens and Martins (2010).

against time-varying cointegration (TVC) in which the cointegrating relationship varies smoothly over time, maintaining the number of cointegration relations as equal to $r < k$. In this test setup, there must exist a fixed certain number of cointegration relations, $r > 0$. Time-varying cointegrating systems do not necessarily generate time-varying cointegration spaces. The conditions for the existence of time-invariant or time-varying cointegration spaces can be found at Martins and Gabriel (2013). Assuming that the function of discrete time β_t is smooth (see Bierens and Martins (2010) for details), it can be written as

$$\beta_t = \beta_m\left(t/T\right) = \sum_{i=0}^{m} \xi_{i,T} P_{i,T}\left(t\right) \tag{2}$$

for some fixed $m < T-1$, where the orthonormal Chebyshev time polynomials $P_{i,T}\left(t\right)$ are defined by $P_{0,T}\left(t\right) = 1$, $P_{i,T}\left(t\right) = \sqrt{2}\cos\left(i\pi\left(t-0.5\right)/T\right)$, $t = 1, 2, ..., T$, $i = 1, 2, 3, ..., m$ and $\xi_{i,T} = \frac{1}{T}\sum_{t=1}^{T}\beta_t P_{i,T}\left(t\right)$ are unknown $k \times r$ matrices.

Similar to Johansen (1988, 1991 and 1995), model (1) can be specified more conveniently as

$$\triangle Y_t = \alpha\xi' Y_{t-1}^{(m)} + \Upsilon X_t + \varepsilon_t, \tag{3}$$

where $\xi' = \left(\xi_0', \xi_1', ..., \xi_m'\right)$ is an $r \times (m+1)k$ matrix of full rank r, $Y_{t-1}^{(m)}$ is defined by $Y_{t-1}^{(m)} = \left(Y_{t-1}', P_{1,T}\left(t\right)Y_{t-1}', P_{2,T}\left(t\right)Y_{t-1}', ..., P_{m,T}\left(t\right)Y_{t-1}'\right)'$ and $\Upsilon = \left(\mu, \Gamma_1, ..., \Gamma_{p-1}\right)$ and $X_t = \left(1, \triangle Y_{t-1}', ..., \triangle Y_{t-p+1}'\right)'$. Under the null hypothesis, $\xi' = \left(\beta', \mathbf{0}_{r,k.m}\right)$, where β is the standard $k \times r$ matrix of TI cointegrating vectors, so that then $\alpha\xi' Y_{t-1}^{(m)} = \alpha\beta' Y_{t-1}^{(0)} \equiv \alpha\beta' Y_{t-1}$. Therefore, given m and r, the LR test takes the form

$$LR_{m,T}^{tvc} = T\sum_{j=1}^{r}\ln\left(\frac{1-\widehat{\lambda}_{0,j}}{1-\widehat{\lambda}_{m,j}}\right), \tag{4}$$

where $1 > \widehat{\lambda}_{m,1} \geq \widehat{\lambda}_{m,2} \geq ... \geq \widehat{\lambda}_{m,r} \geq ... \geq \widehat{\lambda}_{m,(m+1)k}$ are the ordered solutions of the generalized eigenvalue problem $\det\left[\lambda S_{11,T}^{(m)} - S_{10,T}^{(m)}S_{00,T}^{-1}S_{01,T}^{(m)}\right] = 0$ with

$$S_{00,T} = \frac{1}{T}\sum_{t=1}^{T}\triangle Y_t \triangle Y_t' - \left(\frac{1}{T}\sum_{t=1}^{T}\triangle Y_t X_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_t X_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_t \triangle Y_t'\right), \tag{5}$$

$$S_{11,T}^{(m)} = \frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}Y_{t-1}^{(m)'} - \left(\frac{1}{T}\sum_{t=1}^{T}Y_{t-1}^{(m)}X_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_t X_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_t Y_{t-1}^{(m)'}\right), \tag{6}$$

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$$S_{01,T}^{(m)} = \frac{1}{T}\sum_{t=1}^{T}\triangle Y_t Y_{t-1}^{(m)\prime} - \left(\frac{1}{T}\sum_{t=1}^{T}\triangle Y_t X_t'\right)\left(\frac{1}{T}\sum_{t=1}^{T}X_t X_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}X_t Y_{t-1}^{(m)\prime}\right),\tag{7}$$

$$S_{10,T}^{(m)} = \left(S_{01,T}^{(m)}\right)'.\tag{8}$$

The $\widehat{\lambda}_{0,j}$'s are similarly defined by imposing $m = 0$ (standard cointegration).

Assuming that the errors are i.i.d. Gaussian, i.e., $\varepsilon_t \sim i.i.d.\ N_k\left[0,\Omega\right]$, Bierens and Martins (2010) show that, given $m \geq 1$ and $r \geq 1$, under the null hypothesis of standard cointegration, $LR_{m,T}^{tvc}$ is asymptotically χ_{mkr}^2 distributed. They also concluded that, for small T and large m, the test suffers from size distortions and tends to overreject the correct null hypothesis of standard cointegration. Given this result, we propose in the next Section two bootstrap versions of $LR_{m,T}^{tvc}$ along the lines of Cavaliere, Rahbek and Taylor (2010a, 2012) and Swensen (2006).

Following the seminal work by Johansen (1988, 1991, 1995), Bierens and Martins (2010) take the normal distribution for ε_t and from that derive the exact likelihood function and LR statistic. Actually, imposing normality is not necessary for deriving the asymptotic distribution of $LR_{m,T}^{tvc}$ under the null hypothesis. By simply assuming i.i.d. errors, a straightforward but tedious exercise would be to show that the pseudo $LR_{m,T}^{tvc}$ statistic is also asymptotically χ_{mkr}^2 distributed when based on the pseudo Gaussian likelihood function.[3]

In terms of the bootstrap approach, we relax the original assumptions by taking those at Cavaliere, Rahbek and Taylor (2010a) under our context of standard cointegration:

Assumption 1.

(a) For $m = 0$, the usual conditions related to the characteristic roots and the nonsingularity of matrix $\alpha'\left(I_k - \Gamma_1 - ... - \Gamma_{p-1}\right)\beta$ are taken to be true.

(b) The innovations $\{\varepsilon_t\}$ form a martingale difference sequence with respect to the filtration F_t, where $F_{t-1} \subseteq F_t$ for $t = ..., -1, 0, 1, 2, ...$, satisfying (b1) the global homoskedasticity condition

$$\frac{1}{T}\sum_{t=1}^{T}E\left(\varepsilon_t\varepsilon_t'|F_{t-1}\right) \xrightarrow{p} \Omega > 0\tag{9}$$

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[3]In this case, the exact likelihood function and the pseudo Gaussian likelihood function would coincide and therefore the last piece of the proof of Bierens and Martins (2010)' Theorem 1, where a Taylor expansion around the MLE of a particular function, would follow through.

and (b2) $E \left\| \varepsilon_t \right\|^4 \leq K < \infty$.

Remark. Here, $\{\varepsilon_t\}$ is serially uncorrelated and possibly conditionally heteroskedastic. Instead, Cavaliere, Rahbek and Taylor (2012) and Swensen (2006) assume i.i.d. innovations. See Cavaliere, Rahbek and Taylor (2010a) for more details about the underlying differences and the functional central limit theorem and the law of large numbers that apply to martingale difference sequences.

2.2 Bootstrap Tests

The parametric bootstrap versions of the TVC test statistic we consider are the wild bootstrap and the i.i.d. bootstrap. They both are implementations to TVC of the Cavaliere, Rahbek and Taylor (2010a, 2012) (wild and i.i.d.) and Swensen (2006) (i.i.d.) bootstrap procedures for testing the cointegration rank. Next, we define the bootstrap procedures using the unrestricted residuals (and $\widehat{\mu}, \widehat{\Gamma}$) as in Cavaliere, Rahbek and Taylor (2010a) and Swensen (2006) and at the end of this Section we define the alternative procedure based upon the restricted residuals (and $\widehat{\mu}, \widehat{\Gamma}$) as in Cavaliere, Rahbek and Taylor (2012).

As in Cavaliere, Rahbek and Taylor (2010a), we construct the pseudo Gaussian likelihood function and the pseudo LR test by assuming i.i.d. Gaussian errors. For the unrestricted TVC model (3), $m > 0$, let $\left(\widehat{\mu}, \widehat{\Gamma} \right)$ denote the pseudo maximum likelihood (ML) estimate of (μ, Γ) and $\widehat{\varepsilon}_t, t = p + 1, ..., T$, the pseudo ML residuals. Moreover, let $\left(\widehat{\alpha}, \widehat{\beta} \right)$ denote the restricted pseudo ML estimates of (α, β) under the null hypothesis of standard cointegration, $m = 0$. The bootstrap algorithms are described as follows:

Wild bootstrap test for TVC using the unrestricted residuals *First*, generate B bootstrap pseudo-disturbances $\varepsilon_t^b, t = p + 1, ..., T$ and $b = 1, ..., B$, from residuals $\widehat{\varepsilon}_t$ according to $\varepsilon_t^b = \widehat{\varepsilon}_t w_t$ where $\{w_t\}_{t=p+1}^{T}$ is an independent $N(0, 1)$ scalar sequence. *Second*, construct the bootstrap sample $\left\{ \triangle Y_t^b \right\}_{t=1}^{T}$ recursively from the equation

$$\triangle Y_t^b = \widehat{\mu} + \widehat{\alpha} \widehat{\beta}' Y_{t-1}^b + \sum_{j=1}^{p-1} \widehat{\Gamma}_j \triangle Y_{t-j}^b + \varepsilon_t^b, t = 1, ..., T, \tag{10}$$

with initial values $Y_t^b = Y_t, t = 1, ..., p$. *Third*, using the bootstrap sample, $\left\{ \triangle Y_t^b \right\}_{t=1}^{T}$, construct the bootstrap LR test statistic, $LR_{m,T}^b = T \sum_{j=1}^{r} \ln \left(\frac{1 - \widehat{\lambda}_{0,j}^b}{1 - \widehat{\lambda}_{m,j}^b} \right)$, where the $\widehat{\lambda}^b$'s

denote the bootstrap versions of the ordered generalized eigenvalues $\widehat{\lambda}$'s. *Fourth*, the bootstrap 95% percentile of the resulting distribution is used as the 5% critical value for the bootstrap test procedure. In a similar manner, bootstrap p-values $p_{m,T}^b$ for the test statistic $LR_{m,T}^{tvc}$ are computed as $p_{m,T}^b = 1 - F_{m,T}^b \left(LR_{m,T}^{tvc} \right)$, where, conditional on the original data $\{Y_t\}$, $F_{m,T}^b(\cdot)$ denotes the cumulative distribution function of $LR_{m,T}^b$. Because the bootstrap distribution $F_{m,T}^b$ is a complicated function of $\{Y_t\}$, in practice, $p_{m,T}^b$ is obtained through the numerical approximation: $\widetilde{p}_{m,T}^b = \frac{1}{B} \sum_{b=1}^B 1 \left(LR_{m,T}^b > LR_{m,T}^{tvc} \right)$. *Fifth*, for a significance level δ, reject the null hypothesis of standard cointegration if $\widetilde{p}_{m,T}^b < \delta$.

As pointed out by Cavaliere, Rahbek and Taylor (2010a), an alternative could be drawing from equation (10) but without the estimate of the deterministic part, $\widehat{\mu}$, and setting initial values to zero, $Y_t^b = 0, t = 1, ..., p$. See Cavaliere, Rahbek and Taylor (2010a, 2012), Swensen (2006) but also Cavaliere, Rahbek and Trenkler (2013) for a discussion on why the rank statistic is invariant with respect to μ and recall that Bierens and Martins (2010) showed that $LR_{m,T}^{tvc}$ has the same distribution whether or not there is an intercept in the model.

Swensen's i.i.d. bootstrap test for TVC Same as the wild bootstrap approach except that the bootstrap pseudo-disturbances $\varepsilon_t^b, t = p+1, ..., T$, are drawn randomly with replacement from the residuals $\widehat{\varepsilon}_t$. Following the same argument as in Cavaliere, Rahbek and Taylor (2010a), obtain instead the centered residuals $\left\{ \widehat{\varepsilon}_t - \frac{1}{T-p} \sum_{i=p+1}^T \widehat{\varepsilon}_i \right\}_{t=p+1}^T$ if drawing from equation (10) without the intercept term.

The number of bootstrap pseudo-samples B must be "large enough" since $\widetilde{p}_{m,T}^b$ converges to $p_{m,T}^b$ as B increases. In their numerical experiments, Cavaliere, Rahbek and Taylor (2010a, 2012) set $B = 399$ and Swensen (2006) considers $B = 5000$. In the next Theorem we show that the first-order asymptotic distributions of the bootrastrap TVC test statistics and the Bierens and Martins (2010) original TVC test statistic are the same. Moreover, in the next Section, we provide, by means of simulations, evidence that the bootstrap approximation to the finite sample distribution seems to be much more accurate, as opposite to the asymptotic approximation. Hence, we consider the wild and i.i.d. bootstrap versions of the TVC test statistic as valid ones to test for standard cointegration

and recommend using their critical values to ensure correct 5% type-I errors, say.

The proof of Theorem 1 is presented in a similar way as in Cavaliere, Rahbek and Taylor (2010a, 2012) and Swensen (2006). Note that, under the null hypothesis, our model is the same as in Cavaliere, Rahbek and Taylor (2010a, 2012) and Swensen (2006) and the assumptions as in Cavaliere, Rahbek and Taylor (2010a). We also consider the wild and the i.i.d. bootstraps but the difference is that, in our context, we test against TVC and fix r.

Theorem 1. *Let $LR_{m,T}^{wb}$ and $LR_{m,T}^{sb}$ denote the $LR_{m,T}^{b}$ bootstrap test statistics for the wild and i.i.d. cases, respectively. Given $m \geq 1$ and $r \geq 1$, under the null hypothesis of standard cointegration and Assumption 1, the bootstrap LR statistics $LR_{m,T}^{wb}$ and $LR_{m,T}^{sb}$ defined above are asymptotically χ_{mkr}^2 distributed, as $T \to \infty$.*

Proof: For sake of simplicity, and as in Cavaliere, Rahbek and Taylor (2010a), we take the no-drift case, $\mu = 0$, for which Bierens and Martins (2010) showed that $LR_{m,T}^{tvc}$ is also asymptotically χ_{mkr}^2 distributed. To avoid confusion with the "wb/sb" notation, we denote any of the two bootstrap probabilities by P^* and the expectation under P^* by E^* and we do, similarly, with the bootstrap quantities $Y_t^b, \varepsilon_t^b, R_{t,T}^b, ...$

Under standard cointegration, we have from Cavaliere, Rahbek and Taylor (2010a) that (i)

$$Y_t^b = \widehat{C} \sum_{i=p+1}^{t} \varepsilon_i^b + \sqrt{T} R_{t,T}^b, t = p+1, ..., T, \tag{11}$$

where for all $\eta > 0, P^* \left(\max_{p+1 \leq t \leq T} \left| R_{t,T}^b \right| > \eta \right) \xrightarrow{p} 0$ as $T \to \infty$, $|\cdot|$ is the usual Euclidean distance and $\widehat{C} = \widehat{\beta}_\perp \left(\widehat{\alpha}_\perp' \widehat{\Gamma} \widehat{\beta}_\perp \right)^{-1} \widehat{\alpha}_\perp'$, (ii) $\triangle Y_t^b = \sum_{i=0}^{t-1} \widehat{\theta}_i \varepsilon_{t-i}^b$, where $\widehat{\theta}_i$ are exponentially decreasing coefficients, (iii) $E^* \left(\varepsilon_t^b \varepsilon_t^{b\prime} \right) = \widehat{\Omega}^b \xrightarrow{p} \Omega$ as $T \to \infty$, and (iv)

$$\frac{1}{\sqrt{T}} \sum_{i=p+1}^{\lfloor \cdot T \rfloor} \varepsilon_i^b \xrightarrow{w} W(\cdot), \tag{12}$$

where $W(u), u \in [0,1]$ is a $k-$dimensional Wiener process with covariance matrix Ω. Contrary to Cavaliere, Rahbek and Taylor (2010a, 2012) and Swensen (2006), $\widehat{\Gamma}$ and $\widehat{\varepsilon}_t$ are obtained from the TVC model (3) and, as a result, our bootstrap samples $Y_t^b, \varepsilon_i^b, R_{t,T}^b$ are different from theirs. The formulas for $R_{t,T}^b$ are in Cavaliere, Rahbek and Taylor (2010a).

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With the following Lemmas, we show that the TVC test statistic applied to the bootstraped sample, $LR^b_{m,T}$, is also asymptotically χ^2_{mkr} distributed, as $T \to \infty$. The Lemmas are as those at Bierens and Martins (2010) involving the matrices $S^{(m)}_{ij,T}, i, j = 0, 1$, but with bootstrapped data $Y^{b,(m)}_{t-1} = \left(Y^{b\prime}_{t-1}, P_{1,T}(t) Y^{b\prime}_{t-1}, P_{2,T}(t) Y^{b\prime}_{t-1}, ..., P_{m,T}(t) Y^{b\prime}_{t-1} \right)^{\prime}$ and $X^b_t = \left(\triangle Y^{b\prime}_{t-1}, ..., \triangle Y^{b\prime}_{t-p+1} \right)^{\prime}$ and invoking the functional central limit theorem and the law of large numbers that apply to martingale difference sequences instead (Brown, 1971, and Hannan and Heyde, 1972), and where it is shown that the original limiting results still hold.

Lemma 1. *Let $l = 1, ..., p - 1$. As $T \to \infty$,*

$$\frac{1}{T} \sum_{t=1}^{T} \varepsilon^b_t Y^{b,(m)\prime}_{t-1} \xrightarrow{d} \int_0^1 (dW(u)) \widetilde{W}_m(u)^{\prime} (C^{\prime} \otimes I_{m+1}) \tag{13}$$

$$\frac{1}{T} \sum_{t=1}^{T} \triangle Y^b_{t-l} Y^{b,(m)\prime}_{t-1} \xrightarrow{d} C \left(\int_0^1 (dW(u)) \widetilde{W}_m(u)^{\prime} (C^{\prime} \otimes I_{m+1}) \right) + M^b_l \tag{14}$$

$$\frac{1}{T^2} \sum_{t=1}^{T} Y^{b,(m)}_{t-1} Y^{b,(m)\prime}_{t-1} \xrightarrow{d} (C \otimes I_{m+1}) \left(\int_0^1 \widetilde{W}_m(u) \widetilde{W}_m(u)^{\prime} du \right) (C^{\prime} \otimes I_{m+1}), \tag{15}$$

where $W(u)$ is a k-variate standard Wiener process,

$$\widetilde{W}_m(u) = \left(W^{\prime}(u), \sqrt{2} \cos(\pi u) W^{\prime}(u), ..., \sqrt{2} \cos(m\pi u) W^{\prime}(u) \right)^{\prime},$$

the M^b_l's are $k \times k(m+1)$ non-random matrices and $C = \beta_{\perp} \left(\alpha^{\prime}_{\perp} \Gamma \beta_{\perp} \right)^{-1} \alpha^{\prime}_{\perp}$.

Proof: Let $u \in [0, 1]$, fix $j, h = 1, ..., m$ and let $l = 1, ..., p - 1$. For the first result, note that

$$\frac{1}{T} \sum_{t=1}^{T} P_{j,T}(t) \varepsilon^b_t Y^{b\prime}_{t-1} = \sqrt{2} \frac{1}{T} \sum_{t=1}^{T} \cos(j\pi(t - 0.5)/T) \varepsilon^b_t \sum_{i=p+1}^{t-1} \varepsilon^{b\prime}_i \widehat{C}^{\prime}$$

$$+ \sqrt{2} \frac{1}{T} \sum_{t=1}^{T} \cos(j\pi(t - 0.5)/T) \varepsilon^b_t \sqrt{T} R^{b\prime}_{t-1,T}.$$

From Bierens (1994), Lemma 9.6.3., page 200,

$$\frac{1}{T} \sum_{t=1}^{T} \cos(j\pi(t - 0.5)/T) \varepsilon^b_t \sum_{i=p+1}^{t-1} \varepsilon^{b\prime}_i$$

$$= \cos(j\pi(1 - 0.5/T)) \frac{1}{T} \sum_{t=1}^{T} \varepsilon^b_t \sum_{i=p+1}^{t-1} \varepsilon^{b\prime}_i + j\pi \int_0^1 \sin(j\pi(u - 0.5/T)) \left(\frac{1}{T} \sum_{t=1}^{\lfloor uT \rfloor} \varepsilon^b_t \sum_{i=p+1}^{t-1} \varepsilon^{b\prime}_i \right) du$$

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and

$$\frac{1}{T} \sum_{t=1}^{T} \cos\left(j\pi\left(t-0.5\right)/T\right) \varepsilon_t^b \sqrt{T} R_{t-1,T}^{b\prime}$$

$$= \cos\left(j\pi\left(1-0.5/T\right)\right) \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^b \sqrt{T} R_{t-1,T}^{b\prime} + j\pi \int_0^1 \sin\left(j\pi\left(u-0.5/T\right)\right) \left(\frac{1}{T} \sum_{t=1}^{\lfloor uT \rfloor} \varepsilon_t^b \sqrt{T} R_{t-1,T}^{b\prime}\right) du,$$

where the last term vanishes because for all $\eta > 0$,

$$P^*\left(\max_{p+1 \leq t \leq T} \left|\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor uT \rfloor} \varepsilon_t^b R_{t-1,T}^{b\prime}\right| > \eta\right) < P^*\left(\max_{p+1 \leq t \leq T} \left(\left|\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor uT \rfloor} \varepsilon_t^b\right| \left|R_{t-1,T}^{b\prime}\right|\right) > \eta\right)$$

$$< \left|\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor uT \rfloor} \varepsilon_t^b\right| . P^*\left(\max_{p+1 \leq t \leq T} \left|R_{t-1,T}^{b\prime}\right| > \eta\right) \xrightarrow{p} 0 \text{ as } T \to \infty.$$

From (12) and the continuous mapping theorem (CMT), the first term converges in distribution to

$$\cos\left(j\pi\right) \int_0^1 dW\left(u\right) W\left(u\right)' + j\pi \int_0^1 \sin\left(j\pi u\right) \left(\int_0^u dW\left(y\right) W\left(y\right)'\right) du$$

$$= \int_0^1 \left(dW\left(u\right)\right) \cos\left(j\pi u\right) W\left(u\right)',$$

via integration by parts (see Bierens and Martins (2010)). Then,

$$\frac{1}{T} \sum_{t=1}^{T} P_{j,T}\left(t\right) \varepsilon_t^b Y_{t-1}^{b\prime} \xrightarrow{d} \int_0^1 \left(dW\left(u\right)\right) \sqrt{2} \cos\left(j\pi u\right) W\left(u\right)' C' \text{ as } T \to \infty$$

due to convergence in probability of $\widehat{C}$ (see Bierens and Martins (2010) for the limiting properties of the pseudo MLE $\widehat{C}$ and the Proof of Lemma S2 in the Supplement to Swensen (2006), available at Econometrica's website, for why the CMT still holds under P^*) and the result follows.

For the second result,

$$\frac{1}{T} \sum_{t=1}^{T} P_{j,T}\left(t\right) \triangle Y_{t-l}^b Y_{t-1}^{b\prime} = \frac{1}{T} \sum_{t=1}^{T} P_{j,T}\left(t\right) \triangle Y_{t-l}^b \left(Y_{t-1}^{b\prime} - Y_{t-1-l}^{b\prime}\right) + \frac{1}{T} \sum_{t=1}^{T} P_{j,T}\left(t\right) \triangle Y_{t-l}^b Y_{t-1-l}^{b\prime}$$

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where

$$\frac{1}{T} \sum_{t=1}^{T} P_{j,T}(t) \triangle Y_{t-l}^{b} Y_{t-1-l}^{b'}$$

$$= \sqrt{2} \cos\left(j\pi\left(1 - 0.5/T\right)\right) \frac{1}{T} \sum_{t=1}^{T} \triangle Y_{t-l}^{b} Y_{t-1-l}^{b'}$$

$$+ \sqrt{2} j\pi \int_{0}^{1} \sin\left(j\pi\left(u - 0.5/T\right)\right) \left(\frac{1}{T} \sum_{t=1}^{\lfloor uT \rfloor} \triangle Y_{t-l}^{b} Y_{t-1-l}^{b'}\right) du.$$

Cavaliere, Rahbek and Taylor (2010a) show that, under standard cointegration,

$$\frac{1}{T} \sum_{t=1}^{\lfloor uT \rfloor} \triangle Y_{t-l}^{b} Y_{t-1-l}^{b'} \overset{d}{\to} C \left(\int_{0}^{u} dW(y) W(y)'\right) C' + u M_{0}^{b},$$

where M_{0}^{b} is the nonrandom $k \times k$ matrix that defines the limiting variance term $E^{*}\left(\left(\sqrt{T} R_{t,T}^{b}\right)\left(\sqrt{T} R_{t,T}^{b'}\right)\right)$. Hence, $\frac{1}{T} \sum_{t=1}^{T} \cos\left(j\pi\left((t - 0.5)/T\right)\right) \triangle Y_{t-l}^{b} Y_{t-1-l}^{b'}$ converges in distribution to

$$\cos\left(j\pi\right) \left(C \left(\int_{0}^{1} dW(u) W(u)'\right) C' + M_{0}^{b}\right)$$

$$+ j\pi \int_{0}^{1} \sin\left(j\pi u\right) \left(C \left(\int_{0}^{u} dW(y) W(y)'\right) C' + u M_{0}^{b}\right) du$$

$$= C \left(\int_{0}^{1} \left(dW(u)\right) \cos\left(j\pi u\right) W(u)'\right) C',$$

(see Bierens and Martins (2010) for the equality). On the other hand,

$$\frac{1}{T} \sum_{t=1}^{T} P_{j,T}(t) \triangle Y_{t-l}^{b} \left(Y_{t-1}^{b'} - Y_{t-1-l}^{b'}\right)$$

$$= \frac{1}{T} \sum_{t=1}^{T} P_{j,T}(t) \widehat{C} \varepsilon_{t-l}^{b} \varepsilon_{t-l}^{b'} \widehat{C}' + \frac{1}{T} \sum_{t=1}^{T} P_{j,T}(t) \left(\sqrt{T} R_{t-1-l,T}^{b}\right) \left(\sqrt{T} R_{t-1-l,T}^{b'}\right) + o_{p}(1)$$

$$\overset{p}{\to} M_{j,l}^{b}, \text{ say.}$$

Then,

$$\frac{1}{T} \sum_{t=1}^{T} P_{j,T}(t) \triangle Y_{t-l}^{b} Y_{t-1}^{b'} \overset{d}{\to} C \left(\int_{0}^{1} \left(dW(u)\right) \sqrt{2} \cos\left(j\pi u\right) W(u)'\right) C' + M_{j,l}^{b}$$

and the second result follows.

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For the last one, again, by Bierens (1994), Lemma 9.6.3.,

$$\frac{1}{T^2} \sum_{t=1}^{T} P_{h,T}(t) P_{j,T}(t) Y_{t-1}^b Y_{t-1}^{b'}$$

$$= \quad 2 \cos\left(h\pi\left(1 - 0.5/T\right)\right) \cos\left(j\pi\left(1 - 0.5/T\right)\right) \frac{1}{T^2} \sum_{t=1}^{T} Y_{t-1}^b Y_{t-1}^{b'}$$

$$-2 \int_0^1 \frac{d}{du} \cos\left(h\pi\left(u - 0.5/T\right)\right) \cos\left(j\pi\left(u - 0.5/T\right)\right) \left(\frac{1}{T^2} \sum_{t=1}^{\lfloor uT \rfloor} Y_{t-1}^b Y_{t-1}^{b'}\right) du.$$

From Cavaliere, Rahbek and Taylor (2010a),

$$\frac{1}{T^2} \sum_{t=1}^{\lfloor uT \rfloor} Y_{t-1}^b Y_{t-1}^{b'} \xrightarrow{d} C \left(\int_0^u W(y) W(y)' \, dy\right) C',$$

and the result holds given that

$$\frac{1}{T^2} \sum_{t=1}^{T} P_{h,T}(t) P_{j,T}(t) Y_{t-1}^b Y_{t-1}^{b'} \xrightarrow{d} 2C \left(\int_0^1 \cos\left(h\pi u\right) W(u) \cos\left(j\pi u\right) W(u)' \, du\right) C'.$$

(see Bierens and Martins (2010) for the integration by parts). $\blacksquare$

Lemma 2. *Let $\alpha_\perp$ and $\beta_\perp$ be the orthogonal complements of α and β, respectively. The following quantities have the same limiting laws as in Bierens and Martins (2010):*

$$\left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{b,(m)} \left(\beta_\perp' \otimes I_{m+1}\right), \tag{16}$$

$$\sqrt{T} \left(\alpha_\perp' \Omega \alpha_\perp\right)^{-1/2} \alpha_\perp' S_{01,T}^{b,(m)} \left(\beta \otimes I_{m+1}\right), \tag{17}$$

$$\left(\alpha' \Omega^{-1} \alpha\right)^{-1} \alpha' \Omega^{-1} S_{01,T}^{b,(m)} \left(\beta_\perp' \otimes I_{m+1}\right), \tag{18}$$

$$\left(\alpha' \Omega^{-1} \alpha\right)^{-1} \alpha' \Omega^{-1} S_{01,T}^{b,(m)} \left(\beta \otimes I_{m+1}\right), \tag{19}$$

$$\left(\beta' \otimes I_{m+1}, T^{-1/2} \beta_\perp' \otimes I_{m+1}\right) S_{10,T}^{b,(m)} S_{00,T}^{b,-1} S_{01,T}^{b,(m)} \left(\beta \otimes I_{m+1}, T^{-1/2} \beta_\perp' \otimes I_{m+1}\right) \text{ and} \tag{20}$$

$$\left(T^{-1/2} \beta_\perp' \otimes I_{m+1}, \beta' \otimes I_{m+1}\right) S_{11,T}^{b,(m)} \left(T^{-1/2} \beta_\perp' \otimes I_{m+1}, \beta \otimes I_{m+1}\right). \tag{21}$$

Proof: It follows from Lemma 1, given the definition of $S_{ij,T}^{b,(m)}, i,j = 1,2$ (see above) and the sequence of arguments in Bierens and Martins (2010). Notice that these results are related to Lemma A6 at Cavaliere, Rahbek and Taylor (2010a) and Lemma S2 at Swensen (2006) (for i.i.d. errors) where in their case $m = 0$. In our test, m being greater than zero implies deriving the limiting laws of processes that involve the Chebishev polynomials. That is provided by the previous Lemma 1. $\blacksquare$

Lemma 3. *Under the null hypothesis of standard cointegration and Assumption 1 and if* $\eta > 0$,

$$P^* \left(\left\| S_{00,T}^b - \Sigma_{XX} \right\| > \eta \right) \quad \rightarrow \quad 0 \tag{22}$$

$$P^* \left(\left\| S_{01,T}^{b,(m)} \widehat{\beta} - \Sigma_{X\beta} \right\| > \eta \right) \quad \rightarrow \quad 0 \tag{23}$$

$$P^* \left(\left\| \widehat{\beta}' S_{11,T}^{b,(m)} \widehat{\beta} - \Sigma_{\beta\beta} \right\| > \eta \right) \quad \rightarrow \quad 0 \tag{24}$$

in probability, where $\|\cdot\|$ *is the Euclidean norm and*

$$\Sigma_{XX} = p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^T X_t X_t'; \Sigma_{X\beta} = p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^T X_t Y_{t-1}'\beta$$

$$\Sigma_{\beta\beta} = p \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^T \beta' Y_{t-1} Y_{t-1}'\beta.$$

Proof: The results were shown by Cavaliere, Rahbek and Taylor (2010a) (Lemma A7) and Swensen (2006) (Lemma S1, for i.i.d.) for the case of $m = 0$. As in Lemma 2, the results of this Lemma for any $m > 0$ follow from Lemma 1. ∎

Lemma 4. *Under the null hypothesis of standard cointegration and Assumption 1, the* r *largest ordered solutions* $\widehat{\lambda}_{m,1}^b \geq \widehat{\lambda}_{m,2}^b \geq ... \geq \widehat{\lambda}_{m,r}^b$ *of the generalized eigenvalue problem*

$$\det \left[\lambda S_{11,T}^{b,(m)} - S_{10,T}^{b,(m)} S_{00,T}^{-1,b} S_{01,T}^{b,(m)} \right] = 0 \tag{25}$$

converge in probability to constants $1 > \overline{\lambda}_1 \geq ... \geq \overline{\lambda}_r > 0$, *which do not depend on* m. *Thus, these probability limits are the same as in the standard TI cointegration case:*

$$\det \left(\lambda \Sigma_{\beta\beta}^* - \Sigma_{\beta\beta}^* \left(\left(\alpha'\Omega^{-1}\alpha \right)^{-1} + \Sigma_{\beta\beta}^* \right)^{-1} \Sigma_{\beta\beta}^* \right) = 0, \tag{26}$$

where $\Sigma_{\beta\beta}^* = \Sigma_{\beta\beta} - \Sigma_{X\beta}' \Sigma_{XX}^{-1} \Sigma_{X\beta}$. *Moreover,*

$$T \left(\widehat{\lambda}_{m,r+1}^b, \widehat{\lambda}_{m,r+2}^b, ..., \widehat{\lambda}_{m,k}^b \right)' \overset{d}{\rightarrow} \left(\rho_{m,1}, ..., \rho_{m,k-r} \right)', \tag{27}$$

where $\rho_{m,1} \geq \rho_{m,2} \geq .... \geq \rho_{m,k-r}$ *are the* $k - r$ *largest solutions of the generalized eigenvalue problem*

$$\det \left[\rho \left(\begin{array}{cc} \int_0^1 \widetilde{W}_{k-r,m}(u)\widetilde{W}_{k-r,m}'(u)du & O_{(k-r)(m+1),r.m} \\ O_{r.m,(k-r)(m+1)} & I_{r.m} \end{array} \right) \right.$$
$$\left. - \left(\begin{array}{c} \int_0^1 \widetilde{W}_{k-r,m}(u)dW_{k-r}(u)' \\ V \end{array} \right) \left(\int_0^1 \left(dW_{k-r}(u) \right) \widetilde{W}_{k-r,m}(u)', V' \right) \right] = 0, \tag{28}$$

where $W_{k-r}(u)$ is a $k - r$ variate standard Wiener process,

$$\widetilde{W}_{k-r,m}(u) = \left((\alpha'_\perp \Omega \alpha_\perp)^{-1/2} \alpha'_\perp C_0 \otimes I_{m+1} \right) \widetilde{W}_m(u),$$

and with V an $r.m \times (k - r)$ random matrix with i.i.d. $N[0,1]$ elements.

Proof: It follows from applying the previous results to the correspondent Lemmas at Bierens and Martins (2010) (A7, A8 and A9). Just as in Bierens and Martins (2010), we need to define

$$\xi_{\perp,T} = \left(\beta_\perp \otimes I_{m+1}, \left(\begin{array}{c} O_{k,m.r} \\ \sqrt{T} \left(\beta \Sigma_{\beta\beta}^{-1/2} \otimes I_m \right) \end{array} \right) \right)$$

to prevent $T^{-1} \xi'_{\perp,T} S_{11,T}^{b,(m)} \xi_{\perp,T}$ from converging to a singular matrix (see Andersson et al., 1983). See Theorems 3 and 4 at Cavaliere, Rahbek and Taylor (2010a) and Proposition 1 at Swensen (2006) (i.i.d.) for the case of $m = 0$. ∎

Proof of Theorem 1. Following the same route as in Bierens and Martins (2010), the χ^2_{mkr} distribution of the bootstrap TVC test statistic follows from a combination of gaussian and χ^2 processes, as shown by Johansen (1995), once we apply the Taylor expansion of Johansen (1988) around the pseudo MLE to the pseudo LR statistic. Recall that the exact ML function (Bierens and Martins (2010)) is the same as the pseudo ML function once we assume Gaussian disturbances. To that end, and given the previous Lemmas, we state that the pseudo ML estimator of ξ with a bootstrapped sample, $\widehat{\xi}^b = \left(\widehat{\xi}_1^b, ..., \widehat{\xi}_r^b \right)$, where $\widehat{\xi}_i^b$, $i = 1, ..., r$, are the eigenvectors associated with the r largest eigenvalues $\widehat{\lambda}_{m,i}^b$,

$$S_{10,T}^{b,(m)} S_{00,T}^{-1,b} S_{01,T}^{b,(m)} \widehat{\xi}_i^b = \widehat{\lambda}_{m,i}^b S_{11,T}^{b,(m)} \widehat{\xi}_i^b, \ i = 1, .., r, \tag{29}$$

has the same limiting distribution as the one in Bierens and Martins (2010). More specifically, for the normalization $\widetilde{\xi}^b = \widehat{\xi}^b \left(\xi' \widehat{\xi}^b \right)^{-1} \xi' \xi$ with a partition $\widetilde{\xi}^b = \left(\widetilde{\xi}_0^{b\prime}, \widetilde{\xi}_m^{b\prime} \right)$, where $\widetilde{\xi}_m^b$ is a $k.m \times r$ matrix, and under the null hypothesis of standard cointegration,

$$T \left(\widetilde{\xi}_0^b - \beta \right) \xrightarrow{d} (\beta_\perp, O_{k,k.m}) \left(\int_0^1 \widetilde{W}_{k-r,m}(u) \widetilde{W}'_{k-r,m}(u) du \right)^{-1} \tag{30}$$

$$\times \int_0^1 \widetilde{W}_{k-r,m}(u) d\underline{W}_\alpha(u)' \left(\alpha' \Omega^{-1} \alpha \right)^{-1/2}$$

$$\sqrt{T} \widetilde{\xi}_m^b \xrightarrow{d} \left(\beta \Sigma_{\beta\beta}^{-1/2} \otimes I_m \right) \underline{V}_\alpha \left(\alpha' \Omega^{-1} \alpha \right)^{-1/2}, \tag{31}$$

where $\underline{W}_\alpha$ is an r-variate standard Wiener process, $\underline{V}_\alpha$ is a $k.m \times r$ matrix with independent $N[0,1]$ distributed elements, and $\underline{V}_\alpha$, $\underline{W}_\alpha$ and $\widetilde{W}_{k-r,m}$ are independent. The result

can be shown by generalizing Lemma A6 of Cavaliere, Rahbek and Taylor (2010a) and Lemma S2 of Swensen (2006) (Supplementary Material) (i.i.d.) to the case of $m > 0$. That can be done given our Lemma 1 that involves the Chebishev polynomials. See also page 9 of the Supplementary Material to Swensen (2006)'s paper for why the continuous mapping theorem still holds under P^*.∎

Bootstrap test for TVC using the restricted residuals   In the context of the LR cointegration rank test and associated sequential rank determination procedure of Johansen (1995), Cavaliere, Rahbek and Taylor (2012) show that important gains can be achieved once the residuals (and $\widehat{\mu}, \widehat{\Gamma}$) are obtained from the cointegration model under the null hypothesis. In particular, the bootstrap sequential rank determination procedure consistently determines the cointegration rank, in the sense that the probability of choosing a rank smaller than the true one will converge to zero.

As mentioned before, in our test for time varying cointegration, the cointegration rank, r, is fixed and assumed to be known and therefore the "sequential rank determination procedure" is not relevant in our framework. Still, we see no reason why using restricted residuals shouldn't work as well for bootstrap TVC testing and, thus, suggest it as an alternative to the unrestricted procedure defined above.

Let all estimated quantities be exclusively obtained from the standard time invariant cointegration model, $m = 0$: $\widehat{\alpha}, \widehat{\beta}, \widehat{\mu}, \widehat{\Gamma}$ and $\widehat{\varepsilon}_t, t = 1, ..., T$. Following Cavaliere, Rahbek and Taylor (2012), this alternative Wild and i.i.d. bootstrap procedure is the same as the one above but *(i)* the bootstrap pseudo-disturbances ε_t^b are obtained from the restricted residuals $\widehat{\varepsilon}_t$; and *(ii)* the bootstrap sample $\left\{ \triangle Y_t^b \right\}_{t=1}^{T}$ is constructed from the same equation with all estimated coefficients obtained under the null hypothesis. According to the Monte Carlo results shown in the next Section, this bootstrap procedure is surely also asymptotically correctly sized but, apparently, not applicable to some particular models. We suspect that the usual bootstrap procedures using the restricted residuals are not suitable for models with $r > 1$ under conditionally heteroskedastic errors and, thus, alternative methods need to be proposed in the literature.

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3 Monte Carlo Study

In this Section we illustrate the merits of the Bootstrap TVC tests by assessing their finite-sample size performance through numerical simulations. We consider three cointegration models that follow closely the literature and combine distinct values for the number of cointegrating vectors, the number of variables in the system and the VAR lag order.

3.1 The Designs

The data generating processes (DGP's) correspond to the standard cointegrated VAR's ($m = 0$) presented by Bierens and Martins (2010) (denote it by BM), Johansen (2002) and Swensen (2006) (JS) and Engle and Yoo (1987) (EY). In the BM model there is a single cointegrating relationship driving a bivariate system and where $p = 2$; The JS model is instead of dimension 3 and $p = 1$; and, relatively to JS, the EY model assumes that there are two cointegrating vectors. There are no deterministic components and the models parameters are:

$$\text{BM:} \quad \alpha = (-0.5, 0)'; \beta = (1, 1)'; \Gamma = \begin{pmatrix} 0.25 & 0 \\ 0 & 0 \end{pmatrix} \tag{32}$$

$$\text{JS:} \quad \alpha = (-0.4, -0.4, 0)'; \beta = (1, 0, 0)' \tag{33}$$

$$\text{EY:} \quad \alpha = \begin{pmatrix} -0.4 & 0.1 \\ 0.1 & 0.2 \\ 0.1 & 0.3 \end{pmatrix}; \beta = \begin{pmatrix} 1 & 1 \\ -2 & -1/2 \\ 1 & -1/2 \end{pmatrix}. \tag{34}$$

The errors ε_t are independent, Gaussian and with diagonal covariance matrix $\Omega = I_k$, except for EY where $\Omega = 100 I_k$. As in Cavaliere, Rahbek and Taylor (2010a), we also consider a fatter-tails case of independent errors ε_t that follow a t-distribution with five degrees of freedom and the conditional heteroskedasticity case $\varepsilon_{it} \sim h_{it}^{1/2} v_{it}, i = 1, ..., k$, where $v_{it} \sim i.i.d.N(0, 1)$, independent across i, and $h_{it} = 1 + 0.3\varepsilon_{it-1}^2 + 0.65 h_{it-1}, t = 1, ..., T$. These three distinct specifications for the errors ε_t are consistent to our Assumption 1.

The initial values are set equal to zero and the first fifty observations are dropped. All experiments are based on $10,000$ replicas and consider samples of size $T = 50, 100$ and 200. The original and bootstrap TVC tests are calculated for a wide range of Chebyshev time polynomials, with a maximum number of $T/10$ of them: $m = 1, 2, 5$ for $T = 50, m =$

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$1, 2, 5, 10$ for $T = 100$, and $m = 1, 2, 5, 10, 20$ for $T = 200$. Following Cavaliere, Rahbek and Taylor (2010a, 2012), for each replica, the number of bootstrap pseudo-samples B equals 399. For each procedure, we impose a nominal test size of 5%.

3.2   The Results

We seek to estimate the probability of rejecting the null hypothesis of TI cointegration when it is in fact true, for each of the three TVC tests described above and to observe how those results vary with m and T, besides p, r and k which change across DGP's. The Tables 1 (BM model), 2 (JS model) and 3 (EY model) report the finite-sample test-levels for each distributional approximation where tvc denotes the original TVC test against the chi-squared distribution and wb and sb refer to the wild and i.i.d. TVC tests, respectively, against the bootstrap distribution. The results based on the bootstrap procedure using the unrestricted residuals (and $\widehat{\Gamma}$) are presented at the right of label "UR" and those using the restricted ones at the right of "R". The results for tvc under t-distributed and heteroskedastic errors must be somehow read in careful under the original assumption of normality at Bierens and Martins (2010), although we conjecture that the result hold true when relaxing this distributional restriction.

The examination of the empirical properties of the original test illustrates the severity of the size distortions, especially for small T and large m, regardless of the DGP under consideration. For large m, the empirical size approaches the nominal one at the expense of a (much) higher number of observations. In contrast, the bootstrap tests provide in general near-exact levels for any combination of T or m. This fact is robust to any of the considered model specifications except for the conditional heteroskedasticity case with large m where real sizes of the i.i.d. bootstrap test are equal to about 10% or above. Regarding this particular setup, only the wild bootstrap test showed very reasonable results in general. This is somehow consistent with the findings at Cavaliere, Rahbek and Taylor (2010a).

In terms of the unrestricted and restricted approaches, we observe that the empirical size of the latter is always smaller than the former. For most cases, this implies having an empirical size (even) closer to the nominal 5% for the restricted procedure. But on the other hand, the Wild bootstrap under the restricted approach fails to reject the null too

often for m large and models JS and EY. Noticebly, the restricted procedure seems not to work for the EY model under conditionally heteroskedastic errors (contrary to models BM and JS, $r = 2$ at the EY model): the largest solution of the generalized eigenvalue problem under the null and/or under the alternative, $\widehat{\lambda}_{0,1}, \widehat{\lambda}_{m,1}$, is systematically larger than one for the simulated data, thus invalidating the calculation of the test statistic (recall that it must hold true that $1 > \widehat{\lambda}_{m,1} \geq \widehat{\lambda}_{0,1}$).

Hence, we advocate the usage of the bootstrap versions of the TVC test, namely the wild bootstrap using the unrestricted residuals, since these seem to be the only techniques that work well globally when testing for standard cointegration against TVC.

In a recent paper, Cavaliere, Taylor and Trenkler (2013) extend Swensen (2006) and Cavaliere, Rahbek and Taylor (2010a, 2012) by adapting Killian's (1998) bootstrap-after-bootstrap (BaB) framework to their bootstrap-based LR cointegration rank tests. More specifically, bias-corrected VECM parameter estimates, obtained from the bootstrap replications, are used instead to generate the pseudo-data and calculate the test statistic. Because Cavaliere, Taylor and Trenkler's standard cointegration model is the same as ours, we also implemented their algorithm to the TVC unrestricted bootstrap procedures in order to assess whether small sample gains can be obtained. The TVC BaB algorithm is the same as the one previously described with the following difference: The estimated $\widehat{\Gamma}'_j s$ used to obtain the bootstrap sample $\left\{\triangle Y_t^b\right\}_{t=1}^{T}$ in step two, and consequently the LR test statistic in step three, are replaced by the bootstrap bias-corrected estimates. That is,

$$\widehat{\Gamma}_j \equiv \widehat{\Gamma}_j^b = \widetilde{\Gamma}_j^b - \left(\frac{1}{B} \sum_{b=1}^{B} \widetilde{\Gamma}_j^b - \widetilde{\Gamma}_j^b \right), j = 1, ..., p - 1,$$

where $\widetilde{\Gamma}_j^b$ is the estimate based on the $b - th$ bootstrap sample. According to the Monte Carlo results for model BM, there seems not to exist gains by implementing the BaB procedure. In general, the empirical level increases slightly in both wild and i.i.d. bootstrap schemes and for all sample sizes. The only exceptions are: wb with $m = 1$ and $T = 50$ (drops from 5.9 to 5.7) and sb with $m = 10$ and $T = 100$ (a drop from 7.2 to 6.8).

Table 1: Empirical Sizes of Standard and Bootstrap TVC Tests for the BM model

BM		$m=1$			$m=2$			$m=5$			$m=10$			$m=20$		
T		tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$
						Gaussian										
50	UR	10.4	5.9	5.7	14.7	5.8	5.8	35.2	5.9	6.8	-	-	-	-	-	-
	R		5.1	5.8		5.1	5.6		5.5	6.3	-	-	-	-	-	-
100	UR	7.2	6.0	5.7	9.3	6.2	5.7	15.4	5.8	6.0	37.1	6.3	7.2	-	-	-
	R		5.7	5.6		5.9	5.4		5.3	5.4		6.0	6.2	-	-	-
200	UR	5.9	6.0	5.3	6.8	6.0	5.7	8.9	5.6	5.6	15.5	5.7	5.9	44.8	6.6	6.3
	R		5.5	5.5		5.3	6.0		4.9	5.5		5.2	5.4		5.6	4.7
						t_5										
50	UR	10.2	5.4	5.0	15.2	5.8	5.7	38.3	6.6	7.3	-	-	-	-	-	-
	R		5.1	4.7		5.3	5.2		5.8	6.2	-	-	-	-	-	-
100	UR	7.4	5.6	5.7	9.2	6.1	5.4	16.2	5.4	5.9	39.1	6.4	6.5	-	-	-
	R		4.6	6.3		5.2	5.4		4.5	5.5		5.7	5.4	-	-	-
200	UR	5.9	5.4	5.0	6.7	5.3	4.8	10.0	5.9	5.5	17.1	5.8	4.4	48.7	6.7	6.5
	R		5.1	4.9		5.1	4.8		5.8	5.3		5.5	4.2		6.2	4.5
						Heteroskedastic										
50	UR	2.3	6.5	6.8	18.3	6.8	7.5	43.6	8.8	11.5	-	-	-	-	-	-
	R		5.8	7.0		6.2	7.4		6.8	10.3	-	-	-	-	-	-
100	UR	10.1	5.9	7.1	13.0	6.0	7.9	22.9	7.0	10.0	48.7	9.5	13.9	-	-	-
	R		5.7	6.9		5.4	7.7		5.9	9.6		7.4	12.3	-	-	-
200	UR	11.2	6.2	9.4	13.4	6.5	10.5	18.7	6.8	12.0	28.3	7.8	13.1	61.3	10.5	18.6
	R		5.7	9.5		5.8	10.6		6.0	12.4		6.4	12.9		7.3	15.8

Table 2: Empirical Sizes of Standard and Bootstrap TVC Tests for the JS model

JS		m = 1			m = 2			m = 5			m = 10			m = 20		
T		tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$
Gaussian																
50	UR	15.6	5.6	5.9	27.4	5.9	7.0	74.0	6.1	11.0	-	-	-	-	-	-
	R		5.4	5.6		5.3	6.4		4.1	8.9	-	-	-	-	-	-
100	UR	9.0	6.1	5.4	12.9	5.4	5.6	31.1	5.0	7.4	77.5	4.9	10.4	-	-	-
	R		4.9	5.0		4.6	4.9		4.0	6.4		2.6	7.4	-	-	-
200	UR	7.0	5.6	5.4	8.4	5.4	5.5	14.6	4.9	5.8	32.8	3.9	6.9	86.2	4.4	10.1
	R		5.3	5.3		4.9	5.4		4.5	5.3		3.0	5.5		2.3	6.8
t_5																
50	UR	16.1	5.9	6.3	27.5	5.9	7.5	73.4	6.3	10.9	-	-	-	-	-	-
	R		5.0	5.8		4.7	6.6		3.8	8.3	-	-	-	-	-	-
100	UR	9.6	5.3	5.9	12.9	5.1	5.5	30.0	5.0	6.7	76.3	4.6	10.2	-	-	-
	R		4.8	5.8		4.2	5.7		4.0	5.8		3.0	7.1	-	-	-
200	UR	7.0	5.5	5.6	8.1	5.3	5.3	14.1	5.1	6.0	32.0	4.7	6.6	85.3	4.4	10.3
	R		5.2	5.3		5.0	5.4		4.3	5.4		3.3	5.6		1.8	6.3
Heteroskedastic																
50	UR	18.9	6.8	8.0	32.2	7.3	10.0	74.0	8.3	15.0	-	-	-	-	-	-
	R		6.4	7.5		6.0	9.8		5.6	12.1	-	-	-	-	-	-
100	UR	13.2	6.6	8.3	18.4	6.7	10.3	41.2	7.9	14.8	80.7	8.4	18.6	-	-	-
	R		6.4	8.3		6.1	9.7		5.3	13.7		4.8	15.2	-	-	-
200	UR	10.1	6.7	8.3	13.6	6.9	10.1	25.4	7.4	13.4	51.0	7.5	18.9	90.6	9.0	24.7
	R		6.2	8.0		6.1	10.1		5.9	13.2		5.0	17.5		4.1	19.9

Table 3: Empirical Sizes of Standard and Bootstrap TVC Tests for the EY model

EY		m = 1			m = 2			m = 5			m = 10			m = 20		
T		tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$
								Gaussian								
50	UR	10.9	5.7	4.7	17.7	5.0	5.2	57.2	3.9	6.4	-	-	-	-	-	-
	R		4.9	4.9		4.7	5.0		3.0	5.6	-	-	-	-	-	-
100	UR	7.3	5.7	5.1	9.5	5.8	5.1	20.0	4.5	5.4	59.7	3.1	6.0	-	-	-
	R		5.2	5.0		5.0	4.9		3.5	5.2		2.0	5.0	-	-	-
200	UR	5.6	5.2	5.2	6.0	5.3	5.2	9.7	5.1	4.9	21.8	5.0	5.3	71.7	2.8	6.3
	R		5.0	4.8		5.1	5.0		4.5	5.0		3.0	4.9		1.0	4.8
								t_5								
50	UR	11.3	5.8	5.5	17.6	5.6	5.0	56.6	3.7	6.2	-	-	-	-	-	-
	R		4.8	4.9		4.2	5.3		2.4	5.6	-	-	-	-	-	-
100	UR	7.4	5.8	5.4	9.8	5.6	5.2	19.9	4.7	5.5	59.6	3.1	6.1	-	-	-
	R		5.8	5.7		5.2	5.5		3.1	4.7		1.6	4.0	-	-	-
200	UR	6.1	5.8	5.0	7.1	5.7	5.6	10.9	5.9	5.4	22.3	4.4	5.9	72.8	2.7	6.4
	R		5.3	5.3		5.1	5.0		5.1	5.7		3.7	5.5		1.2	5.3
								Heteroskedastic								
50	UR	14.7	7.3	7.3	23.8	8.0	8.9	66.8	7.2	12.6	-	-	-	-	-	-
	R		n/a			n/a			n/a		-	-	-	-	-	-
100	UR	10.3	6.5	7.8	14.5	7.3	9.1	32.7	8.3	13.0	74.6	8.1	18.5	-	-	-
	R		n/a			n/a			n/a			n/a		-	-	-
200	UR	9.7	7.0	8.6	12.6	6.5	10.0	19.9	7.2	13.0	41.0	8.5	18.8	88.1	10.6	29.1
	R		n/a			n/a			n/a			n/a			n/a	

Note: n/a stands for "not available"

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4   Reassessing TV PPP

In Bierens and Martins (2010), the original TVC test statistic was applied to the purchasing power parity (PPP) hypothesis: In its absolute form, it means that the same bundle of goods, measured in real terms, should have the same value across countries. By taking the U.S.A. as the domestic country, Bierens and Martins (2010) concluded that price indices and nominal exchange rates are cointegrated, but in a TV fashion. Given the size distortions of the test and the accurate distribution approximation of the bootstrap versions of the test, in this Section, we reevaluate the PPP hypothesis using the same data as in Bierens and Martins (2010), but now computing the wild and i.i.d. bootstrap TVC test statistics as well, for both unrestricted (UR) and restricted (R) approaches.

4.1   PPP in the Context of TVC

The literature on the PPP hypothesis is recognizably vast and several reviews have been proposed (see, for example, Taylor and Taylor, 2004). In our notation, $Y_t = \left(\ln S_t^f, \ln P_t^n, \ln P_t^f \right)'$, where P_t^n and P_t^f are the price indices in the domestic and foreign economies, respectively, and S_t^f is the nominal exchange rate in home currency per unit of the foreign currency. Taking the symmetry and proportionality restrictions of the absolute version of PPP, $\beta' = (1, -1, 1)$, is not expected to hold in empirical work due to several aspects namely measurement errors of the price indices. Hence, the traditional empirical strategy assumes β to be unknown and estimates the deviation series from PPP, $e_t = \beta' Y_t$, under the Engle and Granger (1987) or Johansen's methodology, $\beta = (1, \beta_2, \beta_3)$. Under the assumption of no transactions costs, PPP requires that $\widehat{e}_t$ follows a stationary process.

Whether it be the existence of transaction costs, nontradable goods, or market structures with imperfect competition, it is highly unlikely that the equilibrium parity condition holds in its traditional representation. Due to the presence of such market frictions or measurement errors of the price indices in equilibrium models of real exchange rate determination, which may imply a nonlinear adjustment process in the PPP relationship, we test for the single constant cointegration hypothesis against our TVC specification. The short run deviations from the PPP due to shocks in the system are measured by

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$e_t = \beta'_t Y_t$, where β_t is an unknown deterministic function of time that is approximated by $\beta_t(m) = \sum_{i=0}^{m} \xi_i P_{i,T}(t)$, where the ξ_i's are the Fourier coefficients. Under the standard PPP hypothesis (time-invariant cointegration), $\xi_i = 0$ for all $i = 1, ..., m$, for a fixed m.

The way to departure from the traditional Engle and Granger and Johansen's approaches might not be consensual. To put it simple, depending on the underlying model specification and assumptions and the properties of the data, one may fit the PPP theory within an I(1) or I(2) framework; assume a linear or nonlinear type of cointegration model; and impose a set of coefficients that are either fixed or time-varying (threshold cointegration, smooth transition, markov-switching, and so forth).

For example, Falk and Wang (2003) considered the PPP relationship within an I(1) framework whereas Johansen et al (2010) and Frydman et al (2008) argue that it fits instead within the I(2) framework and thus making the standard approach possibly misleading. Based on rational expectations hypothesis sticky-price monetary models, the I(1) approach assume that nominal exchange rates and relative good prices are unit-root processes, while the real exchange rate is stationary (or a near-I(1) process). Instead, Johansen et al (2010) question these monetary models, specifying an I(2) model with piecewise linear trends where the *change* in real exchange rate is stationary but highly persistent and apply it to German-USA data in the period 1975-1999.[4] On the other hand, Hong and Phillips (2010) propose a RESET-type test for linear against nonlinear cointegration and applied it to the PPP theory using UK-USA, Mexico-USA, Canada-USA and Japan-USA data from 1971 to 2004. They found little evidence for a linear relationship, except for the Mexico-USA case.

An important branch of the PPP literature assumes a nonlinear adjustment process in the fixed PPP cointegration relationship. It is argued that due to the presence of transactions costs, the deviations from the PPP $e_t = \beta' Y_t$ is a nonlinear process that can very well be characterized in terms of a smooth transition autoregressive model (ESTAR model). In this type of models, regime changes occur gradually rather than suddenly and the speed of adjustment varies with the extent of the deviation from parity. Typically, the deviations from the PPP are obtained (i.e., estimated, $\widehat{e}_t = \widehat{\beta}' Y_t$) using the Engle

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[4]Michael et al. (1997) also found German prices and nominal exchange rates to be I(2) processes whereas non-German series are I(1). That is, German data seems to be a good empirical example where PPP holds within an I(2) approach.

and Granger (see Michael et al., 1997) or the Johansen's cointegration method (see Baum et al, 2001). The results provide strong evidence of mean-reverting behavior for PPP deviations and against the linear framework.

It is known that testing for a linear specification with time varying coefficients against a nonlinear model with fixed parameters, or selecting the best out of the two, is not an easy task. Once we believe in Clive W. J. Granger's assertion that "any non-linear model can be approximated by a time-varying parameter linear model" (Granger, 2008), we cannot reject a priori the relevance of a specification such as the one we are considering in the paper.

As it just so happens with the nonlinear adjustment specification, our model also assumes PPP cointegration in a nonstandard fashion, including the smooth transitions. In particular, the TVECM is able to empirically assess the strongest assumption in PPP theory: The single cointegration vector being of the form $\beta = (1, -1, 1)$ and, correspondingly, the real exchange rate a stationary process. This absolute version of the PPP hypothesis occurs if the null hypothesis of our TVC test is not rejected and, furthermore, the restrictions are also not rejected. The bootstrap TVC test is shown to be a "good" statistical tool to see if those changes around a constant β are significant or not.

Cheung and Lai (1993) claim that, due to transaction costs and measurement errors in prices, if e_t is stationary and $(\beta_2 < 0, \beta_3 > 0)$, PPP holds. The Appendix to Bierens and Martins' paper includes the plots of the estimated $\widehat{\beta}_t$. There, one can see that, in general, $\widehat{\beta}_t$ seems to fluctuate around $(1, -1, 1)$ and, in particular, $\beta_{t2} < 0, \beta_{t3} > 0$ for most t. From an economic point of view, this means that the presence of market frictions and/or measurement errors of the price indices is the cause of time-varying adjustments on the relative importance of each variable in Y_t (nominal exchange rates and prices) to guarantee stationary PPP deviations.[5] That is, contrary to the former two-stage approach (obtain $\widehat{e}_t = \widehat{\beta}' Y_t$ and then fit a STAR model to it), at the TVECM, the PPP equilibrium is directly restored via the cointegration vector, $\widehat{\beta}_t$.

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[5] As a simple example, $\beta_t = \left(1, -1 + \frac{\cos(t)}{T}, 1\right), t = 1, ..., T$, fluctuates very closely to $(1, -1, 1)$ and satisfies $\beta_{t2} < 0, \beta_{t3} > 0$. In this case, slightly smaller importance in relative prices is given to the domestic economy.

4.2 Empirical Results

The data we use to illustrate the usefulness of the bootstrap TVC tests in empirical work is the same as in Falk and Wang (2003), downloaded from the Journal of Applied Econometrics data archives web site. For this particular dataset, they put the PPP relationship within an I(1) framework.[6] The U.S.A. bilateral relationship of study is with the U.K., Japan, Canada, France, Italy, Germany, Belgium, Denmark, the Netherlands, Norway, Spain and Sweden. The data are monthly comprising 324 observations and cover the period from January 1973 to December 1999.

In this empirical application where a drift is included, $k = 3$ and $r = 1$, we consider m ranging from 1 to $32 = \lfloor T/10 \rfloor$. For the bootstrap tests, $B = 399$ and the initial values $Y_t^b, t = -p + 1, ..., 0$, are set equal to the first observation in the sample, Y_1. Just as in Bierens and Martins (2010), the admissible values for the lag order include $p = 1, 6, 10$ and 18. Based on the Hannan-Quinn information criterion (HQ, Hannan and Quinn, 1979), we took $p = 1$ which, for all reasonable m and countries, becomes the selected model over $p = 6, 10, 18$.[7]

The results are presented in Table 2. The column labeled "m_{HQ}^*" indicates the value for m that is selected according to the HQ criterion. At its right, we find results for $1\%, 5\%$ and 10% levels (columns "pvalue<...") for each one of the three tests (tvc, wb and sb). For a given test and significance level, each number corresponds to an integer $\widetilde{m}$ such that one cannot reject the standard PPP hypothesis for all $m < \widetilde{m}$. Also, one can find evidence for TV PPP whenever $m \geq \widetilde{m}$. The pvalues are not always monotonic with respect to m. It is by comparing "m_{HQ}^*" to the $\widetilde{m}'s$, at its right, that we draw conclusions

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[6]We also computed several unit root tests (including some more recent ones not in Falk and Wang, 2003) and confirmed that nominal exchange rates are clearly I(1) and for the case of the price indices some evidence in favor of I(1) is found as well. Results available upon request.

[7]Apparently, in our TVECM, the HQ criterion tends to pick very large models and, consequently, reducing drastically the degrees of freedom, as it happens with the AIC information criterion. In particular, when m is relatively large, HQ chooses models with large values for p. For example, in the UK case HQ selects $p = 1$ rather than $p = 6$ for all m; rather than $p = 10$ for all $m < 23$; and rather than $p = 18$ for all $m < 26$. On the contrary, BIC always picks extremely small models, $p = 1, m = 0$, which becomes non-informative when testing for TVC. Therefore, to prevent from having models with $m = 0$ (BIC) or m close to $\lfloor T/10 \rfloor$ and $p = 18$ (huge loss of degrees of freedom), we set $p = 1$ and adopted the HQ criterion.

about the PPP hypothesis, for each bilateral relationship.

For the U.K., we find some evidence of a standard (TI) type of cointegration between U.K. and U.S.A. prices and the nominal exchange rate. According to Table 4 and for the bootstrap tests at a 1% level, $\widetilde{m} > m^*_{HQ}$ meaning that there is a wide range of possible values for m such that one cannot reject the null hypothesis, including m that corresponds to the selected model, according to HQ. At a 5% level, $\widetilde{m}$ is basically equal to m^*_{HQ}. This conclusion is, nevertheless, not shared by the remaining bilateral relationships where a TV cointegrating system is more likely to be true. That is, for all other countries, for any test and significance level, $\widetilde{m}$ is too small when compared to m^*_{HQ}.

This empirical exercise shows the relevance of computing the bootstrap versions of the TVC test. It seems that the U.S.A. and the U.K. have kept a constant equilibrium relationship of prices and nominal exchange rates during the last quarter of the 20th century.[8] This is contradicted by the original TVC testing procedure.

5   Conclusion

In this paper we have considered two alternative bootstrap algorithms to the time-varying cointegration test proposed by Bierens and Martins (2010), based on a VECM specification where the cointegration vector changes smoothly over time. The original likelihood ratio test and its wild and i.i.d. bootstrap versions have the same first-order asymptotic distribution under the null hypothesis of standard/time-invariant cointegration. According to some extensive Monte Carlo simulations, and contrary to what happens with the original test statistic, the bootstrap procedures did not show severe size distortions. That is, the bootstrap approximation to the finite-sample distribution can be considered very accurate, especially for the wild bootstrap case. We have applied the tests to the purchasing power parity hypothesis of international prices and nominal exchange rates with the U.S. as the home economy, and found evidence of standard cointegration in the U.S.A.-U.K. bilateral relationship and time-varying cointegration in the remaining eleven cases.

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[8]In the standard Johansen's cointegration context, the null hypothesis of symmetry and proportionality restrictions $\beta = (1, -1, 1)$ is clearly rejected and the point estimate is $\widehat{\beta} = (1, -1.413, 0.747)$.

Table 4: Standard and Bootstrap TVC Tests Applied to the PPP Hypothesis

	m^*_{HQ}		$pvalue < 0.1$			$pvalue < 0.05$			$pvalue < 0.01$		
			tvc	wb	$iidb$	tvc	wb	$iidb$	tvc	wb	$iidb$
U.K.	10	UR	3	9	10	3	9	10	3	12	14
		R		10	10		12	10		12	14
Japan	15	UR	3	5	5	3	5	5	5	6	5
		R		5	5		5	5		6	5
Canada	15	UR	1	8	1	1	8	1	1	9	8
		R		8	1		9	1		12	8
France	15	UR	1	1	1	1	1	1	1	6	1
		R		1	1		1	1		7	1
Italy	16	UR	1	10	1	1	10	1	1	10	10
		R		10	1		10	1		12	10
Germany	16	UR	2	9	9	2	9	9	2	10	10
		R		15	8		15	9		16	10
Belgium	18	UR	1	6	6	3	7	7	5	8	8
		R		8	6		8	6		14	7
Denmark	17	UR	1	5	1	1	5	1	1	6	6
		R		7	1		8	1		8	6
Netherlands	12	UR	1	5	1	1	5	1	1	5	1
		R		6	1		6	1		6	1
Norway	16	UR	1	7	11	1	10	11	1	11	12
		R		12	11		13	11		13	13
Spain	16	UR	1	5	1	1	5	1	1	8	1
		R		1	1		6	1		9	2
Sweden	23	UR	2	7	14	3	9	14	4	11	15
		R		15	14		15	15		16	15

Note: Each entry below columns "pvalue<..." represents a value for m (denote it as $\widetilde{m}$), such that for all $m < \widetilde{m}$ one cannot reject the null hypothesis. Similarly, reject for all integers $m \geq \widetilde{m}$.

The simplicity of application of the bootstrap TVC tests and their good performance in finite-samples make the procedures discussed in this paper a valuable tool when addressing the possibility for smooth time-transitions of the equilibrium relationship in several other examples of cointegrated variables. It is important to notice that the LR test setup is conditional on the existence of cointegration. An interesting topic that deserves further attention is how to test for "spurious" regression in our time-varying framework. The work by Park and Hahn (1999) in single-equation time-varying cointegration can be helpful in this respect.

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